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INFORMATION

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ABSTRACT

This study explores an advanced three-parameter generalized log-logistic (GLL) model by incorporating an innovative shape parameter into the conventional log-logistic framework, allowing for greater flexibility in modeling data with increasing, decreasing, and bathtub-shaped failure rates. The estimation challenge under adaptive progressively Type-II censored data is addressed through both classical (likelihood-based) and Bayesian inferential methods. Utilizing observed Fisher information, asymptotic confidence intervals for unknown parameters are derived, while a Markov chain in the Monte Carlo approach is employed to obtain Bayesian point estimates and the highest posterior density intervals. In Bayes' setup, the GLL parameters are presumed to have independent gamma conjugate priors against various symmetric and asymmetric losses. To examine the accuracy of the acquired estimators, an extensive Monte Carlo simulation is used. Four real-life data sets from the physical and biomedical industries are analyzed to illustrate the feasibility of the proposed techniques in an actual-world scenario. Numerical analyses revealed that the suggested model outperforms the other five models in the literature, including the alpha-power exponential, exponentiated exponential, log-logistic, Weibull, and gamma distributions. The findings emphasize the effectiveness of the Bayesian Markov chain Monte Carlo approach over frequentist techniques, reinforcing its practical significance in reliability analysis and survival studies.

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1 Introduction

Recently, Muse et al. [1] created a new three-parameter generalized log-logistic (GLL) distribution by introducing an additional new feature (parameter) to the traditional log-logistic distribution. They demonstrated that the GLL distribution can capture a wide variety of hazard function

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shapes–including increasing, decreasing, bathtub-shaped, and unimodal–which many existing distributions struggle to accommodate simultaneously. They also stated that the GLL model provides greater modeling flexibility for real-life survival data and outperforms several existing lifetime distributions, including log-logistic, exponential, bib28, and Burr-XII distributions. Its adaptability makes it especially useful in practical fields like biomedical research and reliability analysis, where capturing complex hazard behaviors is essential for accurate inference and prediction. Suppose a lifetime random variable X follows the GLL distribution, denoted by GLL($\boldsymbol{\xi}$), where $\boldsymbol{\xi} = (\gamma, \delta, \mu)^{\top}$ for $\gamma, \delta, \mu > 0$. Then, its respective probability density function (PDF), $f(\cdot)$, cumulative distribution function (CDF), $F(\cdot)$, are

$$f(x; \boldsymbol{\xi}) = \gamma\delta(\delta x)^{\gamma-1}[1 + (\mu x)^{\gamma}]^{-(\frac{\delta}{\mu})^{\gamma}-1}, \qquad x > 0, \tag{1}$$

and

$$F(x; \boldsymbol{\xi}) = 1 - [1 + (\mu x)^{\gamma}]^{-(\frac{\delta}{\mu})^{\gamma}}, \tag{2}$$

and its reliability function (RF), $R(\cdot)$, and hazard rate function (HRF), $h(\cdot)$ at time $t > 0$ are defined, respectively, by

$$R(t; \boldsymbol{\xi}) = [1 + (\mu t)^{\gamma}]^{-(\frac{\delta}{\mu})^{\gamma}}, \qquad t > 0, \tag{3}$$

and

$$h(t; \boldsymbol{\xi}) = \frac{\gamma\delta(\delta t)^{\gamma-1}}{[1 + (\mu t)^{\gamma}]}. \tag{4}$$

From (4), the HRF increases (for $\delta^{-1}(\gamma - 1)^{1/\gamma}$, decreases (for $0 < \gamma \leq 1$) and unimodal (for $\gamma > 1$). Fig. 1 exhibits that the PDF (1) has a unimodal shape, while the HRF (4) has a decreasing, increasing, or bathtub shape. Muse et al. [1] also stated that the GLL distribution can be utilized as an extension of the next models:

- Standard log-logistic (when $\delta = \mu = 1$);
- Log-logistic (when $\delta = \mu$);
- Exponential (when $\mu^{\gamma} \to 0$ and $\gamma = 1$);
- Weibull (when $\mu^{\gamma} \to 0$);
- Burr-XII (when $\mu = \delta\theta^{-1/\gamma}$, $\theta > 0$).

Life-testing examinations in the reliability field frequently end before all of the components fail due to time limitations or a shortage of financial resources. The failures that come from these conditions are referred to as censored data. Ng et al. [2] proposed a new censored mechanism called the adaptive progressive-censored Type-II (APC-II) strategy. It has also gotten a lot of focus from a variety of practitioners since it enables the implementation of highly powerful statistical studies. This approach places n independent experimental item units on a test at time zero. According to this process, the number of failures m is decided beforehand, and the examination period may surpass the time T. We also have the removal design $\mathbf{R} = (R_1, R_2, \ldots, R_m)$, but its framework may change as a result of the investigation. If $X_{m:m:m}$ before time T, the test ends at $X_{m:m:m}$. Otherwise, we reassign the removal design as $R_{d+1} = R_{d+2} = \cdots = R_{m-1} = 0$, where d is the size of observed items before T. At mth failure, all remaining units (say, $R_m = n - m - \sum_{i=1}^{d} R_i$) are removed, and the test is ended. Graphically, a schematic representation of the APC-II is depicted in Fig. 2.

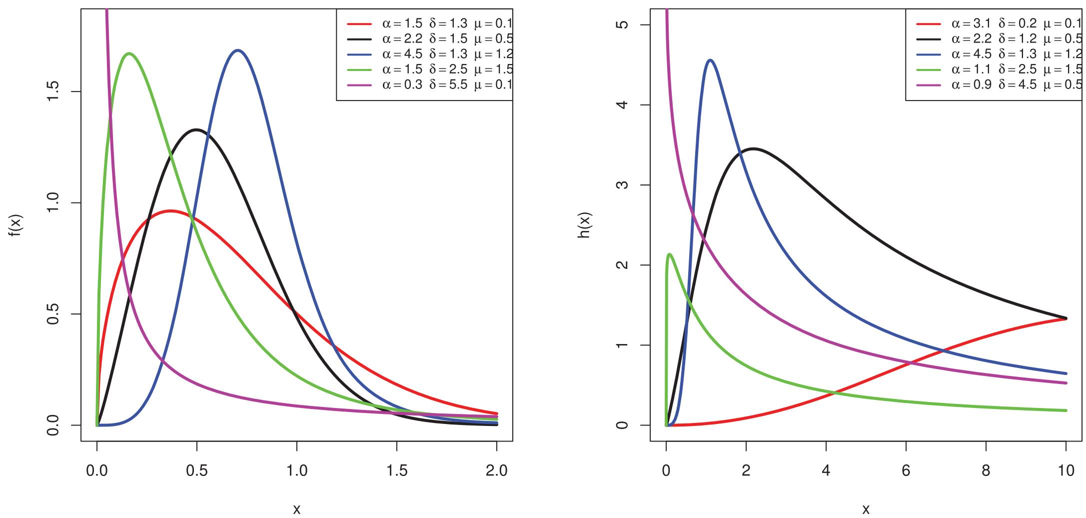


Figure 1: The PDF (left) and HRF (right) shapes of the GLL model

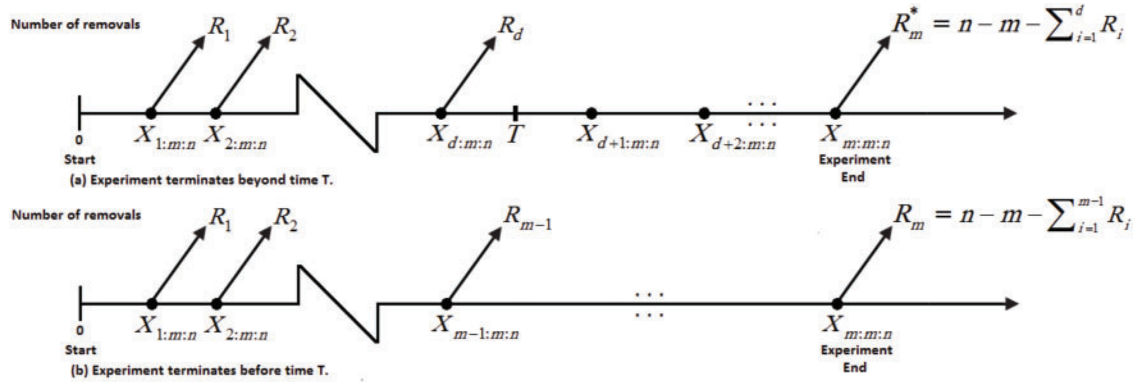


Figure 2: Schematic diagram of the APC-II plan

However, suppose $\mathbf{x} = \{(X_{1:m:n}, R_1), \ldots, (X_{d:m:n}, R_d), (X_{d+1:m:n}, 0), \ldots, (X_{m:m:n}, R_m)\}$ denotes APC-II order statistics from a continuous lifetime population, then the joint PDF of $\mathbf{x}$ is

$$\mathcal{L}(\boldsymbol{\xi}) = Q \prod_{i=1}^{m} f(x_{i:m:n}; \boldsymbol{\theta}) \prod_{i=1}^{d} [R(x_{i:m:n}; \boldsymbol{\theta})]^{R_i} [R(x_{m:m:n}; \boldsymbol{\theta})]^{R_m}, \tag{5}$$

where Q is an independent constant of the parameter vector $\boldsymbol{\theta}$. Based on (5), readers can refer to numerous inferential examinations of various distributions, for example, see Panahi and Moradi [3], Panahi and Asadi [4], Elshahhat and Nassar [5], Alotaibi et al. [6], Ateya et al. [7], Elshahhat et al. [8], Nassar et al. [9], Elshahhat et al. [10], Dutta et al. [11], Dutta et al. [12], among many others.

Also keep in mind that most GLL lifespan model inference studies have only evaluated complete sampling and have not dealt with incomplete (censored) sampling. As a result, the work's novelty derives from the fact that it is the first time that estimation challenges of GLL's parameters or reliability characteristics have been investigated using likelihood and Bayes inferential approaches using APC-II data. The motivation for this study arises from the proposed censoring mechanism's usefulness in improving the efficacy of statistical interpretation when compared to other censored techniques. Even so, we feel inspired to perform this type of work for two separate reasons: (i) The failure rate (4) has a bathtub, decreasing or increasing shape, which is more effective in many practical fields; (ii) The mechanism (5) serves a purpose in practical reliability examinations because it provides portability in terminating research at a certain point in time and dropping the total time of the examination. As far as we know, no offer is taking on the inferential challenges of the GLL parameters, particularly in reliability theory. Therefore, we mainly employ an APC-II strategy to close this gap by providing five objectives in the current study, such as:

1. via maximum likelihood (ML) and Bayes' infer approaches, the problem of estimating γ, δ, μ, $R(t)$, and $h(t)$ of the GLL distribution from an APC-II strategy is investigated.
2. The Bayes estimates against squared-error (SE) and general-entropy (GE) losses are assessed via Markov chain Monte Carlo (MCMC) techniques from independent gamma priors.
3. Two bounds of asymptotic confidence interval (ACI) as well as highest posterior density (HPD) interval estimates of the same unknown quantities are also obtained.
4. All theoretical results of γ, δ, or μ cannot be represented in explicit mathematical expressions, thus via R software, we recommend two programming packages, namely: 'maxLik' and 'coda' to evaluate all offered estimates.
5. Four real-world data set-based applications, consisting of (1) time-to-failure of the turbocharger of the same type of engine; (2) failure times of mechanical components; (3) relief time for arthritis patients; and (4) lifetimes of guinea pigs, highlight the GLL distribution's ability to accommodate a wide range of data types and adjust the presented approaches to actual situations.

The next sections are designed as: Sections 2 and 3 provide the likelihood and Bayes' inferences, respectively. Simulation findings are available in Section 4. Real applications are examined in Section 5. Lastly, Section 6 reports several concluding comments.

2 Likelihood Inference

Using the ML estimation technique, this section looks for point and interval estimates of γ, δ, μ, $R(t)$, and $h(t)$. Setting $x_i = x_{i:m:n}$ (for brevity), the likelihood (5) function becomes

$$\mathcal{L}(\boldsymbol{\xi}) \propto (\gamma\delta^{\gamma})^{m} \prod_{i=1}^{m} x_i^{\gamma} \left[\vartheta(x_i;\gamma,\mu)\right]^{-\left(\frac{\delta}{\mu}\right)^{\gamma}-1} \prod_{i=1}^{d} \left[\vartheta(x_i;\gamma,\mu)\right]^{-R_i\left(\frac{\delta}{\mu}\right)^{\gamma}} \left[\vartheta(x_m;\gamma,\mu)\right]^{-R_m^{*}\left(\frac{\delta}{\mu}\right)^{\gamma}}, \tag{6}$$

and its log-likelihood, $\mathcal{L}^{*}(\cdot) \propto \log \mathcal{L}(\cdot)$, becomes

$$\begin{aligned}\mathcal{L}^{*}(\boldsymbol{\xi}) &\propto m\log(\gamma\delta^{\gamma}) + \gamma\sum_{i=1}^{m}\log(x_i) - \sum_{i=1}^{m}\log(\vartheta(x_i;\gamma,\mu)) \\ &\quad - \left(\frac{\delta}{\mu}\right)^{\gamma}\left\{\sum_{i=1}^{m}\log(\vartheta(x_i;\gamma,\mu)) + \sum_{i=1}^{d}R_i\log(\vartheta(x_i;\gamma,\mu)) + R_m^{*}\log(\vartheta(x_m;\gamma,\mu))\right\},\end{aligned} \tag{7}$$

where $\vartheta(x_i;\gamma,\mu) = 1 + (\mu x_i)^{\gamma}$.

From (7), we get three normal-expressions of γ, δ, and μ, respectively, as

$$\frac{\partial\mathcal{L}^*}{\partial\gamma} = m\gamma^{-1}\left[1+\gamma\log(\delta)\right] + \sum_{i=1}^{m}\log(x_i) - \sum_{i=1}^{m}\frac{\vartheta^\circ(x_i;\gamma,\mu)}{\vartheta(x_i;\gamma,\mu)}$$
$$-\left(\frac{\delta}{\mu}\right)^{\gamma}\left\{\sum_{i=1}^{m}\frac{\vartheta^\circ(x_i;\gamma,\mu)}{\vartheta(x_i;\gamma,\mu)} + \sum_{i=1}^{d}R_i\frac{\vartheta^\circ(x_i;\gamma,\mu)}{\vartheta(x_i;\gamma,\mu)} + R_m^*\frac{\vartheta^\circ(x_m;\gamma,\mu)}{\vartheta(x_m;\gamma,\mu)}\right\}$$
$$-\left(\frac{\delta}{\mu}\right)^{\gamma}\left\{\sum_{i=1}^{m}\log(\vartheta(x_i;\gamma,\mu)) + \sum_{i=1}^{d}R_i\log(\vartheta(x_i;\gamma,\mu)) + R_m^*\log(\vartheta(x_m;\gamma,\mu))\right\}\log\left(\frac{\delta}{\mu}\right), \quad (8)$$

$$\frac{\partial\mathcal{L}^*}{\partial\delta} = m\delta^{-1}$$
$$-\gamma\left(\frac{\delta}{\mu}\right)^{\gamma-1}\left\{\sum_{i=1}^{m}\log(\vartheta(x_i;\gamma,\mu)) + \sum_{i=1}^{d}R_i\log(\vartheta(x_i;\gamma,\mu)) + R_m^*\log(\vartheta(x_m;\gamma,\mu))\right\}, \quad (9)$$

and

$$\frac{\partial\mathcal{L}^*}{\partial\mu} = -\sum_{i=1}^{m}\frac{\vartheta^\bullet(x_i;\gamma,\mu)}{\vartheta(x_i;\gamma,\mu)}$$
$$-\left(\frac{\delta}{\mu}\right)^{\gamma}\left\{\sum_{i=1}^{m}\frac{\vartheta^\bullet(x_i;\gamma,\mu)}{\vartheta(x_i;\gamma,\mu)} + \sum_{i=1}^{d}R_i\frac{\vartheta^\bullet(x_i;\gamma,\mu)}{\vartheta(x_i;\gamma,\mu)} + R_m^*\frac{\vartheta^\bullet(x_m;\gamma,\mu)}{\vartheta(x_m;\gamma,\mu)}\right\}$$
$$+\left(\frac{\delta}{\mu}\right)^{\gamma}\left\{\sum_{i=1}^{m}\log(\vartheta(x_i;\gamma,\mu)) + \sum_{i=1}^{d}R_i\log(\vartheta(x_i;\gamma,\mu)) + R_m^*\log(\vartheta(x_m;\gamma,\mu))\right\}\frac{\gamma}{\mu}, \quad (10)$$

respectively, where $\vartheta^\circ(x_i;\gamma,\mu) = (\mu x_i)^\gamma\log(\mu x_i)$ and $\vartheta^\bullet(x_i;\gamma,\mu) = \gamma x_i(\mu x_i)^{\gamma-1}$.

Due to the fact that the algebraic solutions of (8)–(10) cannot be determined explicitly, we propose employing the Newton-Raphson technique to derive the maximum likelihood estimators (MLEs) of γ, δ, and μ denoted by $\hat{\gamma}$, $\hat{\delta}$, and $\hat{\mu}$, respectively.

Consequently, the offered MLEs $\hat{\gamma}$, $\hat{\delta}$, and $\hat{\mu}$ are substituted in $R(t)$ and $h(t)$ to get the respective MLEs $\hat{R}(t)$ and $\hat{h}(t)$ as

$$\hat{R}(t;\hat{\boldsymbol{\xi}}) = \left[1+(\hat{\mu}t)^{\hat{\gamma}}\right]^{-\left(\frac{\hat{\delta}}{\hat{\mu}}\right)^{\gamma}}, \quad (11)$$

and

$$\hat{h}(t;\hat{\boldsymbol{\xi}}) = \frac{\hat{\gamma}\hat{\delta}(\hat{\delta}t)^{\hat{\gamma}-1}}{\left[1+(\hat{\mu}t)^{\hat{\gamma}}\right]}. \quad (12)$$

To get the $100(1-\varrho)\%$ ACI γ, δ, μ, $R(t)$ or $h(t)$, the 3×3 asymptotic variances and covariances matrix (say $\Sigma(\boldsymbol{\xi})$) must be obtained first. As stated by Lawless [13], under certain regularity conditions, the MLEs $\hat{\boldsymbol{\xi}}$ are asymptotically normally distributed. These conditions include: (i) the true parameter lying in the interior of the parameter space; (ii) the likelihood function being differentiable for the parameters; (iii) the information matrix being non-singular and well-behaved; and (iv) the sample size being sufficiently large. Hence, according to these conditions, the distribution of the MLEs $\hat{\boldsymbol{\xi}}$ is approximately multivariate normal with a mean $\boldsymbol{\xi}$ and variance-covariance matrix $\Sigma(\boldsymbol{\xi})$, where

$$\widehat{\Sigma}(\hat{\boldsymbol{\xi}}) = \begin{bmatrix} \mathcal{L}^*_{11} & \mathcal{L}^*_{12} & \mathcal{L}^*_{13} \\ & \mathcal{L}^*_{22} & \mathcal{L}^*_{23} \\ & & \mathcal{L}^*_{33} \end{bmatrix}^{-1}_{(\hat{\gamma},\hat{\delta},\hat{\mu})} = \begin{bmatrix} \widehat{var}(\hat{\gamma}) & \widehat{cov}(\hat{\gamma},\hat{\delta}) & \widehat{cov}(\hat{\gamma},\hat{\mu}) \\ & \widehat{var}(\hat{\delta}) & \widehat{cov}(\hat{\delta},\hat{\mu}) \\ & & \widehat{var}(\hat{\mu}) \end{bmatrix}, \tag{13}$$

where in Appendix A, the components $\mathcal{L}^*_{ij}$ for $i, j = 1, 2, 3$ are presented.

To justify the asymptotic normality of the MLEs $\hat{\boldsymbol{\xi}}$, depending on the regularity conditions, the score vector is asymptotically normal, we have

$$\sqrt{n}(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}_0) \approx -\left[\frac{1}{n}\frac{\partial^2 \mathcal{L}^*}{\partial \boldsymbol{\xi}\partial \boldsymbol{\xi}^\top}\right]^{-1}\left(\frac{1}{\sqrt{n}}\frac{\partial \mathcal{L}^*}{\partial \boldsymbol{\xi}}\right), \tag{14}$$

and

$$-\frac{1}{n}\frac{\partial^2 \mathcal{L}^*}{\partial \boldsymbol{\xi}\partial \boldsymbol{\xi}^T} \xrightarrow{p} \Sigma^{-1}(\boldsymbol{\xi}_0), \tag{15}$$

where $\boldsymbol{\xi}_0$ is the true parameter value.

Consequently, from (14) and (15), we get

$$\sqrt{n}(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}_0) \xrightarrow{d} \mathcal{N}(0, \Sigma(\boldsymbol{\xi}_0)),$$

or equivalently,

$$\hat{\boldsymbol{\xi}} \approx \mathcal{N}\left(\boldsymbol{\xi}_0, \frac{1}{n}\Sigma(\boldsymbol{\xi}_0)\right).$$

Hence, the $100(1-\varrho)\%$ ACIs of γ, δ, and μ are

$$\left(\hat{\gamma} \mp z_{\frac{\varrho}{2}}\sqrt{\widehat{var}(\hat{\gamma})}\right), \left(\hat{\delta} \mp z_{\frac{\varrho}{2}}\sqrt{\widehat{var}(\hat{\delta})}\right) \text{ and } \left(\hat{\mu} \mp z_{\frac{\varrho}{2}}\sqrt{\widehat{var}(\hat{\mu})}\right),$$

receptively, where $z_{\frac{\varrho}{2}}$ is the $(\varrho/2)$th standard Gaussian point.

On the other hand, the $100(1-\varrho)\%$ ACIs of $R(t)$ and $h(t)$ are acquired by approximating the variance estimate of $\hat{R}(t)$ and $\hat{h}(t)$ using the delta methodology, see Greene [14]. Thus, at $\boldsymbol{\xi} = \hat{\boldsymbol{\xi}}$, the variances of $\hat{R}(t)$ and $\hat{h}(t)$ (denoted by $\widehat{var}(\hat{R}(t))$ and $\widehat{var}(\hat{h}(t))$) are approximated as

$$\widehat{var}(\hat{R}(t)) = \mathcal{U}_R^\top \widehat{\Sigma}(\hat{\boldsymbol{\xi}})\mathcal{U}_R \quad \text{and} \quad \widehat{var}(\hat{h}(t)) = \mathcal{U}_h^\top \widehat{\Sigma}(\hat{\boldsymbol{\xi}})\mathcal{U}_h,$$

receptively, where $\mathcal{U}_R = \left(\frac{\partial R}{\partial \gamma} \quad \frac{\partial R}{\partial \delta} \quad \frac{\partial R}{\partial \mu}\right)$ and $\mathcal{U}_h = \left(\frac{\partial h}{\partial \gamma} \quad \frac{\partial h}{\partial \delta} \quad \frac{\partial h}{\partial \mu}\right)$.

Thus, at a confidence level $(1-\varrho)$, the respective $100(1-\varrho)\%$ ACIs of $R(t)$ and $h(t)$ are

$$\hat{R}(t) \mp z_{\frac{\varrho}{2}}\sqrt{\widehat{var}(\hat{R}(t))} \quad \text{and} \quad \hat{h}(t) \mp z_{\frac{\varrho}{2}}\sqrt{\widehat{var}(\hat{h}(t))},$$

respectively.

3 Bayesian Inference

Based on gamma independent density priors $\gamma \sim Gamma(a_1, b_1)$, $\delta \sim Gamma(a_2, b_2)$, and $\mu \sim Gamma(a_3, b_3)$, using the SE and GE losses, the Bayes estimates of γ, δ, μ, $R(t)$, and $h(t)$ are developed.

Hence, the joint PDF (say, $P(\cdot)$) of γ, δ, and μ becomes

$$P(\delta,\sigma,\gamma) \propto \gamma^{a_1-1}\delta^{a_2-1}\mu^{a_3-1}e^{-(b_1\gamma+b_2\delta+b_3\mu)}, \quad \delta,\sigma,\gamma>0, \tag{16}$$

where the hyperparameters $a_i>0$ and $b_i>0$ for $i=1,2,3$ are known. It is worth noting here that the choice of gamma prior distributions came due to their flexibility and compatibility with the support of GLL model parameters. Compared to alternative prior distributions, gamma priors offer greater adaptability while maintaining analytical tractability. Moreover, the use of independent gamma priors tends to result in relatively simple posterior distributions, thereby mitigating potential computational challenges; for additional details, see Dey et al. [15].

From (6) and (16), the joint posterior distribution, say $P^*(\cdot)$, of γ, δ, and μ is

$$P^*(\boldsymbol{\xi}|\mathbf{x}) \propto \gamma^{m+a_1-1}\delta^{m\gamma+a_2-1}\mu^{a_3-1}e^{-\left(\gamma b_1+\delta b_2+\mu b_3\right)}$$
$$\times\prod_{i=1}^{m}x_i^{\gamma}\left[\vartheta\left(x_i;\gamma,\mu\right)\right]^{-\left(\frac{\delta}{\mu}\right)^{\gamma}-1}\prod_{i=1}^{d}\left[\vartheta\left(x_i;\gamma,\mu\right)\right]^{-R_i\left(\frac{\delta}{\mu}\right)^{\gamma}}\left[\vartheta\left(x_m;\gamma,\mu\right)\right]^{-R_m^*\left(\frac{\delta}{\mu}\right)^{\gamma}}. \tag{17}$$

The SE loss (denoted by $\ell_{SE}(\cdot)$), it is common knowledge that the posterior mean is the Bayes estimator. The GE loss (denoted by $\ell_{GE}(\cdot)$), on the other hand, gives different weights for overestimation and underestimation. However, the SE and GE losses can be defined, respectively, as

$$\ell_{SE}(\psi,\tilde{\psi}) = (\tilde{\psi}-\psi)^2, \tag{18}$$

and

$$\ell_{GE}(\psi,\tilde{\psi}) \propto \left(\frac{\tilde{\psi}}{\psi}\right)^{\tau} - \tau\log\left(\frac{\tilde{\psi}}{\psi}\right) - 1,\ \tau\neq 0, \tag{19}$$

where τ is a parameter that specifies the asymmetry degree and $\tilde{\psi}(\cdot)$ denotes the desired Bayes estimate of $\psi(\cdot)$. To get additional information on SE and GE loss function, one may refer to Martz and Waller [16] and Dey et al. [17], respectively.

However, under the SE and GE losses defined in (18) and (19), the Bayes' estimate of γ, δ, or μ, say $\psi(\boldsymbol{\xi})$, is provided by

$$\tilde{\psi}_{SE}(\boldsymbol{\xi}) = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \psi(\boldsymbol{\xi})P(\boldsymbol{\xi})L(\boldsymbol{\xi})\ \mathrm{d}\gamma\mathrm{d}\delta\mathrm{d}\mu}{\int_0^\infty\int_0^\infty\int_0^\infty P(\boldsymbol{\xi})L(\boldsymbol{\xi})\ \mathrm{d}\gamma\mathrm{d}\delta\mathrm{d}\mu}, \tag{20}$$

and

$$\tilde{\psi}_{GE}(\boldsymbol{\xi}) = \left[\frac{\int_0^\infty\int_0^\infty\int_0^\infty (\psi(\boldsymbol{\xi}))^{-\tau}P(\boldsymbol{\xi})L(\boldsymbol{\xi})\ \mathrm{d}\gamma\mathrm{d}\delta\mathrm{d}\mu}{\int_0^\infty\int_0^\infty\int_0^\infty P(\boldsymbol{\xi})L(\boldsymbol{\xi})\ \mathrm{d}\gamma\mathrm{d}\delta\mathrm{d}\mu}\right]^{-\frac{1}{\tau}}, \tag{21}$$

respectively. Setting $\tau=-1$ in (21), the estimate acquired from (21) coincides with the other acquired from (20).

From (20) and (21), the Bayes estimator $\tilde{\psi}(\boldsymbol{\xi})$ cannot be represented in analytic closed form. As a result, the MCMC approximation idea may be simply included to assess the Bayes' estimates of γ, δ, μ, $R(t)$, and $h(t)$, as well as their HPD intervals.

From (17), the conditional distributions of γ, δ, and μ are

$$\Omega_1(\gamma|\delta,\mu,\mathbf{x}) \propto \gamma^{m+a_1-1}e^{-\gamma b_1} \times \prod_{i=1}^{m} x_i^{\gamma}\left[\vartheta\left(x_i;\gamma,\mu\right)\right]^{-\left(\frac{\delta}{\mu}\right)^{\gamma}-1}\prod_{i=1}^{d}\left[\vartheta\left(x_i;\gamma,\mu\right)\right]^{-R_i\left(\frac{\delta}{\mu}\right)^{\gamma}}\left[\vartheta\left(x_m;\gamma,\mu\right)\right]^{-R_m^*\left(\frac{\delta}{\mu}\right)^{\gamma}}, \tag{22}$$

$$\Omega_2(\delta|\gamma,\mu,\mathbf{x}) \propto \delta^{m\gamma+a_2-1}e^{-\delta b_2} \times \prod_{i=1}^{m}\left[\vartheta\left(x_i;\gamma,\mu\right)\right]^{-\left(\frac{\delta}{\mu}\right)^{\gamma}-1}\prod_{i=1}^{d}\left[\vartheta\left(x_i;\gamma,\mu\right)\right]^{-R_i\left(\frac{\delta}{\mu}\right)^{\gamma}}\left[\vartheta\left(x_m;\gamma,\mu\right)\right]^{-R_m^*\left(\frac{\delta}{\mu}\right)^{\gamma}}, \tag{23}$$

and

$$\Omega_3(\mu|\gamma,\delta,\mathbf{x}) \propto \mu^{a_3-1}e^{-\mu b_3} \times \prod_{i=1}^{m}\left[\vartheta\left(x_i;\gamma,\mu\right)\right]^{-\left(\frac{\delta}{\mu}\right)^{\gamma}-1}\prod_{i=1}^{d}\left[\vartheta\left(x_i;\gamma,\mu\right)\right]^{-R_i\left(\frac{\delta}{\mu}\right)^{\gamma}}\left[\vartheta\left(x_m;\gamma,\mu\right)\right]^{-R_m^*\left(\frac{\delta}{\mu}\right)^{\gamma}}, \tag{24}$$

respectively.

The GLL parameters γ, δ, and μ cannot be stated in terms of a specified distribution from (22)–(24). Thus, the Metropolis-Hastings (M–H) technique is used to generate Makovian variates from (22), (23), and (24). Then produce the Bayes' MCMC or HPD interval estimates of γ, δ, μ, $R(t)$, or $h(t)$. The key challenge while employing the M–H process is deciding on the parameter distribution suggestion.

From GLL(1.2, 0.5, 0.1), by taking $(T, n, m) = (5, 100, 50)$, $\mathbf{R}$ (uniform removal), $(a_1, a_2, a_3) = (12, 5, 1)$ and $b_i = 10$, $i = 1, 2, 3$ (for instance), the plots in Fig. 3 demonstrated that the entire conditional PDFs of γ, δ, and μ provided in (22), (23), and (24), respectively, match the normal distribution.

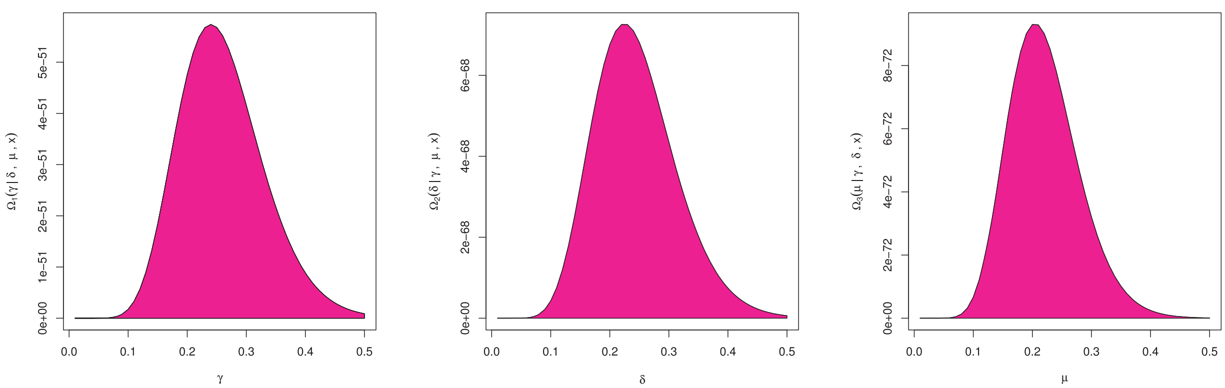


Figure 3: Conditional PDFs of γ (left) and δ (middle) and μ (right)

Numerically, to produce draws of γ, δ, μ, $R(t)$, or $h(t)$ utilizing the M–H idea, do:

Step 1: Set γ, δ, and μ, as $\gamma^{(0)} = \hat{\gamma}$, $\delta^{(0)} = \hat{\delta}$, and $\mu^{(0)} = \hat{\mu}$.

Step 2: Put $\pi = 1$.

Step 3: Obtain γ^*, δ^*, and μ^* using (22), (23), and (24) from the normal $N(\gamma^{(\pi-1)}, var(\gamma^{(\pi-1)}))$, $N(\delta^{(\pi-1)}, var(\delta^{(\pi-1)}))$, and $N(\mu^{(\pi-1)}, var(\mu^{(\pi-1)}))$ distributions, respectively.

Step 4: Generate u_i for $i = 1, 2, 3$ from uniform $U(0, 1)$ distribution.

Step 5: Calculate $\mathcal{D}_i$ for $i = 1, 2, 3$ as

$$\mathcal{D}_1 = \min\left(1, \frac{\Omega_1(\gamma^*|\delta^{(\pi-1)}, \mu^{(\pi-1)}, \mathbf{x})}{\Omega_1(\gamma^{(\pi-1)}|\delta^{(\pi-1)}, \mu^{(\pi-1)}, \mathbf{x})}\right),$$

$$\mathcal{D}_2 = \min\left(1, \frac{\Omega_2(\delta^*|\gamma^{(\pi)}, \mu^{(\pi-1)}, \mathbf{x})}{\Omega_2(\delta^{(\pi-1)}|\gamma^{(\pi)}, \mu^{(\pi-1)}, \mathbf{x})}\right),$$

and

$$\mathcal{D}_3 = \min\left(1, \frac{\Omega_3(\mu^*|\gamma^{(\pi)}, \delta^{(\pi)}, \mathbf{x})}{\Omega_3(\mu^{(\pi-1)}|\gamma^{(\pi)}, \delta^{(\pi)}, \mathbf{x})}\right).$$

Step 6: If $u_i \leq \mathcal{D}_i$, set $\theta^{(\pi)} = \theta^*$, else set $\theta^{(\pi)} = \theta^{(\pi-1)}$. Here, θ refers to γ, δ, or μ.

Step 7: Obtain $R^{(\pi)}(t)$ and $h^{(\pi)}(t)$ for $t > 0$, respectively, as

$$R^{(\pi)}(t) = \left[1 + (\mu^{(\pi)}t)^{\gamma^{(\pi)}}\right]^{-\left(\frac{\delta^{(\pi)}}{\mu^{(\pi)}}\right)^{\gamma^{(\pi)}}},$$

and

$$h^{(\pi)}(t) = \frac{\gamma^{(\pi)}\delta^{(\pi)}(\delta^{(\pi)}t)^{\gamma^{(\pi)}-1}}{\left[1 + (\mu^{(\pi)}t)^{\gamma^{(\pi)}}\right]}.$$

Step 8: Reset $\pi = \pi + 1$.

Step 9: Redo Steps 3-8 $\mathcal{A}$ times to get

$$\left(\gamma^{(1)}, \delta^{(1)}, \mu^{(1)}, R^{(1)}(t), h^{(1)}(t)\right), \ldots, \left(\gamma^{(\mathcal{A})}, \delta^{(\mathcal{A})}, \mu^{(\mathcal{A})}, R^{(\mathcal{A})}(t), h^{(\mathcal{A})}(t)\right).$$

Step 10: Acquire the Bayes' MCMC estimates of γ, δ, μ, $R(t)$, and $h(t)$ (say ψ), after a burn-in (say $\mathcal{A}^\star$), from SE and GE loss functions as

$$\tilde{\psi}_{SE} = \frac{1}{\mathcal{A}^\bullet}\sum_{\pi=\mathcal{A}^\star+1}^{\mathcal{A}} \psi^{(\pi)}.$$

and

$$\tilde{\psi}_{GE} = \left[\frac{1}{\mathcal{A}^\bullet}\sum_{\pi=\mathcal{A}^\star+1}^{\mathcal{A}} (\psi^{(\pi)})^{-\tau}\right]^{-\frac{1}{\tau}},$$

respectively, where $\mathcal{A}^\bullet = \mathcal{A} - \mathcal{A}^\star$.

Step 11: Create the $100(1-\varrho)\%$ HPD interval of ψ as

(a) Arrange the staying Markovian draws of $\psi^{(\pi)}$ for $\pi = Q^\star + 1, Q^\star + 2, \ldots, Q$ as $\psi_{(Q^\star+1)}, \psi_{(Q^\star+2)}, \ldots, \psi_{(Q)}$.

(b) Obtain the $100(1-\varrho)\%$ two-sided HPD interval (see Chen and Shao [18]) of ψ as

$$(\psi_{(\pi*)}, \psi_{(\pi*+(1-\varrho)\mathcal{A}^\bullet)}),$$

where $\pi^* = \mathcal{A}^\star + 1, \mathcal{A}^\star + 2, \ldots, \mathcal{A}$ is chosen as

$$\psi_{(\pi^*+[(1-\varrho)\mathcal{A}^\bullet])} - \psi_{(\pi^*)} = \min_{1\leqslant\pi\leqslant\varrho\mathcal{A}^\bullet}(\psi_{(\pi+[(1-\varrho)\mathcal{A}^\bullet])} - \psi_{(\pi)}).$$

4 Monte Carlo Comparisons

To compare the performance of the proposed estimators of γ, δ, μ, $R(t)$ and $h(t)$, based on 1,000 APC-II samples generated from GLL(1.2, 0.5, 0.1), several Monte Carlo simulations are conducted. Taking $t = 0.1$, the actual values of $(R(t), h(t))$ is (0.97296, 0.32826). The procedure of obtaining an APC-II sample from the GLL distribution is depicted in Algorithm 1.

Algorithm 1: Procedure of simulating APC-II samples

Step 1: Obtain a PC-II (by Balakrishnan and Cramer [19]) as (i)–(iv):

i Create κ independent uniform observations, from $U(0,1)$, of size m (say $\kappa_1, \kappa_2, \ldots, \kappa_m$).

ii For specific n , m , T and R_i, $i = 1, 2, \ldots, m$, set

$$q_i = \kappa_i^{\left(i+\sum_{j=m-i+1}^{m} R_j\right)^{-1}}, \text{ for } i = 1, 2, \ldots, m.$$

iii Let $u_i = 1 - q_m q_{m-1} \cdots q_{m-i+1}$ for $i = 1, 2, \ldots, m$.

iv From (2) set

$$X_i = \frac{1}{\mu}\left[(1-u_i)^{-(\frac{\mu}{\delta})^{\gamma}} - 1\right]^{1/\gamma}, \ i = 1, 2, \ldots, m.$$

Step 2: Find d , where $X_d < T < X_{d+1}$, and discard $X_{d+2}, \ldots, X_m$.

Step 3: From $[1 - F(x_{d+1})]^{-1} f(x)$, get the order statistics $\{X_{d+2}, \ldots, X_m\}$ with size $n - d - \sum_{j=1}^{d} R_j - 1$.

All proposed numerical evaluations are done based on different levels of T, n and m such as T(=1.5, 2.5) and n(=50, 80). Also, for each n, the proposed choices of m are taken as failure percentages (FPs) $\frac{m}{n}$(=40, 80)%. For each (n, m), three progressive scenarios **R** are utilized as

$$\text{Scheme-1}: \ R_1 = n - m, \qquad R_i = 0 \quad \text{for} \quad i \neq 1,$$

$$\text{Scheme-2}: \ R_{\frac{m}{2}} = n - m, \qquad R_i = 0 \quad \text{for} \quad i \neq \frac{m}{2},$$

$$\text{Scheme-3}: \ R_m = n - m, \qquad R_i = 0 \quad \text{for} \quad i \neq m.$$

Once the 1000 APC-II samples are collected, the MLEs along with associated 95% ACIs of γ, δ, μ, $R(t)$, and $h(t)$ are calculated. Following the M-H algorithm, to compute the Bayes MCMC estimates of the same parameters along with their HPD interval estimates, 12,000 MCMC variates are simulated, and then the first 2000 variates (burn-in). Before deriving the Bayes inferences, the convergence of the generated sequences of γ, δ, μ, $R(t)$, and $h(t)$ should be examined first. So, using their remaining 10,000 MCMC samples, both trace and autocorrelation plots (for n[FP%] = 50[40%] and Scheme-1 as an example) are shown in Fig. 4. Fig. 4 displays autocorrelation and trace charts of the 10,000 MCMC iterations of γ, δ, μ, $R(t)$, and $h(t)$ to assess their convergence. It demonstrates that the parameters γ, δ, μ, $R(t)$ and $h(t)$ are effectively blended in the MCMC repetitions, so all estimated results are reasonable. It demonstrates that all unknown parameters are effectively mixed across all of the MCMC iterations, so all acquired estimates are plausible.

Thus, the Bayes values (along with their 95% HPD intervals) of γ, δ, μ, $R(t)$, and $h(t)$ against all SE and GE (for $\tau(= -2, +2)$) loss functions are also produced. To exhibit the gamma density priors' behavior, two sets of (a_i, b_i), $i = 1, 2, 3$ are taken into consideration, according to the prior mean and prior variance conditions, specifically: Prior-1: $(a_1, a_2, a_3) = (6, 2.5, 0.5)$ and $b_i = 5$, $i = 1, 2, 3$; Prior-2: $(a_1, a_2, a_3) = (12, 5, 1)$ and $b_i = 10$, $i = 1, 2, 3$. For clarity, we have assigned the values of (a_i, b_i), $i = 1, 2, 3$ (given in priors 1 and 2) according to the idea presented by Kundu [20].

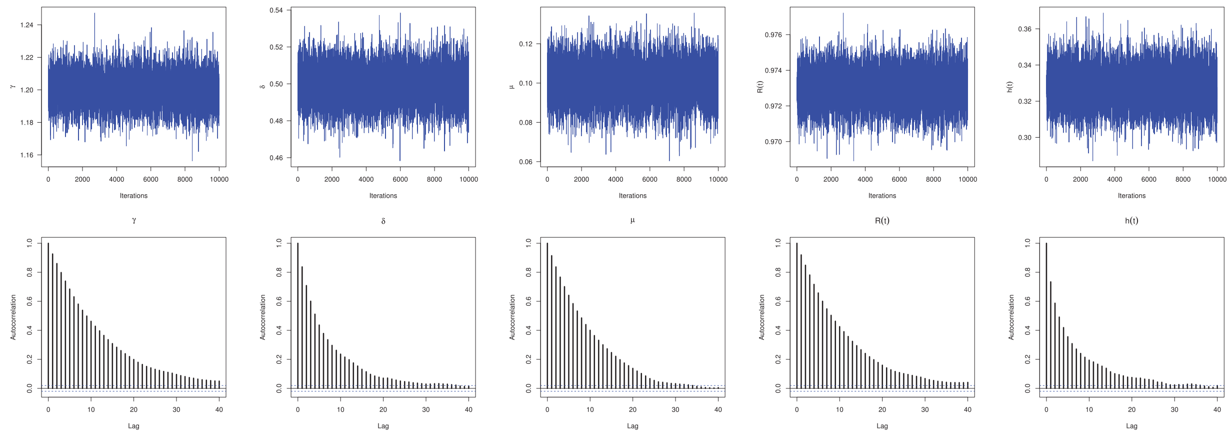


Figure 4: Trace (top) and Autocorrelation (bottom) plots of γ, δ, μ, $R(t)$ and $h(t)$

Via R 4.2.2 programming software, the classical estimates (or their ACIs) as well as the Bayes estimates (or their HPD interval estimates) are evaluated by 'maxLik' and 'maxLik' packages introduced by Henningsen and Toomet [21] and Plummer et al. [22], respectively.

In numerical setup, for $\zeta = 1, 2, 3, 4, 5$, the average estimates (Av.Es) of γ, δ, μ, $R(t)$ and $h(t)$ (say λ) are obtained by

$$\text{Av.E}(\check{\lambda}_\zeta) = \frac{1}{1000}\sum\nolimits_{i=1}^{1000} \check{\lambda}_\zeta^{(i)},$$

where $\check{\lambda}^{(i)}$ is the target estimate of λ at ith sample, $\lambda_1 = \gamma$, $\lambda_2 = \delta$,$\lambda_3 = \mu$, $\lambda_4 = R(t)$ and $\lambda_5 = h(t)$.

Two useful criteria: (i) root mean squared-errors (RMSEs) and (ii) mean relative absolute biases (MRABs) are used to compare the proposed point estimates of λ such as

$$\text{RMSE}(\check{\lambda}_\zeta) = \sqrt{\frac{1}{1000}\sum\nolimits_{i=1}^{1000}\left(\check{\lambda}_\zeta^{(i)} - \lambda_\zeta\right)^2},$$

and

$$\text{MRAB}(\check{\lambda}_\zeta) = \frac{1}{1000}\sum\nolimits_{i=1}^{1000}\frac{1}{\lambda_\zeta}\left|\check{\lambda}_\zeta^{(i)} - \lambda_\zeta\right|,$$

respectively. The other two useful criteria: (i) average interval lengths (AILs) and (ii) coverage percentages (CPs) to compare the interval estimates of λ are also considered as

$$\text{AIL}_{(1-\varrho)\%}(\lambda_\zeta) = \frac{1}{1000}\sum\nolimits_{i=1}^{1000}\left(\mathcal{U}_{\check{\lambda}_k^{(i)}} - \mathcal{L}_{\check{\lambda}_k^{(i)}}\right),$$

and

$$\text{CP}_{(1-\varrho)\%}(\lambda_\zeta) = \frac{1}{1000}\sum\nolimits_{i=1}^{1000} 1^{\bullet}{}_{\left(\mathcal{L}_{\check{\lambda}_k^{(i)}};\mathcal{U}_{\check{\lambda}_k^{(i)}}\right)}\left(\lambda_\zeta\right),$$

where $1^{\bullet}(\cdot)$ is an indicator function and $(\mathcal{L}(\cdot), \mathcal{U}(\cdot))$ denotes the two limits of $(1-\varrho)\%$ ACI (or $(1-\varrho)\%$ HPD interval) of λ_ζ.

All point outcomes of γ, δ, μ, $R(t)$ and $h(t)$ are listed in Tables 1–5, respectively, in such a way that their Av.Es, RMSEs, and MRABs are listed in the first, second, and third columns, respectively.

In Tables 6–10, the AILs and CPs are reported in the first and second columns, respectively. From Tables 1–10, in terms of the lowest (RMSE, MRAB, AIL) values and the highest CP values, the following statements can be easily concluded:

- Generally, all inferential findings of γ, δ, μ, $R(t)$ and $h(t)$ behave satisfactorily in the presence of APC-II data.
- As FP%(or n) grows, all derived estimates of γ, δ, μ, $R(t)$, or $h(t)$ perform well. The same finding is also reached at $\sum_{i=1}^{m} R_i,\ i = 1, 2, \ldots, m$ decreases. This observation satisfies the consistency property of the acquired estimators.
- As T increases, the RMSEs (or MRABs) of all unknown quantities grow; the AILs of γ, δ, μ, and $h(t)$ increase except for $R(t)$ decrease; while the CPs of γ, δ, μ and $h(t)$ decrease except for $R(t)$ increase.
- Due to the gamma prior information, as we anticipated, the Bayes' evaluations of γ, δ, μ, $R(t)$, and $h(t)$ provide good results compared to the others. A similar comment is also noted in the case of their HPD interval estimates.
- The Bayes results obtained using Prior-2 are better than those obtained based on others.
- Asymmetric Bayes estimates of all unknown parameters have overestimates (or underestimates) if τ takes a negative (or positive) value.
- Comparing the performance of the shemes 1, 2, and 3 on the inferential results, the point and interval estimates of γ, δ and μ perform better using Scheme-1 (left-censoring); the estimates of $R(t)$ perform better using Scheme-3; the point estimates of $h(t)$ perform better using Scheme-2 while its interval estimates perform better using Scheme-3 than others.
- To sum up, if someone has to sample from an adaptive Type-II progressive censored system, we advise using the M-H method in a Bayes setup to estimate the GLL distribution parameters or its reliability features.

Table 1: The point results of γ

T	(n,m)	Scheme	MLE			SE			GE					
						Pr.1 Pr.2			Pr.1 Pr.2					
$\tau \rightarrow$									−2			+2		
1.5	(50,20)	1	1.350	0.536	0.247	0.956	0.457	0.240	1.235	0.156	0.061	1.244	0.050	0.038
						0.996	0.435	0.233	1.213	0.150	0.042	1.222	0.030	0.023
		2	1.527	0.598	0.386	0.989	0.475	0.258	1.233	0.159	0.069	1.243	0.059	0.043
						0.959	0.472	0.242	1.213	0.153	0.045	1.221	0.033	0.026
		3	1.770	0.634	0.518	0.964	0.485	0.265	1.258	0.170	0.080	1.268	0.073	0.057
						0.954	0.475	0.248	1.217	0.157	0.046	1.226	0.045	0.035
	(50,40)	1	1.264	0.471	0.212	1.006	0.374	0.183	1.151	0.142	0.052	1.159	0.044	0.032
						0.980	0.363	0.178	1.190	0.137	0.031	1.198	0.027	0.018
		2	1.288	0.483	0.296	0.806	0.398	0.201	1.198	0.142	0.054	1.207	0.045	0.034
						0.960	0.349	0.183	1.231	0.139	0.035	1.240	0.031	0.024
		3	1.458	0.533	0.473	0.971	0.423	0.228	1.138	0.158	0.057	1.148	0.054	0.044
						0.982	0.415	0.202	1.200	0.142	0.041	1.208	0.038	0.029

(Continued)

Table 1 (continued)

T	(n,m)	Scheme	MLE			SE Pr.1 Pr.2			GE Pr.1 Pr.2					
$\tau \rightarrow$									-2			$+2$		
	(80,32)	1	1.263	0.434	0.207	0.994	0.346	0.174	1.172	0.134	0.034	1.165	0.039	0.030
						1.019	0.327	0.151	1.132	0.127	0.029	1.208	0.022	0.017
		2	1.273	0.449	0.260	0.919	0.380	0.194	1.124	0.139	0.044	1.179	0.042	0.032
						0.985	0.313	0.157	1.193	0.131	0.032	1.201	0.030	0.021
		3	1.689	0.472	0.381	1.069	0.399	0.209	1.194	0.142	0.051	1.137	0.052	0.041
						1.058	0.351	0.182	1.158	0.129	0.035	1.149	0.036	0.027
	(80,64)	1	1.236	0.395	0.188	0.959	0.324	0.160	1.143	0.125	0.030	1.238	0.031	0.025
						1.077	0.295	0.138	1.131	0.117	0.022	1.203	0.019	0.012
		2	1.249	0.399	0.221	1.050	0.346	0.168	1.231	0.133	0.041	1.232	0.034	0.027
						1.084	0.275	0.145	1.200	0.130	0.029	1.209	0.026	0.017
		3	1.643	0.448	0.307	0.964	0.377	0.195	1.224	0.139	0.049	1.135	0.047	0.037
						1.097	0.227	0.176	1.200	0.135	0.031	1.105	0.039	0.023
2.5	(50,20)	1	1.634	0.751	0.482	0.989	0.475	0.253	1.231	0.157	0.059	1.240	0.047	0.036
						0.959	0.472	0.241	1.213	0.149	0.042	1.221	0.029	0.023
		2	1.364	0.937	0.551	0.966	0.478	0.265	1.242	0.163	0.067	1.251	0.057	0.044
						0.954	0.475	0.242	1.213	0.151	0.043	1.222	0.030	0.024
		3	1.760	0.995	0.830	0.900	0.514	0.267	1.258	0.170	0.080	1.268	0.073	0.057
						0.965	0.494	0.250	1.217	0.153	0.046	1.226	0.033	0.026
	(50,40)	1	1.266	0.494	0.326	1.009	0.366	0.202	1.241	0.152	0.054	1.172	0.032	0.032
						0.958	0.364	0.181	1.203	0.147	0.034	1.209	0.022	0.017
		2	1.267	0.575	0.361	0.990	0.372	0.215	1.121	0.153	0.057	1.244	0.049	0.038
						0.992	0.374	0.204	1.200	0.148	0.035	1.212	0.024	0.018
		3	1.951	0.790	0.646	0.952	0.480	0.244	1.206	0.158	0.062	1.250	0.055	0.042
						0.951	0.469	0.230	1.179	0.149	0.039	1.184	0.027	0.023
	(80,32)	1	1.270	0.469	0.259	1.048	0.246	0.127	1.195	0.139	0.042	1.224	0.029	0.023
						1.107	0.221	0.120	1.218	0.130	0.029	1.203	0.019	0.015
		2	1.268	0.499	0.320	1.076	0.260	0.128	1.236	0.148	0.052	1.128	0.035	0.030
						1.082	0.251	0.126	1.170	0.135	0.033	1.187	0.022	0.017
		3	1.668	0.517	0.467	1.055	0.346	0.190	1.176	0.150	0.042	1.215	0.047	0.039
						1.021	0.320	0.149	1.203	0.147	0.036	1.210	0.024	0.019
	(80,64)	1	1.235	0.420	0.216	1.080	0.242	0.116	1.175	0.135	0.028	1.183	0.026	0.019
						1.056	0.213	0.114	1.197	0.128	0.025	1.214	0.017	0.014
		2	1.236	0.424	0.268	1.109	0.249	0.122	1.139	0.137	0.039	1.204	0.029	0.022
						1.059	0.236	0.120	1.207	0.131	0.031	1.178	0.018	0.016
		3	1.647	0.473	0.386	0.950	0.275	0.135	1.156	0.142	0.041	1.162	0.041	0.031
						0.986	0.268	0.128	1.167	0.138	0.032	1.144	0.021	0.016

Table 2: The point results of δ

T	(n, m)	Scheme		MLE			SE				GE			
							Pr.1 Pr.2				Pr.1 Pr.2			
$\tau \rightarrow$									−2			+2		
1.5	(50,20)	1	0.535	0.241	0.223	0.563	0.124	0.140	0.573	0.078	0.145	0.549	0.049	0.098
						0.559	0.107	0.118	0.526	0.043	0.099	0.467	0.034	0.066
		2	0.554	0.322	0.278	0.514	0.130	0.148	0.574	0.079	0.147	0.442	0.058	0.115
						0.481	0.116	0.125	0.553	0.060	0.105	0.551	0.051	0.102
		3	0.783	0.772	0.773	0.380	0.162	0.183	0.393	0.118	0.162	0.364	0.136	0.123
						0.441	0.132	0.161	0.450	0.063	0.127	0.431	0.069	0.115
	(50,40)	1	0.510	0.119	0.143	0.454	0.115	0.123	0.542	0.054	0.085	0.472	0.029	0.056
						0.504	0.100	0.112	0.490	0.040	0.072	0.494	0.018	0.015
		2	0.522	0.143	0.184	0.564	0.124	0.132	0.453	0.059	0.109	0.548	0.048	0.096
						0.517	0.108	0.116	0.464	0.049	0.092	0.538	0.039	0.075
		3	0.549	0.571	0.730	0.488	0.132	0.143	0.565	0.072	0.131	0.432	0.068	0.135
						0.444	0.110	0.155	0.540	0.050	0.079	0.458	0.043	0.084
	(80,32)	1	0.499	0.087	0.118	0.532	0.105	0.098	0.478	0.046	0.071	0.523	0.024	0.047
						0.508	0.096	0.056	0.511	0.038	0.046	0.508	0.010	0.016
		2	0.517	0.104	0.137	0.483	0.112	0.117	0.525	0.050	0.086	0.525	0.027	0.050
						0.508	0.096	0.059	0.498	0.041	0.068	0.495	0.017	0.018
		3	0.841	0.487	0.698	0.469	0.116	0.120	0.464	0.056	0.107	0.448	0.053	0.103
						0.495	0.098	0.062	0.512	0.046	0.072	0.487	0.018	0.035
	(80,64)	1	0.500	0.063	0.090	0.547	0.098	0.077	0.504	0.036	0.051	0.548	0.020	0.020
						0.533	0.094	0.043	0.503	0.034	0.040	0.498	0.015	0.010
		2	0.510	0.076	0.104	0.534	0.106	0.081	0.494	0.041	0.055	0.521	0.022	0.042
						0.503	0.092	0.046	0.516	0.036	0.052	0.498	0.016	0.012
		3	0.876	0.420	0.553	0.457	0.096	0.126	0.540	0.051	0.080	0.476	0.025	0.049
						0.497	0.094	0.055	0.516	0.039	0.040	0.485	0.017	0.030
2.5	(50,20)	1	0.566	0.265	0.313	0.573	0.131	0.137	0.579	0.084	0.158	0.557	0.054	0.104
						0.498	0.099	0.070	0.515	0.039	0.056	0.524	0.042	0.074
		2	0.558	0.329	0.285	0.569	0.129	0.150	0.582	0.087	0.164	0.553	0.058	0.107
						0.508	0.099	0.072	0.507	0.043	0.068	0.510	0.045	0.098
		3	0.676	0.816	0.809	0.407	0.139	0.170	0.418	0.093	0.171	0.394	0.106	0.111
						0.509	0.087	0.075	0.540	0.051	0.081	0.499	0.047	0.099
	(50,40)	1	0.528	0.153	0.208	0.539	0.109	0.079	0.541	0.052	0.083	0.528	0.028	0.055
						0.509	0.097	0.062	0.516	0.036	0.043	0.501	0.022	0.029
		2	0.526	0.263	0.197	0.533	0.105	0.089	0.547	0.057	0.094	0.467	0.034	0.067
						0.508	0.096	0.065	0.517	0.041	0.047	0.499	0.025	0.029
		3	0.578	0.583	0.753	0.555	0.111	0.111	0.562	0.070	0.125	0.545	0.045	0.090
						0.533	0.101	0.066	0.483	0.039	0.073	0.522	0.023	0.045
	(80,32)	1	0.508	0.127	0.159	0.529	0.101	0.076	0.535	0.044	0.071	0.483	0.018	0.034
						0.510	0.097	0.059	0.515	0.034	0.042	0.498	0.015	0.019
		2	0.509	0.205	0.164	0.493	0.103	0.081	0.537	0.050	0.073	0.518	0.020	0.046
						0.507	0.094	0.062	0.518	0.038	0.042	0.496	0.018	0.030
		3	0.873	0.548	0.650	0.488	0.112	0.093	0.498	0.045	0.068	0.476	0.025	0.049
						0.495	0.098	0.062	0.504	0.039	0.055	0.487	0.019	0.033

(Continued)

Table 2 (continued)

T	(n, m)	Scheme	MLE			SE			GE					
						Pr.1 Pr.2			Pr.1 Pr.2					
$\tau \rightarrow$									−2			+2		
	(80,64)	1	0.503	0.118	0.120	0.495	0.100	0.063	0.516	0.036	0.045	0.485	0.016	0.030
						0.509	0.096	0.041	0.505	0.031	0.037	0.500	0.015	0.010
		2	0.505	0.138	0.123	0.529	0.093	0.067	0.502	0.037	0.057	0.483	0.017	0.033
						0.493	0.091	0.056	0.500	0.034	0.040	0.485	0.016	0.013
		3	0.862	0.443	0.577	0.496	0.097	0.083	0.504	0.040	0.054	0.519	0.019	0.038
						0.475	0.096	0.062	0.517	0.037	0.039	0.484	0.017	0.017

Table 3: The point results of μ

T	(n, m)	Scheme	MLE			SE			GE					
						Pr.1 Pr.2			Pr.1 Pr.2					
$\tau \rightarrow$									−2			+2		
1.5	(50,20)	1	0.184	0.491	1.005	0.158	0.176	0.517	0.165	0.067	0.545	0.141	0.042	0.410
						0.114	0.156	0.342	0.156	0.058	0.455	0.103	0.037	0.332
		2	0.216	0.591	1.176	0.156	0.178	0.575	0.164	0.067	0.541	0.134	0.058	0.496
						0.149	0.168	0.496	0.124	0.041	0.236	0.108	0.022	0.375
		3	0.715	0.853	1.471	0.304	0.217	0.358	0.313	0.093	0.632	0.195	0.096	0.393
						0.191	0.198	0.215	0.194	0.089	0.555	0.192	0.093	0.320
	(50,40)	1	0.114	0.253	0.826	0.128	0.166	0.321	0.139	0.047	0.291	0.141	0.037	0.278
						0.111	0.159	0.241	0.124	0.035	0.116	0.068	0.032	0.188
		2	0.183	0.415	0.920	0.140	0.172	0.421	0.150	0.056	0.405	0.150	0.050	0.391
						0.117	0.155	0.212	0.126	0.035	0.155	0.107	0.018	0.139
		3	0.525	0.654	1.268	0.076	0.181	0.312	0.232	0.068	0.491	0.061	0.094	0.347
						0.092	0.177	0.176	0.215	0.057	0.379	0.184	0.084	0.241
	(80,32)	1	0.132	0.189	0.595	0.097	0.153	0.228	0.122	0.037	0.140	0.116	0.016	0.129
						0.110	0.150	0.167	0.115	0.031	0.099	0.105	0.013	0.110
		2	0.155	0.301	0.744	0.102	0.163	0.228	0.109	0.036	0.154	0.127	0.028	0.128
						0.115	0.155	0.152	0.119	0.033	0.089	0.086	0.014	0.092
		3	0.377	0.482	0.976	0.094	0.165	0.273	0.107	0.027	0.380	0.266	0.083	0.256
						0.110	0.154	0.172	0.118	0.030	0.214	0.102	0.065	0.133
	(80,64)	1	0.106	0.148	0.411	0.223	0.139	0.193	0.119	0.033	0.089	0.088	0.012	0.084
						0.210	0.120	0.136	0.101	0.018	0.036	0.102	0.011	0.032
		2	0.145	0.239	0.452	0.984	0.153	0.223	0.000	0.025	0.092	0.077	0.024	0.091
						0.095	0.148	0.156	0.104	0.022	0.042	0.101	0.011	0.038
		3	0.252	0.354	0.653	0.110	0.154	0.171	0.087	0.025	0.141	0.083	0.075	0.126
						0.089	0.152	0.135	0.100	0.021	0.069	0.085	0.051	0.066
2.5	(50,20)	1	0.204	0.495	1.146	0.164	0.083	0.542	0.172	0.073	0.327	0.164	0.064	0.438
						0.129	0.058	0.389	0.148	0.043	0.278	0.139	0.039	0.393
		2	0.251	0.598	1.271	0.174	0.090	0.641	0.181	0.082	0.415	0.153	0.066	0.449
						0.130	0.059	0.440	0.137	0.053	0.423	0.122	0.040	0.421
		3	0.696	0.892	1.540	0.237	0.112	0.778	0.187	0.088	0.472	0.149	0.075	0.491
						0.177	0.099	0.677	0.245	0.080	0.455	0.165	0.069	0.451

(Continued)

Table 3 (continued)

T	(n,m)	Scheme	MLE			SE			GE					
						Pr.1 Pr.2			Pr.1 Pr.2					
$\tau \rightarrow$									−2			+2		
	(50,40)	1	0.129	0.272	0.895	0.088	0.071	0.357	0.129	0.041	0.290	0.133	0.033	0.331
						0.113	0.055	0.249	0.123	0.035	0.228	0.072	0.028	0.283
		2	0.137	0.489	0.981	0.147	0.074	0.469	0.157	0.059	0.367	0.066	0.053	0.342
						0.144	0.057	0.304	0.138	0.044	0.351	0.097	0.038	0.354
		3	1.265	0.697	1.374	0.179	0.092	0.650	0.185	0.085	0.449	0.126	0.066	0.362
						0.142	0.074	0.443	0.157	0.058	0.385	0.096	0.054	0.342
	(80,32)	1	0.155	0.219	0.664	0.090	0.067	0.295	0.121	0.035	0.179	0.123	0.024	0.235
						0.111	0.057	0.178	0.117	0.030	0.181	0.100	0.020	0.146
		2	0.165	0.326	0.781	0.094	0.065	0.268	0.107	0.040	0.258	0.077	0.033	0.225
						0.110	0.054	0.192	0.119	0.033	0.189	0.101	0.024	0.132
		3	0.962	0.524	0.995	0.152	0.066	0.229	0.124	0.038	0.276	0.083	0.037	0.166
						0.097	0.061	0.228	0.119	0.035	0.211	0.102	0.030	0.133
	(80,64)	1	0.112	0.157	0.446	0.110	0.054	0.173	0.107	0.027	0.169	0.100	0.022	0.116
						0.106	0.048	0.142	0.114	0.026	0.143	0.101	0.017	0.094
		2	0.115	0.261	0.478	0.104	0.058	0.187	0.106	0.037	0.213	0.106	0.024	0.156
						0.110	0.050	0.167	0.121	0.031	0.180	0.102	0.022	0.100
		3	1.390	0.394	0.737	0.117	0.064	0.240	0.153	0.037	0.230	0.206	0.028	0.136
						0.111	0.053	0.221	0.116	0.033	0.189	0.146	0.025	0.113

Table 4: The point results of $R(t)$

T	(n,m)	Scheme	MLE			SE			GE					
						Pr.1 Pr.2			Pr.1 Pr.2					
$\tau \rightarrow$									−2			+2		
1.5	(50,20)	1	0.973	0.215	0.174	0.969	0.155	0.125	0.953	0.131	0.125	0.962	0.114	0.108
						0.968	0.126	0.110	0.955	0.125	0.117	0.965	0.105	0.090
		2	0.974	0.170	0.151	0.968	0.126	0.118	0.961	0.129	0.119	0.970	0.110	0.103
						0.968	0.116	0.106	0.959	0.125	0.115	0.971	0.104	0.087
		3	0.974	0.171	0.150	0.972	0.120	0.114	0.954	0.116	0.113	0.967	0.107	0.101
						0.969	0.113	0.101	0.961	0.113	0.109	0.971	0.091	0.084
	(50,40)	1	0.973	0.171	0.147	0.970	0.146	0.121	0.962	0.129	0.118	0.972	0.110	0.097
						0.964	0.113	0.109	0.957	0.123	0.114	0.965	0.101	0.088
		2	0.984	0.164	0.136	0.974	0.119	0.112	0.974	0.121	0.116	0.975	0.107	0.094
						0.969	0.105	0.102	0.960	0.118	0.108	0.970	0.098	0.086
		3	0.973	0.156	0.126	0.972	0.113	0.111	0.968	0.113	0.112	0.973	0.104	0.091
						0.965	0.104	0.092	0.960	0.102	0.101	0.966	0.091	0.080
	(80,32)	1	0.973	0.162	0.133	0.961	0.131	0.112	0.960	0.120	0.112	0.970	0.104	0.087
						0.964	0.106	0.102	0.958	0.119	0.108	0.970	0.091	0.080
		2	0.985	0.146	0.131	0.969	0.115	0.111	0.958	0.099	0.109	0.970	0.101	0.091
						0.969	0.099	0.101	0.959	0.093	0.104	0.970	0.091	0.085
		3	0.974	0.141	0.116	0.965	0.111	0.105	0.962	0.096	0.092	0.974	0.095	0.090
						0.970	0.096	0.085	0.960	0.091	0.100	0.971	0.090	0.083

(Continued)

Table 4 (continued)

T	(n, m)	Scheme	MLE			SE			GE					
							Pr.1 Pr.2				Pr.1 Pr.2			
$\tau \rightarrow$										−2			+2	
	(80,64)	1	0.979	0.142	0.123	0.967	0.118	0.109	0.959	0.113	0.109	0.968	0.101	0.084
						0.968	0.096	0.094	0.959	0.108	0.105	0.969	0.090	0.079
		2	0.973	0.133	0.108	0.967	0.112	0.105	0.960	0.096	0.103	0.968	0.091	0.082
						0.968	0.093	0.091	0.958	0.092	0.098	0.969	0.088	0.078
		3	0.973	0.130	0.106	0.968	0.105	0.101	0.966	0.094	0.087	0.969	0.089	0.080
						0.969	0.091	0.088	0.964	0.089	0.084	0.971	0.085	0.076
2.5	(50,20)	1	0.972	0.223	0.169	0.969	0.168	0.128	0.970	0.149	0.146	0.963	0.135	0.126
						0.968	0.154	0.110	0.969	0.138	0.127	0.960	0.129	0.117
		2	0.977	0.176	0.164	0.959	0.146	0.099	0.961	0.145	0.143	0.956	0.134	0.122
						0.966	0.137	0.089	0.967	0.136	0.124	0.952	0.132	0.118
		3	0.985	0.162	0.147	0.966	0.139	0.090	0.967	0.141	0.139	0.959	0.126	0.118
						0.967	0.135	0.084	0.968	0.133	0.122	0.958	0.130	0.115
	(50,40)	1	0.973	0.177	0.150	0.969	0.138	0.119	0.970	0.142	0.136	0.958	0.130	0.121
						0.966	0.133	0.089	0.967	0.134	0.116	0.963	0.123	0.113
		2	0.979	0.163	0.142	0.970	0.129	0.090	0.971	0.139	0.126	0.963	0.124	0.114
						0.966	0.131	0.086	0.968	0.131	0.113	0.958	0.113	0.106
		3	0.973	0.158	0.133	0.970	0.126	0.089	0.971	0.136	0.122	0.965	0.121	0.109
						0.965	0.125	0.082	0.966	0.128	0.107	0.959	0.112	0.104
	(80,32)	1	0.973	0.169	0.134	0.968	0.133	0.109	0.970	0.136	0.114	0.959	0.128	0.119
						0.968	0.127	0.084	0.970	0.123	0.096	0.959	0.122	0.108
		2	0.985	0.147	0.127	0.968	0.126	0.085	0.970	0.126	0.119	0.958	0.115	0.108
						0.968	0.127	0.080	0.970	0.121	0.107	0.959	0.110	0.100
		3	0.973	0.140	0.114	0.972	0.122	0.083	0.974	0.123	0.109	0.962	0.111	0.102
						0.969	0.118	0.081	0.971	0.119	0.095	0.960	0.102	0.090
	(80,64)	1	0.978	0.147	0.126	0.967	0.128	0.101	0.968	0.125	0.106	0.959	0.123	0.115
						0.968	0.120	0.081	0.969	0.118	0.087	0.958	0.115	0.102
		2	0.973	0.141	0.114	0.967	0.124	0.089	0.968	0.122	0.116	0.959	0.111	0.107
						0.967	0.122	0.082	0.969	0.118	0.084	0.958	0.103	0.081
		3	0.973	0.131	0.107	0.967	0.120	0.088	0.968	0.120	0.113	0.964	0.103	0.101
						0.968	0.117	0.079	0.969	0.112	0.083	0.961	0.089	0.085

Table 5: The point results of $h(t)$

T	(n, m)	Scheme	MLE			SE			GE					
							Pr.1 Pr.2				Pr.1 Pr.2			
$\tau \rightarrow$										−2			+2	
1.5	(50,20)	1	0.316	0.178	0.440	0.384	0.071	0.194	0.387	0.059	0.178	0.274	0.055	0.167
						0.359	0.058	0.102	0.362	0.038	0.124	0.361	0.032	0.128
		2	0.304	0.141	0.344	0.299	0.065	0.152	0.353	0.032	0.089	0.287	0.042	0.126
						0.299	0.0542	0.099	0.299	0.029	0.083	0.298	0.031	0.109
		3	0.205	0.186	0.476	0.273	0.136	0.262	0.293	0.072	0.159	0.254	0.075	0.227
						0.356	0.094	0.129	0.349	0.065	0.128	0.254	0.071	0.205

(Continued)

Table 5 (continued)

T	(n,m)	Scheme	MLE			SE			GE					
						Pr.1 Pr.2			Pr.1 Pr.2					
$\tau \rightarrow$									−2			+2		
	(50,40)	1	0.314	0.137	0.394	0.279	0.066	0.178	0.282	0.049	0.141	0.378	0.050	0.153
						0.362	0.038	0.098	0.362	0.034	0.103	0.355	0.029	0.085
		2	0.313	0.132	0.324	0.033	0.061	0.107	0.304	0.030	0.081	0.341	0.028	0.094
						0.300	0.042	0.065	0.302	0.029	0.069	0.297	0.027	0.077
		3	0.308	0.173	0.424	0.327	0.116	0.216	0.265	0.064	0.133	0.282	0.046	0.140
						0.299	0.087	0.107	0.365	0.052	0.116	0.300	0.035	0.115
	(80,32)	1	0.320	0.142	0.349	0.341	0.054	0.094	0.346	0.025	0.056	0.306	0.039	0.129
						0.305	0.031	0.081	0.305	0.023	0.042	0.303	0.026	0.079
		2	0.315	0.115	0.287	0.347	0.051	0.102	0.314	0.028	0.075	0.308	0.021	0.063
						0.305	0.031	0.0611	0.306	0.023	0.054	0.303	0.025	0.058
		3	0.196	0.156	0.409	0.262	0.076	0.166	0.288	0.042	0.122	0.290	0.038	0.117
						0.293	0.043	0.098	0.293	0.035	0.103	0.308	0.021	0.062
	(80,64)	1	0.321	0.114	0.281	0.309	0.034	0.077	0.311	0.022	0.049	0.332	0.025	0.113
						0.311	0.026	0.060	0.310	0.019	0.035	0.310	0.018	0.055
		2	0.320	0.109	0.267	0.310	0.028	0.080	0.310	0.019	0.055	0.309	0.019	0.029
						0.310	0.024	0.057	0.310	0.018	0.031	0.338	0.017	0.026
		3	0.316	0.143	0.356	0.286	0.054	0.139	0.308	0.040	0.106	0.290	0.034	0.105
						0.313	0.030	0.055	0.314	0.015	0.093	0.346	0.019	0.055
2.5	(50,20)	1	0.329	0.188	0.439	0.328	0.085	0.103	0.337	0.049	0.086	0.290	0.038	0.117
						0.313	0.082	0.097	0.305	0.043	0.079	0.348	0.028	0.079
		2	0.311	0.148	0.362	0.395	0.081	0.099	0.397	0.042	0.082	0.303	0.025	0.077
						0.321	0.078	0.093	0.321	0.037	0.061	0.348	0.021	0.060
		3	0.191	0.177	0.470	0.342	0.120	0.144	0.265	0.064	0.193	0.254	0.074	0.125
						0.356	0.079	0.094	0.359	0.057	0.123	0.388	0.060	0.101
	(50,40)	1	0.327	0.169	0.370	0.315	0.082	0.084	0.323	0.033	0.073	0.303	0.027	0.077
						0.336	0.069	0.079	0.317	0.028	0.057	0.305	0.024	0.070
		2	0.316	0.135	0.331	0.306	0.078	0.090	0.314	0.032	0.073	0.306	0.023	0.069
						0.330	0.058	0.072	0.332	0.027	0.053	0.311	0.017	0.053
		3	0.295	0.166	0.433	0.288	0.111	0.124	0.302	0.056	0.160	0.274	0.055	0.116
						0.310	0.075	0.092	0.362	0.044	0.116	0.297	0.032	0.096
	(80,32)	1	0.318	0.143	0.350	0.338	0.069	0.079	0.308	0.027	0.062	0.305	0.023	0.070
						0.305	0.051	0.074	0.311	0.020	0.051	0.305	0.021	0.060
		2	0.320	0.116	0.284	0.339	0.052	0.085	0.343	0.021	0.055	0.312	0.017	0.050
						0.305	0.043	0.068	0.306	0.019	0.049	0.320	0.012	0.025
		3	0.191	0.158	0.415	0.356	0.075	0.116	0.362	0.045	0.104	0.309	0.020	0.060
						0.293	0.050	0.104	0.293	0.035	0.095	0.318	0.014	0.031
	(80,64)	1	0.325	0.120	0.294	0.307	0.060	0.072	0.341	0.022	0.058	0.324	0.019	0.044
						0.311	0.047	0.055	0.338	0.018	0.047	0.332	0.016	0.030
		2	0.323	0.112	0.276	0.311	0.048	0.066	0.312	0.019	0.051	0.328	0.016	0.038
						0.312	0.039	0.053	0.312	0.016	0.038	0.333	0.010	0.021
		3	0.306	0.141	0.352	0.262	0.071	0.096	0.330	0.039	0.069	0.332	0.017	0.053
						0.330	0.042	0.067	0.314	0.024	0.069	0.326	0.013	0.024

Table 6: The interval results of γ

T	(n, m)	Scheme	ACI		HPD			
Prior →					1		2	
1.5	(50,20)	1	1.270	0.924	0.312	0.934	0.246	0.946
		2	1.362	0.915	0.314	0.932	0.250	0.944
		3	1.688	0.904	0.345	0.927	0.255	0.941
	(50,40)	1	0.921	0.929	0.243	0.946	0.215	0.953
		2	0.956	0.922	0.267	0.943	0.221	0.951
		3	0.979	0.915	0.284	0.939	0.231	0.948
	(80,32)	1	0.876	0.937	0.204	0.952	0.198	0.956
		2	0.857	0.928	0.240	0.949	0.205	0.954
		3	0.893	0.924	0.249	0.947	0.227	0.950
	(80,64)	1	0.698	0.945	0.191	0.954	0.177	0.959
		2	0.705	0.936	0.200	0.953	0.187	0.958
		3	0.803	0.932	0.228	0.950	0.201	0.955
2.5	(50,20)	1	1.280	0.922	0.317	0.931	0.256	0.942
		2	1.375	0.913	0.324	0.929	0.259	0.940
		3	1.689	0.904	0.348	0.925	0.260	0.937
	(50,40)	1	0.939	0.927	0.260	0.942	0.234	0.949
		2	0.964	0.920	0.277	0.941	0.238	0.949
		3	1.391	0.909	0.288	0.938	0.246	0.945
	(80,32)	1	0.887	0.934	0.225	0.946	0.203	0.953
		2	0.915	0.926	0.257	0.943	0.230	0.950
		3	1.278	0.912	0.265	0.942	0.235	0.948
	(80,64)	1	0.699	0.943	0.198	0.953	0.199	0.956
		2	0.714	0.933	0.220	0.950	0.218	0.954
		3	1.208	0.918	0.232	0.948	0.219	0.954

Table 7: The interval results of δ

T	(n, m)	Scheme	ACI		HPD			
Prior →					1		2	
1.5	(50,20)	1	0.645	0.914	0.244	0.931	0.181	0.948
		2	0.661	0.909	0.253	0.928	0.186	0.947
		3	1.217	0.893	0.260	0.927	0.188	0.947
	(50,40)	1	0.477	0.926	0.210	0.936	0.161	0.955
		2	0.498	0.918	0.214	0.935	0.168	0.953
		3	1.194	0.899	0.217	0.933	0.184	0.950

(Continued)

Table 7 (continued)

T	(n,m)	Scheme	ACI		HPD			
Prior →					1		2	
	(80,32)	1	0.398	0.932	0.183	0.944	0.155	0.961
		2	0.434	0.925	0.211	0.941	0.158	0.960
		3	1.025	0.908	0.214	0.939	0.164	0.958
	(80,64)	1	0.321	0.935	0.169	0.947	0.139	0.966
		2	0.333	0.934	0.194	0.942	0.153	0.962
		3	0.820	0.917	0.204	0.940	0.159	0.961
2.5	(50,20)	1	0.674	0.911	0.250	0.927	0.187	0.945
		2	0.712	0.903	0.262	0.925	0.189	0.945
		3	1.267	0.887	0.267	0.925	0.216	0.942
	(50,40)	1	0.517	0.921	0.216	0.933	0.174	0.948
		2	0.522	0.914	0.219	0.932	0.187	0.946
		3	1.198	0.897	0.224	0.929	0.200	0.944
	(80,32)	1	0.423	0.928	0.212	0.941	0.161	0.957
		2	0.423	0.923	0.217	0.938	0.173	0.954
		3	1.035	0.906	0.219	0.937	0.181	0.952
	(80,64)	1	0.328	0.934	0.202	0.943	0.153	0.959
		2	0.346	0.932	0.206	0.942	0.158	0.958
		3	0.924	0.913	0.216	0.939	0.159	0.958

Table 8: The interval results of μ

T	(n,m)	Scheme	ACI		HPD			
Prior →					1		2	
1.5	(50,20)	1	1.046	0.921	0.154	0.943	0.114	0.948
		2	1.150	0.916	0.169	0.940	0.117	0.946
		3	1.395	0.908	0.217	0.932	0.129	0.943
	(50,40)	1	0.715	0.931	0.149	0.945	0.102	0.951
		2	0.811	0.927	0.152	0.944	0.110	0.949
		3	1.299	0.913	0.192	0.937	0.114	0.948
	(80,32)	1	0.633	0.935	0.128	0.947	0.097	0.954
		2	0.752	0.927	0.146	0.945	0.108	0.952
		3	0.979	0.920	0.147	0.945	0.111	0.951
	(80,64)	1	0.522	0.939	0.122	0.950	0.084	0.956
		2	0.555	0.937	0.125	0.948	0.092	0.955
		3	0.895	0.929	0.128	0.947	0.101	0.953

(Continued)

Table 8 (continued)

T	(n,m)	Scheme	ACI		HPD			
Prior →					1		2	
2.5	(50,20)	1	1.139	0.918	0.155	0.942	0.122	0.945
		2	1.217	0.909	0.183	0.936	0.124	0.944
		3	1.418	0.905	0.227	0.929	0.132	0.942
	(50,40)	1	0.857	0.925	0.151	0.943	0.107	0.948
		2	0.948	0.920	0.155	0.942	0.113	0.946
		3	1.303	0.911	0.202	0.935	0.116	0.946
	(80,32)	1	0.768	0.931	0.148	0.945	0.105	0.949
		2	0.887	0.924	0.149	0.944	0.112	0.947
		3	1.057	0.916	0.166	0.941	0.113	0.946
	(80,64)	1	0.541	0.937	0.135	0.948	0.094	0.954
		2	0.589	0.935	0.138	0.947	0.105	0.952
		3	0.943	0.927	0.141	0.945	0.110	0.950

Table 9: The interval results of $R(t)$

T	(n,m)	Scheme	ACI		HPD			
Prior →					1		2	
1.5	(50,20)	1	0.083	0.934	0.079	0.937	0.072	0.939
		2	0.082	0.935	0.078	0.938	0.060	0.944
		3	0.081	0.937	0.076	0.940	0.055	0.946
	(50,40)	1	0.081	0.936	0.076	0.939	0.065	0.942
		2	0.078	0.938	0.075	0.940	0.058	0.945
		3	0.077	0.939	0.074	0.941	0.048	0.950
	(80,32)	1	0.077	0.938	0.073	0.940	0.061	0.944
		2	0.073	0.940	0.071	0.942	0.047	0.949
		3	0.070	0.9410	0.067	0.9440	0.045	0.951
	(80,64)	1	0.073	0.939	0.070	0.943	0.046	0.950
		2	0.071	0.941	0.069	0.944	0.045	0.951
		3	0.062	0.944	0.054	0.947	0.042	0.953
2.5	(50,20)	1	0.079	0.937	0.074	0.939	0.065	0.942
		2	0.078	0.937	0.073	0.940	0.059	0.945
		3	0.073	0.939	0.070	0.942	0.055	0.947
	(50,40)	1	0.076	0.938	0.068	0.943	0.059	0.945
		2	0.074	0.939	0.066	0.943	0.056	0.946
		3	0.067	0.941	0.065	0.944	0.053	0.947

(Continued)

Table 9 (continued)

T	(n, m)	Scheme	ACI		HPD			
Prior →					1		2	
	(80,32)	1	0.073	0.940	0.071	0.942	0.051	0.947
		2	0.071	0.942	0.069	0.943	0.048	0.949
		3	0.064	0.945	0.061	0.946	0.043	0.952
	(80,64)	1	0.072	0.941	0.069	0.944	0.048	0.949
		2	0.070	0.942	0.067	0.944	0.045	0.951
		3	0.060	0.946	0.053	0.947	0.032	0.957

Table 10: The interval results of $h(t)$

T	(n, m)	Scheme	ACI		HPD			
Prior →					1		2	
1.5	(50,20)	1	0.644	0.924	0.258	0.932	0.169	0.942
		2	0.523	0.929	0.161	0.943	0.110	0.951
		3	0.496	0.934	0.124	0.945	0.085	0.954
	(50,40)	1	0.577	0.931	0.170	0.941	0.099	0.950
		2	0.510	0.933	0.135	0.946	0.073	0.956
		3	0.402	0.937	0.119	0.948	0.072	0.956
	(80,32)	1	0.517	0.933	0.149	0.943	0.072	0.953
		2	0.429	0.936	0.091	0.956	0.058	0.959
		3	0.389	0.940	0.075	0.958	0.045	0.962
	(80,64)	1	0.441	0.937	0.119	0.950	0.081	0.955
		2	0.417	0.939	0.069	0.959	0.035	0.964
		3	0.287	0.944	0.066	0.961	0.032	0.965
2.5	(50,20)	1	0.662	0.921	0.255	0.931	0.145	0.938
		2	0.558	0.925	0.215	0.937	0.136	0.947
		3	0.524	0.928	0.171	0.940	0.100	0.952
	(50,40)	1	0.544	0.927	0.164	0.938	0.139	0.945
		2	0.528	0.930	0.160	0.943	0.094	0.951
		3	0.497	0.932	0.131	0.945	0.081	0.955
	(80,32)	1	0.464	0.935	0.159	0.940	0.094	0.951
		2	0.431	0.937	0.134	0.945	0.071	0.956
		3	0.422	0.938	0.125	0.947	0.049	0.960
	(80,64)	1	0.453	0.936	0.125	0.946	0.054	0.958
		2	0.362	0.941	0.086	0.954	0.043	0.963
		3	0.297	0.943	0.075	0.958	0.040	0.965

5 Real Data Analysis

In this part, four examples based on data sets gathered from the physical and biomedicine domains are analyzed to highlight the benefit of the presented estimating approaches and to verify the validity of the study outcomes to real circumstances.

5.1 Physical Applications

This subsection describes the analysis of two different sets of physical data. First data set (say Eng-I), by Xu et al. [23], consists of forty observations of the time-to-failure (10^3 h) of the turbocharger of the same type of engine. Other data set (say Eng-II), by Murthy et al. [24], representing the failure times of twenty mechanical units. All data points in Eng-II are multiplied by ten for computational convenience. In Table 11, both Eng-I and Eng-II data sets are sorted and presented.

Table 11: Times-to-failure of turbocharger (Eng-I) and mechanical components (Eng-II)

Data	Times									
Eng-I	1.6	2.0	2.6	3.0	3.5	3.9	4.5	4.6	4.8	5.0
	5.1	5.3	5.4	5.6	5.8	6.0	6.0	6.1	6.3	6.5
	6.5	6.7	7.0	7.1	7.3	7.3	7.3	7.7	7.7	7.8
	7.9	8.0	8.1	8.3	8.4	8.4	8.5	8.7	8.8	9.0
Eng-II	0.67	0.68	0.76	0.81	0.84	0.85	0.85	0.86	0.89	0.98
	0.98	1.14	1.14	1.15	1.21	1.25	1.31	1.49	1.60	4.85

Firstly, to see the superiority of the proposed GLL model, using Eng-I and Eng-II data sets, the GLL distribution is compared to five other common distributions in literature, namely: alpha-power exponential (APE), generalized-exponential (GE), log-logistic (LL), Weibull (W), and gamma (G) distributions. These competitive distributions (for $x > 0$ and $\gamma, \delta, \mu > 0$) are reported in Table 12.

Table 12: Competitive models of the GLL distribution

Dist.	PDF	Author(s)
APE	$\delta(\gamma-1)^{-1}\log(\gamma)e^{-\delta x}\gamma^{1-e^{-\delta x}}$	Mahdavi and Kundu [25]
GE	$\gamma\delta e^{-\delta x}(1-e^{-\delta x})^{\gamma-1}$	Gupta and Kundu [26]
LL	$\frac{\gamma}{\delta}\left(\frac{x}{\delta}\right)^{\gamma-1}\left(1+\left(\frac{x}{\delta}\right)^{\gamma}\right)^{-2}$	Bain [27]
W	$\gamma\delta x^{\gamma-1}e^{-\delta x^{\gamma}}$	Weibull [28]
G	$\frac{\delta^{\gamma}}{\Gamma(\gamma)}x^{\gamma-1}e^{-\delta x}$	Johnson et al. [29]

Using different criteria of distribution selection namely: (1) negative log-likelihood (NL), (2) Akaike (A), (3) consistent Akaike (CA), (4) Bayesian (B), and (5) Hannan–Quinn (HQ) informations, we shall compare the GLL distribution with its competitive models based on Eng-I and Eng-II data sets. Besides them, three popular statistics, namely: (1) Anderson–Darling (AD), (2) Cramér–von

Mises (CvM), and (3) Kolmogorov–Smirnov (KS) (with its p-value) statistics are also considered. All given information criteria or goodness-of-fit statistics are evaluated through the maximum likelihood estimation method. From Eng-I and Eng-II data sets, in Table 13, the fitted values of (A, CA, B, HQ, AD, CvM, NL, KS) as well as the MLEs (with their standard errors (St.Es)) of unknown model parameters are listed. This comparison is made via $\mathcal{R}$ software by installing the 'AdequacyModel' package introduced by Marinho et al. [30]. The findings in Table 13 stated that the GLL model has the highest p-value and the smallest value of A, CA, B, HQ, AD, CvM, and KS statistics. This result shows that the GLL distribution is better than others. Further, using Eng-I and Eng-II data sets, Fig. 5 displays the total time test (TTT), fitted density with histogram and fitted reliability plots of the GLL distribution and all other considered distributions. Fig. 5 shows that the GLL hazard rate has an increasing shape (for Eng-I data) and upside-down bathtub shape (for Eng-II data). It also indicates that the fitted GLL reliability line is the closest to the empirical reliability line, and the fitted GLL density holds the histograms well for both given Eng-I and Eng-II data sets.

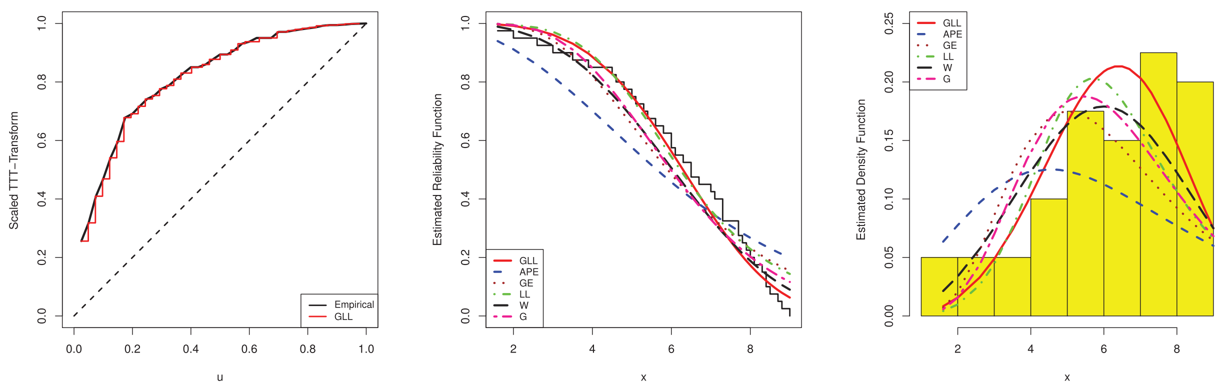
Table 13: Fitting results of GLL and its competitors from Eng-I and Eng-II data sets

Model	MLE St.E			A HQ	AC AD	B CvM	NL	KS	
	γ	δ	μ					Statistic	p-value
				Eng-I					
GLL	3.8726	0.1445	0.0024	170.95	171.62	176.02	82.476	0.108	0.742
	0.5178	0.0062	1.1659	172.78	0.5730	0.0770			
APE	95.659	0.3373	–	193.55	193.88	196.93	94.776	0.211	0.056
	56.013	0.0335	–	194.77	1.2090	0.1793			
GE	9.5141	0.4498	–	184.29	184.61	187.66	90.143	0.154	0.298
	2.8959	0.0578	–	185.51	1.7601	0.2757			
LL	4.8416	6.2249	–	181.41	181.74	184.79	88.707	0.144	0.381
	0.6536	0.3480	–	182.63	1.4072	0.2142			
W	3.1162	0.0026	–	171.37	171.69	174.75	83.685	0.119	0.624
	0.1270	0.0005	–	172.59	0.7081	0.0976			
G	7.7226	0.8096	–	178.82	179.14	182.20	87.410	0.128	0.531
	1.6907	0.1831	–	180.04	1.3616	0.2053			
				Eng-II					
GLL	14.331	1.1797	1.3262	20.128	21.628	23.115	7.0641	0.112	0.962
	7.5484	0.1101	0.1143	20.711	0.3017	0.0420			
APE	49.874	1.7167	–	38.477	39.183	40.468	17.238	0.276	0.096
	41.341	0.2669	–	38.865	1.8465	0.2810			
GE	13.809	2.7741	–	30.151	30.856	32.142	13.075	0.160	0.683
	8.3600	0.6103	–	30.539	1.2513	0.1758			
LL	5.0860	1.0161	–	23.746	24.452	25.737	9.8729	0.119	0.938
	0.9628	0.0755	–	24.134	0.6606	0.0880			

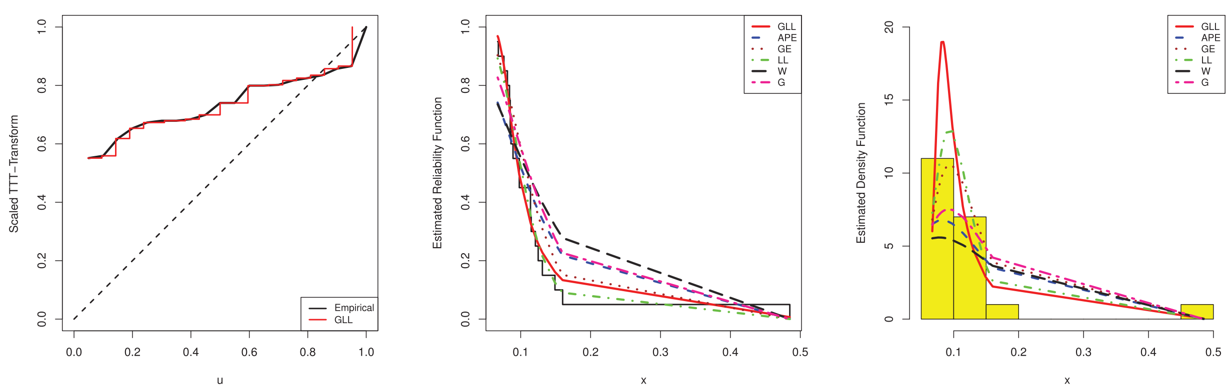
(Continued)

Table 13 (continued)

Model	MLE St.E			A HQ	AC AD	B CvM	NL	KS	
	γ	δ	μ					Statistic	p-value
W	1.6422	0.5921	–	43.258	43.964	45.249	19.629	0.264	0.123
	0.2313	0.1620	–	43.647	2.4520	0.3971			
G	4.2439	0.2864	–	36.322	37.027	38.313	16.161	0.225	0.262
	1.2924	0.0926	–	36.710	1.9068	0.2920			



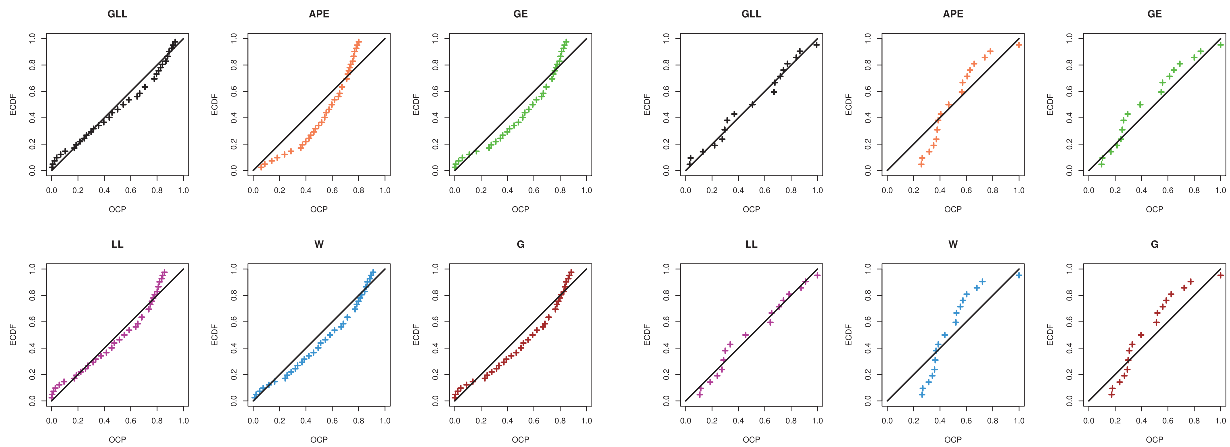
(a) Eng-I



(b) Eng-II

Figure 5: The TTT (left), PDFs (center) and RFs (right) of GLL and its competitors models from Eng-I and Eng-II data sets

The associated probability-probability (PP) plot, which displays the relation between observed cumulative probability (OCP) and expected cumulative distribution function (ECDF), of the GLL and others, is plotted; see Fig. 6. It also backs up identical facts listed in Table 13. Three artificial APC-II samples based on different combinations of m, T and $(R_1, R_2, \ldots, R_m)$ are generated from the complete Eng-I and Eng-II data sets, respectively. By taking $m = 20$ and $\mathbf{R} = (1^{20})$ (for Eng-I data) and $m = 10$ and $\mathbf{R} = (1^{10})$ are created; see Table 14. For example, $\mathbf{R} = (1, 1, 1, 1, 1)$ is symbolized by $\mathbf{R} = (1^{10})$ for short. For each censored sample in Table 14, the acquired frequentist and Bayes estimates (with their St.Es) of γ, δ, μ, $R(t)$, and $h(t)$ at distinct times $t = 5$ (for Eng-I) and $t = 0.5$ (for Eng-II) are acquired, see Table 15. Also, in Table 16, the 95% ACI/HPD intervals (with their lengths) are presented. Based on the improper gamma priors, to calculate the Bayes point (or HPD interval) estimates, we discard the first 10,000 'burn-in' iterations from 50,000 MCMC variates. The Bayes estimates are developed using SE and GE (for $\delta(= -5, -0.05, +5)$) loss functions. As a result, from Tables 15–16, the results of γ, δ, μ, $R(t)$ or $h(t)$ are quite near to each other.



(a) Eng-I (b) Eng-II

Figure 6: The PP diagrams of GLL and its competitive models from Eng-I and Eng-II data sets

Table 14: Artificial APC-II samples from Eng-I and Eng-II data sets

Data[Sample]	Censored data										$T(d)$	R_m
Eng-I[S_1]	1.6	2.0	2.6	3.0	3.5	3.9	4.6	4.8	5.0	5.1	6.4(14)	6
	5.3	5.4	5.6	6.3	6.5	6.7	7.0	7.1	7.3	7.9		
Eng-I[S_2]	1.6	2.0	3.0	3.9	4.6	5.1	5.3	5.4	5.6	5.8	6.9(15)	5
	6.1	6.3	6.5	6.5	6.7	7.3	7.3	7.3	7.8	7.9		
Eng-I[S_3]	1.6	2.0	3.0	3.5	3.9	4.5	5.0	5.1	5.3	5.4	7.1(17)	3
	5.6	5.8	6.0	6.3	6.5	6.5	6.7	7.3	7.3	7.8		
Eng-II[S_1]	0.67	0.68	0.81	0.84	0.85	0.85	0.98	1.14	1.15	1.21	0.9(6)	4
Eng-II[S_2]	0.67	0.76	0.84	0.85	0.86	0.89	0.98	1.14	1.14	1.15	1(7)	3
Eng-II[S_3]	0.67	0.81	0.84	0.85	0.85	0.86	0.89	1.15	1.25	1.31	1.2(8)	2

Table 15: Point estimates of γ, δ, μ, $R(t)$ and $h(t)$ from Eng-I and Eng-II data sets

Data[Sample]	Par.	MLE		SE		GE					
$\tau \rightarrow$						-5		-0.05		$+5$	
		Est.	St.E	Est.	St.E	Est.	St.E	Est.	St.E	Est.	St.E
Eng-I[S_1]	γ	2.92768	0.00088	2.92735	0.01003	2.92742	0.00026	2.92734	0.00034	2.92725	0.00043
	δ	0.13634	0.01041	0.13607	0.00725	0.13684	0.00049	0.13588	0.00046	0.13490	0.00145
	μ	0.00018	0.00113	0.00018	0.00010	0.00018	0.00090	0.00018	0.00050	0.00017	0.00011
	$R(5)$	0.72201	0.05290	0.72242	0.03642	0.72603	0.00403	0.72154	0.00047	0.71673	0.00528
	$h(5)$	0.19072	0.04289	0.19112	0.02972	0.20027	0.00954	0.18893	0.00180	0.17720	0.01352
Eng-I[S_2]	γ	3.65180	0.00762	3.65161	0.01002	3.65166	0.00014	3.65160	0.00021	3.65153	0.00028
	δ	0.13231	0.00810	0.13197	0.00629	0.13257	0.00025	0.13183	0.00048	0.13106	0.00125
	μ	0.00153	0.00639	0.00153	0.00001	0.00153	0.00014	0.00153	2.02147	0.00153	0.00030
	$R(5)$	0.80157	0.05089	0.80187	0.03059	0.80417	0.00261	0.80131	0.00021	0.79827	0.00330
	$h(5)$	0.16154	0.04607	0.16180	0.02802	0.17135	0.00981	0.15950	0.00205	0.14720	0.01435
Eng-I[S_3]	γ	3.59502	0.00314	3.59471	0.01003	3.59477	0.00025	3.59470	0.00032	3.59463	0.00039
	δ	0.14067	0.00875	0.14038	0.00661	0.14099	0.00032	0.14023	0.00045	0.13943	0.00124
	μ	0.00060	0.00009	0.00059	0.00010	0.00062	0.00024	0.00058	0.00017	0.00051	0.00009
	$R(5)$	0.75409	0.04762	0.75436	0.03571	0.75769	0.00360	0.75355	0.00054	0.74912	0.00497
	$h(5)$	0.20293	0.04538	0.20348	0.03427	0.21483	0.01190	0.20074	0.00220	0.18599	0.01695
Eng-II[S_1]	γ	11.9203	8.26081	11.9194	0.09904	11.9211	0.00074	11.9190	0.00129	11.9170	0.00337
	δ	1.13074	0.17359	1.14652	0.05312	1.15100	0.02026	1.14545	0.01471	1.13978	0.00905
	μ	1.29443	0.20659	1.32327	0.07731	1.33102	0.03660	1.32142	0.02700	1.31155	0.01713
	$R(0.5)$	0.99889	0.00280	0.99851	0.00091	0.99851	0.00037	0.99851	0.00038	0.99851	0.00038
	$h(0.5)$	0.02646	0.05891	0.03527	0.02134	0.05792	0.03146	0.03098	0.00452	0.01520	0.01126
Eng-II[S_2]	γ	6.38145	0.08742	6.38457	0.09958	6.38767	0.00622	6.38383	0.00238	6.37991	0.00154
	δ	0.89909	0.04456	0.90384	0.04020	0.90733	0.00824	0.90300	0.00391	0.89842	0.00067
	μ	0.14165	0.14672	0.14444	0.01022	0.14576	0.00411	0.14412	0.00246	0.14238	0.00073
	$R(0.5)$	0.99393	0.00706	0.99352	0.00192	0.99352	0.00041	0.99351	0.00042	0.99351	0.00043
	$h(0.5)$	0.07765	0.08945	0.08299	0.02433	0.09615	0.01850	0.07980	0.00215	0.06244	0.01521
Eng-II[S_3]	γ	16.6830	6.29230	16.6838	0.10033	16.6850	0.00203	16.6835	0.00054	16.6820	0.00099
	δ	1.13158	0.07186	1.13078	0.02205	1.13164	0.00006	1.13058	0.00100	1.12948	0.00210
	μ	1.26372	0.08401	1.26419	0.00999	1.26435	0.00063	1.26415	0.00043	1.26396	0.00023
	$R(0.5)$	0.99993	0.00024	0.99992	0.00026	0.99952	0.00020	0.99922	0.00010	0.99902	0.00010
	$h(0.5)$	0.00249	0.00744	0.00259	0.00085	0.00313	0.00064	0.00246	0.00003	0.00180	0.00069

Table 16: Interval estimates of γ, δ, μ, $R(t)$ and $h(t)$ from Eng-I and Eng-II data sets

Data[Sample]	Par.	ACI			HPD		
		Lower	Upper	Length	Lower	Upper	Length
Eng-I[S_1]	γ	2.92595	2.92941	0.00345	2.90830	2.94737	0.03907
	δ	0.11593	0.15675	0.04082	0.12209	0.15022	0.02813
	μ	0.00000	0.00239	0.00239	0.00011	0.00030	0.00024
	$R(5)$	0.61832	0.82569	0.20738	0.65110	0.79177	0.14067
	$h(5)$	0.10665	0.27479	0.16814	0.13352	0.24779	0.11427

(Continued)

Table 16 (continued)

Data[Sample]	Par.	ACI			HPD		
		Lower	Upper	Length	Lower	Upper	Length
Eng-I[S_2]	γ	3.63687	3.66673	0.02986	3.63182	3.67091	0.03909
	δ	0.11643	0.14819	0.03176	0.11992	0.14444	0.02452
	μ	0.00000	0.01406	0.01406	0.00141	0.00165	0.00024
	$R(5)$	0.70183	0.90130	0.19947	0.74160	0.86086	0.11927
	$h(5)$	0.07126	0.25183	0.18057	0.10952	0.21836	0.10884
Eng-I[S_3]	γ	3.58886	3.60118	0.01232	3.57550	3.61472	0.03922
	δ	0.12352	0.15782	0.03430	0.12744	0.15308	0.02565
	μ	0.00041	0.00078	0.00037	0.00039	0.00079	0.00039
	$R(5)$	0.66076	0.84742	0.18666	0.68463	0.82245	0.13781
	$h(5)$	0.11399	0.29187	0.17788	0.13926	0.27108	0.13182
Eng-II[S_1]	γ	0.00000	28.1112	28.1112	11.7305	12.1229	0.39242
	δ	0.79051	1.47096	0.68050	1.04888	1.24511	0.19624
	μ	0.88952	1.69933	0.80980	1.18382	1.46199	0.27817
	$R(0.5)$	0.99340	0.99890	0.00550	0.99681	0.99965	0.00284
	$h(0.5)$	0.00000	0.14193	0.14193	0.00827	0.07529	0.06702
Eng-II[S_2]	γ	6.21010	6.55280	0.34270	6.19308	6.57916	0.38608
	δ	0.81176	0.98642	0.17466	0.82447	0.97946	0.15499
	μ	0.00000	0.42921	0.42921	0.12553	0.16381	0.03828
	$R(0.5)$	0.98009	0.99940	0.01930	0.98984	0.99695	0.00711
	$h(0.5)$	0.00000	0.25297	0.25297	0.03879	0.12879	0.08999
Eng-II[S_3]	γ	4.35032	29.0157	24.6653	16.4912	16.8820	0.39083
	δ	0.99073	1.27242	0.28169	1.08708	1.17361	0.08653
	μ	1.09907	1.42837	0.32929	1.24587	1.28460	0.03873
	$R(0.5)$	0.99935	0.99990	0.00060	0.99951	0.99980	0.00031
	$h(0.5)$	0.00000	0.01708	0.01708	0.00112	0.00429	0.00317

5.2 *Biomedical Applications*

Two biomedical applications from clinical trials, one based on real human medicine and the other on real veterinary medicine, are discussed in this subsection. First data set (say Biom-I) describes a relief time (in hours) for fifty arthritis patients. For computational purposes, each data point in Biom-I is multiplied by ten. This data was first presented by Wingo [31] and reanalyzed by Wu et al. [32]. The other data set represents the lifetimes (in days) of 72 guinea pigs injected with various doses (per 0.5 mL) of tubercle bacilli as 4×10^6 bacillary units. This data was originally given by Bjerkedal [33] and later discussed by Dube et al. [34] and Xu and Gui [35]. Table 17 presents both Biom-I and Biom-II data sets in order.

Table 17: Survival times of arthritis patients (Biom-I) and guinea pigs (Biom-II)

Data	Times														
Biom-I	2.9	2.9	3.4	3.4	3.5	3.6	3.6	3.6	4.4	4.4	4.6	4.6	4.9	4.9	5.0
	5.0	5.2	5.4	5.5	5.5	5.5	5.6	5.7	5.8	5.9	5.9	6.0	6.0	6.1	6.1
	6.2	6.4	6.8	7.0	7.0	7.1	7.1	7.1	7.2	7.3	7.5	7.5	8.0	8.0	8.1
	8.2	8.4	8.4	8.4	8.7										
Biom-II	12	15	22	24	24	32	32	33	34	38	38	43	44	48	52
	53	54	54	55	56	57	58	58	59	60	60	60	60	61	62
	63	65	65	67	68	70	70	72	73	75	76	76	81	83	84
	85	87	91	95	96	98	99	109	110	121	127	129	131	143	146
	146	175	175	211	233	258	258	263	297	341	341	376			

From Biom-I and Biom-II data sets, to show the GLL distribution has the best fit compared to APE, GE, LL, W and G lifetime models, the proposed criteria called NL, A, CA, B, HQ, AD, CvM and KS (p-value) are computed and presented in Table 18. Additionally, the MLE (with its St.E) of γ, δ or μ is also obtained, see Table 18. It indicates that the GLL distribution offers an adequate fit compared to its competing distributions.

Table 18: Fitting results of GLL and its competitive models from Biom-I and Biom-II data sets

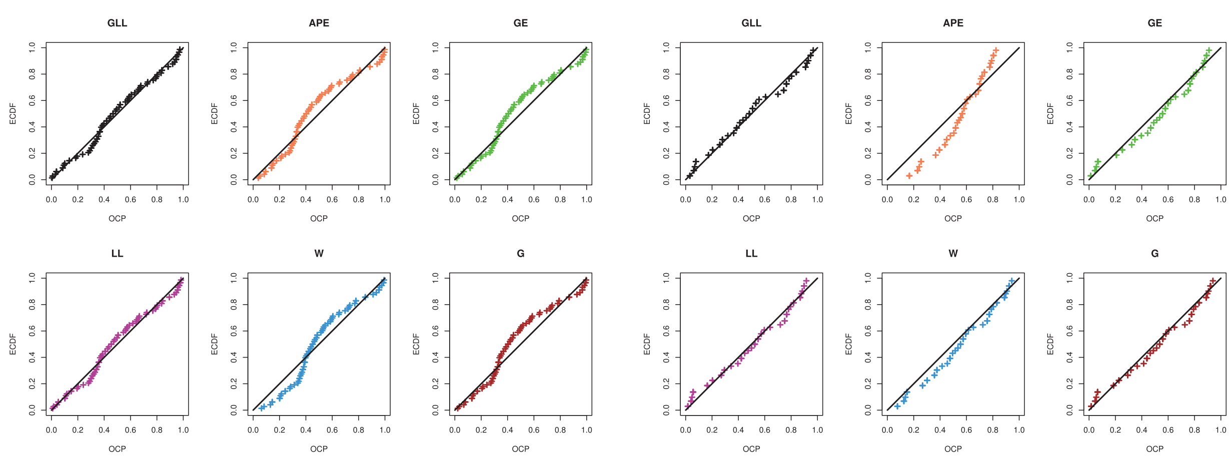
Model	MLE St.E			A HQ	AC AD	B CvM	NL	KS	
	γ	δ	μ					Statistic	p-value
Biom-I									
GLL	4.2457	0.1537	0.0091	193.39	193.91	199.12	93.694	0.084	0.872
	0.4799	0.0054	0.5378	195.57	0.3915	0.0477			
APE	211.36	0.3823	–	225.03	225.28	228.85	110.51	0.207	0.028
	112.33	0.0313		226.48	0.6228	0.0761			
GE	26.459	0.6482	–	198.85	199.10	202.67	97.424	0.108	0.605
	9.5486	0.0732		200.30	0.9989	0.1374			
LL	5.8875	5.8019	–	198.79	199.04	202.61	97.394	0.103	0.661
	0.6906	0.2445		200.24	0.7940	0.1002			
W	3.2688	0.0024	–	196.20	196.45	200.02	96.100	0.117	0.500
	0.1124	0.0004		197.66	0.4037	0.0469			
G	12.457	0.4741	–	194.52	194.78	198.35	95.261	0.101	0.684
	2.4584	0.0955		195.98	0.6296	0.0777			

(Continued)

Table 18 (continued)

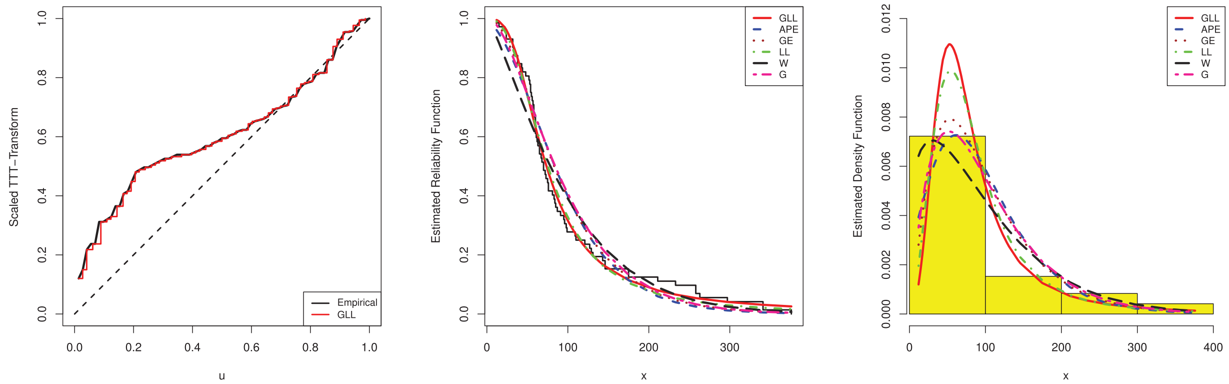
Model	MLE			A	AC	B	NL	KS	
	St.E			HQ	AD	CvM		Statistic	p-value
	γ	δ	μ						
				Biom-II					
GLL	3.0708	0.0146	0.0168	784.95	785.31	791.78	389.48	0.086	0.666
	0.5976	0.0015	0.0031	787.67	0.4658	0.0769			
APE	24.576	0.0189	–	797.16	797.33	801.71	396.58	0.138	0.130
	20.135	0.0025		798.97	2.2788	0.4092			
GE	2.4813	0.0170	–	790.22	790.39	794.77	393.11	0.133	0.157
	0.4788	0.0022		792.03	1.5917	0.2918			
LL	2.5404	75.311	–	783.93	784.10	788.48	389.96	0.086	0.657
	0.2537	6.0220		785.74	0.6557	0.1171			
W	1.2575	0.0029	–	799.70	799.87	804.25	397.85	0.144	0.100
	0.0428	0.0005		801.51	2.2094	0.4019			
G	2.0761	48.106	–	792.50	792.67	797.05	394.25	0.138	0.127
	0.3227	8.4580		794.31	1.8587	0.3395			

The PP, TTT, estimated RFs and estimated PDFs plots of GLL, APE, GE, LL, W and G distributions are displayed in Fig. 7–8. However, Fig. 7 shows that the GLL model is the best when compared to other considered distributions in presence of Biom-I and Biom-II data sets. Also, Fig. 8 demonstrates that the GLL hazard rate increases for both given Biom-I and Biom-II data sets. It also indicates that the GLL density captures the data histograms quite well.

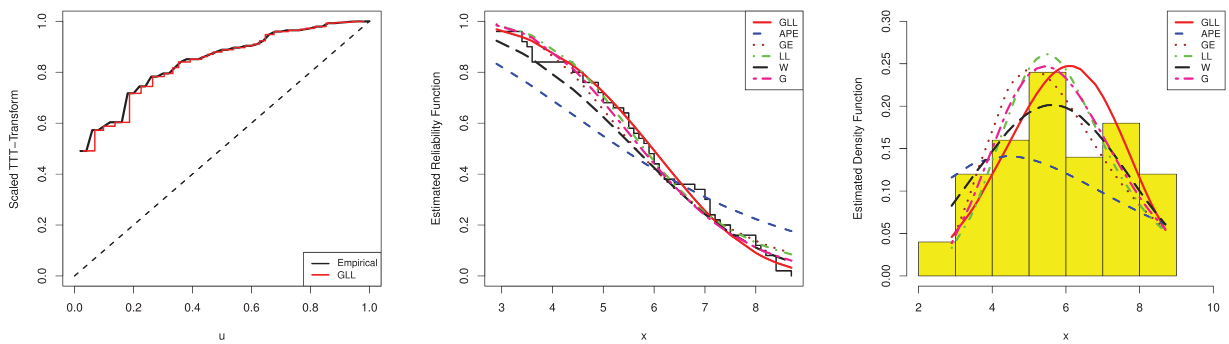


(a) Biom-I (b) Biom-II

Figure 7: The PP diagrams of the GLL and its competitive models from Biom-I and Biom-II data sets



(a) Biom-I



(b) Biom-II

Figure 8: The TTT (left), estimated PDFs (center) and estimated RFs (right) of the GLL and its competitive models from Biom-I and Biom-II data sets

Using the complete Biom-I and Biom-II data sets provided in Table 17, by taking $m = 25$ and $\mathbf{R} = (1^{10})$ (for Biom-I) as well as taking $m = 36$ and $\mathbf{R} = (1^{36})$ (for Biom-II), three artificial APC-II samples are generated and presented in Table 19. For each S_i, $i = 1, 2, 3$ in Biom-I (or Biom-II) data, the point as well as 95% interval estimates are provided in Tables 20–21, respectively. The reliability time parameters $R(t)$ and $h(t)$ from Biom-I and Biom-II data sets are estimated at distinct times $t = 5$ and 15, respectively. Using non-informative priors, based on 40,000 samples after 10,000 iterations as burn-in, all Bayes' calculations are performed. it is clear, from Tables 20 and 21, that the Bayes MCMC (or HPD interval) estimates of γ, δ, μ, $R(t)$, or $h(t)$ are better than those point (or interval) estimates derived from the likelihood approach.

To sum up, if the test sample is obtained from an adaptive Type-II progressively censoring, based on four different real-life data sets taken from physical and biomedical scenarios, we can decide that the investigated approaches to deriving point (or interval) estimates of the generalized log-logistic parameters of life are recommended.

Table 19: Artificial APC-II samples from Biom-I and Biom-II data sets

Data[Sample]	Censored data																		$T(d)$	R_m
Biom-I[S_1]	2.9	2.9	3.4	3.4	3.5	3.6	3.6	3.6	4.4	4.4	4.9	4.9	5.2	5.4	5.6	5.7	6.1	6.4	6.2(17)	8
	6.8	7.0	7.0	7.1	7.2	7.3	7.5													
Biom-I[S_2]	2.9	2.9	3.4	3.4	3.5	3.6	3.6	3.6	4.4	4.4	4.9	4.9	5.0	5.0	5.2	5.5	5.5	5.6	5.9(19)	6
	5.8	6.0	6.1	6.2	6.4	7.1	7.2													
Biom-I[S_3]	2.9	2.9	3.4	3.6	3.6	4.4	4.4	4.6	4.6	5.0	5.0	5.2	5.4	5.5	5.5	5.5	5.8	5.9	6.5(21)	4
	6.0	6.0	6.1	6.8	7.1	7.2	7.3													
Biom-II[S_1]	12	15	22	24	32	33	38	43	44	52	54	54	55	56	57	58	58	59	74(27)	9
	60	61	62	63	68	70	70	72	73	75	76	76	81	83	84	85	87	91		
Biom-II[S_2]	12	22	24	33	34	38	43	52	53	54	56	57	59	60	61	62	63	65	95(30)	6
	65	72	73	75	76	76	81	83	84	85	87	91	95	96	98	99	109	127		
Biom-II[S_3]	12	15	24	32	34	38	44	48	52	53	54	54	55	56	58	59	60	62	115(33)	3
	65	67	68	73	75	83	84	85	87	91	95	96	98	99	110	129	131	143		

Table 20: Point estimates of γ, δ, μ, $R(t)$ and $h(t)$ from Biom-I and Biom-II data sets

Data[Sample]	Par.	MLE		SE		GE					
$\tau \rightarrow$						-5		-0.05		$+5$	
		Est.	St.E	Est.	St.E	Est.	St.E	Est.	St.E	Est.	St.E
Biom-I[S_1]	γ	3.60346	0.00601	3.60363	0.01007	3.60369	0.00023	3.60362	0.00016	3.60355	0.00009
	δ	0.14122	0.00784	0.14223	0.00626	0.14276	0.00154	0.14210	0.00088	0.14141	0.00019
	μ	0.00124	0.00417	0.00127	0.00010	0.00129	0.00005	0.00127	0.00003	0.00125	0.00001
	$R(5)$	0.75175	0.04884	0.74507	0.03481	0.74816	0.00359	0.74432	0.00743	0.74022	0.01153
	$h(5)$	0.20565	0.04658	0.21285	0.03404	0.22316	0.01751	0.21039	0.00474	0.19725	0.00840
Biom-I[S_2]	γ	5.00048	0.00258	5.00058	0.01000	5.00062	0.00013	5.00057	0.00009	5.00052	0.00003
	δ	0.17742	0.00671	0.17471	0.00623	0.17506	0.00236	0.17462	0.00280	0.17416	0.00326
	μ	0.00379	0.00243	0.00380	0.00010	0.00381	0.00002	0.00371	0.00001	0.00379	0.00000
	$R(5)$	0.57731	0.06386	0.60022	0.05390	0.60797	0.03065	0.59831	0.02100	0.58769	0.01037
	$h(5)$	0.54942	0.11052	0.51387	0.08963	0.53996	0.00946	0.50764	0.04178	0.47438	0.07504
Biom-I[S_3]	γ	4.64350	0.00103	3.60363	0.01007	3.60369	0.00023	3.60362	0.00016	3.60355	0.00009
	δ	0.15117	0.00652	0.14223	0.00626	0.14276	0.00154	0.14210	0.00088	0.14141	0.00019
	μ	0.00270	0.00272	0.00127	0.00010	0.00129	0.00005	0.00120	0.00003	0.00125	0.00001
	$R(5)$	0.76137	0.04507	0.74507	0.03481	0.74816	0.00359	0.74432	0.00743	0.74022	0.01153
	$h(5)$	0.25319	0.05493	0.21285	0.03404	0.22316	0.01751	0.21039	0.00474	0.19725	0.00840
Biom-II[S_1]	γ	2.99899	0.00390	2.99900	0.00100	2.99900	0.00001	2.99900	0.00001	2.99900	0.00001
	δ	0.01196	0.00064	0.01207	0.00057	0.01212	0.00016	0.01206	0.00010	0.01199	0.00003
	μ	0.00065	0.00000	0.00071	0.00011	0.00073	0.00008	0.00070	0.00005	0.00066	0.00001
	$R(15)$	0.99424	0.00094	0.99404	0.00084	0.99404	0.00020	0.99404	0.00020	0.99404	0.00020
	$h(15)$	0.00116	0.00019	0.00120	0.00017	0.00124	0.00008	0.00118	0.00003	0.00113	0.00003
Biom-II[S_2]	γ	2.95831	0.64616	2.95831	0.00100	2.95831	0.00006	2.95829	0.00003	2.95832	0.00006
	δ	0.01091	0.00143	0.01102	0.00054	0.01107	0.00016	0.01101	0.00010	0.01094	0.00003
	μ	0.00880	0.00523	0.00880	0.00010	0.00880	0.00004	0.00878	0.00002	0.00879	0.00005
	$R(15)$	0.99529	0.00495	0.99512	0.00071	0.99512	0.00017	0.99512	0.00017	0.99511	0.00017
	$h(15)$	0.00093	0.00106	0.00096	0.00014	0.00100	0.00007	0.00096	0.00002	0.00091	0.00002

(Continued)

Table 20 (continued)

Data[Sample]	Par.	MLE		SE		GE					
$\tau \rightarrow$						-5		-0.05		$+5$	
		Est.	St.E	Est.	St.E	Est.	St.E	Est.	St.E	Est.	St.E
Biom-II[S_3]	γ	4.51601	0.54751	4.51601	0.00015	4.51601	0.00004	4.51596	0.00001	4.51600	0.00004
	δ	0.01499	0.00227	0.01522	0.00027	0.01523	0.00023	0.01520	0.00020	0.01522	0.00023
	μ	0.06902	0.00787	0.06902	0.00001	0.06902	0.00005	0.06902	0.00003	0.06902	0.00005
	$R(15)$	0.99922	0.00163	0.99916	0.00007	0.99916	0.00006	0.99915	0.00004	0.99916	0.00006
	$h(15)$	0.00016	0.00037	0.00018	0.00001	0.00018	0.00002	0.00017	0.00001	0.00018	0.00003

Table 21: Interval estimates of γ, δ, μ, $R(t)$ and $h(t)$ from Biom-I and Biom-II data sets

Data[Sample]	Par.	ACI			HPD		
		Lower	Upper	Length	Lower	Upper	Length
Biom-I[S_1]	γ	3.59167	3.61524	0.02358	3.58312	3.62307	0.03995
	δ	0.12586	0.15658	0.03073	0.13034	0.15436	0.02402
	μ	0.00000	0.00941	0.00941	0.00108	0.00147	0.00039
	$R(5)$	0.65603	0.84746	0.19143	0.67717	0.80992	0.13274
	$h(5)$	0.11436	0.29695	0.18259	0.15158	0.28043	0.12884
Biom-I[S_2]	γ	4.99543	5.00553	0.01010	4.98154	5.02031	0.03877
	δ	0.16427	0.19058	0.02631	0.16363	0.18547	0.02185
	μ	0.00000	0.00855	0.00855	0.00360	0.00399	0.00039
	$R(5)$	0.45215	0.70248	0.25032	0.50648	0.69609	0.18960
	$h(5)$	0.33281	0.76603	0.43322	0.36204	0.67977	0.31773
Biom-I[S_3]	γ	4.64148	4.64551	0.00404	3.58312	3.62307	0.03995
	δ	0.13840	0.16395	0.02556	0.13034	0.15436	0.02402
	μ	0.00000	0.00802	0.00802	0.00108	0.00147	0.00039
	$R(5)$	0.67304	0.84971	0.17667	0.67717	0.80992	0.13274
	$h(5)$	0.14554	0.36085	0.21531	0.15158	0.28043	0.12884
Biom-II[S_1]	γ	2.99136	3.00663	0.01528	2.99698	3.00096	0.00398
	δ	0.01070	0.01322	0.00252	0.01101	0.01317	0.00216
	μ	0.00065	0.00065	0.00000	0.00052	0.00090	0.00038
	$R(15)$	0.99240	0.99608	0.00368	0.99238	0.99556	0.00318
	$h(15)$	0.00079	0.00152	0.00074	0.00089	0.00153	0.00064
Biom-II[S_2]	γ	1.69187	4.22475	2.53288	2.95633	2.96024	0.00391
	δ	0.00811	0.01372	0.00561	0.01002	0.01208	0.00206
	μ	0.00045	0.01860	0.02050	0.00860	0.00900	0.00040
	$R(15)$	0.98558	1.00500	0.01942	0.99376	0.99644	0.00268
	$h(15)$	0.00012	0.00300	0.00218	0.00070	0.00123	0.00053

(Continued)

Table 21 (continued)

Data[Sample]	Par.	ACI			HPD		
		Lower	Upper	Length	Lower	Upper	Length
Biom-II[S_3]	γ	3.44291	5.58912	2.14621	4.51567	4.51628	0.00061
	δ	0.01055	0.01943	0.00888	0.01496	0.01549	0.00053
	μ	0.05359	0.08445	0.03086	0.06900	0.06904	0.00004
	$R(15)$	0.99602	1.00241	0.00639	0.99909	0.99922	0.00013
	$h(15)$	0.00005	0.00088	0.00083	0.00016	0.00019	0.00003

6 Concluding Remarks

In the framework of incomplete data obtained through an adaptive progressive Type-II censoring scheme, this study introduces and investigates a new and flexible generalization of several well-known lifetime models, namely the standard log-logistic, exponential, Burr-XII, and Weibull distributions–referred to as the GLL lifetime model. For classical inference, the MLEs are obtained via the Newton-Raphson iterative method, implemented using the `maxLik` package, ensuring computational stability and convergence. The ACIs for each unknown parameter are also derived from the observed Fisher information matrix to provide additional frequentist insights. In the Bayesian setup, posterior estimates and corresponding HPD intervals are approximated using the M-H algorithm with independent gamma priors, executed through the `coda` package. A comprehensive Monte Carlo simulation study is conducted under multiple scenarios to evaluate the finite-sample behavior and robustness of the proposed estimation methods, considering various censoring levels and parameter settings. Furthermore, four real-world data applications–including reliability engineering and biomedical case studies–are analyzed to illustrate the practical utility of the proposed distribution. Comparative model assessment using standard goodness-of-fit criteria demonstrates that the proposed GLL model provides superior or comparable performance relative to competing models such as the alpha-power exponential, exponentiated exponential, log-logistic, bib28, and gamma distributions. The main limitation of this work is estimating the GLL parameters of life using samples affected by censoring. Still, future research can expand the suggested methods to more complicated reliability situations, such as accelerated life testing, competing risks, and different censoring methods.

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Appendix A

From (7), the Fisher items $\mathcal{L}_{ij}^*$ for $i, j = 1, 2, 3$ are

$$\mathcal{L}_{11}^* = -\frac{m}{\sigma^2} - \left(\frac{\delta}{\mu}\right)^{\gamma} \log\left(\frac{\delta}{\mu}\right) \sum_{i=1}^{m} \left[\vartheta(x_i;\mu,\gamma)\log\left(\frac{\delta}{\mu}\right) + \xi(x_i;\mu,\gamma)\log(\mu x_i)\right]$$
$$-\left[1+\left(\frac{\delta}{\mu}\right)^{\gamma}\right]\sum_{i=1}^{m}\xi(x_i;\mu,\gamma)(1-\xi(x_i;\mu,\gamma))(\log(\mu x_i))^2$$
$$-\left(\frac{\delta}{\mu}\right)^{\gamma}\log\left(\frac{\delta}{\mu}\right)\left[\log\left(\frac{\delta}{\mu}\right)\sum_{j=1}^{d}R_j\vartheta(x_j;\mu,\gamma)-\sum_{j=1}^{d}R_j\xi(x_j;\mu,\gamma)\log(\mu x_i)\right]$$
$$-\left(\frac{\delta}{\mu}\right)^{\gamma}\sum_{j=1}^{d}R_j\xi(x_j;\mu,\gamma)(1-\xi(x_j;\mu,\gamma))\log^2(\mu x_i)$$
$$\times\left(\frac{\delta}{\mu}\right)^{\gamma}\log\left(\frac{\delta}{\mu}\right)R_m^*\left[\log\left(\frac{\delta}{\mu}\right)\vartheta(x_m;\mu,\gamma)-\xi(x_m;\mu,\gamma)\log(\mu x_m)\right]$$
$$-\left(\frac{\delta}{\mu}\right)^{\gamma}R_m^*\xi(x_m;\mu,\gamma)(1-\xi(x_m;\mu,\gamma))(\log(\mu x_m))^2,$$

$$\mathcal{L}_{22}^* = -\frac{m\gamma}{\delta^2} - \frac{\gamma(\gamma-1)}{\mu^2}\left(\frac{\delta}{\mu}\right)^{\gamma-2} Q_1(x;\mu,\gamma),$$

$$\mathcal{L}_{33}^* = \frac{1}{\mu^2}\left[1+\gamma+\left(\frac{\delta}{\mu}\right)^{\gamma}\right]\sum_{i=1}^{m}\xi(x_i;\mu,\gamma) - \left(\frac{\gamma}{\mu}\right)^2\left(1+\left(\frac{\delta}{\mu}\right)^{\gamma}\right)\sum_{i=1}^{m}\xi(x_i;\mu,\gamma)(1-\xi(x_i;\mu,\gamma))$$
$$-\frac{2\gamma}{\mu^2}\left(\frac{\delta}{\mu}\right)^{\gamma}\sum_{j=1}^{m}(R_j-1)\vartheta(x_j;\mu,\gamma) - \frac{\gamma^2}{\mu^3}(\gamma-1)\left(\frac{\delta}{\mu}\right)^{\gamma}\sum_{j=1}^{d}(R_j-1)\vartheta(x_j;\mu,\gamma)$$
$$-\left(\frac{\gamma}{\mu}\right)^2\left(\frac{\delta}{\mu}\right)^{\gamma}\sum_{i=1}^{m}\xi(x_i;\mu,\gamma) + \left(\frac{\gamma}{\mu}\right)^2\left(\frac{\delta}{\mu}\right)^{\gamma}\sum_{j=1}^{m}(R_j-1)\xi(x_j;\mu,\gamma)$$
$$+\frac{\gamma}{\mu^2}\left(\frac{\delta}{\mu}\right)^{\gamma}(1+\gamma)\sum_{i=1}^{m}\xi(x_i;\mu,\gamma) - \left(\frac{\gamma}{\mu}\right)^2\left(\frac{\delta}{\mu}\right)^{\gamma-1}(1+\gamma)\sum_{i=1}^{m}\xi(x_i;\mu,\gamma)(1-\xi(x_i;\mu,\gamma))$$
$$+R_m^*\left[-\frac{\gamma}{\mu^3}\left(\frac{\delta}{\mu}\right)^{\gamma-1}\{2\delta+\gamma(\gamma-1)\}\vartheta(x_m;\mu,\gamma) - \frac{\delta\gamma^2}{\mu^3}\left(\frac{\delta}{\mu}\right)^{\gamma}\xi(x_m;\mu,\gamma)\right]$$
$$-R_m^*\left[-\frac{\gamma}{\mu^2}\left(\frac{\delta}{\mu}\right)^{\gamma}\{\gamma+1\}\xi(x_m;\mu,\gamma) + \left(\frac{\gamma^2}{\mu}\right)^2\left(\frac{\delta}{\mu}\right)^{\gamma}\xi(x_m;\mu,\gamma)(1-\xi(x_m;\mu,\gamma))\right],$$

$$\mathcal{L}_{12}^* = -\frac{m}{\delta} - \frac{\gamma}{\mu}\left(\frac{\delta}{\mu}\right)^{\gamma-1}\left[\left(\frac{1}{\gamma}+log\left(\frac{\delta}{\mu}\right)\right)Q_1(x;\mu,\gamma) - Q_3(x;\mu,\gamma)\right],$$

$$\mathcal{L}_{13}^{*} = \frac{\gamma\delta}{\mu^2}\left(\frac{\delta}{\mu}\right)^{\gamma-1}\log\left(\frac{\delta}{\mu}\right)\sum_{i=1}^{m}[\vartheta(x_i;\mu,\gamma)-\xi(x_i;\mu,\gamma)]$$
$$+\frac{1}{\mu}\left(\frac{\delta}{\mu}\right)^{\gamma}\sum_{i=1}^{m}[\vartheta(x_i;\mu,\gamma)-\xi(x_i;\mu,\gamma)\log(\mu x_i)]$$
$$-\left[1+\left(\frac{\delta}{\mu}\right)^{\gamma}\right]\sum_{i=1}^{m}\left[\frac{\gamma}{\mu}\xi(x_i;\mu,\gamma)(1-\xi(x_i;\mu,\gamma))\log(\mu x_i)-\frac{1}{\mu}\xi(x_i;\mu,\gamma)\right]$$
$$+\frac{1}{\mu}\left(\frac{\delta}{\mu}\right)^{\gamma}\left[1+\gamma\log\left(\frac{\delta}{\mu}\right)\right]\sum_{j=1}^{d}R_j\vartheta(x_j;\mu,\gamma)-\frac{\gamma}{\mu}\left(\frac{\delta}{\mu}\right)^{\gamma}\log\left(\frac{\delta}{\mu}\right)\sum_{i=1}^{m}\xi(x_i;\mu,\gamma)$$
$$-\frac{1}{\mu}\left(\frac{\delta}{\mu}\right)^{\gamma}\sum_{j=1}^{d}R_j\left[\gamma\{1+\xi(x_j;\mu,\gamma)(1-\xi(x_j;\mu,\gamma))\}\log(\mu x_j)+\xi(x_j;\mu,\gamma)\right]$$
$$-\frac{1}{\mu}\left(\frac{\delta}{\mu}\right)^{\gamma}R_m^{*}\left[\{\vartheta(x_m;\mu,\gamma)-\xi(x_m;\mu,\gamma)\}\log\left(\frac{\delta}{\mu}\right)-\vartheta(x_m;\mu,\gamma)\right]$$
$$+\frac{\gamma}{\mu}\left(\frac{\delta}{\mu}\right)^{\gamma}R_m^{*}\left[1-\xi(x_m;\mu,\gamma)(1-\xi(x_m;\mu,\gamma)\log(\mu x_m)-\frac{1}{\gamma}\xi(x_m;\mu,\gamma)\right]$$

and

$$\mathcal{L}_{23}^{*} = \frac{\gamma^2}{\mu^2}\left(\frac{\delta}{\mu}\right)^{\gamma-1}[Q_1(x;\mu,\gamma)-Q_2(x;\mu,\gamma)],$$

where $\xi(x_i;\mu,\gamma)=\log\vartheta(x_i;\mu,\gamma)$

$$Q_1(x;\mu,\gamma)=\sum_{i=1}^{m}\vartheta(x_i;\mu,\gamma)-\left(\frac{\delta}{\mu}\right)^{\gamma}\left[\sum_{j=1}^{d}R_j\vartheta(x_j;\mu,\gamma)-R_m^{*}\vartheta(x_m;\mu,\gamma)\right],$$

$$Q_2(x;\mu,\gamma)=\sum_{i=1}^{m}\xi(x_i;\mu,\gamma)+\sum_{j=1}^{d}R_j\xi(x_j;\mu,\gamma)-R_m^{*}\xi(x_m;\mu,\gamma),$$

and

$$Q_3(x;\mu,\gamma)=\sum_{i=1}^{m}\xi(x_i;\mu,\gamma)\log(\mu x_i)+\sum_{j=1}^{d}R_j\xi(x_j;\mu,\gamma)\log(\mu x_j)-R_m^{*}\xi(x_m;\mu,\gamma)\log(\mu x_m).$$