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CLUSTERING-BASED PARAMETRIC SURROGATE MODELING OF VIBROACOUSTIC PROBLEMS ASSISTED BY NEURAL NETWORKS AND ACTIVE SUBSPACE METHOD

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Summary. This contribution presents a combined framework to perform parametric surrogate modeling of vibroacoustic problems that enables efficient training of large-scale problems. The proposed framework combines the active subspace method to perform dimensionality reduction of high-dimensional problems and thereafter a clustering-based approach within the identified active subspace region to yield smaller training clusters. Finally, a trained neural network assists the cluster classification task for any desired parameter point so as to query the parametric system response during the online phase.

1 INTRODUCTION

Parametric reduced-order modeling delivers significant advantages in computational costs for multi-query problems, where the expensive high-fidelity simulations are replaced with the low-fidelity solutions obtained from the respective reduced-order model representing the desired parameter domain (pROM). However, the convergence of the pROM depends on the difficulty to capture the characteristics of the underlying system. Practical problems in structural dynamics and vibroacoustics, for instance the simulation of an aircraft involving coupled subsystems [1], possess system responses that are challenging for existing parametric model order reduction (PMOR) techniques.

In this contribution, we address the above issue for vibroacoustic problems by adopting an existing clustering-based PMOR modeling approach from [2] thereby producing pROMs for each of the identified clusters. With this approach, we produce local ROMs at a number of parameter sample points and clusters are formed according to the dissimilarity of the underlying Krylov subspace using the Grassmannian metric. Once the clusters are established, pROMs are generated for each of the clusters using a classical

Greedy algorithm. Finally in the PMOR online phase, the desired system response at any parametric setting can be obtained from the respective valid cluster pROM in real-time. In addition, neural networks (NN) are deployed to learn and predict the expensive cluster assignment. The major contribution of this paper is the formulation and extension of the above approach to a high-dimensional parametric setting. The curse of dimensionality is addressed using the dimensionality reduction technique of the active subspace method (ASM). As a result, we propose a combined framework of PMOR training in active subspaces and clustering-based PMOR assisted by a NN to alleviate the computational demand to generate a parametric surrogate for complex problems. In this paper, we present the adaptive algorithms of the proposed framework and demonstrate its potential using a generic example from vibroacoustics.

2 FUNDAMENTALS

The current section deals with the underlying theory and concepts which are then used in a combined framework presented later in Section 3.

2.1 Vibroacoustic problem

Modeling structure-borne and air-borne sound is of high interest to adopt necessary strategies that can enhance acoustic comfort for instance in passenger cabins of an automobile and aircraft. This includes the modeling of wave propagation through various physical domains: structural domain, acoustic fluid medium (air) and furthermore the interaction between them popularly termed as fluid-structure interaction (FSI).

Using the finite element method (FEM), one obtains a second-order dynamic system of equations that enables solving of complex models for their behavior under various working conditions. Here for vibroacoustic analysis, we perform analysis in the desired frequency domain and the coupled system of equations for a simple structural domain coupled to a fluid medium that is excited on the structural part takes the form:

$$\left(-\omega^2 \begin{bmatrix} \mathbf{M}_s & \mathbf{0} \\ \mathbf{K}_c^T & \mathbf{M}_f \end{bmatrix} + \begin{bmatrix} \mathbf{K}_s & -\mathbf{K}_c \\ \mathbf{0} & \mathbf{K}_f \end{bmatrix}\right) \begin{bmatrix} \mathbf{x}_s \\ \mathbf{x}_f \end{bmatrix} = \begin{bmatrix} \mathbf{f}_s \\ \mathbf{0} \end{bmatrix}, \tag{1}$$

where the system matrices $\mathbf{M}, \mathbf{K} \in \mathbb{C}^{n \times n}$ represent the mass and stiffness matrices, and the subscripts s, f, c denote the structural domain, fluid domain and coupling interface respectively. The solution vector $\mathbf{x} = (\mathbf{x}_s, \mathbf{x}_f) \in \mathbb{C}^n$ is obtained by solving the system of equations for an excitation represented as $\mathbf{f} = (\mathbf{f}_s, \mathbf{0}) \in \mathbb{R}^n$. Also, ω is the angular frequency in rad/s. Due to the wave-resolving nature of FEM, the system matrices are often huge and increase at higher frequency regions where the wavelength becomes small.

In addition, a number of design and material parameters influence the system which are of modeling interest so as to perform optimization or uncertainty quantification that delivers further insights about the model behavior. The underlying material modeling and finite element types are not discussed further for simplicity and can be referred to classical FEM literature. As a result, obtaining an accurate parametric surrogate for such a high-dimensional parametric setting is of high importance to mitigate the high computational cost involved for the mentioned multi-query problems.

2.2 Parametric model order reduction with subspace interpolation

A typical second-order system depending on a number of parameters $\mathbf{p} \in \mathcal{P}^{n_p} \subset \mathbb{R}^{n_p}$ can be expressed as:

$$\Sigma(\omega, \mathbf{p}) : \begin{cases} \left(-\omega^2 \mathbf{M}(\mathbf{p}) + \mathbf{K}(\mathbf{p})\right) \mathbf{x}(\omega, \mathbf{p}) = \mathbf{B}\mathbf{u}(\omega, \mathbf{p}) \\ \mathbf{y}(\omega, \mathbf{p}) = \mathbf{C}\mathbf{x}(\omega, \mathbf{p}) \end{cases}, \tag{2}$$

where $\mathbf{B} \in \mathbb{R}^{n \times p}$ and $\mathbf{C} \in \mathbb{R}^{q \times n}$ are the input and output system matrices, $\mathbf{u} \in \mathbb{R}^p$ are the p inputs and $\mathbf{y} \in \mathbb{C}^q$ contain the q outputs. The system of equations for a generic vibroacoustic problem in (1) can be translated into the above parametric equation (2) and constitute the expensive full-order model $\Sigma(\omega, \mathbf{p})$.

PMOR techniques can be deployed to obtain low-fidelity approximations of the expensive model in a desired parametric region. Various approaches exist where the localized approach using local ROMs is practical to handle large-scale models when compared to global approaches. The local approach identifies a few local ROMs produced at different parameter points to represent the desired parameter region. As a result, a query towards a desired parameter setting can be easily obtained for example using interpolation techniques applied to the identified local ROMs. An overview of the various techniques can be referred to in [3]. In this contribution, we present results where interpolation is performed for the underlying subspaces that have been shown to produce accurate pROMs with the least number of local ROMs.

The aim with PMOR and subspace interpolation, outlined below, is to produce a parametric surrogate Σ_R of the full-order model Σ using projection-based model order reduction (MOR) techniques. More details on MOR using Krylov-subspace and moment matching can be referred to in [4, 5]. The expression for the parametric surrogate follows:

$$\Sigma_R(\omega, \hat{\mathbf{p}}) : \begin{cases} \left(-\omega^2 \mathbf{M}_R(\hat{\mathbf{p}}) + \mathbf{K}_R(\hat{\mathbf{p}})\right) \mathbf{x}_R(\omega, \hat{\mathbf{p}}) = \mathbf{B}_R \mathbf{u}(\omega, \hat{\mathbf{p}}) \\ \mathbf{y}_R(\omega, \hat{\mathbf{p}}) = \mathbf{C}_R \mathbf{x}_R(\omega, \hat{\mathbf{p}}) \end{cases}, \tag{3}$$

where the ROM matrices at any desired parameter point $\hat{\mathbf{p}}$ can be obtained with classical Galerkin projection $[\,\cdot\,]_R = \mathbf{V}^T(\hat{\mathbf{p}})\,[\,\cdot\,]\,\mathbf{V}(\hat{\mathbf{p}})$ with $[\,\cdot\,] : \{\mathbf{M}(\hat{\mathbf{p}}), \mathbf{K}(\hat{\mathbf{p}})\}$. $\mathbf{V}(\hat{\mathbf{p}})$ is the projection basis that is obtained by interpolating subspaces.

As mentioned before, local approaches identify a number of ROMs with projection bases $\mathbf{V}_1, \cdots, \mathbf{V}_k \in \mathbb{C}^{n \times r}$. Interpolation is then performed on the underlying subspace [6], rather than on the projection matrix directly, based on the Grassmann manifold and its tangent space. Interpolation of the mapped subspaces can be then performed within this tangent space for any parameter point $\hat{\mathbf{p}}$. Finally, the orthonormal basis for the new parameter point $\hat{\mathbf{p}}$ can be obtained by performing a reverse mapping. The method is relatively accurate due to the interpolation of underlying system characteristics populated within the projection basis. However, in comparison to other interpolation techniques, the online phase using subspace interpolation requires access to the system matrices which may add to the computational cost. For the presented example in Section 4, subspace interpolation is used as a compromise between training effort and yielded accuracy.

2.3 Dimensionality reduction using active subspace method

Dimensionality reduction of high-dimensional parametric problems is essentially a requirement for surrogate modeling. A large number of parameters leads to the curse of dimensionality issue which drastically increases the computational demand and makes the pROM training unfeasible. The ASM, see [7], is an efficient way to realize a low-dimensional parametric space where there is a strong anisotropy between the various parameters. Application to vibroacoustic models and training within the identified active subspace has shown to yield accurate pROMs with reduced training cost [8].

The gradient-based ASM computes a covariance-style matrix or the active subspace matrix $\mathbf{C} \in \mathbb{C}^{n_p \times n_p}$, which can be expressed as for a function $f(\mathbf{p})$ with probability density function $\rho(\mathbf{p})$ and corresponding gradient in respective parametric directions $\nabla_{\mathbf{p}} f(\mathbf{p})$:

$$\mathbf{C} = \int (\nabla_{\mathbf{p}} f)(\nabla_{\mathbf{p}} f)^T \rho(\mathbf{p}) \mathrm{d}\mathbf{p} \approx \tilde{\mathbf{C}} = \frac{1}{m} \Sigma_{j=1}^{m} (\nabla_{\mathbf{p}} f_j)(\nabla_{\mathbf{p}} f_j)^T. \tag{4}$$

Monte Carlo integration for gradients $\nabla_{\mathbf{p}} f_j$ evaluated at m parameter points is used to construct the active subspace matrix. An eigenvalue decomposition is then performed on matrix $\mathbf{C}$ to identify the most dominant eigenpairs defining the active subspace as follows:

$$\tilde{\mathbf{C}} = \mathbf{W} \boldsymbol{\Lambda} \mathbf{W}^T \text{ where } \boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_1 & \\ & \boldsymbol{\Lambda}_2 \end{bmatrix} \text{ and } \mathbf{W} = \begin{bmatrix} \mathbf{W}_1 & \mathbf{W}_2 \end{bmatrix}. \tag{5}$$

Classifying the eigenvalues $\boldsymbol{\Lambda}$ and eigenvectors $\mathbf{W}$ as per (5) defines the transformation into active parameters $\boldsymbol{\phi}$ and inactive parameters $\boldsymbol{\psi}$ as: $\boldsymbol{\phi} = \mathbf{W}_1^T \mathbf{p}$, $\boldsymbol{\psi} = \mathbf{W}_2^T \mathbf{p}$. Finally, the original function $f(\mathbf{p})$ depending on the physical parameters can be approximated with function evaluation in the active subspace $g(\boldsymbol{\phi})$ such that $f(\mathbf{p}) \approx g(\boldsymbol{\phi})$.

2.4 Clustering and cluster assignment using neural networks

The clustering of local ROMs is based on the dissimilarity measure of the Grassmannian metric that accounts for the distance between the underlying ROM's subspace. Details of the same can be referred to [2]. The idea is to compute the Grassmannian distance between all the local ROMs and to form a distance matrix. The distance matrix $\mathbf{D} \in \mathbb{R}^{n_{\text{train}} \times n_{\text{train}}}$ for n_{train} local ROMs and respective projection bases can be expressed as [2]:

$$\mathbf{D}_{(i,j)} = d_{Gr(\infty,\infty)}(\mathbf{V}_i, \mathbf{V}_j) = \frac{1}{4} \left(|r_i - r_j| + \Sigma_{k=1}^{\min(r_i, r_j)} \alpha_k^2 \right)^{\frac{1}{2}}, \tag{6}$$

where r_i, r_j are the ROM dimensions of the i^{th} and j^{th} local ROM or the number of columns in $\mathbf{V}_i$ and $\mathbf{V}_j$. The scalars α_k denote the principle angles between the two considered subspaces obtained from the singular values $\mathbf{S}_{kk}$ by performing a singular value decomposition (SVD) on $\mathbf{V}_i^T \mathbf{V}_j = \mathbf{U} \mathbf{S} \mathbf{W}^T$ delivering $\alpha_k = \arccos(\mathbf{S}_{kk})$.

Then we perform clustering using the k-medoids algorithm with $\mathbf{D}$ as input to form a few clusters where the PMOR algorithm can be deployed separately to obtain a local pROM for every cluster [2]. An optimal number of clusters can be determined based on for example the Silhouette analysis. During the PMOR training phase, the cluster assignment for the considered training ROMs is known. However to perform the PMOR

online phase, it is necessary to assign the desired parameter point $\hat{\mathbf{p}}$ to the belonging cluster. Hence, a NN is deployed to learn the cluster assignment at the PMOR training phase and predict the cluster assignment during the online phase at negligible costs. The NNs are trained upon the clustered information, where the inputs to the NN are the parameter points and the prediction is the belonging cluster label.

3 PROPOSED TRAINING METHODOLOGY

The proposed framework is an extension to the training methodology presented in [8], which performs PMOR training in the identified active subspace. The current contribution adds the clustering-based framework after the ASM procedure to perform training on each of the identified clusters with an objective to enhance the accuracy of the pROM for complex models and to reduce the computational effort in contrast to training on a conventional grid. The proposed framework is presented as pseudocode in Algorithm 1.

Algorithm 1 Clustering-based PMOR training algorithm

function performClusteringBasedPmorTraining(Parametrized full system Σ, frequency range $\omega = [\omega_{\min} : \omega_{\max}]$, parameter domain $\mathcal{P} \in [-1, 1]$, error tolerance σ_{tol})
  Φ = constructActiveSubspace(Σ, $\mathcal{P}$) ▷ See [8]
  Ξ = generateSamples(n_{train}, Φ) ▷ Generates n_{train} samples
  $\mathbf{D}$ = computeGrassmannianDistanceForAllRoms(Ξ) ▷ Distance matrix (6)
  $\{\Xi^1_{\text{cluster}}, \cdots, \Xi^{n_c}_{\text{cluster}}\}$, labels = performKMedoidsClustering(Ξ, $\mathbf{D}$, n_c)
  NN = trainNeuralNetworkAndExport($\{\Xi^1_{\text{cluster}}, \cdots, \Xi^{n_c}_{\text{cluster}}\}$, labels)
  for Ξ^i_{cluster} in $\{\Xi^1_{\text{cluster}}, \cdots, \Xi^{n_c}_{\text{cluster}}\}$ **do**
    $\Sigma^i_{R,\text{cluster}}$ = performPmorTrainingGreedy(Ξ^i_{cluster}, ω, σ_{tol}) ▷ See Eqn. (3)
  end for
  return Local pROMs for n_c clusters $\Sigma_{R,\text{cluster}}$, trained NN
end function

Firstly, the active subspace is identified by evaluating the gradients at m random samples for Monte Carlo integration in (4). Training samples Ξ are then generated for the active parameter space $\boldsymbol{\phi} \in \Phi$ where we deploy the k-medoids clustering algorithm from scikit-learn [9] to form n_c optimal number of clusters. The clustering algorithm receives the distance matrix containing the Grassmannian distance between each pair of training ROM. The clustering algorithm delivers the cluster labels to which the considered training ROMs are classified thereby yielding clustered training samples $\Xi = \cup_{i=1}^{n_c} \Xi_{\text{cluster},i}$. For the online phase, a less expensive classifier is required to predict the cluster assignment as mentioned in Section 2.4. It is to be noted that the clustering was performed with the Grassmanian distance matrix as input, which is to be replaced in the online phase with the classifier that takes the desired parameter coordinates as input to predict the cluster assignment. In the case of a simple classification task, a suitable multi-label classification algorithm can be used. However, we choose a NN to perform the classification task with a view of more complex problems where the relationship between a large number of input parameters and the clustered labels can be accurately predicted by a trained NN. Also,

deep classifiers based on deep NNs may be deployed for other applications where complex clustering situations occur. In this contribution, a fully connected NN with a single hidden layer of size 100 is trained using the machine learning library Keras [10] to incorporate the relation between the active parameters and the corresponding identified cluster label. For the numerical example in Section 4, a stratified 10-fold cross-validation accuracy of 98.34% is achieved for the classification task. The trained NN is then exported for serving the online phase where it predicts the cluster for any parameter point in real-time. As a final step in Algorithm 1, the PMOR algorithm based on subspace interpolation is performed on each of the clusters with training samples $\Xi_{\text{cluster},i}$. Hence, a subset of ROMs is identified in a greedy manner that can represent the individual cluster region.

4 NUMERICAL RESULTS

The presented vibroacoustic benchmark example is a rectangular plate-cavity problem also considered in [11] and [8]. The model is presented in Fig. 1, where the plate and fluid medium are discretized with quadratic elements accounting for a total of 10395 degrees of freedom. We consider a simple input-output configuration of exciting the plate at a single point and observing the pressure response at a node inside the fluid domain. The model accounts for 8 parameters that include the material and design parameters with 5% parameter variation from the mean value: (1) in plate domain: Young's modulus $E = 70$ GPa, Poisson's ratio $\nu = 0.3$, plate density $\rho_s = 2700$ kg/m^3, thickness $t = 0.003$ m, damping constant $\eta_s = 0.05$, and (2) from the fluid domain: bulk modulus $K_f = 142360$ Pa, fluid density $\rho_f = 1.21$ kg/m^3 and damping constant $\eta_f = 0.05$. The damping constants account for losses numerically by introducing imaginary values to the parameters $\hat{E} = E(1 + i\eta_s)$ and $\hat{K_f} = K_f(1 + i\eta_f)$ accordingly. Here, higher values are chosen for the damping constants to obtain relatively smoother frequency response functions (FRF). Finally, the analysis is performed in the frequency range of $[\omega_{\min}, \omega_{\max}] = 2\pi[100, 300]$ rad/s for every 1 Hz frequency step.

A local ROM can be generated using a projection basis computed with Krylov-subspace methods yielding a ROM of size 120 converging with a maximum relative error norm below 10^{-6} for all generated ROMs. PMOR training is then performed using Algorithm 1 where at first the dimensionality reduction using ASM delivers two active parameters. A full grid is generated in the two-dimensional active subspace 30×30 leading to a total of 900 samples that are used for training. Adaptive strategies can be used to replace the expensive training grid and are interesting for future research. With the help of Silhouette analysis, the choice of 6 clusters is identified as optimal and the resulting cluster formation is plotted in Fig. 2. The cluster pattern is simple and can be easily learned by a shallow NN for usage in PMOR online phase. As a final step, the various clusters are trained individually using interpolation of subspaces presented in Section 2.2 where the interpolated ROMs are identified in a Greedy manner. The resulting number of interpolated ROMs in each cluster are $17, 15, 15, 14, 13$ and 13 respectively converged for a relative error tolerance of 10^{-3} yielding good approximation in the active subspace.

Accuracy in the online phase is evaluated by evaluating FRFs of 100 random samples and the sample with maximum relative error is plotted in Fig. 3. We observe good fitting

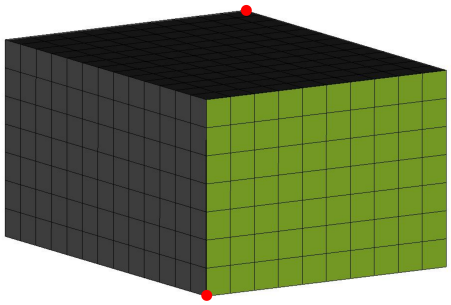

Figure 1: Discretized plate-cavity model with plate domain in green and fluid medium in grey. Points ● are the excitation and observation nodes.

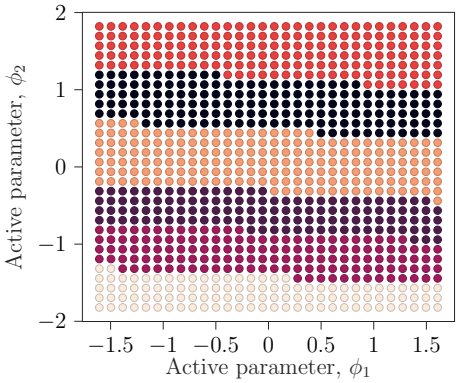

Figure 2: Clustered training ROMs based on Grassmannian distance containing 900 ROMs plotted using active subspace coordinates.

of FRFs qualitatively, however, the considered coupled plate-cavity model suffers from high modal density that makes it difficult to preserve exact system behavior for samples outside the active subspace. In Fig. 3, slight deviations in the interpolated FRF at certain frequencies can be attributed to high response variations at these frequencies and also the frequency dimension being not considered for active subspace modeling. Increasing the number of active parameters can also be considered for improved approximations.

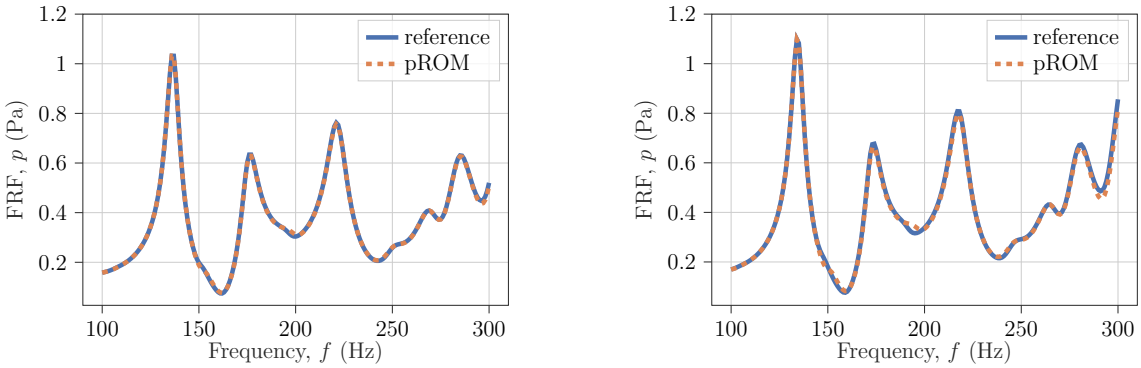

Figure 3: Comparison of approximated FRFs or the pressure response p obtained from the PMOR surrogate and the original FRF computed for a sample (out of 100 random samples) with highest error within the trained active subspace region (left) and the entire original parameter region (right).

5 CONCLUSIONS

This contribution presented a combined framework of clustering-based PMOR using ASM for handling the curse of dimensionality of high-dimensional parametric problems and NNs to perform the cluster assignment task during the online phase. For the considered vibroacoustic example, the approach has shown good results and is observed to have great potential to reduce the computational demand for large-scale models so as to have a better span and accuracy over the desired parametric space. Areas for future research include better handling of coupled vibroacoustic models that possess challenging system

responses to be preserved in a parametric surrogate. Also, reduction of high-dimensional parametric problems can also benefit from the usage of error-controlled adaptive sparse-grids [11] in addition to ASM for better convergence.

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