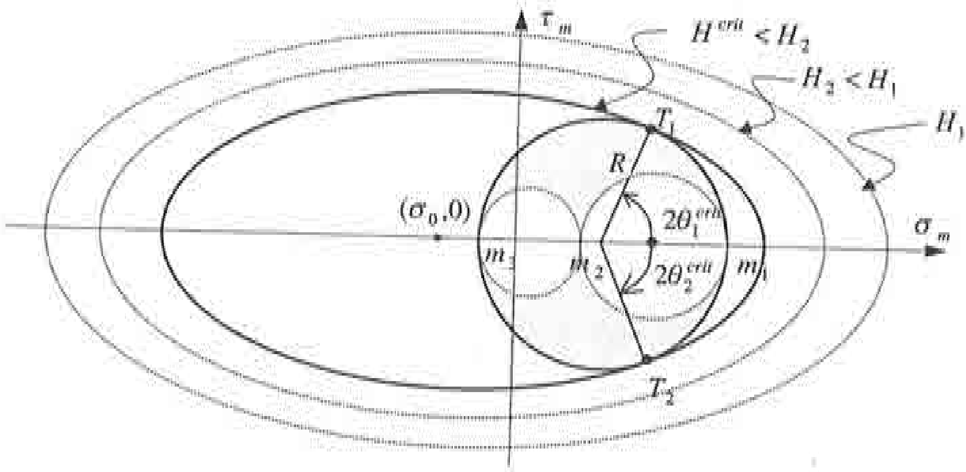
Topics on Failure Mechanics

X. Oliver



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Introduction

Failure Mechanics

Definition:

Set of methods and techniques to determine:

- *When*
- *How*
- *Where*

structural failure takes place.

Failure modes

In structural analysis the following failure modes are considered:

- *Material failure:* Material provides responses that lead to a reduction of the carrying capacity of the structure *(material instability)*. In quasi-brittle materials (steel, concrete, rocks, soils) this entails *loss of uniqueness, strong ellipticity and stability* phenomena that produce cracks, fractures, shear bands etc.

- *Geometrical failure*: The structure undertakes geometrical configurations that drastically reduce its loading capacity.

Determination of mode and type of failure is then associated to determination of the ultimate load of the structure.

Goal of the work

Analyze constitutive models (constitutive equations) in front of material failure and provide tools to capture failure modes.

Contents

1 Uniqueness, ellipticity and material stability

1.1 Boundary value problem

The general non-linear quasistatic Solid Mechanics Problem can be stated as (see Figure 1-1):

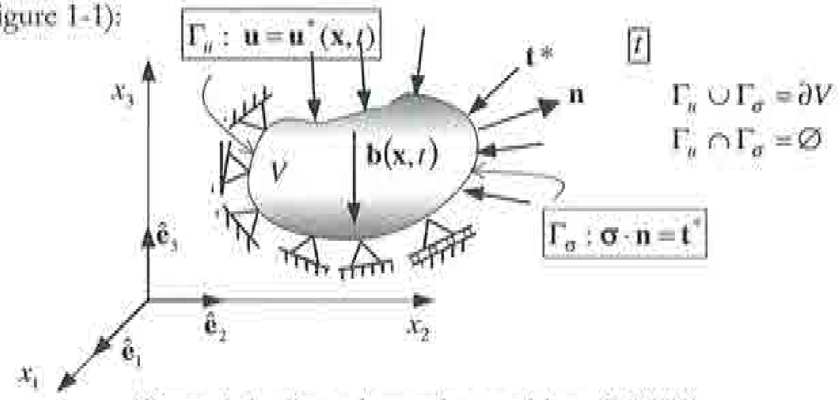


Figure 1-1– Boundary value problem (B.V.P.)

Find:

$$
\begin{aligned}
&\mathbf{u}(\mathbf{x},t): V\times[0,T]\rightarrow\mathbb{R}^3 &&\Rightarrow \textit{Displacements}\\
&\boldsymbol{\varepsilon}(\mathbf{x},t): V\times[0,T]\rightarrow\mathbb{R}^6 &&\Rightarrow \textit{Strains}\\
&\boldsymbol{\sigma}(\mathbf{x},t): V\times[0,T]\rightarrow\mathbb{R}^6 &&\Rightarrow \textit{Stresses}
\end{aligned}
\tag{1.1}
$$

Fulfilling $\forall \mathbf{x}\in V$ and for any $t\in[0,T]$ (the time interval of interest):

NOTE

We shall restrict the analysis to the quasistatic case (neglecting the inertial term: $\rho\dfrac{\partial^2\mathbf{u}}{\partial t^2}$)

$$
\begin{aligned}
&\nabla\cdot\boldsymbol{\sigma}+\rho\,\mathbf{b}=\mathbf{0} &&\rightarrow \text{Momentum balance (equilibrium)}\\
&\boldsymbol{\sigma}=\boldsymbol{f}(\boldsymbol{\varepsilon}) &&\rightarrow \text{Constitutive equation}\\
&\boldsymbol{\varepsilon}=\nabla^S\mathbf{u}=\frac{1}{2}(\mathbf{u}\otimes\nabla+\nabla\otimes\mathbf{u}) &&\rightarrow \text{Kinematic equation}
\end{aligned}
\tag{1.2}
$$

$$
\left.\begin{aligned}
&\Gamma_u:\ \mathbf{u}=\mathbf{u}^* \quad \forall\,\mathbf{x}\in\Gamma_u\\
&\Gamma_\sigma:\ \boldsymbol{\sigma}\cdot\mathbf{n}=\mathbf{t}^* \ \forall\,\mathbf{x}\in\Gamma_\sigma
\end{aligned}\right\}\rightarrow \text{Boundary conditions}
\tag{1.3}
$$

The B.V.P can then be interpreted in terms of action-response (see Figure 1-2):

$$\underbrace{\left.\begin{matrix}\mathbf{b}(\mathbf{x},t)\\ \mathbf{t}^{*}(\mathbf{x},t)\\ \mathbf{u}^{*}(\mathbf{x},t)\end{matrix}\right\}}_{\substack{\text{not}\\ \text{Action}=\mathbb{A}(\mathbf{x},t)}} \Rightarrow \left\langle \begin{matrix}\text{MATHEMATICAL}\\ \text{MODEL}\\ \text{P.D.E's + b.c.}\end{matrix}\right\rangle \Rightarrow \underbrace{\left\{\begin{matrix}\mathbf{u}(\mathbf{x},t)\\ \boldsymbol{\varepsilon}(\mathbf{x},t)\\ \boldsymbol{\sigma}(\mathbf{x},t)\end{matrix}\right.}_{\substack{\text{not}\\ \text{Response}=\mathbb{R}(\mathbf{x},t)}} \tag{1.4}$$

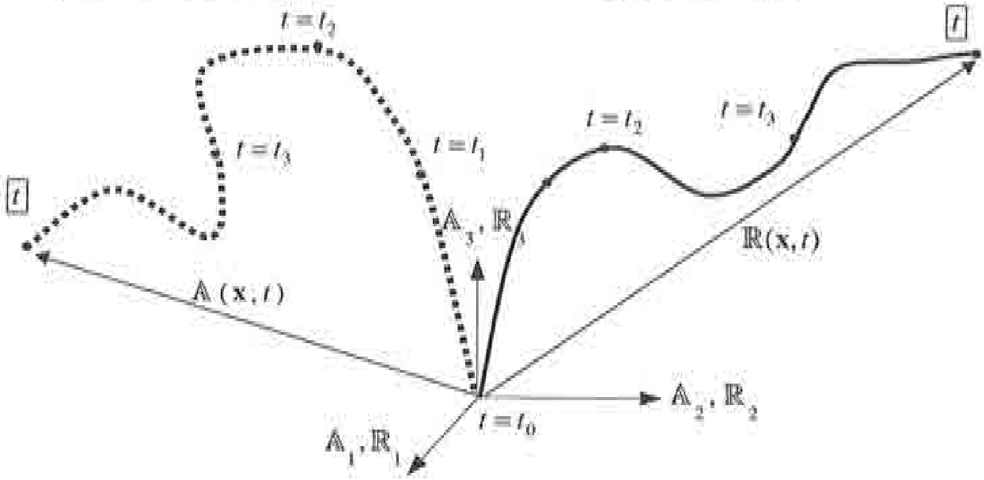


Figure 1-2– Action-Response for the B.V.P.

1.2 Virtual work principle

For the problem of Figure 1-3 let us define the space of test functions (virtual displacements):

$$\left.\begin{matrix}\text{Space of}\\ \text{test functions}\\ \text{(Virtual displacements)}\end{matrix}\right\} \rightarrow \mathbb{V} := \{\boldsymbol{\eta}(\mathbf{x}) : V \rightarrow \mathbb{R}^{3} \;\; | \; \boldsymbol{\eta}(\mathbf{x})|_{\mathbf{x}\in\Gamma_u} = \mathbf{0}\} \tag{1.5}$$

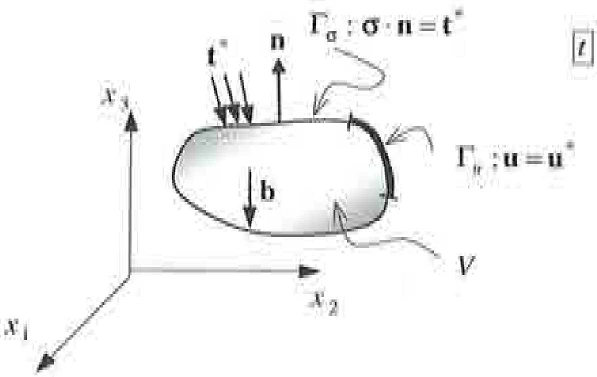


Figure 1-3

The Virtual Work Principle (V.W.P) reads:

Virtual Work Principle

$$\underbrace{\int_V \boldsymbol{\sigma} : \underbrace{\nabla^S \boldsymbol{\eta}}_{\delta\boldsymbol{\varepsilon}} \, dV}_{\substack{\text{Internal Virtual} \\ \text{Work } (\delta\mathbb{W}^{int})}} = \underbrace{\int_V \rho\, \mathbf{b} \cdot \underbrace{\boldsymbol{\eta}}_{\delta\mathbf{u}} \, dV + \int_{\Gamma_\sigma} \mathbf{t}^* \cdot \underbrace{\boldsymbol{\eta}}_{\delta\mathbf{u}} \, d\Gamma}_{\substack{\text{External Virtual} \\ \text{Work } (\delta\mathbb{W}^{ext})}} \quad \forall \boldsymbol{\eta} \in \mathbb{V} \tag{1.6}$$

and can be interpreted, in terms of a virtual configuration, as shown in Figure 1-4.

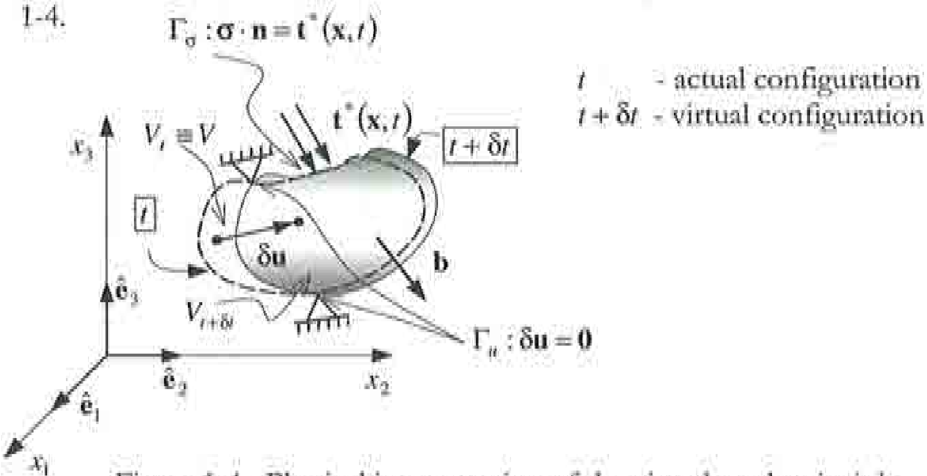


Figure 1-4– Physical interpretation of the virtual work principle

Remark 1-1

The virtual work principle (V.W.P.) is completely equivalent to the momentum balance equation:

$$\nabla \cdot \boldsymbol{\sigma} + \rho\, \mathbf{b} = \mathbf{0} \quad \forall \mathbf{x} \in V$$

Remark 1-2

- No assumption is done on the type of constitutive equation $\boldsymbol{\sigma} = \boldsymbol{f}(\boldsymbol{\varepsilon})$ (it can be any linear or nonlinear constitutive equation).
- No assumption is done on the size of the virtual displacements $\boldsymbol{\eta} \equiv \delta\mathbf{u}$ and virtual strains $\nabla^s \boldsymbol{\eta} \equiv \delta\boldsymbol{\varepsilon}$ (they can be large).

1.2.1 Rate form of the Virtual Work Principle

The rate form of the B.V.P. of equations (1.1) to (1.3)reads:

$$\nabla\cdot\dot{\boldsymbol{\sigma}}+\rho\,\dot{\mathbf{b}}=\mathbf{0} \quad\rightarrow\quad \text{Momentum balance (equilibrium)}$$

$$\dot{\boldsymbol{\sigma}}=\boldsymbol{f}(\boldsymbol{\varepsilon},\dot{\boldsymbol{\varepsilon}}) \quad\rightarrow\quad \text{Constitutive equation} \qquad (1.7)$$

$$\dot{\boldsymbol{\varepsilon}}=\nabla^{S}\dot{\mathbf{u}}=\frac{1}{2}(\dot{\mathbf{u}}\otimes\nabla+\nabla\otimes\dot{\mathbf{u}}) \quad\rightarrow\quad \text{Kinematic equation}$$

$$\left.\begin{array}{l}\Gamma_u:\ \dot{\mathbf{u}}=\dot{\mathbf{u}}^{*}\ \forall\mathbf{x}\in\Gamma_u\\ \Gamma_\sigma:\ \dot{\mathbf{t}}^{*}=\dot{\boldsymbol{\sigma}}\cdot\mathbf{n}\ \forall\ \mathbf{x}\in\Gamma_\sigma\end{array}\right\} \quad\rightarrow\quad \text{Boundary conditions} \qquad (1.8)$$

where $\dot{(\bullet)}(\mathbf{x},t)\overset{def}{=}\dfrac{\partial(\bullet)/\mathbf{x},t)}{\partial t}$. The corresponding form of the Virtual Work Principle (1.6) is:

Virtual Work Principle (rate form)

$$\int_V\dot{\boldsymbol{\sigma}}:\underbrace{\nabla^{s}\boldsymbol{\eta}}_{\delta\varepsilon}\,dV=\int_V\rho\,\dot{\mathbf{b}}\cdot\boldsymbol{\eta}\,dV+\int_{\Gamma_\sigma}\dot{\mathbf{t}}^{*}\cdot\boldsymbol{\eta}\,d\Gamma\quad\forall\boldsymbol{\eta}\in\mathbb{V}$$
$$\mathbb{V}:=\{\boldsymbol{\eta}(\mathbf{x}):V\rightarrow\mathbb{R}^{3}\ \mid\boldsymbol{\eta}(\mathbf{x})|_{\mathbf{x}\in\Gamma_u}=\mathbf{0}\} \qquad (1.9)$$

1.3 Bifurcation of the fundamental solution

The solution $\mathbb{R}(\mathbf{x},t)$ of the boundary problem can be either unique or have a fundamental solution $\mathbb{R}^{(1)}(\mathbf{x},t)$ and different (bifurcated) solutions $\mathbb{R}^{(2)}(\mathbf{x},t)$ (see (1.5)).

The time at which the instantaneous solution begins to be non-unique is the bifurcation time t_B.

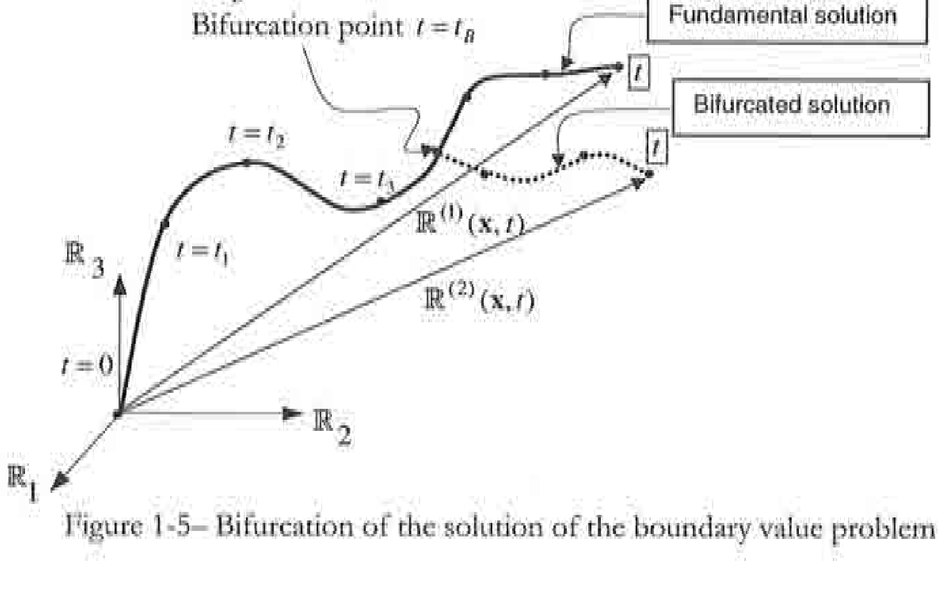


Figure 1-5– Bifurcation of the solution of the boundary value problem

> **Question:**
>
> What condition has the constitutive equation to accomplish to provide a unique response $\mathbb{R}(\mathbf{x},t)$ to the given actions $\mathbb{A}(\mathbf{x},t)$?

If the solution of the problem is not unique $\Rightarrow \exists\, \mathbb{R}^{(1)}(\mathbf{x},t)$ and $\mathbb{R}^{(2)}(\mathbf{x},t)$ ($\mathbb{R}^{(2)} \neq \mathbb{R}^{(1)}$ for $t > t_B$). The bifurcation time t_B is characterized by:

$$\left.\begin{array}{l} \dot{\mathbb{R}}_B^{(1)}(\mathbf{x}) \overset{not}{=} \left.\dfrac{\partial \mathbb{R}^{(1)}(\mathbf{x},t)}{\partial t}\right|_{t=t_B} \\ \dot{\mathbb{R}}_B^{(2)}(\mathbf{x}) \overset{not}{=} \left.\dfrac{\partial \mathbb{R}^{(2)}(\mathbf{x},t)}{\partial t}\right|_{t=t_B} \end{array}\right. \rightarrow \begin{array}{l} \mathbb{R}_B^{(1)}(\mathbf{x}) = \mathbb{R}_B^{(2)}(\mathbf{x}) \\ \dot{\mathbb{R}}_B^{(1)}(\mathbf{x}) \neq \dot{\mathbb{R}}_B^{(2)}(\mathbf{x}) \text{ for, at least some } \mathbf{x} \in V \end{array} \tag{1.10}$$

That is:

$$\left.\begin{array}{l} \mathbf{u}_B^{(1)}(\mathbf{x}) = \mathbf{u}_B^{(2)}(\mathbf{x}) \\ \boldsymbol{\varepsilon}_B^{(1)}(\mathbf{x}) = \boldsymbol{\varepsilon}_B^{(2)}(\mathbf{x}) \\ \boldsymbol{\sigma}_B^{(1)}(\mathbf{x}) = \boldsymbol{\sigma}_B^{(2)}(\mathbf{x}) \end{array}\right\} \forall \mathbf{x} \in V \text{ and } \left.\begin{array}{l} \dot{\mathbf{u}}_B^{(1)}(\mathbf{x}) \neq \dot{\mathbf{u}}_B^{(2)}(\mathbf{x}) \\ \dot{\boldsymbol{\varepsilon}}_B^{(1)}(\mathbf{x}) \neq \dot{\boldsymbol{\varepsilon}}_B^{(2)}(\mathbf{x}) \\ \dot{\boldsymbol{\sigma}}_B^{(1)}(\mathbf{x}) \neq \dot{\boldsymbol{\sigma}}_B^{(2)}(\mathbf{x}) \end{array}\right\} \text{ for, at least some } \mathbf{x} \in V \tag{1.11}$$

Defining:

$$\begin{array}{lcl} \Delta \mathbf{u}_B(\mathbf{x}) = \mathbf{u}_B^{(2)} - \mathbf{u}_B^{(1)} & & \Delta \dot{\mathbf{u}}_B(\mathbf{x}) = \dot{\mathbf{u}}_B^{(2)} - \dot{\mathbf{u}}_B^{(1)} \\ \Delta \boldsymbol{\varepsilon}_B(\mathbf{x}) = \boldsymbol{\varepsilon}_B^{(2)} - \boldsymbol{\varepsilon}_B^{(1)} & \text{and} & \Delta \dot{\boldsymbol{\varepsilon}}_B(\mathbf{x}) = \dot{\boldsymbol{\varepsilon}}_B^{(2)} - \dot{\boldsymbol{\varepsilon}}_B^{(1)} \\ \Delta \boldsymbol{\sigma}_B(\mathbf{x}) = \boldsymbol{\sigma}_B^{(2)} - \boldsymbol{\sigma}_B^{(1)} & & \Delta \dot{\boldsymbol{\sigma}}_B(\mathbf{x}) = \dot{\boldsymbol{\sigma}}_B^{(2)} - \dot{\boldsymbol{\sigma}}_B^{(1)} \end{array} \tag{1.12}$$

at the bifurcation time t_B:

$$\boxed{\left.\begin{array}{l} \Delta \mathbf{u}_B(\mathbf{x}) = \mathbf{0} \\ \Delta \boldsymbol{\varepsilon}_B(\mathbf{x}) = \mathbf{0} \\ \Delta \boldsymbol{\sigma}_B(\mathbf{x}) = \mathbf{0} \end{array}\right\}} \text{ and } \boxed{\left.\begin{array}{l} \Delta \dot{\mathbf{u}}_B(\mathbf{x}) \neq \mathbf{0} \\ \Delta \dot{\boldsymbol{\varepsilon}}_B(\mathbf{x}) \neq \mathbf{0} \\ \Delta \dot{\boldsymbol{\sigma}}_B(\mathbf{x}) \neq \mathbf{0} \end{array}\right\} \text{ for, at least some } \mathbf{x} \in V} \tag{1.13}$$

NOTE

Strictly speaking only one of inequalities
$\Delta \dot{\mathbf{u}}_B(\mathbf{x}) \neq \mathbf{0}$
$\Delta \dot{\boldsymbol{\varepsilon}}_B(\mathbf{x}) \neq \mathbf{0}$
$\Delta \dot{\boldsymbol{\sigma}}_B(\mathbf{x}) \neq \mathbf{0}$
is required.

and, in general:

$$\boxed{\begin{array}{l} \left.\begin{array}{l} \Delta \mathbf{u}(\mathbf{x},t) = \mathbf{0} \\ \Delta \boldsymbol{\varepsilon}(\mathbf{x},t) = \mathbf{0} \\ \Delta \boldsymbol{\sigma}(\mathbf{x},t) = \mathbf{0} \end{array}\right\} \forall \mathbf{x} \in V \quad \forall t \in [0, t_B] \\ \\ \left.\begin{array}{l} \Delta \dot{\mathbf{u}}(\mathbf{x},t) \neq \mathbf{0} \\ \Delta \dot{\boldsymbol{\varepsilon}}(\mathbf{x},t) \neq \mathbf{0} \\ \Delta \dot{\boldsymbol{\sigma}}(\mathbf{x},t) \neq \mathbf{0} \end{array}\right\} \forall \mathbf{x} \in \Delta V \subset V \quad \forall t \in [t_B, T] \end{array}} \tag{1.14}$$

where ΔV is the set of material points of V that have experienced the bifurcation.

> **Remark 1-3**
>
> If there is no bifurcation time in the time interval of interest $[0,T]$, then the solution is unique.

1.4 Uniqueness conditions. Positive material

Let's consider two different solutions of the B.V.P, $\mathbb{R}^{(1)}(\mathbf{x},t)$ and $\mathbb{R}^{(2)}(\mathbf{x},t)$. Applying the rate form of the V.W.P of equation (1.9) to both solutions:

NOTE

Actions $\dot{\mathbf{b}}$ and $\dot{\mathbf{t}}^*$ are the same for both solutions.

$$\begin{aligned}&\int_V \dot{\boldsymbol{\sigma}}^{(1)} : \nabla^S \boldsymbol{\eta}\, dV = \int_V \rho\, \dot{\mathbf{b}} \cdot \boldsymbol{\eta}\, dV + \int_{\Gamma_\sigma} \dot{\mathbf{t}}^* \cdot \boldsymbol{\eta}\, d\Gamma \quad \forall \boldsymbol{\eta} \in \mathbb{V}\\ &\int_V \dot{\boldsymbol{\sigma}}^{(2)} : \nabla^S \boldsymbol{\eta}\, dV = \int_V \rho\, \dot{\mathbf{b}} \cdot \boldsymbol{\eta}\, dV + \int_{\Gamma_\sigma} \dot{\mathbf{t}}^* \cdot \boldsymbol{\eta}\, d\Gamma \quad \forall \boldsymbol{\eta} \in \mathbb{V}\\ &\mathbb{V} := \{\boldsymbol{\eta}(\mathbf{x}) : V \to \mathbb{R}^3 \;\; | \; \boldsymbol{\eta}(\mathbf{x})|_{\mathbf{x}\in\Gamma_u} = \mathbf{0}\}\end{aligned} \tag{1.15}$$

and subtracting them:

$$\int_V \Delta\dot{\boldsymbol{\sigma}} : \nabla^S \boldsymbol{\eta}\, dV = \mathbf{0} \quad \forall \boldsymbol{\eta} \in \mathbb{V} \tag{1.16}$$

Now notice that if both $\mathbb{R}^{(1)}(\mathbf{x},t)$ and $\mathbb{R}^{(2)}(\mathbf{x},t)$ are solutions of the problem then from equation (1.8):

$$\left.\begin{aligned}\dot{\mathbf{u}}^{(1)} = \dot{\mathbf{u}}^* \;\; \forall \mathbf{x} \in \Gamma_u\\ \dot{\mathbf{u}}^{(2)} = \dot{\mathbf{u}}^* \;\; \forall \mathbf{x} \in \Gamma_u\end{aligned}\right\} \Rightarrow \Delta\dot{\mathbf{u}}(\mathbf{x},t) = \mathbf{0} \;\; \forall \mathbf{x} \in \Gamma_u \;\; \forall t \in [0,T] \tag{1.17}$$

so, in view of equation (1.16):

$$\Delta\dot{\mathbf{u}}(\mathbf{x},t) \in \mathbb{V} \tag{1.18}$$

Therefore we can consider equation (1.16) with $\boldsymbol{\eta} \equiv \Delta\dot{\mathbf{u}}(\mathbf{x},t)$:

$$\int_V \Delta\dot{\boldsymbol{\sigma}} : \underbrace{\nabla^S \Delta\dot{\mathbf{u}}}_{\Delta\dot{\boldsymbol{\varepsilon}}}\, dV = \int_V \Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}}\, dV = \mathbf{0} \tag{1.19}$$

Thus, a **necessary** condition for non-uniqueness is:

$$\begin{aligned}&\exists\, \Delta\dot{\boldsymbol{\sigma}}(\mathbf{x},t) \neq \mathbf{0}\\ &\exists\, \Delta\dot{\boldsymbol{\varepsilon}}(\mathbf{x},t) \neq \mathbf{0}\end{aligned} \quad \text{such that} \quad \int_V \Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}}\, dV = \mathbf{0} \quad \forall t \in [t_B, T] \tag{1.20}$$

and a **sufficient** condition for uniqueness is:

NOTE

$\Delta\dot{\boldsymbol{\sigma}}(\mathbf{x},t)$ and $\Delta\dot{\boldsymbol{\varepsilon}}(\mathbf{x},t)$ refer to *possible* perturbations of the homogeneous solution of the B.V.P.

Uniqueness condition (global or *in the large*)

$$\int_V \Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}}\, dV > \mathbf{0} \quad \forall\; \Delta\dot{\boldsymbol{\sigma}}(\mathbf{x},t) \neq \mathbf{0} \quad \forall\; \Delta\dot{\boldsymbol{\varepsilon}}(\mathbf{x},t) \neq \mathbf{0} \tag{1.21}$$

A more restrictive, and therefore **sufficient**, condition for uniqueness is:

Uniqueness condition (local or *in the small*)
$\Delta\dot{\boldsymbol{\sigma}}:\Delta\dot{\boldsymbol{\varepsilon}}>\mathbf{0} \quad \forall\ \Delta\dot{\boldsymbol{\sigma}}\neq\mathbf{0} \quad \forall\ \Delta\dot{\boldsymbol{\varepsilon}}\neq\mathbf{0}$

(1.22)

since fulfillment of equation (1.22) necessarily implies fulfillment of equation (1.21).

Remark 1-4

A material fulfilling equation (1.22) :
$\Delta\dot{\boldsymbol{\sigma}}:\Delta\dot{\boldsymbol{\varepsilon}}>\mathbf{0} \quad \forall\ \Delta\dot{\boldsymbol{\sigma}}\neq\mathbf{0} \quad \forall\ \Delta\dot{\boldsymbol{\varepsilon}}\neq\mathbf{0}$

is called a *positive material*

Remark 1-5

Notice that, equation (1.18), can be extended to:

$$\Delta\mathbf{u}(\mathbf{x},t)\in\mathbb{V}\ ;\quad \Delta\dot{\mathbf{u}}(\mathbf{x},t)\in\mathbb{V}\quad ;\ \Delta\ddot{\mathbf{u}}(\mathbf{x},t)\in\mathbb{V}\ ......$$

and that the Virtual Work Principle (1.6) can be written for higher order time derivatives:

$$\int_V \boldsymbol{\sigma}:\nabla^s\boldsymbol{\eta}\,dV=\int_V \rho\ \mathbf{b}\cdot\boldsymbol{\eta}\ dV+\int_{\Gamma_\sigma}\mathbf{t}^*\cdot\boldsymbol{\eta}\,d\Gamma \quad \forall\boldsymbol{\eta}\in\mathbb{V}$$

$$\int_V \dot{\boldsymbol{\sigma}}:\nabla^s\boldsymbol{\eta}\,dV=\int_V \rho\ \dot{\mathbf{b}}\cdot\boldsymbol{\eta}\ dV+\int_{\Gamma_\sigma}\dot{\mathbf{t}}^*\cdot\boldsymbol{\eta}\,d\Gamma \quad \forall\boldsymbol{\eta}\in\mathbb{V}$$

$$\int_V \ddot{\boldsymbol{\sigma}}:\nabla^s\boldsymbol{\eta}\,dV=\int_V \rho\ \ddot{\mathbf{b}}\cdot\boldsymbol{\eta}\ dV+\int_{\Gamma_\sigma}\ddot{\mathbf{t}}^*\cdot\boldsymbol{\eta}\,d\Gamma \quad \forall\boldsymbol{\eta}\in\mathbb{V}$$

$$\mathbb{V}:=\{\boldsymbol{\eta}(\mathbf{x}):V\rightarrow\mathbb{R}^3\ \ |\ \boldsymbol{\eta}(\mathbf{x})|_{\mathbf{x}\in\Gamma_u}=\mathbf{0}\}$$

so that, by combination of them, additional sufficient conditions for uniqueness could be derived i.e.:

$$\Delta\boldsymbol{\sigma}:\Delta\boldsymbol{\varepsilon}>\mathbf{0} \quad \forall\ \Delta\boldsymbol{\sigma}\neq\mathbf{0} \quad \forall\ \Delta\boldsymbol{\varepsilon}\neq\mathbf{0}$$
$$\Delta\dot{\boldsymbol{\sigma}}:\Delta\dot{\boldsymbol{\varepsilon}}>\mathbf{0} \quad \forall\ \Delta\dot{\boldsymbol{\sigma}}\neq\mathbf{0} \quad \forall\ \Delta\dot{\boldsymbol{\varepsilon}}\neq\mathbf{0}$$
$$\Delta\ddot{\boldsymbol{\sigma}}:\Delta\ddot{\boldsymbol{\varepsilon}}>\mathbf{0} \quad \forall\ \Delta\ddot{\boldsymbol{\sigma}}\neq\mathbf{0} \quad \forall\ \Delta\ddot{\boldsymbol{\varepsilon}}\neq\mathbf{0}$$
.....

1.5 Discontinuous bifurcation

1.5.1 Maxwell's compatibility conditions

The Maxwell's conditions refer to fields that are *continuous* in a domain Ω and whose first derivatives are discontinuous across a line (for 2D) or a surface S (for 3D). The discontinuous interface S divides Ω in Ω^+ (pointed by the normal $\mathbf{n}$) and Ω^-, see Figure 1-6.

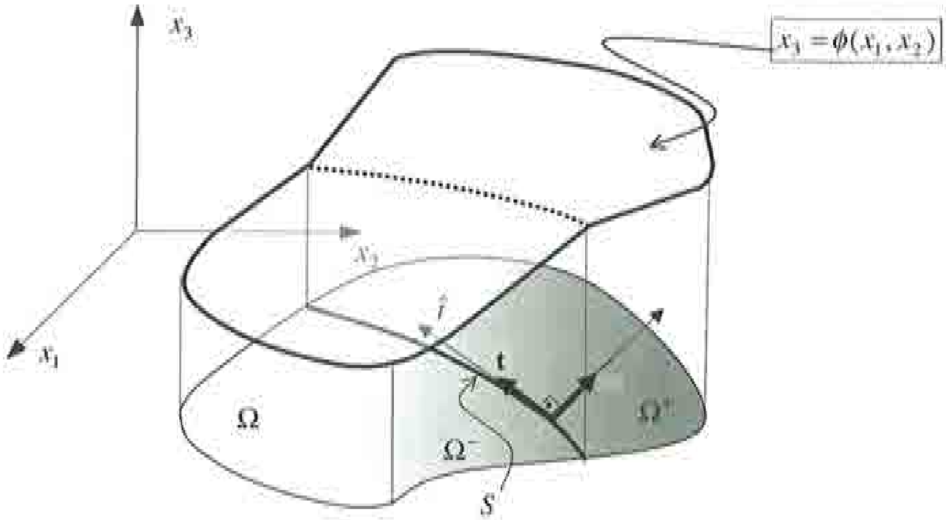


Figure 1-6– Continuous scalar field with discontinuous gradients

> **NOTE**
> We define
> $[[(\bullet)]] \overset{not}{=} (\bullet)_S^+ - (\bullet)_S^-$

- Scalar field $\phi(x_1, x_2): \mathbb{R}^2 \to \mathbb{R}$

$$\left.\begin{array}{l} \nabla\phi^+\big|_S = \dfrac{\partial\phi^+}{\partial n}\Big|_S \mathbf{n} + \dfrac{\partial\phi^+}{\partial \hat{t}}\Big|_S \mathbf{t} \\ \nabla\phi^-\big|_S = \dfrac{\partial\phi^-}{\partial n}\Big|_S \mathbf{n} + \dfrac{\partial\phi^-}{\partial \hat{t}}\Big|_S \mathbf{t} \end{array}\right\} \Rightarrow [[\nabla\phi]] \overset{def}{=} \left[\left[\frac{\partial\phi}{\partial n}\right]\right]_S \mathbf{n} + \left[\left[\frac{\partial\phi}{\partial \hat{t}}\right]\right]_S \mathbf{t} \tag{1.23}$$

Since the field ϕ is continuous across S the tangential derivative $\dfrac{\partial\phi}{\partial\hat{t}}$ does not jump:

$$\frac{\partial\phi^+}{\partial\hat{t}}\Big|_S = \frac{\partial\phi^-}{\partial\hat{t}}\Big|_S = \frac{d\phi(\mathbf{x}(\hat{t}))}{d\hat{t}}\Big|_{\mathbf{x}\in S} \Rightarrow \left[\left[\frac{\partial\phi}{\partial\hat{t}}\right]\right]_S = 0 \tag{1.24}$$

and from equations (1.24) and (1.23):

$$\boxed{[[\nabla\phi]] = \left[\left[\frac{\partial\phi}{\partial n}\right]\right]_S \mathbf{n} = \beta\,\mathbf{n}} \tag{1.25}$$

> **NOTE**
> From now on we shall use Einstein's notation:
> $\sum_{i=1}^{i=3} u_i\ \hat{\mathbf{e}}_i \overset{not}{=} u_i\ \hat{\mathbf{e}}_i$

- Vectorial field $\mathbf{u}(\mathbf{x}) = u_i(\mathbf{x})\,\hat{\mathbf{e}}_i$

$$[[\nabla\otimes\mathbf{u}]] = \underbrace{[[\nabla u_i]]}_{\beta_i\mathbf{n}} \otimes \hat{\mathbf{e}}_i = \beta_i\,\mathbf{n}\otimes\hat{\mathbf{e}}_i = \mathbf{n}\otimes\underbrace{\beta_i\ \hat{\mathbf{e}}_i}_{\boldsymbol{\beta}} = \mathbf{n}\otimes\boldsymbol{\beta} \tag{1.26}$$

$$[[\nabla^s\mathbf{u}]] = \left[\left[\frac{1}{2}(\nabla\otimes\mathbf{u} + \mathbf{u}\otimes\nabla)\right]\right] = \frac{1}{2}(\mathbf{n}\otimes\boldsymbol{\beta} + \boldsymbol{\beta}\otimes\mathbf{n}) = (\boldsymbol{\beta}\otimes\mathbf{n})^s \tag{1.27}$$

If $\mathbf{u}(\mathbf{x})$ stands for a displacement field then, $\nabla^s\mathbf{u} = \boldsymbol{\varepsilon}$, and from equation (1.27):

Maxwells' s compatibility condition

$$[[\boldsymbol{\varepsilon}]] \overset{def}{=} \boldsymbol{\varepsilon}^+ - \boldsymbol{\varepsilon}^- = [[\nabla^s\mathbf{u}]] = (\boldsymbol{\beta}\otimes\mathbf{n})^s \tag{1.28}$$

1.5.2 Material ellipticity

Let us consider the *continuous fields* in Ω, $\boldsymbol{\varepsilon}(\mathbf{x},t)$, and $\boldsymbol{\sigma}(\mathbf{x},t)$. Let us look for the conditions that allow $\dot{\boldsymbol{\varepsilon}}(\mathbf{x},t)$ and $\dot{\boldsymbol{\sigma}}(\mathbf{x},t)$ to be discontinuous but compatible with the necessary traction ($\dot{\mathcal{T}} = \dot{\boldsymbol{\sigma}} \cdot \mathbf{n}$) continuity across S, expressed as:

$$[\![\dot{\mathcal{T}}]\!](\mathbf{x}) = [\![\dot{\boldsymbol{\sigma}}]\!] \cdot \mathbf{n} = \dot{\boldsymbol{\sigma}}^{+} \cdot \mathbf{n} - \dot{\boldsymbol{\sigma}}^{-} \cdot \mathbf{n} = \mathbf{0} \quad \forall \mathbf{x} \in S \tag{1.29}$$

According to equation (1.28) we can write for the strains:

$$[\![\dot{\boldsymbol{\varepsilon}}]\!] = (\boldsymbol{\beta} \otimes \mathbf{n})^{s} \Rightarrow \dot{\boldsymbol{\varepsilon}}^{+}(\mathbf{x}) = \dot{\boldsymbol{\varepsilon}}^{-}(\mathbf{x}) + (\boldsymbol{\beta} \otimes \mathbf{n})^{s}(\mathbf{x}) \quad for\ \mathbf{x} \in S \tag{1.30}$$

Let us consider now the very general case for the rate form of the constitutive equation (1.7):

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{f}(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \dot{\boldsymbol{\varepsilon}}) = \mathbb{E}_T(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}) : \dot{\boldsymbol{\varepsilon}} + \mathbf{g}(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}) \tag{1.31}$$

where $\mathbb{E}_T$ is the tangent constitutive tensor. Then, from equations (1.30) and (1.31):

NOTE

We assume that $\mathbb{E}_T$ has, at least, minor symmetries: $\mathbb{E}_{T\,ijkl} = \mathbb{E}_{T\,ijlk}$

$$[\![\dot{\boldsymbol{\sigma}}]\!] = \mathbb{E}_T : [\![\dot{\boldsymbol{\varepsilon}}]\!] + \underbrace{[\![\mathbf{g}]\!]}_{=\mathbf{0}} = \mathbb{E}_T : (\boldsymbol{\beta} \otimes \mathbf{n})^{s} = (\mathbb{E}_T \cdot \mathbf{n}) \cdot \boldsymbol{\beta} \tag{1.32}$$

and inserting equation (1.32) into (1.29):

$$[\![\dot{\mathcal{T}}]\!] = \mathbf{n} \cdot [\![\dot{\boldsymbol{\sigma}}]\!] = \underbrace{(\mathbf{n} \cdot \mathbb{E}_T \cdot \mathbf{n})}_{\mathbb{Q}_T} \cdot \boldsymbol{\beta} = \mathbf{0} = \mathbb{Q}_T \cdot \boldsymbol{\beta} = \mathbf{0} \tag{1.33}$$

where $\mathbb{Q}_T$ is the *localization tensor*. In order equation (1.33) to have a non trivial solution ($\boldsymbol{\beta} \neq \mathbf{0}$):

$$\det \mathbb{Q}_T = \det(\mathbf{n} \cdot \mathbb{E}_T \cdot \mathbf{n}) = 0 \tag{1.34}$$

Remark 1-6

Condition $\det \mathbb{Q}_T(\mathbf{x}) = 0$ is a necessary condition for the appearance, at the material point $\mathbf{x}$, of discontinuous rate of strain fields as in equation (1.30). Condition

Strong ellipticity condition
$\det \mathbb{Q}_T > 0$

is, then, a sufficient condition to preclude the appearance of such discontinuous (rate of) strain fields. This condition is also called the *strong ellipticity condition*.

1.5.3 Material positivity and strong ellipticity

Let us consider a positive material, according to definition of Remark 1-4, and the following particular cases of bifurcation of the solution of the B.V.P.:

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^{(1)}(\mathbf{x},t)\Big|_{t=t_B} \overset{not}{=} \dot{\boldsymbol{\varepsilon}}_B^{(1)}(\mathbf{x}) \rightarrow \text{fundamental solution (smooth)} \\ \dot{\boldsymbol{\varepsilon}}_B^{(2)}(\mathbf{x}) = \dot{\boldsymbol{\varepsilon}}_B^{(1)}(\mathbf{x}) + \mu_S(\mathbf{x})(\boldsymbol{\beta}\otimes\mathbf{n})^s(\mathbf{x}) \rightarrow \begin{bmatrix} \text{bifurcated solution} \\ (\text{discontinuous at time } t_B) \end{bmatrix} \\ \left(\dot{\boldsymbol{\varepsilon}}_B^{(2)} - \dot{\boldsymbol{\varepsilon}}_B^{(1)}\right)_{\mathbf{x}\in S} \overset{not}{=} \Delta\dot{\boldsymbol{\varepsilon}}_B(\mathbf{x}) = \mu_S(\mathbf{x})(\boldsymbol{\beta}\otimes\mathbf{n})^s(\mathbf{x}) \end{cases} \tag{1.35}$$

where $\mu_S(\mathbf{x})$ is a collocation function on the discontinuity domain S:

$$\mu_S(\mathbf{x}) = \begin{cases} 1 \ \forall \ \mathbf{x}\in S \\ 0 \ \forall \ \mathbf{x}\notin S \end{cases} \tag{1.36}$$

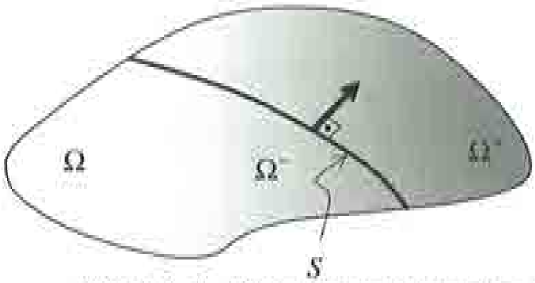


Figure 1-7– Discontinuous interface S

For the family of materials following equation (1.31) ($\dot{\boldsymbol{\sigma}} = \mathbb{E}_T(\boldsymbol{\varepsilon},\boldsymbol{\sigma}):\dot{\boldsymbol{\varepsilon}} + \mathbf{g}(\boldsymbol{\varepsilon},\boldsymbol{\sigma})$) from equations (1.22) and (1.31):

$$\begin{aligned} &\Delta\dot{\boldsymbol{\sigma}}_B = \mathbb{E}_T : \Delta\dot{\boldsymbol{\varepsilon}}_B \Rightarrow \Delta\dot{\boldsymbol{\sigma}}_B : \Delta\dot{\boldsymbol{\varepsilon}}_B = \underbrace{\Delta\dot{\boldsymbol{\varepsilon}}_B}_{(\boldsymbol{\beta}\otimes\mathbf{n})^s} : \mathbb{E}_T : \Delta\dot{\boldsymbol{\varepsilon}}_B = \\ &= (\boldsymbol{\beta}\otimes\mathbf{n})^s : \mathbb{E}_T : (\boldsymbol{\beta}\otimes\mathbf{n})^s = \boldsymbol{\beta}\cdot\underbrace{(\mathbf{n}\cdot\mathbb{E}_T\cdot\mathbf{n})}_{\mathbb{Q}_T}\cdot\boldsymbol{\beta} > 0 \quad \forall\boldsymbol{\beta}\neq\mathbf{0} \end{aligned} \tag{1.37}$$

$$\Rightarrow \boldsymbol{\beta}\cdot\mathbb{Q}_T\cdot\boldsymbol{\beta} > 0 \quad \forall\boldsymbol{\beta}\neq\mathbf{0} \Rightarrow \det\mathbb{Q}_T > 0 \rightarrow \textit{strong elipticity} \tag{1.38}$$

Remark 1-7

- For the family of materials following equation (1.31), the **positive character** of the material **implies strong ellipticity.**
- Materials fulfilling the strong ellipticity condition $\det\mathbb{Q}_T > \mathbf{0}$ preclude any type of bifurcation involving discontinuous strain fields. Therefore they **exclude local discontinuous bifurcation modes** (material bifurcation modes) and reduce the possible bifurcation modes to the global ones (geometrical bifurcation modes).

1.6 Stability

1.6.1 Referenced boundary value problem

Let's now consider the B.V.P. of equations (1.1) to (1.3) with the actions given in terms of a reference action $\mathbb{A}_0(\mathbf{x})$, which is scaled by means a *load factor* $\lambda(t)$ to give the actual actions $\mathbb{A}(\mathbf{x},t)$ (see Figure 1-8):

$$\mathbb{A}(\mathbf{x},t)=\lambda(t)\,\mathbb{A}_0(\mathbf{x}) \quad ; \quad \mathbb{A}_0(\mathbf{x}) \rightarrow \text{reference action}$$

$$\mathbb{A}_0(\mathbf{x})=\begin{bmatrix}\mathbf{b}_0(\mathbf{x})\\ \mathbf{t}_0^*(\mathbf{x})\\ \mathbf{u}_0^*(\mathbf{x})\end{bmatrix} \Rightarrow \mathbb{A}(\mathbf{x},t)\equiv\begin{bmatrix}\mathbf{b}(\mathbf{x},t)\\ \mathbf{t}^*(\mathbf{x},t)\\ \mathbf{u}^*(\mathbf{x},t)\end{bmatrix}=\begin{bmatrix}\lambda(t)\mathbf{b}_0(\mathbf{x})\\ \lambda(t)\mathbf{t}_0^*(\mathbf{x})\\ \lambda(t)\mathbf{u}_0^*(\mathbf{x})\end{bmatrix} \tag{1.39}$$

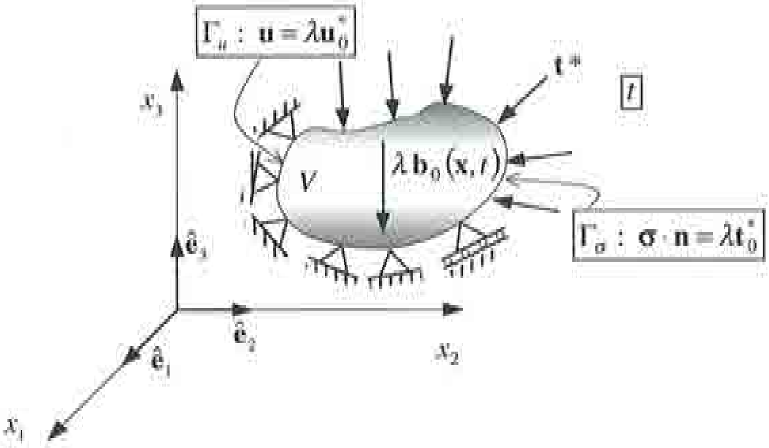


Figure 1-8– Referenced boundary value problem

The B.V.P. reads:

Find:

$$\begin{aligned}&\mathbf{u}(\mathbf{x},t): V\times[0,T]\rightarrow\mathbb{R}^3 &&\Rightarrow \textit{Displacements}\\ &\boldsymbol{\varepsilon}(\mathbf{x},t): V\times[0,T]\rightarrow\mathbb{R}^6 &&\Rightarrow \textit{Strains}\\ &\boldsymbol{\sigma}(\mathbf{x},t): V\times[0,T]\rightarrow\mathbb{R}^6 &&\Rightarrow \textit{Stresses}\end{aligned} \tag{1.40}$$

Fulfilling $\forall\mathbf{x}\in V$ and for any $t\in[0,T]$:

$$\nabla\cdot\boldsymbol{\sigma}+\rho\,\lambda\mathbf{b}_0=0 \quad \rightarrow \text{ Momentum balance (equilibrium)}$$

$$\boldsymbol{\sigma}=\boldsymbol{f}(\boldsymbol{\varepsilon}) \quad \rightarrow \text{ Constitutive equation} \tag{1.41}$$

$$\boldsymbol{\varepsilon}=\nabla^S\mathbf{u}=\frac{1}{2}(\mathbf{u}\otimes\nabla+\nabla\otimes\mathbf{u}) \quad \rightarrow \text{ Kinematic equation}$$

$$\left.\begin{aligned}&\Gamma_u:\ \mathbf{u}=\lambda\mathbf{u}_0^*\ \ \forall\mathbf{x}\in\Gamma_u\\ &\Gamma_\sigma:\boldsymbol{\sigma}\cdot\mathbf{n}=\lambda\mathbf{t}_0^*\ \ \forall\ \mathbf{x}\in\Gamma_\sigma\end{aligned}\right\} \rightarrow \text{ Boundary conditions} \tag{1.42}$$

The corresponding rate version of the B.V.P is now given by:

$\nabla \cdot \dot{\boldsymbol{\sigma}} + \rho \dot{\lambda} \mathbf{b}_0 = 0$	$\rightarrow$ Momentum balance (equilibrium)	
$\dot{\boldsymbol{\sigma}} = \boldsymbol{f}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}})$	$\rightarrow$ Constitutive equation	(1.43)
$\dot{\boldsymbol{\varepsilon}} = \nabla^S \dot{\mathbf{u}} = \frac{1}{2}(\dot{\mathbf{u}} \otimes \nabla + \nabla \otimes \dot{\mathbf{u}})$	$\rightarrow$ Kinematic equation	
$\left.\begin{aligned} &\dot{\mathbf{u}} = \dot{\lambda} \mathbf{u}_0^* \quad \forall \mathbf{x} \in \Gamma_u \\ &\dot{\boldsymbol{\sigma}} \cdot \mathbf{n} = \dot{\lambda} \mathbf{t}_0^* \quad \forall \mathbf{x} \in \Gamma_\sigma \end{aligned}\right\}$	$\rightarrow$ Boundary conditions	(1.44)

The action-response curves along time can now be given in terms of the load factor $\lambda(t)$ vs. $\mathbb{R}$. Points characterized by $\dot{\lambda} = 0$ and $\dot{\mathbb{R}} \neq \mathbf{0}$ are termed **limit points** or **instability points** (see Figure 1-9).

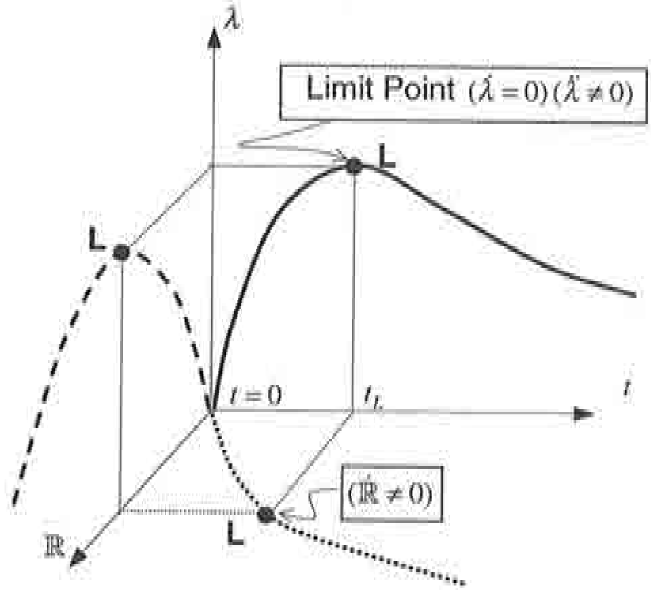


Figure 1-9

The rate form of the Virtual Work Principle (1.9) now reads:

$$\int_V \dot{\boldsymbol{\sigma}} : \nabla^s \boldsymbol{\eta} \, dV = \int_V \rho \, \dot{\lambda} \mathbf{b}_0 \cdot \boldsymbol{\eta} \, dV + \int_{\Gamma_\sigma} \dot{\lambda} \mathbf{t}_0^* \cdot \boldsymbol{\eta} \, d\Gamma] \quad \forall \boldsymbol{\eta} \in \mathbb{V}$$
$$\mathbb{V} := \{\boldsymbol{\eta}(\mathbf{x}) : V \rightarrow \mathbb{R}^3 \;\; | \; \boldsymbol{\eta}(\mathbf{x})|_{\mathbf{x} \in \Gamma_u} = \mathbf{0}\} \tag{1.47}$$

which, for the particular case of an instability point ($\dot{\lambda} = 0$) yields to:

$$\int_V \dot{\boldsymbol{\sigma}}_L : \nabla^s \boldsymbol{\eta} \, dV = 0 \quad \forall \boldsymbol{\eta} \in \mathbb{V} \quad \textit{with} \;\; \dot{\boldsymbol{\sigma}}_L \neq \mathbf{0}$$
$$\mathbb{V} := \{\boldsymbol{\eta}(\mathbf{x}) : V \rightarrow \mathbb{R}^3 \;\; | \; \boldsymbol{\eta}(\mathbf{x})|_{\mathbf{x} \in \Gamma_u} = \mathbf{0}\} \tag{1.48}$$

The solution in terms of the displacement at the limit point fulfills:

$$\Gamma_u : \dot{\mathbf{u}}(\mathbf{x},t)\big|_{t=t_L} \overset{def}{=} \dot{\mathbf{u}}_L(\mathbf{x})\big|_{t=t_L} = \underbrace{\dot{\lambda}\big|_{t=t_L}}_{=0} \mathbf{u}_0^*(\mathbf{x}) = \mathbf{0} \quad \Rightarrow \dot{\mathbf{u}}_L(\mathbf{x}) \in \mathbb{V} \tag{1.49}$$

and substitution in equation (1.48) reads:

$$\int_V \dot{\boldsymbol{\sigma}}_L : \underbrace{\nabla^s \dot{\mathbf{u}}_L}_{\dot{\boldsymbol{\varepsilon}}_L} dV = \int_V \dot{\boldsymbol{\sigma}}_L : \dot{\boldsymbol{\varepsilon}}_L \, dV = 0 \quad with \begin{cases} \dot{\boldsymbol{\varepsilon}}_L \neq \mathbf{0} \\ \dot{\boldsymbol{\sigma}}_L \neq \mathbf{0} \end{cases} \tag{1.50}$$

Equation (1.50) is a necessary equation for instability at time t_L. Therefore a sufficient condition for satiability along the time interval of interest $[0,T]$ is:

$$\boxed{\begin{array}{l}\text{Stability condition (global or in the large)}\\ \int_V \dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} \, dV > 0 \quad \forall t \in [0,T] \quad \forall \begin{cases} \dot{\boldsymbol{\varepsilon}} \neq \mathbf{0} \\ \dot{\boldsymbol{\sigma}} \neq \mathbf{0} \end{cases}\end{array}} \tag{1.51}$$

A more restrictive and, therefore sufficient, condition is:

NOTE

This is also referenced in the literature as the Hill's stability condition.

$$\boxed{\begin{array}{l}\text{Stability condition (local or in the small)}\\ \dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} > 0 \quad \forall \dot{\boldsymbol{\varepsilon}} \neq \mathbf{0} \; \forall \dot{\boldsymbol{\sigma}} \neq \mathbf{0}\end{array}} \tag{1.52}$$

1.7 Constitutive models uniqueness and stability

Let us now check the properties of different families of materials (constitutive models) in terms of uniqueness (material positivity), discontinuous bifurcation (material ellipticity) and stability (material stability).

1.7.1 Linear elastic models

NOTE

$\mathbb{E}$ is a positive definite fourth order tensor, therefore, for any given second order tensor $\mathbf{a} \neq \mathbf{0} \rightarrow \mathbf{a} : \mathbb{E} : \mathbf{a} > 0$

$$\boxed{\text{Constitutive equation} \rightarrow \begin{cases} \boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\varepsilon} \\ \dot{\boldsymbol{\sigma}} = \mathbb{E} : \dot{\boldsymbol{\varepsilon}} \end{cases}} \tag{1.53}$$

- Positivity

$$\begin{array}{l} \left.\begin{array}{l} \dot{\boldsymbol{\sigma}}^{(1)} = \mathbb{E} : \dot{\boldsymbol{\varepsilon}}^{(1)} \\ \dot{\boldsymbol{\sigma}}^{(2)} = \mathbb{E} : \dot{\boldsymbol{\varepsilon}}^{(2)} \end{array}\right\} \Rightarrow \Delta\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}^{(2)} - \dot{\boldsymbol{\sigma}}^{(1)} = \mathbb{E} : (\dot{\boldsymbol{\varepsilon}}^{(2)} - \dot{\boldsymbol{\varepsilon}}^{(1)}) = \mathbb{E} : \Delta\dot{\boldsymbol{\varepsilon}} \\ \Delta\dot{\boldsymbol{\sigma}} = \mathbb{E} : \Delta\dot{\boldsymbol{\varepsilon}} \Rightarrow \Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}} = \Delta\dot{\boldsymbol{\varepsilon}} : \mathbb{E} : \Delta\dot{\boldsymbol{\varepsilon}} > 0 \quad \forall \Delta\dot{\boldsymbol{\varepsilon}} \neq 0 \rightarrow \text{pass} \end{array} \tag{1.54}$$

- Ellipticity $\rightarrow$ pass (see Remark 1-7
- Stability

$$\dot{\boldsymbol{\sigma}} = \mathbb{E} : \dot{\boldsymbol{\varepsilon}} \Rightarrow \dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}} : \mathbb{E} : \dot{\boldsymbol{\varepsilon}} > 0 \quad \forall \dot{\boldsymbol{\varepsilon}} \neq 0 \rightarrow \text{pass} \tag{1.55}$$

1.7.2 Non linear hyperelastic models

$$\text{Constitutive equation} \rightarrow \begin{cases} \boldsymbol{\sigma} = \dfrac{\partial W(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \\ \dot{\boldsymbol{\sigma}} = \underbrace{\dfrac{\partial^2 W(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}}}_{\mathbb{E}_T} : \dot{\boldsymbol{\varepsilon}} = \mathbb{E}_T(\boldsymbol{\varepsilon}) : \dot{\boldsymbol{\varepsilon}} \end{cases} \tag{1.56}$$

where $\mathbb{E}$ is a symmetric but not necessarily positive definite fourth order tensor. As $\mathbb{E}_T(\boldsymbol{\varepsilon})$ is **positive definite**, the model enjoys the same properties than the linear model:

- Positivity

$$\left.\begin{array}{l} \dot{\boldsymbol{\sigma}}^{(1)} = \mathbb{E}_T(\boldsymbol{\varepsilon}) : \dot{\boldsymbol{\varepsilon}}^{(1)} \\ \dot{\boldsymbol{\sigma}}^{(2)} = \mathbb{E}_T(\boldsymbol{\varepsilon}) : \dot{\boldsymbol{\varepsilon}}^{(2)} \end{array}\right\} \Rightarrow \Delta\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}^{(2)} - \dot{\boldsymbol{\sigma}}^{(1)} = \mathbb{E}_T : (\dot{\boldsymbol{\varepsilon}}^{(2)} - \dot{\boldsymbol{\varepsilon}}^{(1)}) = \mathbb{E}_T : \Delta\dot{\boldsymbol{\varepsilon}} \tag{1.57}$$

$$\Delta\dot{\boldsymbol{\sigma}} = \mathbb{E}_T : \Delta\dot{\boldsymbol{\varepsilon}} \Rightarrow \Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}} = \Delta\dot{\boldsymbol{\varepsilon}} : \mathbb{E}_T : \Delta\dot{\boldsymbol{\varepsilon}} > 0 \quad \forall \Delta\dot{\boldsymbol{\varepsilon}} \neq 0 \rightarrow \text{pass} \tag{1.58}$$

- Ellipticity $\rightarrow$ pass (see Remark 1-7)

- Stability

$$\dot{\boldsymbol{\sigma}} = \mathbb{E}_T : \dot{\boldsymbol{\varepsilon}} \Rightarrow \dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}} : \mathbb{E}_T : \dot{\boldsymbol{\varepsilon}} > 0 \quad \forall \dot{\boldsymbol{\varepsilon}} \neq 0 \rightarrow \text{pass} \tag{1.59}$$

> **Remark 1-8**
>
> Non-linear elastic models with a *non positive definite* tangent constitutive tensor $\mathbb{E}_T$ do not assure uniqueness, ellipticity and stability

1.7.3 Elasto-plastic, associative, strain hardening models

$$\text{Constitutive equation} \rightarrow \begin{cases} \boldsymbol{\sigma} = \dfrac{\partial W(\boldsymbol{\varepsilon}, \boldsymbol{\alpha})}{\partial \boldsymbol{\varepsilon}} \\ \dot{\boldsymbol{\sigma}} = \mathbb{E}_T(\boldsymbol{\sigma}) : \dot{\boldsymbol{\varepsilon}} \rightarrow \textit{loading} \\ \dot{\boldsymbol{\sigma}} = \mathbb{E} : \dot{\boldsymbol{\varepsilon}} \rightarrow \textit{unloading} \end{cases} \tag{1.60}$$

Let us consider the case of hardening. Therefore $\mathbb{E}_T$ is positive definite.

- Positivity

NOTE

Only perturbations keeping *the loading character* (loading or unloading) of the fundamental solution are considered here.

$$\begin{aligned} &\textit{Loading} \rightarrow \left.\begin{matrix} \dot{\boldsymbol{\sigma}}^{(1)} = \mathbb{E}_T(\boldsymbol{\sigma}) : \dot{\boldsymbol{\varepsilon}}^{(1)} \\ \dot{\boldsymbol{\sigma}}^{(2)} = \mathbb{E}_T(\boldsymbol{\sigma}) : \dot{\boldsymbol{\varepsilon}}^{(2)} \end{matrix}\right\} \Rightarrow \Delta\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}^{(2)} - \dot{\boldsymbol{\sigma}}^{(1)} = \\ &\qquad\qquad = \mathbb{E}_T : (\dot{\boldsymbol{\varepsilon}}^{(2)} - \dot{\boldsymbol{\varepsilon}}^{(1)}) = \mathbb{E}_T : \Delta\dot{\boldsymbol{\varepsilon}} \\ &\textit{Unoading} \rightarrow \left.\begin{matrix} \dot{\boldsymbol{\sigma}}^{(1)} = \mathbb{E} : \dot{\boldsymbol{\varepsilon}}^{(1)} \\ \dot{\boldsymbol{\sigma}}^{(2)} = \mathbb{E} : \dot{\boldsymbol{\varepsilon}}^{(2)} \end{matrix}\right\} \Rightarrow \Delta\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}^{(2)} - \dot{\boldsymbol{\sigma}}^{(1)} = \\ &\qquad\qquad = \mathbb{E} : (\dot{\boldsymbol{\varepsilon}}^{(2)} - \dot{\boldsymbol{\varepsilon}}^{(1)}) = \mathbb{E} : \Delta\dot{\boldsymbol{\varepsilon}} \end{aligned} \tag{1.61}$$

$$\begin{aligned} &\textit{Loading} \rightarrow \Delta\dot{\boldsymbol{\sigma}} = \mathbb{E}_T : \Delta\dot{\boldsymbol{\varepsilon}} \Rightarrow \Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}} = \Delta\dot{\boldsymbol{\varepsilon}} : \mathbb{E}_T : \Delta\dot{\boldsymbol{\varepsilon}} > 0 \quad \forall \Delta\dot{\boldsymbol{\varepsilon}} \neq 0 \rightarrow \text{pass} \\ &\textit{Unloading} \rightarrow \Delta\dot{\boldsymbol{\sigma}} = \mathbb{E} : \Delta\dot{\boldsymbol{\varepsilon}} \Rightarrow \Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}} = \Delta\dot{\boldsymbol{\varepsilon}} : \mathbb{E} : \Delta\dot{\boldsymbol{\varepsilon}} > 0 \quad \forall \Delta\dot{\boldsymbol{\varepsilon}} \neq 0 \rightarrow \text{pass} \end{aligned} \tag{1.62}$$

- Ellipticity $\rightarrow$ pass (see Remark 1-7)

- Stability

$$\begin{aligned} &\textit{Loading} \rightarrow \dot{\boldsymbol{\sigma}} = \mathbb{E}_T : \dot{\boldsymbol{\varepsilon}} \Rightarrow \dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}} : \mathbb{E}_T : \dot{\boldsymbol{\varepsilon}} > 0 \quad \forall \dot{\boldsymbol{\varepsilon}} \neq 0 \rightarrow \text{pass} \\ &\textit{Unloading} \rightarrow \dot{\boldsymbol{\sigma}} = \mathbb{E} : \dot{\boldsymbol{\varepsilon}} \Rightarrow \dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}} : \mathbb{E} : \dot{\boldsymbol{\varepsilon}} > 0 \quad \forall \dot{\boldsymbol{\varepsilon}} \neq 0 \rightarrow \text{pass} \end{aligned} \tag{1.63}$$

1.7.4 Continuum damage associative hardening models

$$\text{Constitutive equation} \rightarrow \begin{cases} \boldsymbol{\sigma} = \dfrac{\partial W(\boldsymbol{\varepsilon}, d)}{\partial \boldsymbol{\varepsilon}} = (1-d)\mathbb{E} : \boldsymbol{\varepsilon} \\ \dot{\boldsymbol{\sigma}} = \mathbb{E}_T(\boldsymbol{\varepsilon}) : \dot{\boldsymbol{\varepsilon}} \rightarrow \textit{loading} \\ \dot{\boldsymbol{\sigma}} = (1-d)\,\mathbb{E} : \dot{\boldsymbol{\varepsilon}} \rightarrow \textit{unloading} \end{cases} \tag{1.64}$$

Let us consider the case of hardening. Therefore $\mathbb{E}_T$ is positive definite. Also $(1-d) > 0$ and, thus $(1-d)\mathbb{E}$, is positive definite.

- Positivity

NOTE

Only perturbations keeping *the loading character* (loading or unloading) of the fundamental solution are considered here.

$$\begin{aligned}
&\textit{Loading} \to \left.\begin{array}{l}\dot{\sigma}^{(1)} = \mathbb{E}_T(\varepsilon):\dot{\varepsilon}^{(1)} \\ \dot{\sigma}^{(2)} = \mathbb{E}_T(\varepsilon):\dot{\varepsilon}^{(2)}\end{array}\right\} \Rightarrow \Delta\dot{\sigma} = \dot{\sigma}^{(2)} - \dot{\sigma}^{(1)} = \\
&\qquad\qquad = \mathbb{E}_T:(\dot{\varepsilon}^{(2)} - \dot{\varepsilon}^{(1)}) = \mathbb{E}_T:\Delta\dot{\varepsilon} \\
&\textit{Unoading} \to \left.\begin{array}{l}\dot{\sigma}^{(1)} = (1-d)\mathbb{E}:\dot{\varepsilon}^{(1)} \\ \dot{\sigma}^{(2)} = (1-d)\mathbb{E}:\dot{\varepsilon}^{(2)}\end{array}\right\} \Rightarrow \Delta\dot{\sigma} = \dot{\sigma}^{(2)} - \dot{\sigma}^{(1)} = \\
&\qquad\qquad = (1-d)\mathbb{E}:(\dot{\varepsilon}^{(2)} - \dot{\varepsilon}^{(1)}) = (1-d)\mathbb{E}:\Delta\dot{\varepsilon}
\end{aligned} \tag{1.65}$$

$$\begin{aligned}
&\textit{Loading} \to \Delta\dot{\sigma} = \mathbb{E}_T:\Delta\dot{\varepsilon} \Rightarrow \\
&\qquad \Delta\dot{\sigma}:\Delta\dot{\varepsilon} = \Delta\dot{\varepsilon}:\mathbb{E}_T:\Delta\dot{\varepsilon} > 0 \qquad \forall\Delta\dot{\varepsilon} \neq 0 \to \text{pass} \\
&\textit{Unloading} \to \Delta\dot{\sigma} = (1-d)\mathbb{E}:\Delta\dot{\varepsilon} \Rightarrow \\
&\qquad \Delta\dot{\sigma}:\Delta\dot{\varepsilon} = (1-d)\Delta\dot{\varepsilon}:\mathbb{E}:\Delta\dot{\varepsilon} > 0 \qquad \forall\Delta\dot{\varepsilon} \neq 0 \to \text{pass}
\end{aligned} \tag{1.66}$$

- Ellipticity $\to$ pass (see Remark 1-7)
- Stability

$$\begin{aligned}
&\textit{Loading} \to \dot{\sigma} = \mathbb{E}_T:\dot{\varepsilon} \Rightarrow \dot{\sigma}:\dot{\varepsilon} = \dot{\varepsilon}:\mathbb{E}_T:\dot{\varepsilon} > 0 \quad \forall\dot{\varepsilon} \neq 0 \to \text{pass} \\
&\textit{Unloading} \to \dot{\sigma} = (1-d)\mathbb{E}:\dot{\varepsilon} \Rightarrow \\
&\qquad \dot{\sigma}:\dot{\varepsilon} = (1-d)\dot{\varepsilon}:\mathbb{E}:\dot{\varepsilon} > 0 \qquad \forall\dot{\varepsilon} \neq 0 \to \text{pass}
\end{aligned} \tag{1.67}$$

Remark 1-9

For plasticity o damages models (or, in general, rate independent models) with either *strain softening or non-associative flows*, uniqueness, ellipticity and stability are not assured.

1.7.5 Rate dependent (viscoelastic, viscoplastic etc.) models

$$\text{Constitutive equation} \to \begin{cases} \sigma = \dfrac{\partial W(\varepsilon,\alpha)}{\partial \varepsilon} \\ \dot{\sigma} = \mathbb{E}:\dot{\varepsilon} + \mathbf{g}(\sigma,\alpha) \end{cases} \tag{1.68}$$

- Positivity

$$\Delta\dot{\sigma} = \mathbb{E}:\Delta\dot{\varepsilon} \Rightarrow \Delta\dot{\sigma}:\Delta\dot{\varepsilon} = \Delta\dot{\varepsilon}:\mathbb{E}:\Delta\dot{\varepsilon} > 0 \quad \forall\Delta\dot{\varepsilon} \neq 0 \to \text{pass} \tag{1.69}$$

- Strong ellipticity $\to$ pass (see Remark 1-7)

- Stability

$$\dot{\sigma} = \mathbb{E} : \dot{\varepsilon} + g(\sigma, \alpha) \Rightarrow \dot{\sigma} : \dot{\varepsilon} = \underbrace{\dot{\varepsilon} : \mathbb{E} : \dot{\varepsilon}}_{>0} + \underbrace{g(\sigma, \alpha) : \dot{\varepsilon}}_{\substack{unknown\\sign}} = \begin{cases} > 0 \\ < 0 \end{cases} \rightarrow \text{do not pass} \tag{1.70}$$

Remark 1-10

Rate dependent models, independently of their hardening/softening or associative/nonassociative character, *provide always a unique solution and keep the ellipticity of the material*. However, they do not assure stability.

2 Strong discontinuity approach (1D)

2.1 Motivation

2.1.1 One-dimensional elasto-plastic model

Let us consider the following one-dimensional elasto-plastic model with strain softening (see Figure 2-1 and Figure 2-2):

Free energy:	$\psi(\varepsilon,\varepsilon^p,\alpha)=\underbrace{\frac{1}{2}E(\varepsilon-\varepsilon^p)^2}_{\psi^e(\varepsilon-\varepsilon^p)}+\underbrace{\sigma_y\alpha+\frac{1}{2}H\alpha^2}_{\psi^p(\alpha)}$	(2.1)
Constitutive equation:	$\sigma=\frac{\partial\psi}{\partial\varepsilon}=E(\varepsilon-\varepsilon^p)$	(2.2)
Yield function:	$f(\sigma,q)=\lvert\sigma\rvert-q$	(2.3)
Flow rule:	$\dot{\varepsilon}^p=\lambda\frac{\partial f}{\partial\sigma}=\lambda\,sign(\sigma)$	(2.4)
Internal variable	$\dot{\alpha}=\lambda\frac{\partial f}{\partial q}=\lambda$	(2.5)
Hardening/ softening law	$q=\frac{\partial\psi}{\partial\alpha}=\sigma_y+H\alpha \;\Rightarrow\; \dot{q}=H\dot{\alpha}$	(2.6)
Loading/ unloading conditions	$\lambda\geq 0\;\; f\leq 0\;\; \lambda f=0\;\; (Kuhn-Tucker)$ $f=0\Rightarrow\;\lambda\dot{f}=0\quad (persistency)$	(2.7)
Tangent constitutive equation	$\dot{\sigma}=E_T\dot{\varepsilon}\;\;;\;\;E_T=\frac{EH}{E+H}\;\;(loading)$ $\dot{\sigma}=E\dot{\varepsilon}\;\;;\;\;(unloading)$	(2.8)

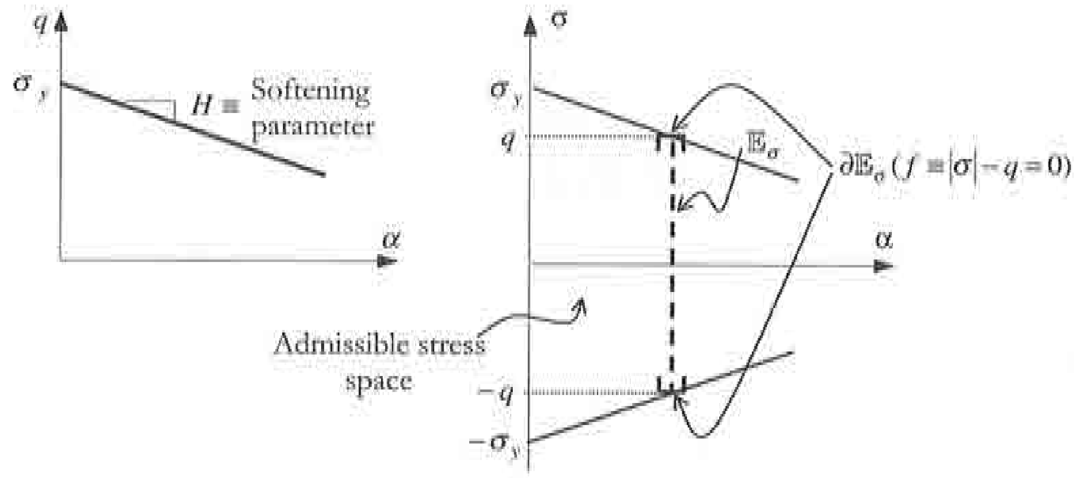


Figure 2-1– Elasto-plastic model: softening law and elastic domain

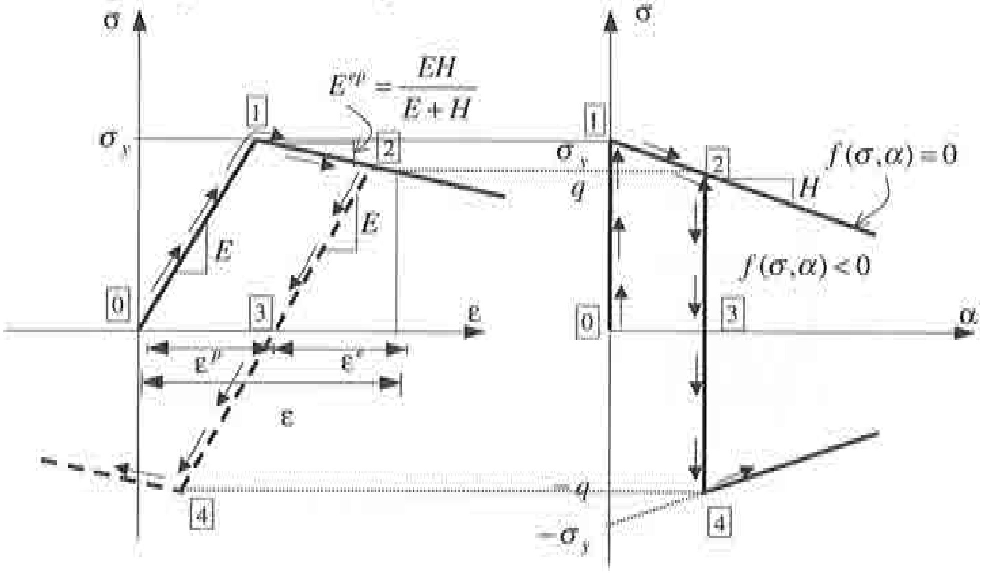


Figure 2-2– Elasto-plastic model: loading-unloading cycle

2.1.2 One-dimensional strain-localization problem

Let us consider the one-dimensional problem of the bar of Figure 2-3. The constitutive model is the elasto-plastic one given in section 2.1.1. The bar, of length ℓ and cross section A is homogeneous and characterized by a Young modulus E, elastic strength σ_y and softening parameter $H<0$. Under increasing imposed displacements at the right end, the bar experiences the following load-displacement response:

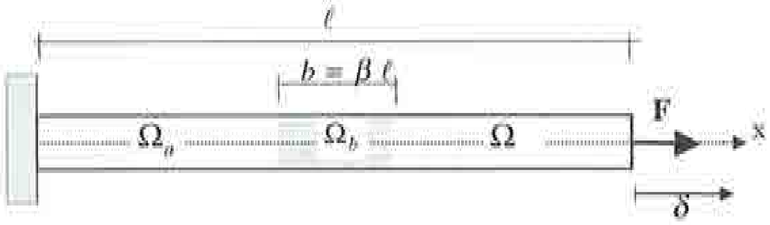


Figure 2-3– One dimensional localization problem

1) Elastic regime : $(\delta \le \frac{\sigma_y}{E}\ell)$:

$$\left.\begin{aligned}\varepsilon(x) &= \frac{\delta}{\ell} \quad \forall x \\ \sigma(x) &= E\varepsilon = E\,\frac{\delta}{\ell} \quad \forall x\end{aligned}\right\} \Rightarrow \boxed{F = A\sigma = \frac{EA}{\ell}\delta} \tag{2.12}$$

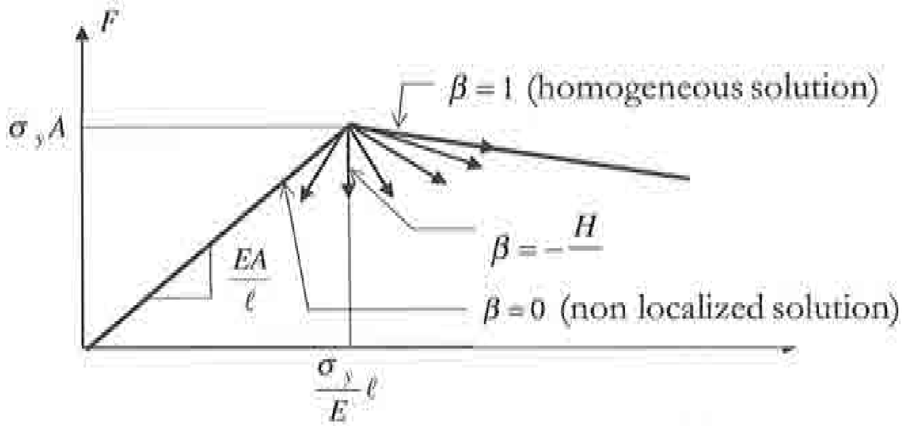


Figure 2-4– Solution of the 1D localization problem

2) Elasto-plastic regime: $(\delta > \frac{\sigma_y}{E}\ell)$

At the peak stress, $(\sigma = \sigma_y)$ let us consider that loading $(\dot\sigma = E_T\,\dot\varepsilon)$ takes place in the domain Ω_b, of length $b = \beta\ell$ $(\beta \in [0,1])$, whereas unloading $(\dot\sigma = E\dot\varepsilon)$ occurs at the rest of the bar (domain Ω_a). Let $\dot\varepsilon_a$ and $\dot\varepsilon_b$ $(\dot\varepsilon_a \ne \dot\varepsilon_b)$ the rate of the strains at both domains.

Equilibrium requires that:,

$$\dot\sigma_a = E\,\dot\varepsilon_a = \dot\sigma_b = E_T\dot\varepsilon_b \Rightarrow \dot\varepsilon_b = \frac{E}{E_T}\dot\varepsilon_a = \frac{E+H}{H}\dot\varepsilon_a \tag{2.13}$$

NOTE

This is a typical case of discontinuous bifurcation: $\dot\varepsilon_b \ne \dot\varepsilon_a$; $[\![\dot\sigma]\!] = \dot\sigma_b - \dot\sigma_a = 0$

and compatibility requires:

$$\dot\varepsilon_a(\ell - b) + \dot\varepsilon_b(b) = \dot\delta \tag{2.14}$$

and substituting equation (2.13) into (2.14):

$$\dot{\varepsilon}_u(\ell - b) + \dot{\varepsilon}_u \frac{E+H}{H} b = \dot{\delta} \Rightarrow$$
$$\dot{\delta} = (\ell + \frac{E}{H} b)\dot{\varepsilon}_u = (\ell + \frac{E}{H} \underbrace{b}_{\beta\ell})\frac{\dot{\sigma}}{E} = (1 + \frac{E}{H}\beta)\ell\frac{\dot{\sigma}}{E} \tag{2.15}$$

$$\dot{\sigma} = \frac{E}{(1 + \frac{E}{H}\beta)\ell}\dot{\delta} \quad \Rightarrow \quad \boxed{\dot{F} = \frac{EA}{(1 + \frac{E}{H}\beta)\ell}\dot{\delta}} \tag{2.16}$$

Remark 2-1

Inclusion of strain softening in the constitutive equation results in an infinite number of solutions of the problem after bifurcation at the peak-load. The indetermination of the size of the strain localization zone (β is undetermined) is responsible for that. This fact could be expected since the material is not positive (see Chapter 1). Some additional ingredients are then required to make strain-softening constitutive models available for actual problems.

2.2 Strong discontinuity kinematics

Let us consider de body Ω of Figure 2-5 experiencing a displacement discontinuity (strong discontinuity) at section S, which splits it into Ω^+ and Ω^-.

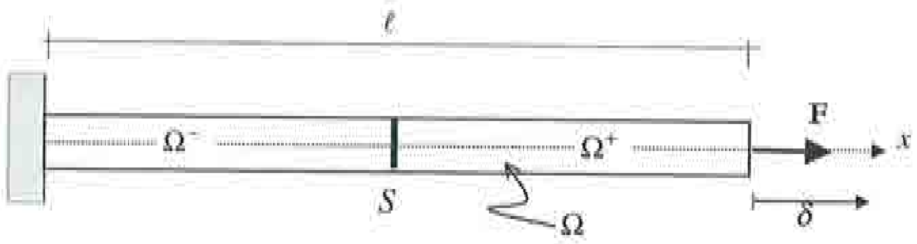


Figure 2-5– One-dimensional strong discontinuity problem

The mathematical expression of the displacement and strain fields read:

$$\begin{aligned} \dot{u}(x,t) &= \dot{\hat{u}}(x,t) + H_S(x)[[\dot{u}]](t) \\ [[\dot{u}]] &\overset{def}{=} \dot{u}(x_S,t)|_{x\in\Omega^+} - \dot{u}(x_S,t)|_{x\in\Omega^-} \\ H_S(x) &= \begin{cases} 1 & \forall x \in \Omega^+ \\ 0 & \forall x \in \Omega^- \end{cases} \rightarrow \text{Heaviside's (step) function} \end{aligned} \tag{2.17}$$

$$\dot{\varepsilon}(x,t) = \frac{\partial \dot{u}}{\partial x} = \underbrace{\frac{\partial \dot{\bar{u}}}{\partial x}}_{\dot{\bar{\varepsilon}}} + \frac{\partial H_S}{\partial x}[[\dot{u}]] = \underbrace{\dot{\bar{\varepsilon}}(x,t)}_{\text{regular}} + \underbrace{\delta_S(x)[[\dot{u}]](t)}_{\text{singular}} \tag{2.18}$$

$$\delta_S = \frac{\partial H_S}{\partial x} \rightarrow \text{Dirac's delta function}$$

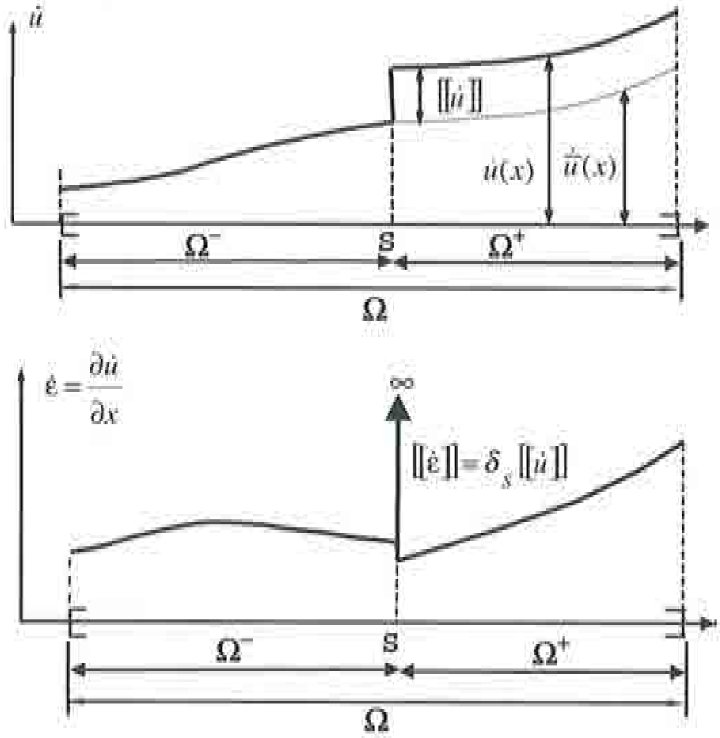


Figure 2-6– Strong discontinuity kinematics

2.3 Regularized strong discontinuity kinematics

Instead of the strong discontinuity kinematics of equations (2.17) and (2.18) we shall consider a regularized version of them, by assuming that the displacement jump takes place across a very thin *discontinuity band* Ω^k, of bandwidth k, (see Figure 2-7).

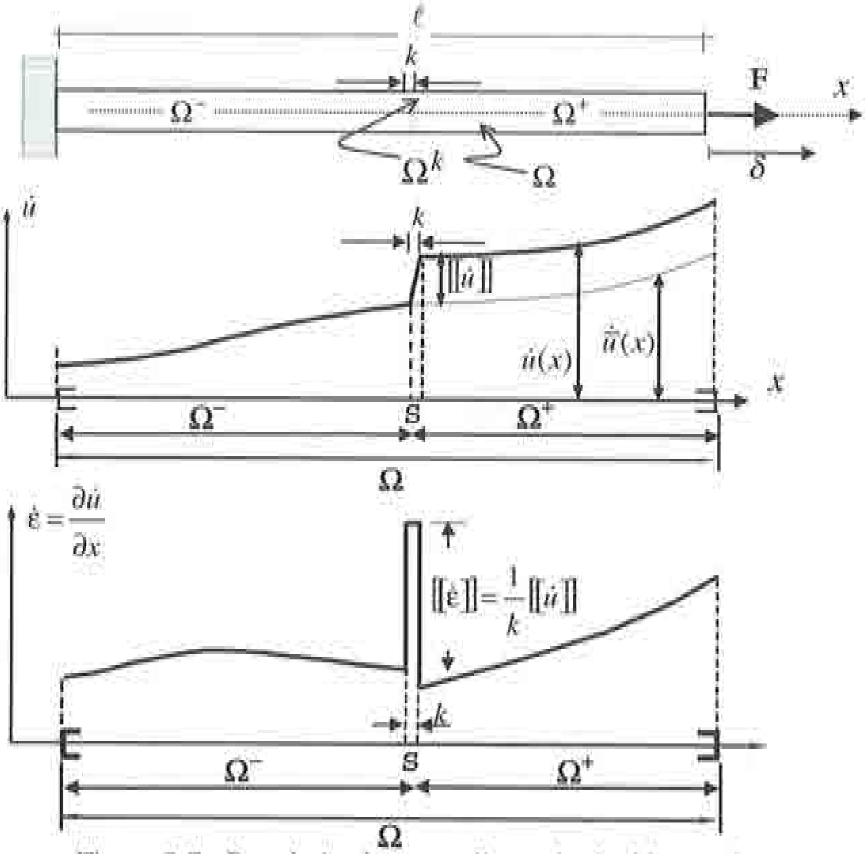


Figure 2-7– Regularized strong discontinuity kinematics

The mathematical description of such kinematics then read:

$$\begin{aligned} &\dot{u}(x,t) = \dot{\bar{u}}(x,t) + H_S^k(x)[\![\dot{u}]\!](t) \\ &H_S^k \rightarrow \text{Ramp function on } \Omega^k \end{aligned} \tag{2.48}$$

$$\begin{aligned} &\dot{\varepsilon}(x,t) = \frac{\partial \dot{u}}{\partial x} = \underbrace{\dot{\bar{\varepsilon}}(x,t)}_{\text{regular}} + \underbrace{\mu_S(x)\frac{1}{k}[\![\dot{u}]\!](t)}_{\text{unbounded when } k\rightarrow 0} \Rightarrow \\ &\varepsilon(x,t) = \underbrace{\bar{\varepsilon}(x,t)}_{\text{regular}} + \underbrace{\mu_S(x)\frac{1}{k}[\![u]\!](t)}_{\text{unbounded when } k\rightarrow 0} \end{aligned} \tag{2.49}$$

$$\mu_S(x) = \begin{cases} 1 & \forall x \in \Omega^k \\ 0 & \forall x \notin \Omega^k \end{cases} \rightarrow \text{Collocation function on } S$$

$$\lim_{k\rightarrow 0} \mu_S(x)\frac{1}{k} = \delta_S \rightarrow \text{Regularized Dirac's delta function on } S$$

Remark 2-2
When the regularization parameter $k \rightarrow 0$, the strong discontinuity kinematics (2.18) is recovered.

2.4 Strong discontinuity analysis

2.4.1 Stress boundedness

Although the strains are unbounded at the discontinuous interface S, as stated by equation (2.49):

$$\dot{\varepsilon}(x,t)\big|_{x\in S} \overset{not}{=} \dot{\varepsilon}_S = \bar{\dot{\varepsilon}} + \underbrace{\frac{1}{k}[\![\dot{u}]\!](t)}_{\substack{\text{unbounded} \\ \text{when } k\to 0}} \tag{2.50}$$

NOTE

We assume that the continuum constitutive model return bounded stresses from bounded strains, therefore: $\sigma_{\Omega/S}(\varepsilon_{\Omega/S}) = \text{bounded}$:

in virtue of the equilibrium across a cross section (momentum balance principle)

$$\dot{\sigma}(x,t)\big|_{x\in S} \overset{not}{=} \dot{\sigma}_S = \dot{\sigma}(x,t)\big|_{\notin S} \overset{not}{=} \dot{\sigma}_{\Omega/S} \tag{2.51}$$

and, since we assume that the strains are unbounded at Ω / S :

$$\dot{\sigma}_S = \dot{\sigma}_{\Omega/S} \rightarrow \text{bounded} \tag{2.52}$$

Remark 2-3

- From the traction continuity, the stresses σ_S (and the rate of stresses $\dot{\sigma}_S$) are bounded at the discontinuous interface, even when the strains ε_S are unbounded.

- So are the stress-like hardening variable ($q_S \in [0, \sigma_y] \Rightarrow$ bounded, see Figure 2-1) and its rate $\dot{q}_S$, since in virtue of the consistency equation (2.7):

$$\text{loading} \rightarrow \dot{f}_S = sign(\sigma_S)\dot{\sigma}_S - \dot{q}_S = 0 \Rightarrow \dot{q}_S = sign(\sigma_S)\dot{\sigma}_S \Rightarrow \text{bounded}$$

$$\text{unloading} \rightarrow \dot{q}_S = \lambda H = 0 \quad (\lambda = 0) \Rightarrow \text{bounded}$$

2.4.2 Strong discontinuity analysis (SDA)

The strong discontinuity analysis is devoted to extract those features of the continuum constitutive model that make it compatible with the unbounded strains, keeping the bounded character of the stress and the stress-like variables.

Considering the elasto-plastic model of section 2.1.1 and the rate version of equations (2.2) and (2.4) at the discontinuity interface S:

$$\left.\begin{aligned} \dot{\sigma} &= E(\dot{\varepsilon} - \dot{\varepsilon}^p) \\ \dot{\varepsilon}^p &= \lambda \frac{\partial f}{\partial \sigma} = \lambda\, sign(\sigma) \end{aligned}\right\} \Rightarrow \dot{\sigma}_S = E(\dot{\varepsilon}_S - \lambda_S\, sign(\sigma_S)) \tag{2.53}$$

And substituting the strong discontinuity kinematics (2.50):

$$\dot{\sigma}_S = E(\dot{\bar{\varepsilon}}_S + \frac{1}{k}[\![\dot{u}]\!] - \lambda_S \, sign(\sigma_S)) \tag{2.54}$$

$$k\dot{\sigma}_S = E(k\dot{\bar{\varepsilon}}_S + [\![\dot{u}]\!] - k\lambda_S \, sign(\sigma_S)) \Rightarrow$$

$$\underbrace{\lim_{k\to 0}(k \underbrace{\dot{\sigma}_S}_{\text{bounded}})}_{=0} = \underbrace{\lim_{k\to 0}(Ek \underbrace{\dot{\bar{\varepsilon}}_S}_{\text{bounded}})}_{=0} + E[\![\dot{u}]\!] - \lim_{k\to 0}(Ek\lambda_S \underbrace{sign(\sigma_S)}_{\text{bounded}}) = 0 \Rightarrow \tag{2.55}$$

$$[\![\dot{u}]\!] = \lim_{k\to 0}(k\lambda_S) \, sign(\sigma_S) \tag{2.56}$$

In order to have displacement jump ($[\![\dot{u}]\!] \neq 0$), from equation (2.56):

$$\lim_{k\to 0}(k\lambda_S) \neq 0 \tag{2.57}$$

which can be accomplished by defining a new bounded variable:

$$\boxed{\bar{\lambda} \overset{def}{=} k\lambda_S \rightarrow \text{discrete plastic multiplier (bounded)}} \tag{2.58}$$

By substituting equation (2.58) into ((2.57)) the following relationship is obtained:

$$\boxed{[\![\dot{u}]\!] = \bar{\lambda} \, sign(\sigma_S) \rightarrow \text{discrete constitutive equation}} \tag{2.59}$$

> **Remark 2-4**
>
> Equation (2.59) relates the displacement, $[\![u]\!]$, at the discontinuous interface S with the interfacial stress σ_S. It can be then recognized as a discrete (o fracture mechanics-like) constitutive equation that, as will be shown below in section 2.4.3, is part of a complete discrete constitutive model.

Also, substituting equation (2.58) into (2.5) ($\dot{\alpha} = \lambda$):

$$\dot{\alpha}_S = \lambda_S = \frac{1}{k}\bar{\lambda} \overset{def}{=} \frac{1}{k}\dot{\bar{\alpha}} \Rightarrow$$

$$\boxed{\dot{\bar{\alpha}} = \bar{\lambda} = k\dot{\alpha}_S \rightarrow \text{discrete internal variable (bounded)}} \tag{2.60}$$

Finally, taking into account equation (2.60) into the rate version of equation (2.6) ($\dot{q} = H\dot{\alpha}$) and considering the bounded character of q_S (see Remark 2-3):

$$\underbrace{\dot{q}_S}_{\text{bounded}} = H\dot{\alpha}_S = H\frac{1}{k}\underbrace{\dot{\bar{\alpha}}}_{\text{bounded}} \Rightarrow \lim_{k\to 0} H\frac{1}{k} = \bar{H} \text{ (bounded)} \tag{2.61}$$

NOTE

Here $\overline{H}$ is considered constant (linear softening). Generalization to non-linear softening is straightforward.

Condition (2.61), and subsidiary conditions (2.58) an (2.60), can be achieved by imposing the following structure to the continuum softening parameter H

Softening regularization condition

$$H = k\overline{H}$$

$\overline{H} \rightarrow$ discrete (or intrinsic) softening parameter (2.62)

And, now, by substituting in equation (2.61):

$$\dot{q}_S = \overline{H}\dot{\overline{\alpha}} \Rightarrow \quad q_S = \sigma_y + \overline{H}\overline{\alpha} \quad \rightarrow \text{discrete softening law} \tag{2.63}$$

Remark 2-5

The intrinsic softening parameter $\overline{H}$ is material variable that will be related to the fracture properties of the material (see section 2.4.3). By imposing the softening regularization condition (2.62) (and, thus, considering the continuum softening parameter H given in terms of the discrete one $\overline{H}$) the bounded character of the discrete variables $\dot{\overline{\alpha}}$ and $\overline{\lambda}$ and, eventually, the bounded character of σ_S and q_S is assured.

2.4.3 Discrete free energy.

From equation (2.59) and the flow rule (2.4) ($\dot{\varepsilon}^p = \lambda\, sign(\sigma)$):

$$[\![\dot{u}]\!] = \overline{\lambda}\, sign(\sigma_S) \Rightarrow \frac{1}{k}[\![\dot{u}]\!] = \underbrace{\frac{1}{k}\overline{\lambda}}_{\lambda_S}\, sign(\sigma_S) = \lambda_S\, sign(\sigma_S) = \dot{\varepsilon}_S^p \Rightarrow$$

$$\boxed{\dot{\varepsilon}_S^p = \frac{1}{k}[\![\dot{u}]\!]} \tag{2.64}$$

and from the strong discontinuity kinematics (2.49):

$$\dot{\varepsilon}_S = \dot{\overline{\varepsilon}}_S + \underbrace{\frac{1}{k}[\![\dot{u}]\!]}_{\dot{\varepsilon}_S^p} = \dot{\varepsilon}_S^e + \dot{\varepsilon}_S^p \Rightarrow \begin{cases} \dot{\varepsilon}_S^e = \dot{\overline{\varepsilon}}_S \\ \varepsilon_S^e = \underbrace{\overline{\varepsilon}_S}_{\text{bounded}} \end{cases} \Rightarrow \boxed{\lim_{k\to 0} k\varepsilon_S^e = 0} \tag{2.65}$$

Remark 2-6

Equation (2.64) sates that the plastic strain translates entirely into displacement jump.

Let us consider the continuum density (per unit of volume) of free energy (2.1):

$$\psi(\varepsilon,\varepsilon^p,\alpha)=\underbrace{\frac{1}{2}E(\overbrace{\varepsilon-\varepsilon^p}^{\varepsilon^e})^2}_{\psi^e(\varepsilon^e)}+\underbrace{\sigma_y\alpha+\frac{1}{2}H\alpha^2}_{\psi^p(\alpha)} \tag{2.66}$$

and, from it, let us consider the free energy at the discontinuous interface, $\bar{\psi}$, *per unit of surface*. Both are related by (see Figure 2-8):

$$\underbrace{\frac{\text{Free energy}}{\text{unit surface}}}_{\bar{\psi}}=\underbrace{\frac{\text{Free energy}}{\text{unit volume}}}_{\psi}\cdot\underbrace{\frac{\text{volume}}{\text{surface}}}_{k}\Rightarrow\boxed{\bar{\psi}=\lim_{k\to 0}(k\psi_S)} \tag{2.67}$$

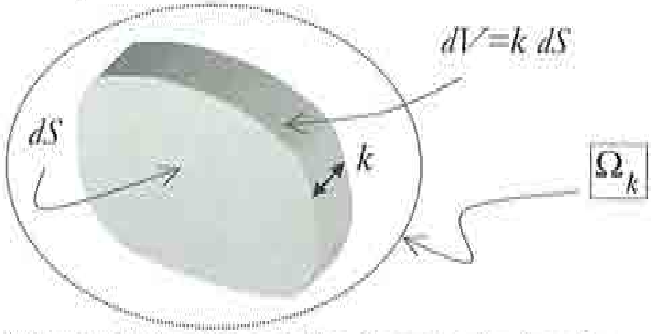


Figure 2-8– Differential of volume at the discontinuity interface

Now, by considering equation (2.67) and the result in equation (2.65):

$$\begin{aligned}\bar{\psi}&=\lim_{k\to 0}(k\psi_S)=\lim_{k\to 0}k\frac{1}{2}E\varepsilon^{e^2}+\lim_{k\to 0}k(\sigma_y\alpha+\frac{1}{2}H\alpha^2)=\\&=\lim_{k\to 0}\frac{1}{2}E\underbrace{k\varepsilon^e}_{=0}\varepsilon^e+\lim_{k\to 0}k(\sigma_y\underbrace{\alpha}_{\frac{1}{k}\bar{\alpha}}+\frac{1}{2}\underbrace{\overbrace{H}^{k\bar{H}}\alpha^2}_{\frac{1}{k}(\bar{H}\bar{\alpha}^2)})=\\&=\lim_{k\to 0}k(\frac{1}{k}\sigma_y\bar{\alpha}+\frac{1}{k}\bar{H}\bar{\alpha}^2)=\sigma_y\bar{\alpha}+\frac{1}{2}\bar{H}\bar{\alpha}^2\end{aligned} \tag{2.68}$$

where equations (2.58) to (2.62) have been considered. Therefore, the discrete (per unit of surface) free energy $\bar{\psi}$ can be written as:

$$\bar{\psi}([\![u]\!],\bar{\alpha})=\underbrace{0}_{\bar{\psi}^e([\![u]\!])}+\underbrace{\sigma_y\bar{\alpha}+\frac{1}{2}\bar{H}\bar{\alpha}^2}_{\bar{\psi}^p(\bar{\alpha})}\rightarrow\text{discrete free energy} \tag{2.69}$$

> **Remark 2-7**
> Notice, from equations (2.69) and (2.63) that:
> $$\frac{\partial\bar{\psi}([\![u]\!],\bar{\alpha})}{\partial\bar{\alpha}}=\sigma_y+\bar{H}\bar{\alpha}=q_S$$
> that states that q_S and $\bar{\alpha}$ are thermodynamically conjugated variables with respect to the discrete free energy $\bar{\psi}$.

Finally notice that loading unloading conditions (2.7) at the discontinuous interface, can be rephrased from equations (2.58) as:

$$
\begin{aligned}
& f(\sigma,q)\big|_{\in S} \overset{def}{=} \bar{f}(\sigma_{S},q_S) \equiv |\sigma_S| - q_S = 0 \ \rightarrow \text{discrete yield function} \\
& \left.\begin{aligned} & \bar{f} \le 0 \quad \bar{\lambda} \ge 0 \quad \bar{\lambda}\bar{f} = 0 \\ & \dot{\bar{f}} = 0 \quad \bar{\lambda}\dot{\bar{f}} = 0 \end{aligned}\right\} \rightarrow \text{discrete loading/unloading conditions}
\end{aligned} \tag{2.70}
$$

in such a way that the evolution equation (2.59) ($[\![\dot{u}]\!] = \bar{\lambda}\, sign(\sigma_S)$) can be integrated as:

1) Loading

$$
\begin{aligned}
& \bar{\lambda} \ne 0 \Rightarrow \bar{f}(\sigma_{S},q_S) = 0 \Rightarrow |\sigma_S| = q_S \\
& \dot{\bar{f}}(\sigma_{S},q_S) = 0 \Rightarrow sign(\sigma_S)\dot{\sigma}_S = \dot{q}_S = \overline{H}\,\bar{\lambda} = \overline{H}\,\dot{\bar{\alpha}} \Rightarrow \\
& \qquad \dot{\bar{\alpha}} = \bar{\lambda} = \frac{1}{\overline{H}}\, sign(\sigma_S)\dot{\sigma}_S
\end{aligned} \tag{2.71}
$$

and substituting equation (2.71) into (2.59):

$$
[\![\dot{u}]\!] = \bar{\lambda}\, sign(\sigma_S) = \frac{1}{\overline{H}} \underbrace{sign^2(\sigma_S)}_{=1} \dot{\sigma}_S \Rightarrow \quad \boxed{\begin{aligned} & [\![\dot{u}]\!] = \frac{1}{\overline{H}}\dot{\sigma}_S \\ & \dot{\sigma}_S = \overline{H}\,[\![\dot{u}]\!] \end{aligned}} \tag{2.72}
$$

2) Elastic unloading

$$
\bar{\lambda} = 0 \Rightarrow [\![\dot{u}]\!] = \bar{\lambda}\, sign(\sigma_S) = 0 \Rightarrow \qquad \boxed{[\![\dot{u}]\!] = 0} \tag{2.73}
$$

The above results and their implications in a loading-unloading cycle are sketched in Figure 2-9.

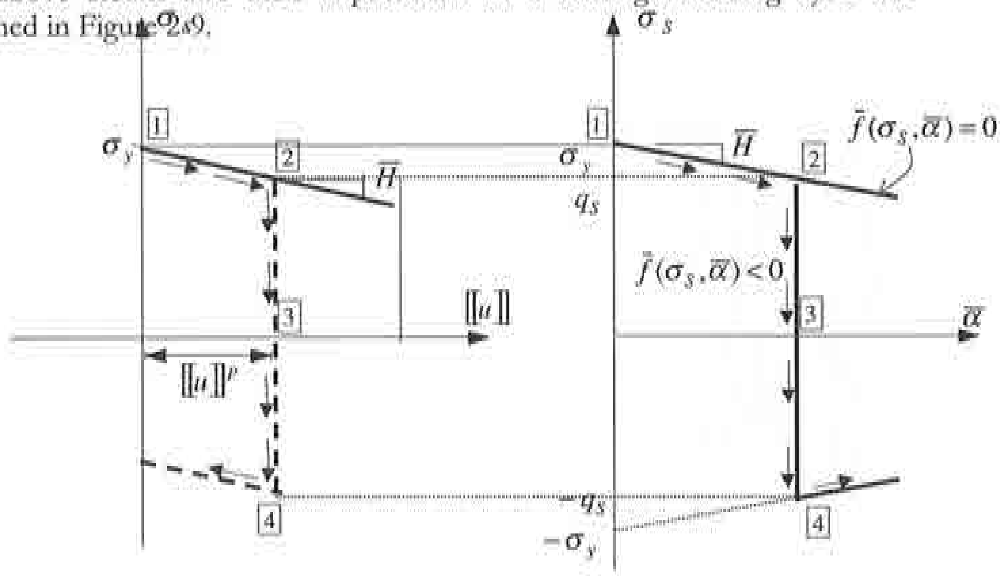


Figure 2-9– Elasto-plastic discrete model: loading-unloading cycle

2.4.4 Discrete elasto-plastic model.

The results in the preceding sections define a complete discrete constitutive model (relating the interface stress σ_S with the displacement jump $[\![u]\!]$) that can be summarized as follows:

Free energy:	$\bar{\psi}([\![u]\!],\bar{\alpha}) = \underbrace{0}_{\bar{\psi}^{e}([\![u]\!])} \underbrace{+\sigma_y\bar{\alpha} + \frac{1}{2}\bar{H}\bar{\alpha}^2}_{\bar{\psi}^{p}(\bar{\alpha})}$	(2.74)
Yield function:	$\bar{f}(\sigma_S, q_S) \equiv \lvert\sigma_S\rvert - q_S$	(2.75)
Flow rule:	$[\![\dot{u}]\!] = \bar{\lambda}\ sign(\sigma_S)$	(2.76)
Internal variable	$\dot{\bar{\alpha}} = \bar{\lambda}$	(2.77)
Hardening/ softening law	$q_S = \frac{\partial\bar{\psi}}{\partial\bar{\alpha}} = \sigma_y + \bar{H}\bar{\alpha}\ ;\ \bar{H} < 0$	(2.78)
Loading/ unloading conditions	$\bar{\lambda} \geq 0\ \ \bar{f} \leq 0\ \ \bar{\lambda}\bar{f} = 0\ \ (Kuhn-Tucker)$ $\bar{f} = 0 \Rightarrow\ \ \bar{\lambda}\dot{\bar{f}} = 0\ \ \ (persistency)$	(2.79)
Tangent constitutive equation	$\dot{\sigma}_S = \bar{E}_T\,[\![\dot{u}]\!]\ \ ;\ \ \bar{E}_T = \bar{H}\ \ \ \ (loading)$ $\dot{\sigma}_S = \bar{E}\,[\![\dot{u}]\!]\ \ ;\ \ \bar{E} = \infty\ \ (\Rightarrow [\![\dot{u}]\!] = 0)\ \ (unloading)$	(2.80)

Remark 2-8

The discrete elasto-plastic constitutive model of equations (2.74) to (2.80) is characterized by an infinite elastic stiffness $\bar{E}$ (see equation (2.80)). It could be then appropriately termed as a *rigid-plastic model*. The classical decomposition of elastic and plastic (displacement jumps) counterparts reads:

$$[\![u]\!] = [\![u]\!]^e + [\![u]\!]^p$$

and the corresponding elastic free energy:

$$\bar{\psi}([\![u]\!],\alpha) = \underbrace{\frac{1}{2}\bar{E}([\![u]\!] - [\![u]\!]^p)^2}_{\bar{\psi}^{e}([\![u]\!]^e)} \underbrace{+\sigma_y\bar{\alpha} + \frac{1}{2}\bar{H}\bar{\alpha}^2}_{\bar{\psi}^{p}(\bar{\alpha})}$$

The constitutive equation is then given by:

$$\sigma_S = \frac{\partial\bar{\psi}([\![u]\!],\bar{\alpha})}{\partial[\![u]\!]} = \bar{E}([\![u]\!] - [\![u]\!]^p)$$

where $\bar{E} = \infty \Rightarrow [\![u]\!] = [\![u]\!]^p \Rightarrow [\![u]\!]^e = 0$

Remark 2-9

It can also be observed the one to one correspondences between the variables of the continuum model and the ones of the induced discrete model according to the following table:

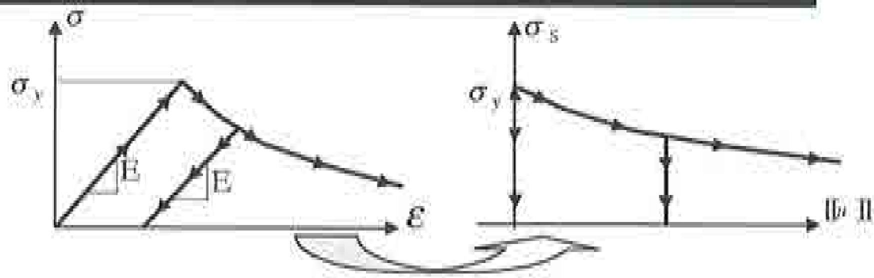
Continuum	ε	ε^e	ε^p	ψ	α	q	σ	λ	H
Discrete	$[[u]]$	$[[u]]^e = 0$	$[[u]]^p$	$\bar{\psi}$	$\bar{\alpha}$	q_S	σ_s	$\bar{\lambda}$	$\bar{H}$

Remark 2-10

The discrete elasto plastic model is automatically induced from the continuum one, by introducing only two ingredients (see Figure 2-10):

- The strong discontinuity kinematics of equation (2.49)
- The softening regularization of equation (2.62).

Although it can be explicitly derived (as has been done here) it comes out automatically from the continuum model if both conditions are imposed.



$$\dot{\varepsilon} = \dot{\bar{\varepsilon}} + \frac{1}{k}[[\dot{u}]]$$
$$H = k\bar{H}$$

Figure 2-10– Original continuum and induced discrete elasto-plastic models

2.4.5 Strong discontinuity solution of the bar problem

Let us consider again the one-dimensional problem of the bar of Figure 2-11 but now in the light of the strong discontinuity approach. Therefore an strong discontinuity is assumed to take place at cross section S. The solution can be now sketched as follows:

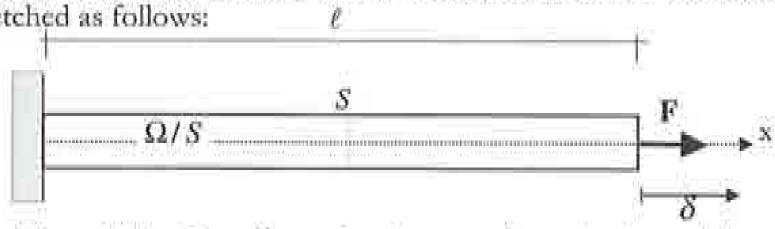


Figure 2-11– One-dimensional strong discontinuity problem

1) Elastic regime : $(\delta \le \frac{\sigma_y}{E}\ell)$:

$$\left.\begin{aligned} \varepsilon(x) &= \frac{\delta}{\ell} \quad \forall x \\ \sigma(x) &= E\varepsilon = E\frac{\delta}{\ell} \quad \forall x \end{aligned}\right\} \Rightarrow \boxed{F = A\sigma = \frac{EA}{\ell}\delta} \tag{2.81}$$

2) Elasto-plastic regime: ($\delta > \frac{\sigma_y}{E}\ell$)

At the peak stress, ($\sigma = \sigma_y$) let us consider that loading ($\dot{\sigma} = E_T \dot{\varepsilon}$) takes place in the discontinuous interface S whereas unloading ($\dot{\sigma} = E\dot{\varepsilon}$) occurs at the rest of the domain Ω / S). From the strong discontinuity kinematics of equation (2.49) ($\dot{\varepsilon}(x,t) = \dot{\bar{\varepsilon}} + \mu_S(x)\frac{1}{k}[\![\dot{u}]\!]$) it results that $\dot{\varepsilon}_{\Omega/S} = \dot{\bar{\varepsilon}}$ since $\mu_S(x_{\Omega/S}) = 0$. Equilibrium requires that:,

$$\dot{\sigma}_{\Omega/S} = E\dot{\bar{\varepsilon}} = \dot{\sigma}_S = \overline{H}[\![\dot{u}]\!] \Rightarrow [\![\dot{u}]\!] = \frac{E}{\overline{H}}\dot{\bar{\varepsilon}} \tag{2.82}$$

and compatibility requires:

$$\begin{aligned} &\dot{\delta} = \dot{\bar{\varepsilon}}\,\ell + [\![\dot{u}]\!] = \dot{\bar{\varepsilon}}\,\ell + \frac{E}{\overline{H}}\dot{\bar{\varepsilon}} = (\ell + \frac{E}{\overline{H}})\dot{\bar{\varepsilon}} = (\ell + \frac{E}{\overline{H}})\frac{\dot{\sigma}}{E} \Rightarrow \\ &\dot{\sigma} = \frac{E\overline{H}}{\overline{H}\ell + E}\dot{\delta} \end{aligned} \tag{2.83}$$

where equation (2.82) has been taken into account. Finally we obtain:

$$\boxed{\dot{F} = A\dot{\sigma} = \frac{EA\overline{H}}{\overline{H}\ell + E}\dot{\delta} = \frac{EA}{\ell + \frac{E}{\overline{H}}}\dot{\delta}} \tag{2.84}$$

The complete result is presented in Figure 2-12.

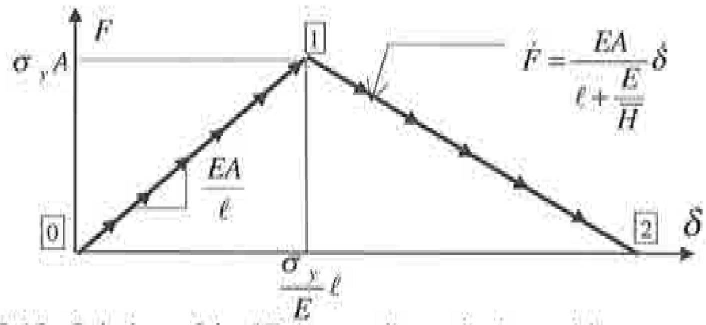


Figure 2-12– Solution of the 1D strong discontinuity problem

> **Remark 2-11**
>
> Unlike the strain-localization solution of Figure 2-4 the strong discontinuity approach provides a unique solution for the problem. Observe that the strain localization solution and the strong discontinuity ones coincide with each other, if it is set in the former:
>
> $$H = \overline{H}\underbrace{\beta\ell}_{b} = \overline{H}b$$

2.4.6 Physical interpretation of the discrete softening parameter. The fracture energy.

Let us compute the energy consumed in the formation of the strong discontinuity in the problem of Figure 2-11. In virtue of the theorem of the expended power (and neglecting the kinetic energy) the external power entering into the system is spent in stress power production:

$$\delta W_{ext} = F\dot{\delta} = \int_{\Omega} \sigma \cdot \dot{\varepsilon}\, d\Omega \tag{2.85}$$

NOTE

Here we use the following property of the Dirac's delta function:

$\int_a^b \delta_S(x) \cdot \phi(x) dx = \phi(x_S)$

Considering the strong discontinuity kinematics (2.49) ($\dot{\varepsilon} = \dot{\bar{\varepsilon}} + \delta_S [\![\dot{u}]\!]$) it results:

$$\begin{aligned}\delta W_{ext} &= \int_{\Omega} \sigma \cdot (\dot{\bar{\varepsilon}} + \delta_S [\![\dot{u}]\!]) \underbrace{d\Omega}_{A dx} = A\int_0^l \sigma \cdot (\dot{\bar{\varepsilon}} + \delta_S [\![\dot{u}]\!]) dx = \\ &= A\int_0^l \sigma \cdot \underbrace{\dot{\bar{\varepsilon}}}_{\frac{\dot{\sigma}}{E}} dx + A\sigma_S [\![\dot{u}]\!] = A\int_0^l \frac{1}{E} \sigma \cdot \dot{\sigma}\, dx + A\sigma_S [\![\dot{u}]\!]\end{aligned} \tag{2.86}$$

$$\delta W_{ext} = A\int_0^l \frac{d}{dt}\left(\frac{1}{2}\frac{\sigma^2}{E}\right) dx + A\sigma_S [\![\dot{u}]\!] \tag{2.87}$$

If we now consider the whole loading process $\boxed{0-1-2}$ in Figure 2-12 and compute the total energy consumption:

$$\begin{aligned}W_{ext} &= \int_0^2 \delta W_{ext} dt = A\int_0^2 \left[\int_0^l \frac{d}{dt}\left(\frac{1}{2}\frac{\sigma^2}{E}\right) dx \right] dt + A\int_o^2 \sigma_S [\![\dot{u}]\!] dt = \\ &= A\int_0^l \left[\int_0^2 \frac{d}{dt}\left(\frac{1}{2}\frac{\sigma^2}{E}\right) dt \right] dx + A\int_o^2 \sigma_S [\![\dot{u}]\!] dt = \\ &= \underbrace{Al\left[\frac{1}{2}\frac{\sigma^2}{E}\right]_0^2}_{=0-0} + A\int_o^2 \sigma_S [\![\dot{u}]\!] dt = A\int_o^2 \sigma_S [\![\dot{u}]\!] dt\end{aligned} \tag{2.88}$$

So that the total consumption of energy per unit of cross section is given by:

$$\int_o^2 \sigma_S [\![\dot{u}]\!] dt = G_f \rightarrow \text{Fracture energy} \tag{2.89}$$

NOTE

In nonlinear Fracture Mechanics, the Fracture energy is defined as *the mechanical energy required for the formation of a unit of fracture surface*, and it is considered a material property.

The integral in equation (2.89) is the area under the stress-strain curve in Figure 2-13 and can be readily computed from that figure as:

$$G_f = -\frac{1}{2}\frac{\sigma_y^2}{\bar{H}} \Rightarrow \quad \boxed{\bar{H} = -\frac{1}{2}\frac{\sigma_y^2}{G_f}} \tag{2.90}$$

Equation (2.90) allows to interpret the discrete softening parameter $\overline{H}$ as a material property that can be obtained from the mechanical values σ_y and G_f.

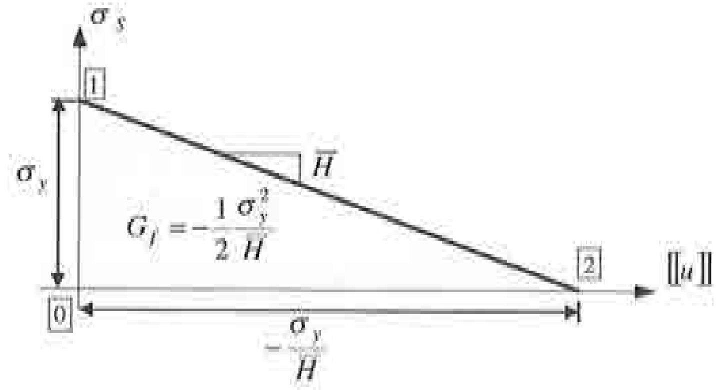


Figure 2-13– Solution of the 1D strong discontinuity problem

3 Strong discontinuity approach (3D)

3.1 Kinematics

3.1.1 Strong discontinuity kinematics

Let us consider the body Ω experiencing a jump in the displacement field $[\![\mathbf{u}]\!](\mathbf{x},t)$ across a fixed (material) surface S, whose normal (pointing towards a fixed side of S, Ω^+) is denoted by $\mathbf{n}$.

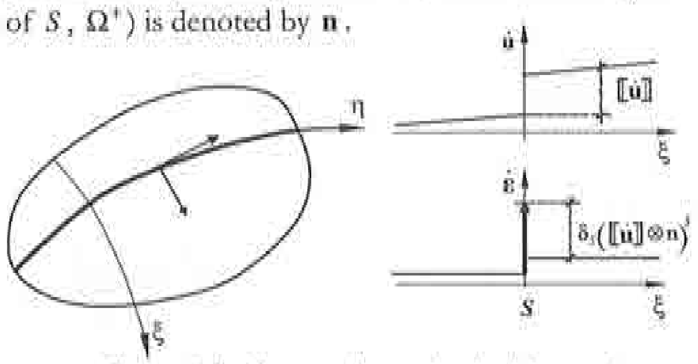


Figure 3-1– Strong discontinuity kinematics

The displacement and displacement fields read:

$$
\begin{aligned}
&\dot{\mathbf{u}}(\mathbf{x},t) = \dot{\bar{\mathbf{u}}}(\mathbf{x},t) + H_S(\mathbf{x})[\![\dot{\mathbf{u}}]\!](\mathbf{x},t) \\
&[\![\dot{\mathbf{u}}]\!](\mathbf{x},t) \overset{def}{=} \dot{\mathbf{u}}(\mathbf{x}_S,t)\big|_{\mathbf{x}\in\Omega^+} - \dot{\mathbf{u}}(\mathbf{x}_S,t)\big|_{\mathbf{x}\in\Omega^-} \\
&H_S(\mathbf{x}) = \begin{cases} 1 & \forall \mathbf{x}\in\Omega^+ \\ 0 & \forall \mathbf{x}\in\Omega^- \end{cases} \rightarrow \text{Heaviside's step function}
\end{aligned}
\tag{3.1}
$$

In the continuum format the (infinitesimal) strains are given by:

$$
\begin{aligned}
\dot{\varepsilon}(\mathbf{x},t) &= \frac{1}{2}(\nabla \otimes \dot{\mathbf{u}} + \dot{\mathbf{u}} \otimes \nabla) \overset{not}{=} \nabla^S \dot{\mathbf{u}} = \\
&= \underbrace{\nabla^S \dot{\bar{\mathbf{u}}} + H_S \nabla^S [\![\dot{\mathbf{u}}]\!]}_{\dot{\bar{\varepsilon}}} + ([\![\dot{\mathbf{u}}]\!] \otimes \underbrace{\nabla H_S}_{\delta_S \otimes \mathbf{n}})^S = \\
&= \underbrace{\dot{\bar{\varepsilon}}}_{\text{regular}} + \underbrace{\delta_S ([\![\dot{\mathbf{u}}]\!] \otimes \mathbf{n})^S}_{\text{singular (unbounded)}}
\end{aligned}
\tag{3.2}
$$

$$
\delta_S(\mathbf{x}) = \frac{\partial H_S}{\partial n} \rightarrow \text{Dirac's delta function}
$$

3.1.2 Regularized strong discontinuity kinematics. Weak-strong discontinuities.

Instead of the strong discontinuity kinematics of equations (3.1) and (3.2) and we shall consider a regularized version, by assuming that the displacement jump takes place across a very thin *discontinuity band* Ω^h, of bandwidth h, (see Figure 3-2

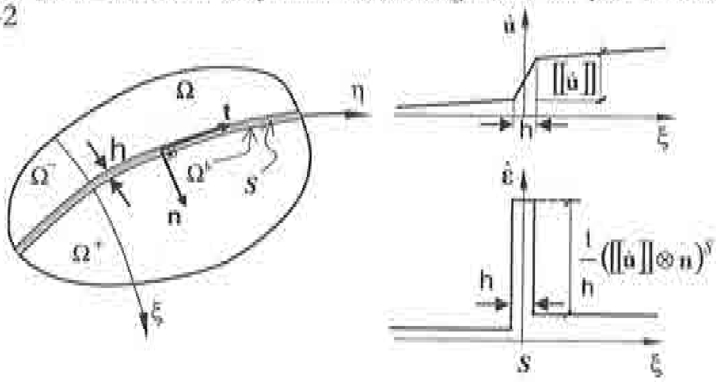


Figure 3-2– Regularized strong discontinuity kinematics

The mathematical description of such kinematics then read:

$$
\begin{aligned}
&\dot{\mathbf{u}}(\mathbf{x},t) = \dot{\bar{\mathbf{u}}}(\mathbf{x},t) + H_S^h(\mathbf{x})[\![\dot{\mathbf{u}}]\!](\mathbf{x},t) \\
&H_S^h \rightarrow \text{Ramp function on } S
\end{aligned}
\tag{3.3}
$$

$$
\dot{\varepsilon} = \underbrace{\dot{\bar{\varepsilon}}}_{\substack{\text{regular} \\ \text{(bounded)}}} + \underbrace{\mu_S \frac{1}{h}([\![\dot{\mathbf{u}}]\!] \otimes \mathbf{n})^S}_{\substack{\text{unbounded} \\ \text{when } h \rightarrow 0}}
$$

$$
\mu_S(\mathbf{x}) = \begin{cases} 1 & \forall \mathbf{x} \in \Omega^h \\ 0 & \forall \mathbf{x} \notin \Omega^h \end{cases} \rightarrow \text{Collocation function on } \Omega^h
\tag{3.4}
$$

$$
\lim_{h \rightarrow 0} \mu_S(\mathbf{x}) \frac{1}{h} = \delta_S^h \rightarrow \text{Regularized Dirac's delta function on } \Omega^h
$$

Remark 3-1

When the bandwidth $h \rightarrow 0$, in equation (3.4), the concept of strong discontinuity is recovered.

The kinematics of equation (3.4) allows extending the previously defined concept of strong discontinuity to the more general case of weak discontinuity characterized by a non-zero bandwidth:

Discontinous kinematics

$$\dot{\varepsilon} = \dot{\bar{\varepsilon}} + \mu_S \frac{1}{h}([\![\dot{\mathbf{u}}]\!] \otimes \mathbf{n})^s \qquad (3.5)$$

$$\begin{cases} h \neq 0 \longrightarrow \text{weak discontinuity} \\ h = 0 \longrightarrow \text{strong discontinuity} \end{cases}$$

Remark 3-2

According to the previous definitions we can characterize two families of discontinuities:

- *Weak discontinuities:* continuous displacement fields and discontinuous (but bounded) strain fields (see Figure 3-2).
- *Strong discontinuities:* discontinuous displacement fields and unbounded strain fields (see Figure 3-3).

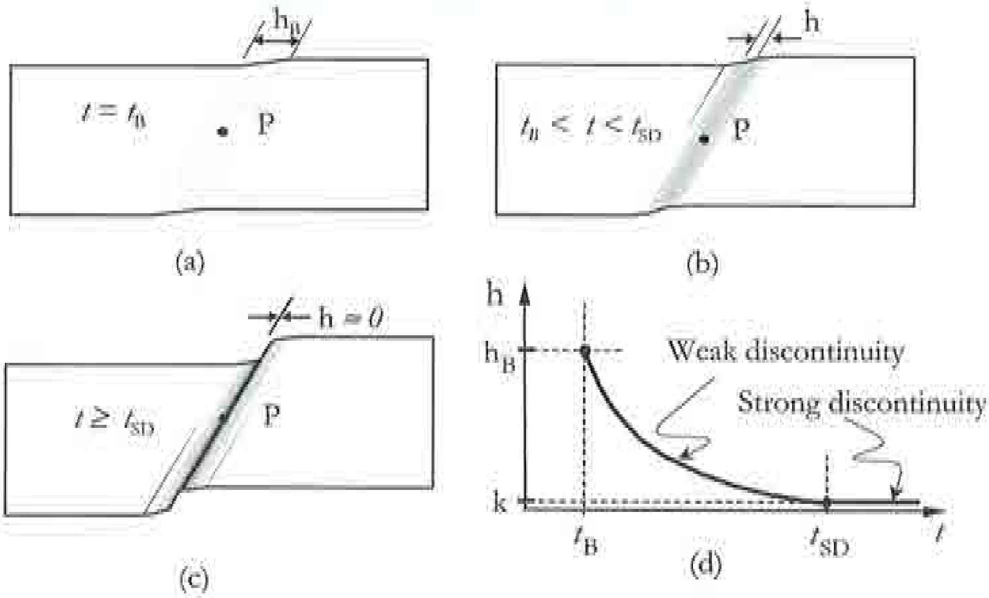


Figure 3-3-- (a) to (c): mechanism of formation of a strong discontinuity. (d) variable bandwidth model

> **Remark 3-3**
>
> The kinematics of equation (3.5) allows imagining the following process of formation of a strong discontinuity at a material point $\mathcal{P}$ of the body:
>
> *1)* A discontinuous bifurcation procedure (see section 1.5.3 in Chapter 1) induces the formation of a weak discontinuity at time t_B (the *bifurcation time*), according to the kinematics of equation (3.5), characterized by a bandwidth $h_B \neq 0$. (see Figure 3-3(a)).
>
> *2)* At subsequent times ($t > t_B$) the bandwidth decreases (see Figure 3-3 (b)-(c), ruled by a certain (material property) bandwidth evolution law (see Figure 3-3(d), up to reach a null value (for computational purposes, a very small parameter k) at time t_{SD} (the *strong discontinuity time*) which characterizes the onset of the strong discontinuity.
>
> *3)* Therefore, during the time interval $[t_B, t_{SD}]$ a weak discontinuity develops at $\mathcal{P}$ (*weak discontinuity regime*) that collapses into a strong discontinuity at time t_{SD}. Finally for $t > t_{SD}$ a full strong discontinuity develops (*strong discontinuity regime*

Equation (3.5) can then be integrated along time as follows:

$$\boldsymbol{\varepsilon}(\mathbf{x},t)\big|_{t>t_{SD}} = \underbrace{\int_0^t \dot{\bar{\boldsymbol{\varepsilon}}}\, dt + \mu_S \int_0^{t_{SD}} \frac{1}{\mathsf{h}} ([\![\dot{\mathbf{u}}]\!] \otimes \mathbf{n})^S dt}_{\bar{\boldsymbol{\varepsilon}}\ \text{(bounded when h} \to 0)} + \mu_S \underbrace{\frac{1}{\mathsf{h}}}_{=k} (\underbrace{\int_{t_{SD}}^t [\![\dot{\mathbf{u}}]\!] dt}_{\Delta[\![\mathbf{u}]\!]} \otimes \mathbf{n})^S = \tag{3.6}$$

$$\boxed{\begin{aligned} &\boldsymbol{\varepsilon}(\mathbf{x},t)\big|_{t>t_{SD}} = \underbrace{\bar{\boldsymbol{\varepsilon}}}_{\text{(bounded)}} + \underbrace{\mu_S \frac{1}{\mathsf{h}} (\Delta[\![\mathbf{u}]\!] \otimes \mathbf{n})^S}_{\text{(unbounded)}} \\ &\Delta[\![\mathbf{u}]\!](\mathbf{x},t) \overset{def}{=} [\![\mathbf{u}]\!](\mathbf{x},t) - [\![\mathbf{u}]\!](\mathbf{x},t_{SD}) \quad \forall t \geq t_{SD} \end{aligned}} \tag{3.7}$$

3.2 Three-dimensional elasto-plastic model

Let us consider the general three-dimensional elasto-plastic model with strain softening (see Figure 3-4):

Free energy:	$\psi(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p,\alpha)=\underbrace{\frac{1}{2}\overbrace{(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)}^{\boldsymbol{\varepsilon}^e}:\mathbb{E}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)}_{\psi^e(\boldsymbol{\varepsilon}^e)}+\psi^p(\alpha)$	(3.8)
Constitutive equation:	$\boldsymbol{\sigma}=\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}=\mathbb{E}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)$	(3.9)
Yield function:	$f(\boldsymbol{\sigma},q)=\phi(\boldsymbol{\sigma})-q \quad ; \ \phi(\boldsymbol{\sigma})\geq 0$	(3.10)
Flow rule:	$\dot{\boldsymbol{\varepsilon}}^p=\lambda\frac{\partial f}{\partial\boldsymbol{\sigma}}=\lambda\underbrace{\frac{\partial\phi}{\partial\boldsymbol{\sigma}}}_{\mathbf{m}}=\lambda\mathbf{m}(\boldsymbol{\sigma})$	(3.11)
Evolution law	$\dot{\alpha}=-\lambda\frac{\partial f}{\partial q}=\lambda$	(3.12)
Hardening/ softening law	$q(\alpha)=\frac{\partial\psi^p(\alpha)}{\partial\alpha} \quad ; \quad \left.\begin{matrix} q(0)=\sigma_y \\ q(\infty)=0\end{matrix}\right\\} \Rightarrow q\in[0,\sigma_y]$ $H=\frac{\partial q(\alpha)}{\partial\alpha}\leq 0 \quad \Rightarrow \quad \dot{q}=H\dot{\alpha}$	(3.13)
Loading/ unloading conditions	$\lambda\geq 0 \ \ f\leq 0 \ \ \lambda f=0 \quad (Kuhn-Tucker)$ $f=0\Rightarrow \ \lambda\dot{f}=0 \quad (persistency/consistency)$	(3.14)
Tangent constitutive equation	$\dot{\boldsymbol{\sigma}}=\mathbb{E}^{ep}:\dot{\boldsymbol{\varepsilon}} \quad ; \quad \mathbb{E}^{ep}=\mathbb{E}-\frac{\mathbb{E}:\mathbf{m}\otimes\mathbf{m}:\mathbb{E}}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}} \quad (loading)$ $\dot{\boldsymbol{\sigma}}=\mathbb{E}:\dot{\boldsymbol{\varepsilon}} \quad ; \quad (unloading)$	(3.15)

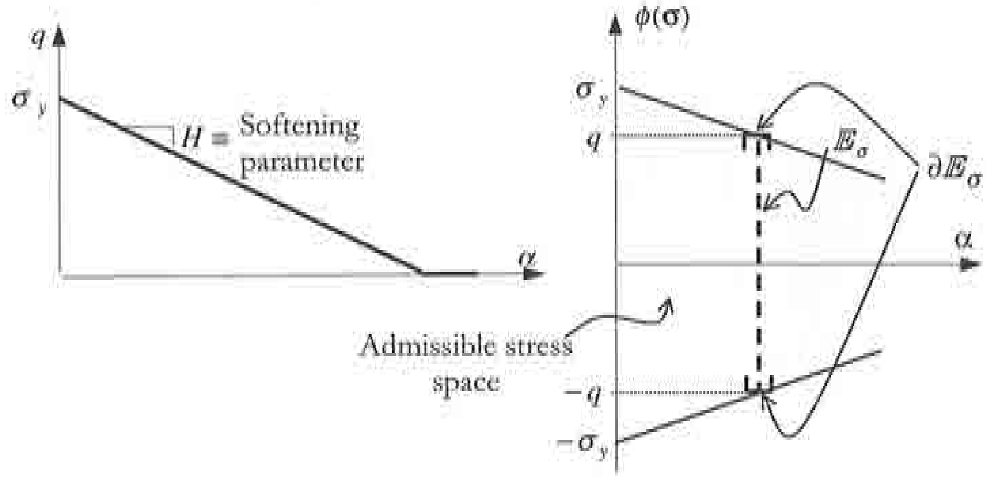


Figure 3-4– Elasto-plastic 3D model: softening law and elastic domain

where $\mathbb{E}=\lambda\mathbf{1}\otimes\mathbf{1}+2G\mathbb{I}$ is the elastic constitutive tensor and $\mathbb{E}^{ep}$ the (fourth order) tangent constitutive tensor, $\phi(\boldsymbol{\sigma}):\mathbb{R}^6\rightarrow\mathbb{R}$ is the uniaxial equivalent stress, and $\mathbf{m}=\frac{\partial\phi(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}=\frac{\partial f(\boldsymbol{\sigma},q)}{\partial\boldsymbol{\sigma}}$ is the flow tensor.

NOTE

For uniaxial stress states $\phi(\boldsymbol{\sigma}) = \sigma$.

Remark 3-4

The uniaxial equivalent stress $\phi(\boldsymbol{\sigma})$ is a continuous function that remains bounded (and also $\mathbf{m} = \dfrac{\partial \phi(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}$) for bounded values of its arguments $\boldsymbol{\sigma}$. It defines the yield surface, $\partial \mathbb{E}_\sigma$, and the elastic domain, $\mathbb{E}_\sigma$, in the stress space (see Figure 3-5). whose size is determined by the value of the hardening/softening variable q.

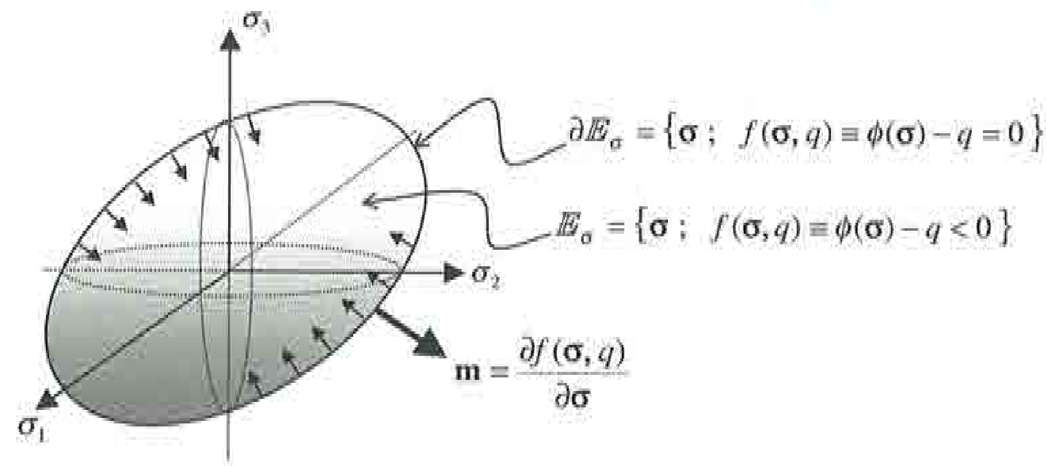


Figure 3-5– Elasto-plastic 3D model: yield surface and elastic domain in the stress space

3.3 Strong discontinuity analysis

3.3.1 Traction continuity. Stress boundedness.

NOTE

Traction continuity across any surface can be argued from the momentum balance principle.

We impose the traction $\mathcal{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$ to be continuous across the discontinuity line S (see Figure 3-6) all along the analysis, i.e.:

$$\left.\begin{aligned} \mathcal{T}_{\Omega/S} &\overset{def}{=} \boldsymbol{\sigma}_{\Omega/S} \cdot \mathbf{n} \\ \mathcal{T}_S &\overset{def}{=} \boldsymbol{\sigma}_S \cdot \mathbf{n} \end{aligned}\right\} \rightarrow \mathcal{T}_{\Omega/S} = \mathcal{T}_S \Rightarrow \tag{3.16}$$

Traction continuity

$$[\![\mathcal{T}]\!] = \boldsymbol{\sigma}_{\Omega/S} \cdot \mathbf{n} - \boldsymbol{\sigma}_S \cdot \mathbf{n} = [\![\boldsymbol{\sigma}]\!] \cdot \mathbf{n} = \mathbf{0} \tag{3.17}$$

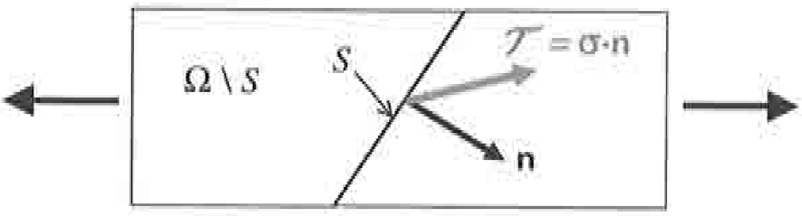


Figure 3-6– Traction continuity across the discontinuous interface

Equation (3.17) can be written as:

$$\begin{aligned} \boldsymbol{\sigma}_{\Omega/S} \cdot \mathbf{n} = \boldsymbol{\sigma}_S \cdot \mathbf{n} = \mathcal{T} \\ \dot{\boldsymbol{\sigma}}_{\Omega/S} \cdot \mathbf{n} = \dot{\boldsymbol{\sigma}}_S \cdot \mathbf{n} = \dot{\mathcal{T}} \end{aligned} \tag{3.18}$$

From equation (3.18) the following arguments can be stated:

1) The strains (and the rate of the strains) at the regular (continuous) part of the body, Ω/S, are bounded from equation (3.4).

$$\dot{\boldsymbol{\varepsilon}}_{\Omega/S} = \dot{\bar{\boldsymbol{\varepsilon}}} + \underbrace{\mu_S}_{=0\,\forall \mathbf{x}\in\Omega/S} \frac{1}{h}([\![\dot{\mathbf{u}}]\!]\otimes\mathbf{n})^s = \dot{\bar{\boldsymbol{\varepsilon}}} \rightarrow \text{bounded} \tag{3.19}$$

2) The continuum (elasto-plastic) model of section 3.2 returns bounded stresses (and also bounded rate of stresses) for bounded strains and rate of strains:

$$\begin{aligned} &\boldsymbol{\sigma}_{\Omega/S}(\bar{\boldsymbol{\varepsilon}}) \rightarrow \text{bounded} \\ &\dot{\boldsymbol{\sigma}}_{\Omega/S}(\bar{\boldsymbol{\varepsilon}}, \dot{\bar{\boldsymbol{\varepsilon}}}) \rightarrow \text{bounded} \end{aligned} \tag{3.20}$$

3) Therefore the traction (and the traction's rate) are bounded at Ω/S, and from equation (3.18) so are in S:

$$\begin{cases} \mathcal{T} = \boldsymbol{\sigma}_{\Omega/S} \cdot \mathbf{n} \rightarrow \text{bounded} \\ \dot{\mathcal{T}} = \dot{\boldsymbol{\sigma}}_{\Omega/S} \cdot \mathbf{n} \rightarrow \text{bounded} \end{cases} \Rightarrow \begin{cases} \mathcal{T} = \boldsymbol{\sigma}_S \cdot \mathbf{n} \rightarrow \text{bounded} \\ \dot{\mathcal{T}} = \dot{\boldsymbol{\sigma}}_S \cdot \mathbf{n} \rightarrow \text{bounded} \end{cases} \tag{3.21}$$

4) In principal stress directions, equation (3.21) read:

$$\left.\begin{aligned} &\boldsymbol{\sigma}_S = \sigma_1\, \mathbf{p}_1 \otimes \mathbf{p}_1 + \sigma_2\, \mathbf{p}_2 \otimes \mathbf{p}_2 + \sigma_3\, \mathbf{p}_3 \otimes \mathbf{p}_3 \\ &\mathbf{n} = n_1\, \mathbf{p}_1 + n_2\, \mathbf{p}_2 + n_3\, \mathbf{p}_3 \end{aligned}\right\} \Rightarrow \\ [\mathcal{T}] = [\sigma_1\, n_1, \sigma_2\, n_2, \sigma_3\, n_3]^T \rightarrow \text{bounded} \tag{3.22}$$

where $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a unit ($\|\mathbf{p}_i\| = 1$) cartesian basis in the principal stress directions. Therefore and since $\{n_1, n_2, n_3\}$ are bounded, from equation (3.22):

$$[\sigma_1, \sigma_2, \sigma_3]^T \rightarrow \text{bounded} \Rightarrow \boxed{\boldsymbol{\sigma}_S \rightarrow \text{bounded}} \tag{3.23}$$

The same reasoning than in equations (3.21) and (3.22) for the rate of strains leads to :

$$\boxed{\dot{\boldsymbol{\sigma}}_S \rightarrow \text{bounded}} \tag{3.24}$$

Remark 3-5

In order to fulfill the traction continuity (momentum balance principle), the stresses $\boldsymbol{\sigma}_S$ (and the rate of stresses $\dot{\boldsymbol{\sigma}}_S$) are bounded at the discontinuous interface, even when the strains $\boldsymbol{\varepsilon}_S$ are unbounded. This applies not only to the traction vector components, but to all the components of $\boldsymbol{\sigma}_S$ an $\dot{\boldsymbol{\sigma}}_S$ in any basis.

Considering now the stress-like internal variable q we can argue the following:

1) Since the stress field $\boldsymbol{\sigma}_S$ is bounded, so is the plastic flow tensor $\mathbf{m}_S = \dfrac{\partial\phi(\boldsymbol{\sigma}_S)}{\partial\boldsymbol{\sigma}}$ (see Remark 3-4).

2) From the loading-unloading and persistency conditions (3.14):

$$\left.\begin{array}{l}\text{Loading}\rightarrow\lambda_S>0\Rightarrow\dot{f}_S\equiv\dot{\phi}_S-\dot{q}_S=0\Rightarrow\dot{q}_S=\dot{\phi}(\boldsymbol{\sigma}_S)\\ \text{Unloading}\rightarrow\lambda_S=0\Rightarrow\dot{q}_S=\lambda_S H=0\end{array}\right\}\Rightarrow\dot{q}_S\in\left[0,\dot{\phi}(\boldsymbol{\sigma}_S)\right]$$
$$\dot{\phi}(\boldsymbol{\sigma}_S)=\frac{\partial\phi(\boldsymbol{\sigma}_S)}{\partial\boldsymbol{\sigma}_S}:\dot{\boldsymbol{\sigma}}_S=\underbrace{\mathbf{m}_S}_{\text{bounded}}:\underbrace{\dot{\boldsymbol{\sigma}}_S}_{\text{bounded}}=\text{bounded} \tag{3.25}$$

and consequently, from equation (3.25):

$$\boxed{\dot{q}_S\rightarrow\text{bounded}} \tag{3.26}$$

3.3.2 Strong discontinuity analysis (SDA)

Let us consider the rate version of the constitutive equation (3.9):

$$\left.\begin{array}{l}\dot{\boldsymbol{\sigma}}=\mathbb{E}:(\dot{\boldsymbol{\varepsilon}}-\dot{\boldsymbol{\varepsilon}}^{p})\\ \dot{\boldsymbol{\varepsilon}}^{p}=\lambda\mathbf{m}\end{array}\right\}\Rightarrow\dot{\boldsymbol{\sigma}}_S=\mathbb{E}:(\dot{\boldsymbol{\varepsilon}}_S-\lambda_S\,\mathbf{m}(\boldsymbol{\sigma}_S)) \tag{3.27}$$

And substituting the strong discontinuity kinematics (3.4):

$$\dot{\boldsymbol{\sigma}}_S=\mathbb{E}(\dot{\bar{\boldsymbol{\varepsilon}}}_S+\frac{1}{\mathsf{h}}([\![\dot{\mathbf{u}}]\!]\otimes\mathbf{n})^s-\lambda_S\,\mathbf{m}(\boldsymbol{\sigma}_S)) \tag{3.28}$$

$$\begin{array}{c}\mathsf{h}\dot{\boldsymbol{\sigma}}_S=\mathbb{E}:(\mathsf{h}\dot{\bar{\boldsymbol{\varepsilon}}}_S+([\![\dot{\mathbf{u}}]\!]\otimes\mathbf{n})^s-\mathsf{h}\lambda_S\,\mathbf{m}(\boldsymbol{\sigma}_S))\Rightarrow\\ \underbrace{\lim_{\mathsf{h}\to0}(\mathsf{h}\underbrace{\dot{\boldsymbol{\sigma}}_S}_{\text{bounded}})}_{=0}=\underbrace{\lim_{\mathsf{h}\to0}(\mathbb{E}:\mathsf{h}\underbrace{\dot{\bar{\boldsymbol{\varepsilon}}}_S}_{\text{bounded}})}_{=0}+\mathbb{E}:([\![\dot{\mathbf{u}}]\!]\otimes\mathbf{n})^s-\\ -\lim_{\mathsf{h}\to0}(\mathsf{h}\lambda_S)\underbrace{\mathbb{E}:\mathbf{m}(\boldsymbol{\sigma}_S)}_{\text{bounded}}=\mathbf{0}\ \Rightarrow\end{array} \tag{3.29}$$

$$\mathbb{E}:([\![\dot{\mathbf{u}}]\!]\otimes\mathbf{n})^s-\mathbb{E}:\lim_{\mathsf{h}\to0}(\mathsf{h}\lambda_S)\mathbf{m}(\boldsymbol{\sigma}_S)=\mathbf{0}\Rightarrow \tag{3.30}$$

$$([\![\dot{\mathbf{u}}]\!]\otimes\mathbf{n})^{s}=\lim_{\mathsf{h}\to 0}(\mathsf{h}\lambda_{S})\underbrace{\mathbf{m}(\boldsymbol{\sigma}_{S})}_{\text{bounded}} \tag{3.31}$$

In order to have a non-trivial solution ($[\![\dot{\mathbf{u}}]\!]\neq\mathbf{0}$), in equation (3.31), the following condition emerges:

$$\lim_{\mathsf{h}\to 0}(\mathsf{h}\lambda_{S})\neq 0 \tag{3.32}$$

which can be accomplished by defining a new bounded variable:

$$\boxed{\bar{\lambda}\overset{def}{=}\mathsf{h}\lambda_{S}\longrightarrow\text{discrete plastic multiplier (bounded when } \mathsf{h}\to 0)} \tag{3.33}$$

By substituting equation (3.33) into ((3.31)) the following relationship is obtained:

$$\boxed{\begin{array}{c}\text{Strong discontinuity equation}\\ ([\![\dot{\mathbf{u}}]\!]\otimes\mathbf{n})^{s}=\bar{\lambda}\,\mathbf{m}(\boldsymbol{\sigma}_{S})\end{array}} \tag{3.34}$$

Equation (3.34) termed the *strong discontinuity equation* plays an important role in the subsequent analysis. Also, substituting equation (3.33) into equation (3.12) ($\dot{\alpha}=\lambda$):

$$\dot{\alpha}_{S}=\lambda_{S}=\frac{1}{\mathsf{h}}\bar{\lambda}\overset{def}{=}\frac{1}{\mathsf{h}}\dot{\bar{\alpha}}\Rightarrow\dot{\bar{\alpha}}=\bar{\lambda}$$
$$\boxed{\bar{\alpha}\longrightarrow\text{discrete internal variable (bounded when } \mathsf{h}\to 0)} \tag{3.35}$$

Equation (3.35) can be integrated for $t>t_{SD}$ as:

$$\alpha_{S}\big|_{t>t_{SD}}=\int_{0}^{t}\frac{1}{\mathsf{h}}\dot{\bar{\alpha}}\,dt=\underbrace{\int_{0}^{t_{SD}}\frac{1}{\mathsf{h}}\dot{\bar{\alpha}}\,dt}_{\alpha_{SD}}+\underbrace{\frac{1}{\mathsf{h}}}_{=\mathsf{k}}\underbrace{\int_{t_{SD}}^{t}\dot{\bar{\alpha}}\,dt}_{\Delta\bar{\alpha}}=\alpha_{SD}+\frac{1}{\mathsf{h}}\Delta\bar{\alpha}$$
$$\Delta\bar{\alpha}(\mathbf{x},t)=\bar{\alpha}(\mathbf{x},t)-\bar{\alpha}(\mathbf{x},t_{SD})\quad\forall t\geq t_{SD} \tag{3.36}$$

Finally, taking into account equation (3.35) into the rate version of equation (3.13) ($\dot{q}=H\dot{\alpha}$) and considering the bounded character of $\dot{q}_{S}$ (see equation (3.26)):

$$\underbrace{\dot{q}_{S}}_{\text{bounded}}=H\dot{\alpha}_{S}=H\frac{1}{\mathsf{h}}\underbrace{\dot{\bar{\alpha}}}_{\text{bounded}}\Rightarrow\lim_{\mathsf{h}\to 0}H\frac{1}{\mathsf{h}}=\bar{H}\ \text{(bounded)} \tag{3.37}$$

Therefore, condition (3.37), and subsidiary conditions (3.33) and (3.35), can be achieved by imposing the following structure to the continuum softening parameter H:

$$\boxed{\begin{array}{l}\text{Softening regularization condition}\\ H=\mathsf{h}\bar{H}\\ \bar{H}\rightarrow\text{discrete (or intrinsic) softening parameter}\end{array}} \tag{3.38}$$

Remark 3-6

According to the preceding analysis the softening regularization condition (3.38) is only strictly necessary at the strong discontinuity regime (when the bandwidth $h \equiv k \to 0$. However, for subsequent purposes, we shall extend it to the complete process of formation of the discontinuity *including the weak discontinuity regime* (see Remark 3-3 and Figure 3-3).

Now, by substitution of equation (3.38) into equation (3.37)

$$\dot{q}_S = \overline{H}\dot{\overline{\alpha}} \tag{3.39}$$

NOTE

Here $\overline{H}$ is considered constant (linear discrete softening). Generalization to non-linear softening is straightforward.

and integrating along time:

$$q_S\big|_{t>t_{SD}} = \int_0^t \dot{q}_S dt = \underbrace{\int_0^{t_{SD}} \dot{q}_S dt}_{q_{SD}} + \int_{t_{SD}}^t \overline{H}\dot{\overline{\alpha}}\, dt = q_{SD} + \overline{H}\underbrace{\int_{t_{SD}}^t \dot{\overline{\alpha}} dt}_{\Delta\overline{\alpha}} \tag{3.40}$$

$q_S = q_{SD} + \overline{H}\Delta\overline{\alpha}$ → discrete softening law (linear)

Remark 3-7

As in the 1D case, the intrinsic softening parameter $\overline{H}$ is material variable that will be related to the fracture properties of the material . By imposing the softening regularization condition (3.38) (and, thus, considering the continuum softening parameter H given in terms of the discrete one $\overline{H}$) the bounded character of the discrete variables $\dot{\overline{\alpha}}$ and $\overline{\lambda}$ and, eventually, the bounded character of $\boldsymbol{\sigma}_S$ and q_S is guaranteed.

Let us now go back to the components of the strong discontinuity equation (3.34) into the specific orthogonal basis constituted by the normal $\mathbf{n}$ and two unit vectors $\hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ parallel to the discontinuity interface (see Figure 3-7)

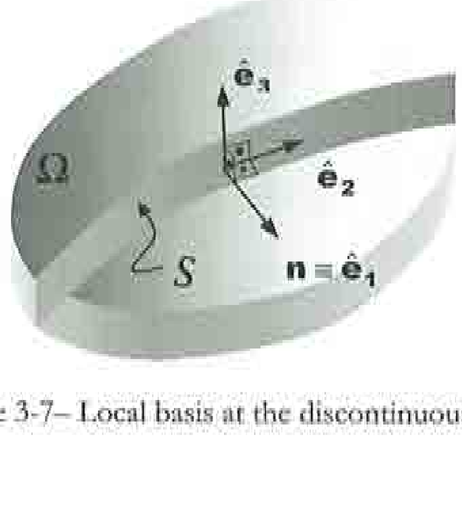


Figure 3-7– Local basis at the discontinuous interface

$$(\llbracket \dot{\mathbf{u}} \rrbracket \otimes \mathbf{n})^s = \bar{\lambda}\, \mathbf{m}(\boldsymbol{\sigma}_S) \rightarrow$$

$$\underbrace{\begin{bmatrix} \llbracket \dot{u} \rrbracket_1 & \frac{1}{2}\llbracket \dot{u} \rrbracket_2 & \frac{1}{2}\llbracket \dot{u} \rrbracket_3 \\ \frac{1}{2}\llbracket \dot{u} \rrbracket_2 & 0 & 0 \\ \frac{1}{2}\llbracket \dot{u} \rrbracket_3 & 0 & 0 \end{bmatrix}}_{\left[(\llbracket \dot{\mathbf{u}} \rrbracket \otimes \mathbf{n})^s\right]} = \bar{\lambda} \underbrace{\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix}}_{[\mathbf{m}]} \tag{3.41}$$

Equation (3.41) provides six independent equations (due to its symmetry) three of them (components $(\bullet)_{11}$, $(\bullet)_{12}$, $(\bullet)_{13}$) provide the value of the displacement jumps:

Discrete constitutive equation

$$\left.\begin{aligned} \llbracket \dot{u} \rrbracket_1 &= \bar{\lambda}\, m_{11} \\ \llbracket \dot{u} \rrbracket_2 &= \bar{\lambda}\, 2m_{12} \\ \llbracket \dot{u} \rrbracket_3 &= \bar{\lambda}\, 2m_{13} \end{aligned}\right\} \Rightarrow \llbracket \dot{\mathbf{u}} \rrbracket = \bar{\lambda}\, \mathbf{m}^* \qquad [\mathbf{m}^*] = \begin{bmatrix} m_{11} \\ 2m_{12} \\ 2m_{13} \end{bmatrix}_S \qquad \forall t \geq t_{SD} \tag{3.42}$$

The remaining components (components $(\bullet)_{22}$, $(\bullet)_{23}$, $(\bullet)_{33}$) provide a set of restrictions on the stress field $\boldsymbol{\sigma}_S$ to be fulfilled at the strong discontinuity regime:

Strong discontinuity conditions

$$\mathcal{R}(\boldsymbol{\sigma}_S) \equiv \begin{bmatrix} m_{22}(\boldsymbol{\sigma}_S) = \dfrac{\partial \phi(\boldsymbol{\sigma}_S)}{\partial \sigma_{22}} \\ m_{23}(\boldsymbol{\sigma}_S) = \dfrac{\partial \phi(\boldsymbol{\sigma}_S)}{\partial \sigma_{23}} \\ m_{33}(\boldsymbol{\sigma}_S) = \dfrac{\partial \phi(\boldsymbol{\sigma}_S)}{\partial \sigma_{33}} \end{bmatrix} = \mathbf{0} \qquad \forall t \geq t_{SD} \tag{3.43}$$

Remark 3-8

The strong discontinuity conditions, $\mathcal{R}(\boldsymbol{\sigma}_S) = \mathbf{0}$, are not, in general, fulfilled at the bifurcation time t_B. Therefore they preclude the strong discontinuity to originate directly by discontinuous bifurcation of the stress-strain field. This justifies the introduction of a transition mechanism like the variable bandwidth model sketched in section 3.1.2 (see Figure 3-8).

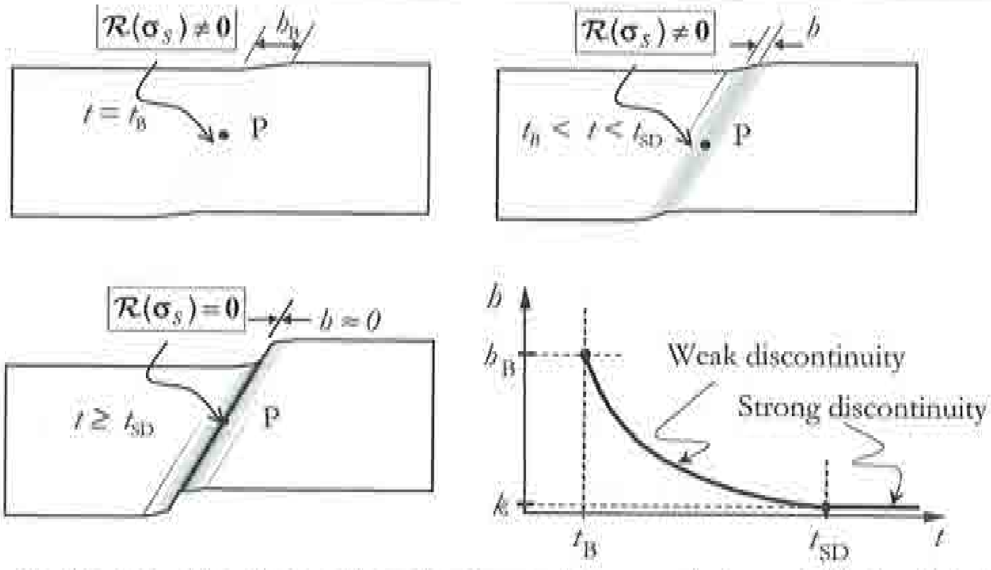


Figure 3-8– The strong discontinuity conditions and the variable bandwidth model

> **Remark 3-9**
>
> The strong discontinuity conditions (3.43) also state that, at the strong discontinuity regime (for $t \geq t_{SD}$) the uniaxial equivalent stress, $\phi(\boldsymbol{\sigma}_S)$, does not depend on components $\sigma_{22}, \sigma_{23}, \sigma_{33}$. Therefore it only depends on the remaining components of the stress field $\boldsymbol{\sigma}_S$ $(\sigma_{11}, \sigma_{12}, \sigma_{13})$ which, in fact, are the components of the traction vector in the chosen basis: $[\mathcal{T}] = [\sigma_{11}, \sigma_{12}, \sigma_{13}]^T$.

$$\phi(\boldsymbol{\sigma}_S) \equiv \bar{\phi}(\mathcal{T}) \quad \forall t \geq t_{SD} \Rightarrow f(\boldsymbol{\sigma}_S, q) = \bar{\phi}(\mathcal{T}) - q_S \quad \forall t \geq t_{SD} \Rightarrow \tag{3.44}$$

$$f(\boldsymbol{\sigma}_S, q_S) \equiv \mathcal{F}(\mathcal{T}, q_S) = \bar{\phi}(\mathcal{T}) - q_S \quad \forall t \geq t_{SD} \rightarrow \text{discrete yield function} \tag{3.45}$$

and, in view of equation (3.45), equation (3.42)can be written as:

$$\left[\mathbf{m}^*\right] = \begin{bmatrix} m_{11} \\ 2m_{12} \\ 2m_{13} \end{bmatrix}_S = \begin{bmatrix} \dfrac{\partial \phi}{\partial \sigma_{11}} \\ \dfrac{\partial \phi}{\partial \sigma_{12}} + \dfrac{\partial \phi}{\partial \sigma_{21}} \\ \dfrac{\partial \phi}{\partial \sigma_{13}} + \dfrac{\partial \phi}{\partial \sigma_{31}} \end{bmatrix}_S = \begin{bmatrix} \dfrac{\partial \mathcal{F}}{\partial \sigma_{11}} \\ \dfrac{\partial \mathcal{F}}{\partial \sigma_{12}} \\ \dfrac{\partial \mathcal{F}}{\partial \sigma_{13}} \end{bmatrix}_S = \left[\dfrac{\partial \mathcal{F}}{\partial \mathcal{T}}\right] \Rightarrow$$
$$\mathbf{m}^* = \frac{\partial \mathcal{F}}{\partial \mathcal{T}} \Rightarrow \tag{3.46}$$

$$[\![\dot{\mathbf{u}}]\!] = \dot{\bar{\lambda}}\, \mathbf{m}^* = \dot{\bar{\lambda}} \frac{\partial \mathcal{F}}{\partial \mathcal{T}} \quad \forall t \geq t_{SD} \rightarrow \text{discrete flow rule} \tag{3.47}$$

3.3.3 Discrete free energy.

From equation (3.34) $(([\![\dot{\mathbf{u}}]\!] \otimes \mathbf{n})^s = \dot{\bar{\lambda}}\, \mathbf{m}(\boldsymbol{\sigma}_S)$ we notice that that for $t > t_{SD}$:

$$(\llbracket \dot{\mathbf{u}} \rrbracket \otimes \mathbf{n})^s = \bar{\lambda}\mathbf{m}(\boldsymbol{\sigma}_S) \Rightarrow \frac{1}{h}(\llbracket \dot{\mathbf{u}} \rrbracket \otimes \mathbf{n})^s = \underbrace{\frac{1}{h}\bar{\lambda}}_{\lambda}\mathbf{m}(\boldsymbol{\sigma}_S) = \lambda\mathbf{m}(\boldsymbol{\sigma}_S) = \dot{\boldsymbol{\varepsilon}}_S^p$$

$$\boxed{\dot{\boldsymbol{\varepsilon}}_S^p = \frac{1}{h}(\llbracket \dot{\mathbf{u}} \rrbracket \otimes \mathbf{n})^s \quad \forall t > t_{SD}} \tag{3.48}$$

$$\dot{\boldsymbol{\varepsilon}}_S = \dot{\bar{\boldsymbol{\varepsilon}}} + \underbrace{\frac{1}{h}(\llbracket \dot{\mathbf{u}} \rrbracket \otimes \mathbf{n})^s}_{\dot{\boldsymbol{\varepsilon}}_S^p} = \dot{\boldsymbol{\varepsilon}}_S^e + \dot{\boldsymbol{\varepsilon}}_S^p \Rightarrow \quad \boxed{\dot{\boldsymbol{\varepsilon}}_S^e = \dot{\bar{\boldsymbol{\varepsilon}}} \quad \forall t > t_{SD}} \tag{3.49}$$

> **Remark 3-10**
>
> Equations (3.48) and (3.49) state that at the strong discontinuity regime *the plastic flow translates entirely into displacement jump* and *that the regular strain is purely elastic.*

Integrating equation (3.49) along time:

$$\boldsymbol{\varepsilon}_S^e = \int_0^t \dot{\bar{\boldsymbol{\varepsilon}}}\, dt = \underbrace{\int_0^{t_{SD}} \dot{\bar{\boldsymbol{\varepsilon}}}\, dt}_{\bar{\boldsymbol{\varepsilon}}_{SD}} + \underbrace{\int_{t_{SD}}^t \dot{\bar{\boldsymbol{\varepsilon}}}\, dt}_{\Delta\bar{\boldsymbol{\varepsilon}}} = \underbrace{\bar{\boldsymbol{\varepsilon}}_{SD} + \Delta\bar{\boldsymbol{\varepsilon}}}_{\text{bounded}} \Rightarrow \quad \boxed{\lim_{h\to 0} h\boldsymbol{\varepsilon}_S^e = \mathbf{0} \qquad \forall t > t_{SD}} \tag{3.50}$$

Let us now consider the free density energy (3.8) at the strong discontinuity regime ($t > t_{SD}$ and $\equiv k \to 0$):

$$\psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \alpha) = \underbrace{\frac{1}{2}\overbrace{(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)}^{\boldsymbol{\varepsilon}^e} : \mathbb{E} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)}_{\psi^e(\boldsymbol{\varepsilon}^e)} + \psi^p(\alpha) \tag{3.51}$$

and, from it let us consider the free energy at the discontinuous interface, $\bar{\psi}$, *per unit of surface.* Both are related by (see Figure 3-9):

$$\underbrace{\frac{\text{Incremental free energy}}{\text{unit surface}}}_{\bar{\psi}} = \underbrace{\frac{\text{Free energy}}{\text{unit volume}}}_{\psi_S} \cdot \underbrace{\frac{\text{volume}}{\text{surface}}}_{h} \Rightarrow \quad \boxed{\bar{\psi} = \lim_{h\to 0}(h\psi_S)} \tag{3.52}$$

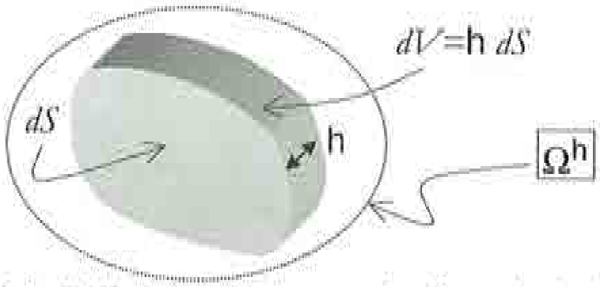


Figure 3-9– Differential of volume at the discontinuity interface

Now, by considering equation (3.52) and inserting the strong discontinuity kinematics (3.7)

$$\begin{aligned}\bar{\psi} &= \lim_{h\to 0}(h\psi_S) = \lim_{h\to 0} h(\frac{1}{2}\boldsymbol{\varepsilon}^e:\mathbb{E}:\boldsymbol{\varepsilon}^e) + \lim_{h\to 0} h\psi^p(\alpha_S) = \\ &= \lim_{h\to 0}(\frac{1}{2}\underbrace{(h\boldsymbol{\varepsilon}^e}_{=\mathbf{0}}:\mathbb{E}:\boldsymbol{\varepsilon}^e) + \underbrace{\lim_{h\to 0} h\psi^p(\alpha_{SD} + \frac{1}{h}\Delta\bar{\alpha})}_{\bar{\psi}^p(\Delta\bar{\alpha})} = \bar{\psi}^p(\Delta\bar{\alpha})\end{aligned} \tag{3.53}$$

where equations (3.38) and (3.36) have been considered. Therefore, the discrete (per unit of surface) free energy $\bar{\psi}$ can be written as:

$$\boxed{\bar{\psi}(\Delta[[\mathbf{u}]], \Delta\alpha) = \underbrace{0}_{\bar{\psi}^e(\Delta[[\mathbf{u}]])} + \underbrace{\lim_{h\to 0} h\psi^p(\alpha_S)}_{\bar{\psi}^p(\Delta\bar{\alpha})} \rightarrow \text{discrete free energy}} \tag{3.54}$$

So that the elastic counterpart of the discrete free energy is zero. On the other hand, from equations (3.54), (3.36) and (3.13):

$$\frac{\partial\bar{\psi}^p(\Delta\bar{\alpha})}{\partial\Delta\bar{\alpha}} = \lim_{h\to 0}\frac{\partial(h\psi^p(\alpha_S))}{\partial\alpha_S}\underbrace{\frac{\partial\alpha_S}{\partial\Delta\bar{\alpha}}}_{\frac{1}{h}} = \lim_{h\to 0}\underbrace{\frac{\partial\psi^p(\alpha_S)}{\partial\alpha_S}}_{q_S} = q_S \Rightarrow$$

$$\boxed{q_S = \frac{\partial\bar{\psi}^p(\Delta\bar{\alpha})}{\partial\Delta\bar{\alpha}} \quad \forall t > t_{SD}} \tag{3.55}$$

and, from equations (3.55) and (3.40) ($q_S = q_{SD} + \bar{H}\Delta\bar{\alpha}$) the explicit format of $\bar{\psi}^p(\Delta\bar{\alpha})$ can be found as:

$$q_S = \frac{\partial\bar{\psi}^p(\Delta\bar{\alpha})}{\partial\Delta\bar{\alpha}} = q_{SD} + \bar{H}\Delta\bar{\alpha} \quad \Rightarrow \bar{\psi}^p(\Delta\bar{\alpha}) = q_{SD}\Delta\bar{\alpha} + \frac{1}{2}\bar{H}(\Delta\bar{\alpha})^2 \tag{3.56}$$

3.3.4 Discrete elasto-plastic model.

The results in the sections 3.3.2 and 3.3.3define a complete discrete constitutive model (relating the traction at the interface $\mathcal{T}$ and the incremental displacement jump $\Delta[[\mathbf{u}]]$) see figure (Figure 3-10) that can be summarized as follows:

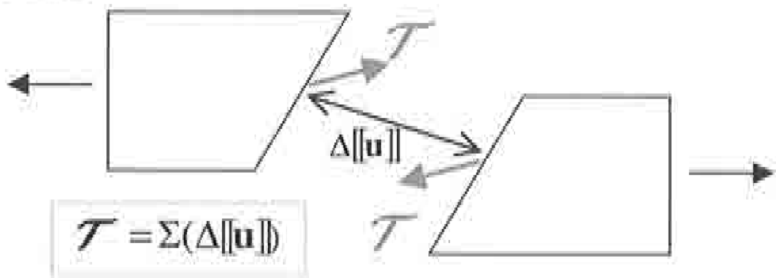


Figure 3-10– Discrete constitutive model at the discontinuity interface

Free energy:	$\bar{\psi}(\Delta[[\mathbf{u}]],\Delta\alpha)=\underbrace{0}_{\bar{\psi}^{e}(\Delta[[\mathbf{u}]])}+\underbrace{\lim_{h\to 0} h\psi^{p}(\alpha_S)}_{\bar{\psi}^{p}(\Delta\bar{\alpha})}$	(3.57)
Yield function:	$f(\sigma_S,q_S)\equiv\mathcal{F}(\mathcal{T},q_S)=\bar{\phi}(\mathcal{T})-q_S$	(3.58)
Flow rule:	$[[\dot{\mathbf{u}}]]=\bar{\lambda}\,\mathbf{m}^{*}=\bar{\lambda}\dfrac{\partial\mathcal{F}}{\partial\mathcal{T}}$	(3.59)
Evolution law	$\dot{\bar{\alpha}}=\bar{\lambda}$	(3.60)
Hardening/ softening law	$q_S(\Delta\bar{\alpha})=\dfrac{\partial\bar{\psi}^{p}(\Delta\bar{\alpha})}{\partial\bar{\alpha}}\;;\;\left.\begin{matrix}q(0)=q_{SD}\\ q(\infty)=0\end{matrix}\right\}\Rightarrow q\in[0,q_{SD}]$ $\bar{H}=\dfrac{\partial q(\Delta\bar{\alpha})}{\partial\bar{\alpha}}\le 0\;\Rightarrow\;\dot{q}_S=\bar{H}\dot{\bar{\alpha}}$	(3.61)
Loading/ unloading conditions	$\bar{\lambda}\ge 0\;\;\mathcal{F}\le 0\;\;\bar{\lambda}\mathcal{F}=0\;\;(Kuhn-Tucker)$ $\mathcal{F}=0\Rightarrow\;\bar{\lambda}\dot{\mathcal{F}}=0\quad(consistency)$	(3.62)
Tangent constitutive equation	$[[\dot{\mathbf{u}}]]=\dfrac{1}{\bar{H}}(\mathbf{m}^{*}\otimes\mathbf{m}^{*})\cdot\dot{\mathcal{T}}\;\;;\;\;(loading)$ $[[\dot{\mathbf{u}}]]=\mathbf{0}\qquad(unloading)$	(3.63)

Remark 3-11

The discrete elasto-plastic constitutive model of equations (3.57)to (3.63) is a *rigid-plastic model* since it could be characterized by and infinity elastic stiffness $\bar{\mathbb{E}}$ as follows:

$$\Delta[[\mathbf{u}]]=\Delta[[\mathbf{u}]]^{e}+\Delta[[\mathbf{u}]]^{p}$$

and the corresponding elastic free energy:

$$\bar{\psi}=\underbrace{\frac{1}{2}(\Delta[[\mathbf{u}]]-\Delta[[\mathbf{u}]]^{p}):\bar{\mathbb{E}}:(\Delta[[\mathbf{u}]]-\Delta[[\mathbf{u}]]^{p})}_{\bar{\psi}^{e}([[\mathbf{u}]]^{e})}+\bar{\psi}^{p}(\Delta\bar{\alpha})$$

The constitutive equation is then given by:

$$\mathcal{T}=\frac{\partial\bar{\psi}}{\partial\Delta[[\mathbf{u}]]}=\bar{\mathbb{E}}:(\Delta[[\mathbf{u}]]-\Delta[[\mathbf{u}]]^{p})$$

where $\bar{\mathbb{E}}=\infty\Rightarrow\Delta[[\mathbf{u}]]=\Delta[[\mathbf{u}]]^{p}\Rightarrow\Delta[[\mathbf{u}]]^{e}=\mathbf{0}$

Remark 3-12

It can be also observed a one to one correspondence between the variables of the continuum model and the ones of the induced discrete model according to the following table:

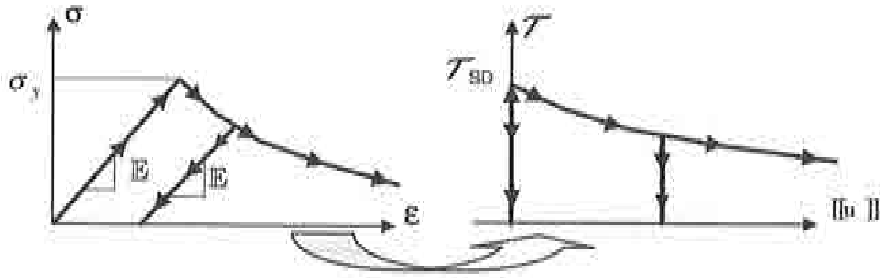
Continuum	$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}^e$	$\boldsymbol{\varepsilon}^p$	ψ	α	q	$\boldsymbol{\sigma}$	λ	H
Discrete	$\Delta[[\mathbf{u}]]$	$\underbrace{\Delta[[\mathbf{u}]]^e}_{=0}$	$\Delta[[\mathbf{u}]]^p$	$\bar{\psi}$	$\Delta\bar{\alpha}$	q_S	$\mathcal{T}$	$\bar{\lambda}$	$\bar{H}$

Remark 3-13

The discrete elasto plastic model is automatically induced from the continuum one, by introducing only two ingredients (see Figure 3-11):

- The strong discontinuity kinematics of equation (3.5)
- The softening regularization in equation (3.38).

Although it can be explicitly derived (as done here) it comes out automatically from the continuum model if both conditions are introduced.



$$\dot{\boldsymbol{\varepsilon}} = \dot{\bar{\boldsymbol{\varepsilon}}} + \frac{1}{h}([[\dot{\mathbf{u}}]] \otimes \mathbf{n})$$
$$H = h\bar{H}$$

Figure 3-11– Original continuum and induced discrete elasto-plastic models

3.4 Application to different plasticity models

3.4.1 Rankine-type models (mode I of fracture)

3.4.1.1 Continuum model

Uniaxial equivalent stress	$\phi(\boldsymbol{\sigma}) = \sigma_1$ $\sigma_1 \geq \sigma_2 \geq \sigma_3 \rightarrow$ principal stresses $\quad \boldsymbol{\sigma} = \sum_1^3 \sigma_i \mathbf{p}_i \otimes \mathbf{p}_i$ $\mathbf{p}_i \; i \in \{1,2,3\} \rightarrow$ eigenvalues	(3.64)
Flow tensor	$\mathbf{m} = \frac{\partial \phi(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = \frac{\partial \sigma_1}{\partial \boldsymbol{\sigma}} = \mathbf{p}_1 \otimes \mathbf{p}_1 \qquad (\mathbf{n} = \mathbf{p}_1)$ $[\mathbf{m}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	(3.65)

NOTE

It can be shown from the discontinuous bifurcation analysis (see Chapter 4) that $\mathbf{n} = \mathbf{p}_1$

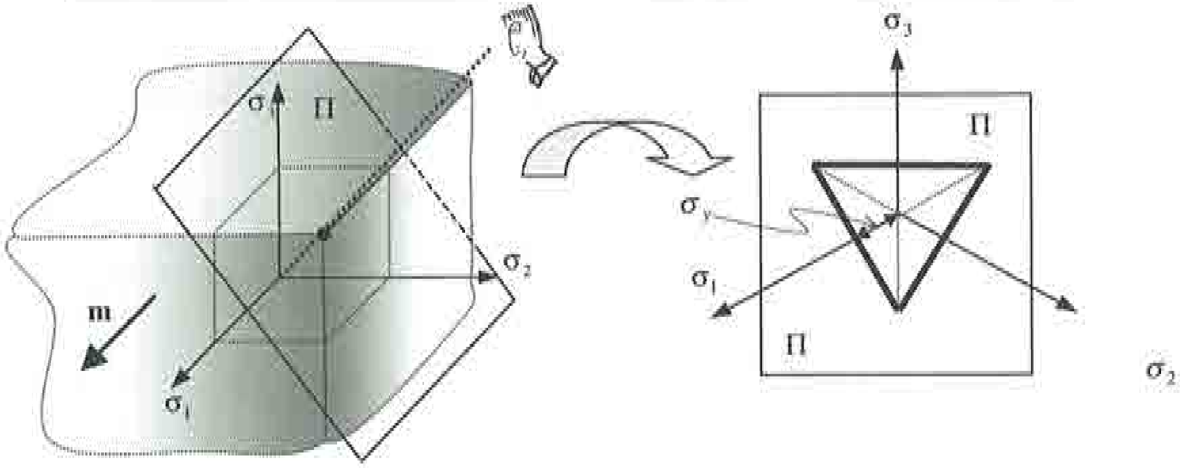


Figure 3-12– Rankine plasticity model

3.4.1.2 Discrete induced model

Discrete flow tensor	$[\mathbf{m}^*]=\begin{bmatrix}m_{11}\\2m_{12}\\2m_{13}\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$	(3.66)
Strong discontinuity conditions	$\left.\begin{matrix}m_{22}=0\Rightarrow\ 0=0\\m_{33}=0\Rightarrow\ 0=0\\m_{23}=0\Rightarrow\ 0=0\end{matrix}\right\}\rightarrow$ (Identically fulfilled)	(3.67)
Tangent constitutive tensor (loading)	$[\![\dot{\mathbf{u}}]\!]=\frac{1}{H}(\mathbf{m}^*\otimes\mathbf{m}^*)\cdot\dot{\mathcal{T}}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\dot{\mathcal{T}}_1(=\dot{\sigma}_{11})\\\dot{\mathcal{T}}_2(=\dot{t}_{12})\\\dot{\mathcal{T}}_3(=\dot{t}_{13})\end{bmatrix}\Rightarrow$ $\left.\begin{matrix}[\![\dot{u}_1]\!]=\frac{1}{H}\dot{\mathcal{T}}_1=\frac{1}{H}\dot{\sigma}_{11}\\ [\![\dot{u}_2]\!]=0\\ [\![\dot{u}_3]\!]=0\end{matrix}\right\}\rightarrow$ Mode I of fracture	(3.68)

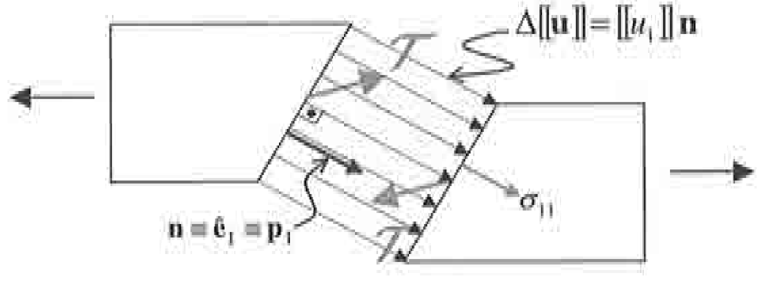


Figure 3-13– Mode I of fracture

Remark 3-14

Notice, from equation (3.67), that the strong discontinuity conditions are always identically accomplished independently of the stress state. This provides to the Rankine-type plasticity models the interesting property of being able to induce a strong discontinuity directly by bifurcation of the stress-strain field (see Remark 3-8). No transition (variable bandwidth) mechanism is, in this case, needed.

This fact, and also that they induce a mode I (no tangential jump) fracture mode, could explain why so many discrete constitutive models, based on Rankine-like plasticity, have been traditionally (and successfully) used in discrete fracture mechanics for rocks and concrete.

3.4.2 J2-type models (mode II of fracture)

3.4.2.1 Continuum model

Uniaxial equivalent stress	$\phi(\boldsymbol{\sigma})=\sqrt{\frac{3}{2}\mathbf{s}:\mathbf{s}}\quad ; \quad \mathbf{s}=dev(\boldsymbol{\sigma})$	(3.69)
Flow tensor	$\mathbf{m}=\frac{\partial\phi(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}=\sqrt{\frac{3}{2}}\frac{\mathbf{s}}{\lVert\mathbf{s}\rVert}$ $[\mathbf{m}]=\sqrt{\frac{3}{2}}\frac{1}{\lVert\mathbf{s}\rVert}\begin{bmatrix}-(s_{22}+s_{33}) & \tau_{12} & \tau_{13}\\ \tau_{12} & s_{22} & \tau_{23}\\ \tau_{13} & \tau_{23} & s_{33}\end{bmatrix}\quad (tr(\mathbf{s})=0)$	(3.70)

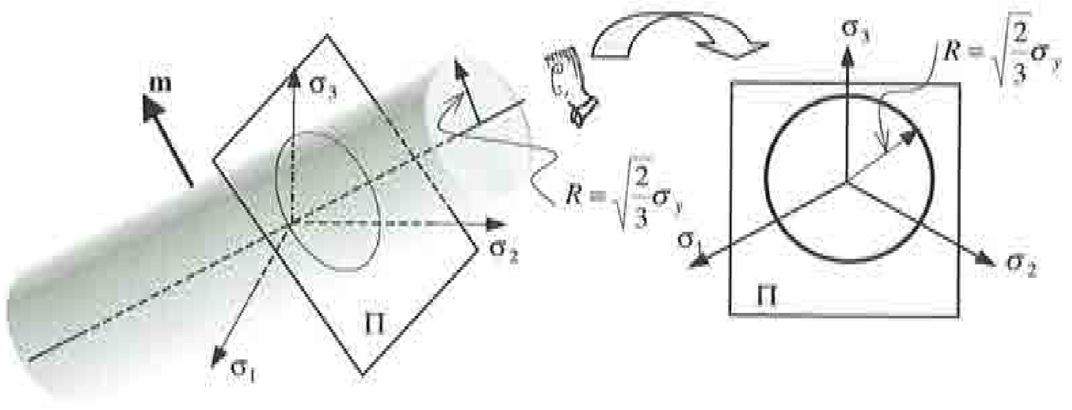


Figure 3-14– J_2 (von Mises) plasticity model

3.4.2.2 Discrete induced model

Discrete flow tensor	$[\mathbf{m}^*]=\begin{bmatrix} m_{11} \\ 2m_{12} \\ 2m_{13}\end{bmatrix}=\sqrt{\frac{3}{2}}\frac{1}{\lVert\mathbf{s}\rVert}\begin{bmatrix}-(s_{22}+s_{33}) \\ 2\tau_{12} \\ 2\tau_{13}\end{bmatrix}$	(3.71)
Strong discontinuity conditions	$\left.\begin{matrix}\left.\begin{matrix} m_{22}=0\Rightarrow s_{22}=0 \\ m_{33}=0\Rightarrow s_{33}=0\end{matrix}\right\}\Rightarrow m_{11}=0 \\ m_{23}=0\Rightarrow \tau_{23}=0\end{matrix}\right\}\Rightarrow \mathbf{s}=\begin{bmatrix}0 & \tau_{12} & \tau_{13} \\ \tau_{12} & 0 & 0 \\ \tau_{13} & 0 & 0\end{bmatrix}$ $\mathbf{m}=\frac{\sqrt{3}}{2\sqrt{(\tau_{12}^2+\tau_{13}^2)}}\begin{bmatrix}0 & \tau_{12} & \tau_{13} \\ \tau_{12} & 0 & 0 \\ \tau_{13} & 0 & 0\end{bmatrix}$ $[\mathbf{m}^*]=\frac{\sqrt{3}}{\sqrt{(\tau_{12}^2+\tau_{13}^2)}}\begin{bmatrix}0 \\ \tau_{12} \\ \tau_{13}\end{bmatrix}$	(3.72)
Tangent constitutive tensor (loading)	$[[\dot{\mathbf{u}}]]=\frac{1}{H}(\mathbf{m}^*\otimes\mathbf{m}^*)\cdot\dot{\mathcal{T}}=$ $=\frac{3}{H}\frac{1}{(\tau_{12}^2+\tau_{13}^2)}\begin{bmatrix}0 & 0 & 0 \\ 0 & \tau_{12}^2 & \tau_{12}\tau_{13} \\ 0 & \tau_{12}\tau_{13} & \tau_{13}^2\end{bmatrix}\begin{bmatrix}\dot{\mathcal{T}}_1(=\dot{\sigma}_{11}) \\ \dot{\mathcal{T}}_2(=\dot{t}_{12}) \\ \dot{\mathcal{T}}_3(=\dot{t}_{13})\end{bmatrix}\Rightarrow$ $\left.\begin{matrix}[[\dot{u}_1]]=0 \\ [[\dot{u}_2]]=\frac{3}{H}\dot{t}_{12} \\ [[\dot{u}_3]]=\frac{3}{H}\dot{t}_{13}\end{matrix}\right\}\rightarrow$ Mode II of fracture	(3.73)

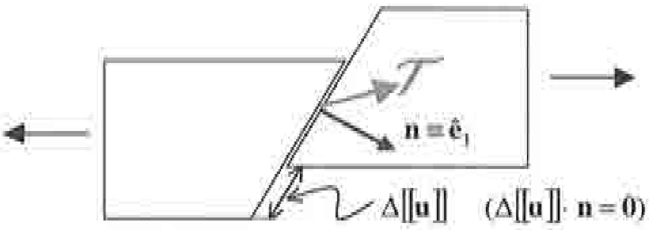


Figure 3-15– Mode II of fracture

> **Remark 3-15**
>
> Unlike for Rankine plasticity models, J_2 plasticity models have non trivial strong discontinuity conditions, which require a *pure shear structure* to the deviatoric stress field $\mathbf{s}_S$ at the strong discontinuity regime (see equations (3.72)). Obtaining such a particular stress-state will require, in general, the transition mechanism provided by the variable bandwidth model.

Remark 3-16

Notice that the induced discrete constitutive equations (3.73) are pure mode II fracture modes (no normal jump is induced), with uncoupled behavior in two orthogonal directions on the discontinuous interface. J_2 plasticity models are, then, suitable for modeling *slip lines in soils* or *shear bands* in metals.

4 Discontinuous bifurcation analysis

4.1 Onset of a discontinuity

In Chapter 3, it was presented a mechanism for the formation of a strong discontinuity as a weak discontinuity collapsing to a null bandwidth strong discontinuity (see Figure 4-1). According to this mechanism, then initial step is the discontinuous bifurcation of the stress-strain field into a weak discontinuity. This will provide both the direction of propagation of the discontinuity and the bandwidth h_B associated to the weak discontinuity at the bifurcation time t_B.

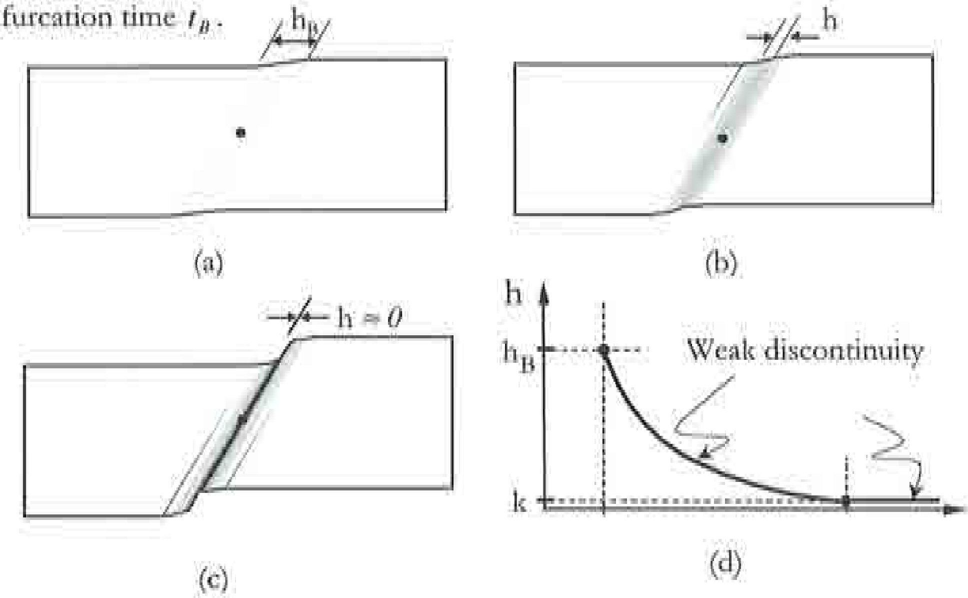


Figure 4-1– Mechanism of formation of a strong discontinuity: (a) Discontinuous bifurcation into a weak discontinuity. (b)-(c) Collapse into a strong discontinuity of null bandwidth.

4.2 Elasto-plastic softening model

Let us consider again the general three-dimensional elasto-plastic model with strain softening:

Free energy:	$\psi(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p,\alpha)=\underbrace{\frac{1}{2}\overbrace{(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)}^{\boldsymbol{\varepsilon}^e}:\mathbb{E}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)}_{\psi^e(\boldsymbol{\varepsilon}^e)}+\psi^p(\alpha)$	(4.1)
Constitutive equation:	$\boldsymbol{\sigma}=\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}=\mathbb{E}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)$	(4.2)
Yield function:	$f(\boldsymbol{\sigma},q)=\phi(\boldsymbol{\sigma})-q \quad ; \ \phi(\boldsymbol{\sigma})\geq 0$	(4.3)
Flow rule:	$\dot{\boldsymbol{\varepsilon}}^p=\lambda\frac{\partial f}{\partial\boldsymbol{\sigma}}=\lambda\underbrace{\frac{\partial\phi}{\partial\boldsymbol{\sigma}}}_{\mathbf{m}}=\lambda\mathbf{m}(\boldsymbol{\sigma})$	(4.4)
Evolution law	$\dot{\alpha}=\lambda$	(4.5)
Hardening/ softening law	$q(\alpha)=\frac{\partial\psi^p(\alpha)}{\partial\alpha} \ ; \ \left.\begin{matrix} q(0)=\sigma_y \\ q(\infty)=0\end{matrix}\right\} \Rightarrow q\in[0,\sigma_y]$ $H=\frac{\partial q(\alpha)}{\partial\alpha}\leq 0 \quad \Rightarrow \ \dot{q}=H\dot{\alpha}$	(4.6)
Loading/ unloading conditions	$\lambda\geq 0 \ \ f\leq 0 \ \ \lambda f=0 \ \ (Kuhn-Tucker)$ $f=0\Rightarrow \ \lambda\dot{f}=0 \quad (persistency\,/\,consistency)$	(4.7)
Tangent constitutive equation	$\dot{\boldsymbol{\sigma}}=\mathbb{E}^{ep}:\dot{\boldsymbol{\varepsilon}} \quad ; \quad \mathbb{E}^{ep}=\mathbb{E}-\frac{\mathbb{E}:\mathbf{m}\otimes\mathbf{m}:\mathbb{E}}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}} \quad (loading)$ $\dot{\boldsymbol{\sigma}}=\mathbb{E}:\dot{\boldsymbol{\varepsilon}} \quad ; \quad (unloading)$	(4.8)

and examine in more detail the loading unloading conditions. Considering equations (4.1) to(4.8):

1) Loading (L):

$$\boxed{\lambda>0} \quad \rightarrow\begin{cases} f=0 \\ \lambda>0\end{cases}\Rightarrow(\lambda\dot{f}=0)\rightarrow \quad \dot{f}=0 \ \text{(persistency)} \tag{4.9}$$

$$\left.\begin{matrix} f(\boldsymbol{\sigma},q)=\phi(\boldsymbol{\sigma})-q \\ \dot{f}=0\end{matrix}\right\}\Leftrightarrow\underbrace{\frac{\partial\phi}{\partial\boldsymbol{\sigma}}}_{\mathbf{m}}:\dot{\boldsymbol{\sigma}}-\underbrace{\dot{q}}_{H\lambda}=\mathbf{m}:\mathbb{E}:(\dot{\boldsymbol{\varepsilon}}-\dot{\boldsymbol{\varepsilon}}^p)-H\lambda=0$$

$$\mathbf{m}:\mathbb{E}:(\dot{\boldsymbol{\varepsilon}}-\underbrace{\dot{\boldsymbol{\varepsilon}}^p}_{\lambda\mathbf{m}})-H\lambda=\mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}-\lambda\mathbf{m}:\mathbb{E}:\mathbf{m}-\lambda H=0\Leftrightarrow \tag{4.10}$$

$$\lambda=\frac{\mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}}>0\Leftrightarrow \tag{4.11}$$

NOTE

It is assumed here that $H+\mathbf{m}:\mathbb{E}:\mathbf{m}>0$

$$\boxed{\mathbf{m}:\underbrace{\mathbb{E}:\dot{\boldsymbol{\varepsilon}}}_{\overset{def}{=}\dot{\boldsymbol{\sigma}}^{\text{trial}}}=\mathbf{m}:\dot{\boldsymbol{\sigma}}^{\text{trial}}>0 \quad \begin{cases}\dot{\boldsymbol{\sigma}}=\mathbb{E}^{ep}:\dot{\boldsymbol{\varepsilon}} \\ \mathbb{E}^{ep}=\mathbb{E}-\frac{\mathbb{E}:\mathbf{m}\otimes\mathbf{m}:\mathbb{E}}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}}\end{cases}} \tag{4.12}$$

2) Neutral loading (NL):

$$\boxed{\dot\lambda=0} \quad \rightarrow \begin{cases} f=0 \\ \dot\lambda=0 \end{cases} \Rightarrow (\dot\lambda \dot f=0) \rightarrow \quad \dot f=0 \ \text{(persistency)} \tag{4.13}$$

$$\dot\lambda=\frac{\mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}}=0 \Leftrightarrow$$

$$\boxed{\mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}=\mathbf{m}:\dot{\boldsymbol{\sigma}}^{\text{trial}}=0 \quad \begin{cases} \dot{\boldsymbol{\sigma}}=\mathbb{E}^{ep}:\dot{\boldsymbol{\varepsilon}}=\mathbb{E}:\dot{\boldsymbol{\varepsilon}} \\ \mathbb{E}^{ep}=\mathbb{E}-\dfrac{\mathbb{E}:\mathbf{m}\otimes\mathbf{m}:\mathbb{E}}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}} \end{cases}} \tag{4.14}$$

3) Elastic unloading (U):

$$\boxed{\dot\lambda=0} \quad \rightarrow \begin{cases} f=0 \\ \dot\lambda=0 \end{cases} \Rightarrow (\dot\lambda \dot f=0) \rightarrow \quad \dot f<0 \tag{4.15}$$

$$\left.\begin{matrix} f(\boldsymbol{\sigma},q)\equiv\phi(\boldsymbol{\sigma})-q \\ \dot f<0 \end{matrix}\right\} \Leftrightarrow \underbrace{\frac{\partial\phi}{\partial\boldsymbol{\sigma}}}_{\mathbf{m}}:\dot{\boldsymbol{\sigma}}-\underbrace{\dot q}_{H\dot\lambda=0}=\mathbf{m}:\mathbb{E}:(\dot{\boldsymbol{\varepsilon}}-\dot{\boldsymbol{\varepsilon}}^p)<0$$

$$\mathbf{m}:\mathbb{E}:(\dot{\boldsymbol{\varepsilon}}-\underbrace{\dot{\boldsymbol{\varepsilon}}^p}_{\dot\lambda\mathbf{m}=\mathbf{0}})=\mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}<0 \Leftrightarrow \tag{4.16}$$

$$\boxed{\begin{matrix} \mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}=\mathbf{m}:\dot{\boldsymbol{\sigma}}^{\text{trial}}<0 \\ \dot{\boldsymbol{\sigma}}=\mathbb{E}:\dot{\boldsymbol{\varepsilon}} \end{matrix}}$$

(see Figure 4-2):

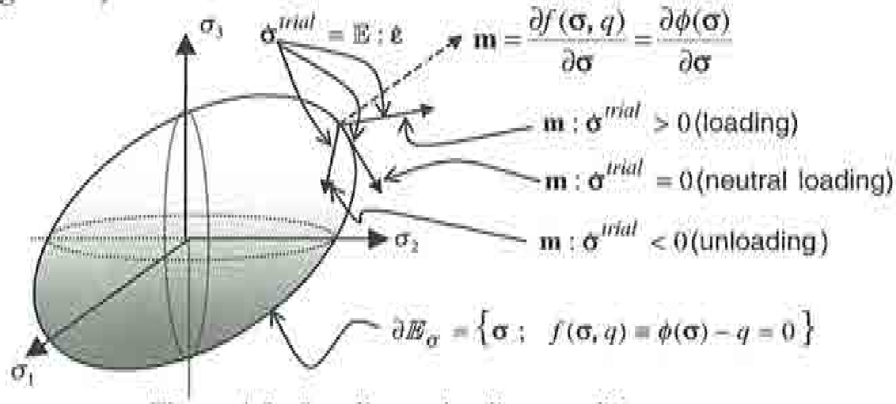


Figure 4-2– Loading-unloading conditions

4.3 Discontinuous bifurcation

4.3.1 Bifurcation scenarios

Let us consider the domain Ω of Figure 4-3 and the bifurcation of the smooth strain field $\bar{\boldsymbol{\varepsilon}}(\mathbf{x},t)$ resulting in the weak discontinuity kinematics:

$$\left.\begin{matrix} \dot{\boldsymbol{\varepsilon}}\big|_{\mathbf{x}\in\Omega/S} \overset{not}{=} \dot{\boldsymbol{\varepsilon}}_{\Omega/S}=\dot{\bar{\boldsymbol{\varepsilon}}}(\mathbf{x}) \\ \dot{\boldsymbol{\varepsilon}}\big|_{\mathbf{x}\in S} \overset{not}{=} \dot{\boldsymbol{\varepsilon}}_S=\dot{\bar{\boldsymbol{\varepsilon}}}(\mathbf{x})+\dfrac{1}{h}([\![\dot{\mathbf{u}}]\!]\otimes\mathbf{n})^s(\mathbf{x}) \end{matrix}\right\} \Rightarrow [\![\dot{\boldsymbol{\varepsilon}}]\!]=\dot{\boldsymbol{\varepsilon}}_S-\dot{\boldsymbol{\varepsilon}}_{\Omega/S}=\frac{1}{h}([\![\dot{\mathbf{u}}]\!]\otimes\mathbf{n})^s \tag{4.17}$$

which has to be compatible with the traction continuity condition:

$$[\![\dot{\mathcal{T}}]\!](\mathbf{x}) = [\![\dot{\boldsymbol{\sigma}}]\!]\cdot\mathbf{n} = \dot{\boldsymbol{\sigma}}_S\cdot\mathbf{n} - \dot{\boldsymbol{\sigma}}_{\Omega/S}\cdot\mathbf{n} = \mathbf{0} \quad \forall\mathbf{x}\in S \tag{4.18}$$

We notice from equations (4.17) and (4.18) that:

$$[\![\dot{\boldsymbol{\sigma}}]\!]:[\![\dot{\boldsymbol{\varepsilon}}]\!] = [\![\dot{\boldsymbol{\sigma}}]\!]:\frac{1}{h}([\![\dot{\mathbf{u}}]\!]\otimes\mathbf{n})^S = \frac{1}{h}\underbrace{\mathbf{n}\cdot[\![\dot{\boldsymbol{\sigma}}]\!]}_{=\mathbf{0}}\cdot[\![\dot{\mathbf{u}}]\!] = 0 \tag{4.19}$$

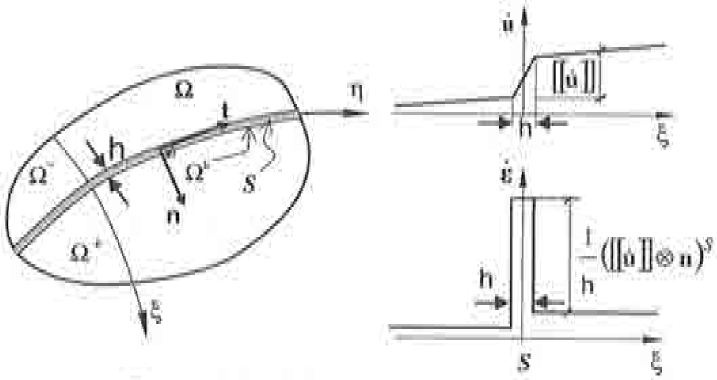


Figure 4-3– Bifurcation of the strain field

With regard to the loading-unloading conditions of Section 4.2 let us now consider the following possible scenarios for the bifurcation:

I) **L-NL at S and L-NL at Ω/S (loading–loading):**

$$\left.\begin{aligned}\dot{\boldsymbol{\sigma}}_S &= \mathbb{E}^{ep}:\dot{\boldsymbol{\varepsilon}}_S\\ \dot{\boldsymbol{\sigma}}_{\Omega/S} &= \mathbb{E}^{ep}:\dot{\boldsymbol{\varepsilon}}_{\Omega/S}\end{aligned}\right\}\Rightarrow [\![\dot{\boldsymbol{\sigma}}]\!] = \mathbb{E}^{ep}:[\![\dot{\boldsymbol{\varepsilon}}]\!] \tag{4.20}$$

From equation (4.19) and (4.20) we have:

$$0 = [\![\dot{\boldsymbol{\varepsilon}}]\!]:[\![\dot{\boldsymbol{\sigma}}]\!] = \underbrace{[\![\dot{\boldsymbol{\varepsilon}}]\!]}_{\frac{1}{h}([\![\dot{\mathbf{u}}]\!]\otimes\mathbf{n})^S}:\mathbb{E}^{ep}:\underbrace{[\![\dot{\boldsymbol{\varepsilon}}]\!]}_{\frac{1}{h}([\![\dot{\mathbf{u}}]\!]\otimes\mathbf{n})^S} = \frac{1}{h^2}[\![\dot{\mathbf{u}}]\!]\cdot\underbrace{(\mathbf{n}\cdot\mathbb{E}^{ep}\cdot\mathbf{n})}_{\mathbf{Q}^{ep}}\cdot[\![\dot{\mathbf{u}}]\!] \tag{4.21}$$

$$\Rightarrow \boxed{[\![\dot{\mathbf{u}}]\!]\cdot\mathbf{Q}^{ep}\cdot[\![\dot{\mathbf{u}}]\!] = 0} \tag{4.22}$$

where $\mathbf{Q}^{ep} = \mathbf{n}\cdot\mathbb{E}^{ep}\cdot\mathbf{n}$ is the elasto-plastic localization tensor.

II) **Loading at S and unloading in Ω/S (loading–unloading):**

$$\left.\begin{aligned}\dot{\boldsymbol{\sigma}}_S &= \mathbb{E}^{ep}:\dot{\boldsymbol{\varepsilon}}_S = \left(\mathbb{E} - \frac{\mathbb{E}:\mathbf{m}\otimes\mathbf{m}:\mathbb{E}}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}}\right):\dot{\boldsymbol{\varepsilon}}_S\\ \dot{\boldsymbol{\sigma}}_{\Omega/S} &= \mathbb{E}:\dot{\boldsymbol{\varepsilon}}_{\Omega/S}\end{aligned}\right\}\Rightarrow$$
$$[\![\dot{\boldsymbol{\sigma}}]\!] = \mathbb{E}:[\![\dot{\boldsymbol{\varepsilon}}]\!] - \frac{\mathbb{E}:\mathbf{m}\otimes\mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_S}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}} \tag{4.23}$$

From equations (4.14) and (4.16):

$$\left.\begin{array}{r}\text{Loading at } S \rightarrow \mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_S > 0 \\ \text{Unloading at } \Omega/S \rightarrow \mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_{\Omega/S} < 0\end{array}\right\}\Rightarrow$$

$$0<\mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_S < \underbrace{\mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_S - \mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_{\Omega/S}}_{\mathbf{m}:\mathbb{E}:[\![\dot{\boldsymbol{\varepsilon}}]\!]} \tag{4.24}$$

$$\boxed{0<\mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_S < \mathbf{m}:\mathbb{E}:[\![\dot{\boldsymbol{\varepsilon}}]\!]}$$

and from equations (4.19) and (4.23) taking into account equation (4.24) we have:

$$0=[\![\dot{\boldsymbol{\varepsilon}}]\!]:[\![\dot{\boldsymbol{\sigma}}]\!]=[\![\dot{\boldsymbol{\varepsilon}}]\!]:\mathbb{E}:[\![\dot{\boldsymbol{\varepsilon}}]\!]-\underbrace{[\![\dot{\boldsymbol{\varepsilon}}]\!]:\mathbb{E}:\mathbf{m}}_{>0}\frac{\overbrace{\mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_S}^{<\mathbf{m}:\mathbb{E}:[\![\dot{\boldsymbol{\varepsilon}}]\!]}}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}}>$$

$$>[\![\dot{\boldsymbol{\varepsilon}}]\!]:\mathbb{E}:[\![\dot{\boldsymbol{\varepsilon}}]\!]-[\![\dot{\boldsymbol{\varepsilon}}]\!]:\mathbb{E}:\mathbf{m}\frac{\mathbf{m}:\mathbb{E}:[\![\dot{\boldsymbol{\varepsilon}}]\!]}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}}= \tag{4.25}$$

$$=[\![\dot{\boldsymbol{\varepsilon}}]\!]:\underbrace{(\mathbb{E}-\frac{\mathbb{E}:\mathbf{m}\otimes\mathbf{m}:\mathbb{E}}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}})}_{\mathbb{E}^{ep}}:[\![\dot{\boldsymbol{\varepsilon}}]\!]=[\![\dot{\boldsymbol{\varepsilon}}]\!]:\mathbb{E}^{ep}:[\![\dot{\boldsymbol{\varepsilon}}]\!]\Rightarrow$$

$$\underbrace{[\![\dot{\boldsymbol{\varepsilon}}]\!]}_{\frac{1}{h}([\![\dot{\mathbf{u}}]\!]\otimes\mathbf{n})^s}:\mathbb{E}^{ep}:\underbrace{[\![\dot{\boldsymbol{\varepsilon}}]\!]}_{\frac{1}{h}([\![\dot{\mathbf{u}}]\!]\otimes\mathbf{n})^s}<0\Rightarrow\frac{1}{h^2}[\![\dot{\mathbf{u}}]\!]\cdot\underbrace{(\mathbf{n}\cdot\mathbb{E}^{ep}\cdot\mathbf{n})}_{\mathbf{Q}^{ep}}\cdot[\![\dot{\mathbf{u}}]\!]<0 \tag{4.26}$$

$$\Rightarrow\quad\boxed{[\![\dot{\mathbf{u}}]\!]\cdot\mathbf{Q}^{ep}\cdot[\![\dot{\mathbf{u}}]\!]<0} \tag{4.27}$$

III) Unloading at S and loading in Ω/S (unloading–loading):

$$\left.\begin{array}{l}\dot{\boldsymbol{\sigma}}_S=\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_{\Omega/S\,S} \\ \dot{\boldsymbol{\sigma}}_{\Omega/S}=\mathbb{E}^{ep}:\dot{\boldsymbol{\varepsilon}}_{\Omega/S}=\left(\mathbb{E}-\dfrac{\mathbb{E}:\mathbf{m}\otimes\mathbf{m}:\mathbb{E}}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}}\right):\dot{\boldsymbol{\varepsilon}}_{\Omega/S}\end{array}\right\}\Rightarrow \tag{4.28}$$

$$[\![\dot{\boldsymbol{\sigma}}]\!]=\mathbf{E}:[\![\dot{\boldsymbol{\varepsilon}}]\!]+\frac{\mathbb{E}:\mathbf{m}\otimes\mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_{\Omega/S}}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}}$$

From equations (4.14) and (4.28):

$$\left.\begin{array}{r}\text{Unloading at } S \rightarrow \mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_S < 0 \\ \text{Loading at } \Omega/S \rightarrow \mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_{\Omega/S} > 0\end{array}\right\}\Rightarrow$$

$$0<\mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_{\Omega/S} < \underbrace{\mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_{\Omega/S} - \mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_S}_{-\mathbf{m}:\mathbb{E}:[\![\dot{\boldsymbol{\varepsilon}}]\!]} \tag{4.29}$$

$$\boxed{0<\mathbf{m}:\mathbb{E}:\dot{\boldsymbol{\varepsilon}}_S < -\mathbf{m}:\mathbb{E}:[\![\dot{\boldsymbol{\varepsilon}}]\!]}$$

and from equations (4.19) and (4.23) taking into account equation (4.24) we have:

$$
\begin{aligned}
0 &= [\![\dot{\varepsilon}]\!] : [\![\dot{\sigma}]\!] = [\![\dot{\varepsilon}]\!] : \mathbb{E} : [\![\dot{\varepsilon}]\!] - \underbrace{(-[\![\dot{\varepsilon}]\!] : \mathbb{E} : \mathbf{m})}_{>0} \frac{\overbrace{\mathbf{m} : \mathbb{E} : \dot{\varepsilon}_S}^{< -\mathbf{m} : \mathbb{E} : [\![\dot{\varepsilon}]\!]}}{H + \mathbf{m} : \mathbb{E} : \mathbf{m}} > \\
&> [\![\dot{\varepsilon}]\!] : \mathbb{E} : [\![\dot{\varepsilon}]\!] - [\![\dot{\varepsilon}]\!] : \mathbb{E} : \mathbf{m} \frac{\mathbf{m} : \mathbb{E} : [\![\dot{\varepsilon}]\!]}{H + \mathbf{m} : \mathbb{E} : \mathbf{m}} = \\
&= [\![\dot{\varepsilon}]\!] : \underbrace{(\mathbb{E} - \frac{\mathbb{E} : \mathbf{m} \otimes \mathbf{m} : \mathbb{E}}{H + \mathbf{m} : \mathbb{E} : \mathbf{m}})}_{\mathbb{E}^{ep}} : [\![\dot{\varepsilon}]\!] = [\![\dot{\varepsilon}]\!] : \mathbb{E}^{ep} : [\![\dot{\varepsilon}]\!] \Rightarrow
\end{aligned}
\tag{4.30}
$$

$$
\underbrace{[\![\dot{\varepsilon}]\!]}_{\frac{1}{h}([\![\dot{u}]\!]\otimes\mathbf{n})^s} : \mathbb{E}^{ep} : \underbrace{[\![\dot{\varepsilon}]\!]}_{\frac{1}{h}([\![\dot{u}]\!]\otimes\mathbf{n})^s} < 0 \Rightarrow \frac{1}{h^2} [\![\dot{\mathbf{u}}]\!] \cdot \underbrace{(\mathbf{n} \cdot \mathbb{E}^{ep} \cdot \mathbf{n})}_{\mathbf{Q}^{ep}} \cdot [\![\dot{\mathbf{u}}]\!] < 0 \tag{4.31}
$$

$$
\Rightarrow \boxed{[\![\dot{\mathbf{u}}]\!] \cdot \mathbf{Q}^{ep} \cdot [\![\dot{\mathbf{u}}]\!] < 0} \tag{4.32}
$$

IV) U-NL at S and U-NL at Ω / S (unloading–unloading):

$$
\left.
\begin{aligned}
\dot{\sigma}_S &= \mathbb{E} : \dot{\varepsilon}_S \\
\dot{\sigma}_{\Omega/S} &= \mathbb{E} : \dot{\varepsilon}_{\Omega/S}
\end{aligned}
\right\} \Rightarrow [\![\dot{\sigma}]\!] = \mathbb{E} : [\![\dot{\varepsilon}]\!] \tag{4.33}
$$

and from equations (4.21) and (4.33):

$$
0 = [\![\dot{\varepsilon}]\!] : [\![\dot{\sigma}]\!] = \underbrace{[\![\dot{\varepsilon}]\!]}_{\frac{1}{h}([\![\dot{u}]\!]\otimes\mathbf{n})^s} : \mathbb{E} : \underbrace{[\![\dot{\varepsilon}]\!]}_{\frac{1}{h}([\![\dot{u}]\!]\otimes\mathbf{n})^s} = \frac{1}{h^2} [\![\dot{\mathbf{u}}]\!] \cdot \underbrace{(\mathbf{n} \cdot \mathbb{E} \cdot \mathbf{n})}_{\mathbf{Q}^e} \cdot [\![\dot{\mathbf{u}}]\!] = 0 \tag{4.34}
$$

$$
\Rightarrow \boxed{[\![\dot{\mathbf{u}}]\!] \cdot \mathbf{Q}^{e} \cdot [\![\dot{\mathbf{u}}]\!] = 0} \tag{4.35}
$$

where $\mathbf{Q}^e = \mathbf{n} \cdot \mathbb{E} \cdot \mathbf{n}$ is the elastic acoustic tensor.

Remark 4-1

From the structure of the elastic constitutive tensor $\mathbb{E} = \lambda \mathbf{1} \otimes \mathbf{1} + 2G\mathbb{I}$ the acoustic (second order) tensor results: $\mathbf{Q}^e = \lambda \mathbf{n} \otimes \mathbf{n} + 2G\mathbf{1}$ and the corresponding eigenvalues are: $\gamma_1 = \lambda + 2G > 0$ and $\gamma_2 = \gamma_3 = 2G > 0$. Consequently the acoustic tensor $\mathbf{Q}^e$ is always positive definite and, then from equation (4.35) in scenarios IV we conclude that $[\![\dot{\mathbf{u}}]\!] = \mathbf{0}$ and therefore no bifurcation would take place. This allows discarding such a scenario.

Remark 4-2

- Regarding scenarios I , and IV, in the former equations (4.22) and (4.35) implies that, at least, one eigenvalue of $\mathbf{Q}^{ep}$ is zero, whereas in scenarios II and III, equations (4.27) and (4.32) imply that, at least, one eigenvalue of $\mathbf{Q}^{ep}$ is negative.
- Since in the elastic regime all the eigenvalues of $\mathbf{Q} \equiv \mathbf{Q}^e$ are positives (see Remark 4-1), in the context of a continuous evolution of those eigenvalues along the deformation process, scenarios I (one null eigenvalue) will come first than scenarios II or III (one negative eigenvalue).
- Therefore we conclude that scenario I (loading-neutral loading at S and loading-neutral loading at Ω / S) *is the one determining the first, and therefore the actual, discontinuous bifurcation.*

4.3.2 Bifurcation equation. Determination of the normal and the bifurcation bandwidth

Once determined that the discontinuous bifurcation takes place in the loading-loading scenario, from equations (4.18) and (4.20) we get:

$$\mathbf{0} = \mathbf{n} \cdot [\![\dot{\boldsymbol{\sigma}}]\!] = \frac{1}{\mathrm{h}} \mathbf{n} \cdot (\mathbb{E}^{ep} \cdot \mathbf{n}) \cdot [\![\dot{\mathbf{u}}]\!] = \frac{1}{\mathrm{h}} \underbrace{(\mathbf{n} \cdot \mathbb{E}^{ep} \cdot \mathbf{n})}_{\mathbf{Q}^{ep}} \cdot [\![\dot{\mathbf{u}}]\!] \Rightarrow \mathbf{Q}^{ep} \cdot [\![\dot{\mathbf{u}}]\!] = \mathbf{0} \tag{4.36}$$

In order that equation (4.36) has solutions different from the trivial one ($[\![\dot{\mathbf{u}}]\!] = \mathbf{0}$) it is necessary that the localization tensor $\mathbf{Q}^{ep}$ is singular:

$$\boxed{\begin{array}{c}\text{Bifurcation equation}\\ \det(\mathbf{Q}^{ep}) = \det\left[\mathbf{n} \cdot \mathbb{E}^{ep}(H) \cdot \mathbf{n}\right] = 0\end{array}} \tag{4.37}$$

Therefore equation (4.37) is a necessary condition for the bifurcation to take place. Let us, then, consider the set $\mathbb{G}$ of values of the hardening/softening parameter

$\hat{H}$, and their associated values $\hat{\mathbf{n}}$, satisfying equation (4.37):

$$\mathbb{G} := \{\hat{H} \in \mathbb{R} \mid \exists\ \hat{\mathbf{n}} \to \det\left[\hat{\mathbf{n}} \cdot \mathbb{E}^{ep}(\hat{H}) \cdot \hat{\mathbf{n}}\right] = \mathbf{0}\} \tag{4.38}$$

In the context of a decreasing evolution of the softening parameter $H(t)$ we then define:

$$H^{crit} = \max_{\hat{H} \in \mathbb{G}} \hat{H} \qquad \mathbf{n}^{crit} \to \det\left[\mathbf{n}^{crit} \cdot \mathbb{E}^{ep}(H^{crit}) \cdot \mathbf{n}^{crit}\right] = \mathbf{0} \tag{4.39}$$

and state that bifurcation takes place, for a given material point $\mathbf{x}$, at time $t_B(\mathbf{x})$ as soon as:

$$\boxed{t_B \;:\; H(t_B) \overset{def}{=} H_B \overset{not}{=} H^{crit}(\mathbf{x}, t)\Big|_{t=t_B} \quad \to \quad \mathbf{n}(\mathbf{x})\Big|_{t \geq t_B} \overset{def}{=} \mathbf{n}^{crit}(\mathbf{x}, t)\Big|_{t=t_B}} \tag{4.40}$$

NOTE

Notice that $\mathbf{n}$ is kept fixed once computed at $t = t_B$.

Once the time if bifurcation is determined, and extending the softening regularization condition ($H = \mathrm{h}\overline{H}$) to the weak discontinuity regime, we can determine the bifurcation bandwidth in the variable bandwidth law of Figure 4-4 as:

$$H_B = \mathrm{h}_B \overline{H} \quad \Rightarrow \quad \boxed{\mathrm{h}_B = \frac{H_B}{\overline{H}}} \tag{4.41}$$

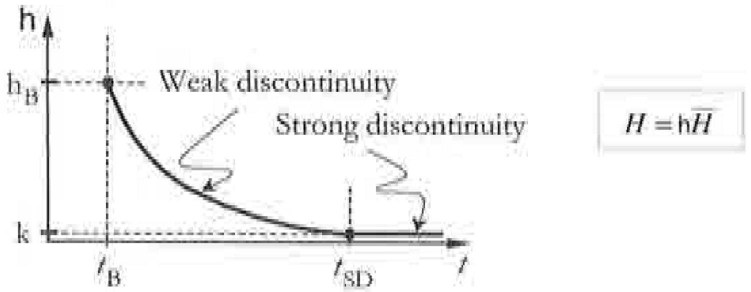


Figure 4-4– Variable bandwidth law

4.4 Resolution of the bifurcation equation

Let us consider the bifurcation equation (4.37):

Bifurcation equation

$$\det(\mathbf{Q}^{ep}) = \det\left[\mathbf{n} \cdot \mathbb{E}^{ep}(H) \cdot \mathbf{n}\right] = 0 \tag{4.42}$$

where $\mathbb{E}^{ep}$ is the elasto-plastic tangent operator (4.14):

$$\mathbb{E}^{ep} = \mathbb{E} - \frac{\mathbb{E} : \mathbf{m} \otimes \mathbf{m} : \mathbb{E}}{H + \mathbf{m} : \mathbb{E} : \mathbf{m}} \tag{4.43}$$

Substitution of (4.43) into (4.42) leads to:

$$\begin{aligned} \mathbf{Q}^{ep} &= \mathbf{n} \cdot \mathbb{E}^{ep} \cdot \mathbf{n} = \underbrace{\mathbf{n} \cdot \mathbb{E} \cdot \mathbf{n}}_{\mathbf{Q}^e} - \frac{\mathbf{n} \cdot \mathbb{E} : \mathbf{m} \otimes \overbrace{\mathbf{m} : \mathbb{E} \cdot \mathbf{n}}^{\overset{def}{=} \mathbf{e}}}{H + \mathbf{m} : \mathbb{E} : \mathbf{m}} = \\ &= \mathbf{Q}^e - \frac{1}{H + \mathbf{m} : \mathbb{E} : \mathbf{m}} \mathbf{e} \otimes \mathbf{e} \quad ; \quad \mathbf{e} = \mathbf{n} \cdot \mathbb{E} : \mathbf{m} \end{aligned} \tag{4.44}$$

where $\mathbf{Q}^e$ is the so called *acoustic tensor*:

$$\mathbf{Q}^e = \mathbf{n} \cdot \mathbb{E}^{ep} \cdot \mathbf{n} = \lambda\, \mathbf{n} \otimes \mathbf{n} + 2G\,\mathbf{1} \rightarrow \text{acoustic tensor} \tag{4.45}$$

Since $\mathbf{Q}^e$ is a positive definite second order tensor (see Remark 4-1) then $\det(\mathbf{Q}^e) > 0$, and equation (4.42) is equivalent to:

$$0 = \det(\mathbf{Q}^{e^{-1}} \cdot \mathbf{Q}^{ep}) = \underbrace{\det(\mathbf{Q}^{e^{-1}})}_{\neq 0} \det(\mathbf{Q}^{ep}) = \underbrace{\mathbf{1} - \frac{1}{H + \mathbf{m} : \mathbb{E} : \mathbf{m}} \mathbf{Q}^{e^{-1}} \cdot \mathbf{e} \otimes \mathbf{e}}_{\mathbf{B}} \tag{4.46}$$

where equation (4.44) have been considered. The eigenvalues of tensor **B** in equation (4.46) are:

> **NOTE**
> A result in tensorial algebra states that the eigenvalues of: $\mathbf{1}-\mathbf{a}\otimes\mathbf{b}$ are: $\lambda^{(1)}=\lambda^{(2)}=1$, $\lambda^{(3)}=1-\mathbf{a}\cdot\mathbf{b}$

$$\lambda_B^1=1 \quad ; \quad \lambda_B^2=1 \quad ; \quad \lambda_B^3=1-\underbrace{\frac{1}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}}}_{\geq 0}\underbrace{\mathbf{e}\cdot\mathbf{Q}^{e^{-1}}\cdot\mathbf{e}}_{\geq 0} \tag{4.47}$$

so the smaller eigenvalue is λ_B^3. Therefore equations (4.42) and (4.46) are equivalent to:

$$\lambda_B^3=1-\frac{1}{H+\mathbf{m}:\mathbb{E}:\mathbf{m}}\mathbf{e}\cdot\mathbf{Q}^{e^{-1}}\cdot\mathbf{e}=0\Rightarrow \tag{4.48}$$

$$\boxed{H(\mathbf{n})=\mathbf{e}(\mathbf{n})\cdot\mathbf{Q}^{e^{-1}}\cdot\mathbf{e}(\mathbf{n})-\mathbf{m}:\mathbb{E}:\mathbf{m}} \tag{4.49}$$

4.4.1 Geometric bifurcation condition

Lets consider the symmetric second order tensor $\mathbf{m}(\boldsymbol{\sigma})$ (the flow tensor) and the Mohr's coordinates associated to that tensor:

$$\sigma_m=\mathbf{n}\cdot\mathbf{m}\cdot\mathbf{n} \quad ; \quad \tau_m=\left((\mathbf{n}\cdot\mathbf{m})\cdot(\mathbf{n}\cdot\mathbf{m})-(\sigma_m)^2\right)^{\frac{1}{2}} \tag{4.50}$$

> **NOTE**
> Here it is considered $\mathbf{m}=\sum_1^3 m_i\,\mathbf{p}_i\otimes\mathbf{p}_i$

in such a way that the locus of all the values of (σ_m,τ_m) obtained for all the possible orientations $\mathbf{n}$, keeping the constraint $\|\mathbf{n}\|=1$, is determined by the three Mohr's circles in the $(\sigma-\tau)$ space (see Figure 4-5) in terms of the eigenvalues of $\mathbf{m}$ i.e. $m_1\geq m_2\geq m_3$:

> **NOTE**
> The Mohr's circles and their properties can be stated for *any symmetric second order tensor.*

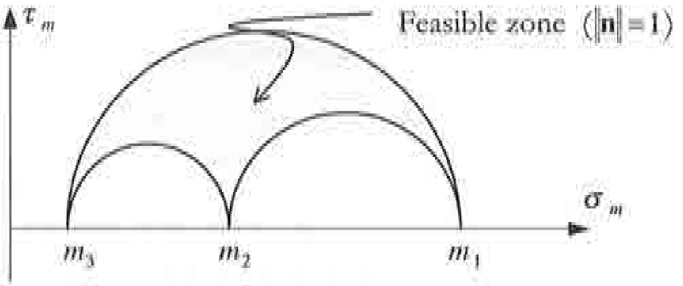


Figure 4-5– Mohr's circles

On the other hand, after some algebraic manipulation, the bifurcation condition (4.49) can be also written, in terms of the Mohr's coordinates (σ_m,τ_m) in equation (4.50), for the flow tensor $\mathbf{m}$ as:

$$\boxed{\frac{[\sigma_m-\sigma_0]^2}{A^2}+\frac{\tau_m^2}{B^2}=1} \qquad \begin{aligned} \sigma_0 &= -\frac{\nu}{1-2\nu}I_1^{(m)} \\ B^2 &= \frac{1}{4G}H+J_2^{(m)}+\frac{1+\nu}{6(1-2\nu)}\left(I_1^{(m)}\right)^2 \\ A^2 &= 2\underbrace{\frac{1-\nu}{1-2\nu}}_{\geq 2}B^2=g^2(\nu)B^2 \end{aligned} \tag{4.51}$$

where $I_1^{(m)}$ and $J_2^{(m)}$ are the corresponding first and second invariants of the flow tensor $\mathbf{m}$:

$$I_1^{(m)} \overset{def}{=} I_1(\mathbf{m}) = tr(\mathbf{m}) \quad ; \quad J_2^{(m)} \overset{def}{=} J_2(\mathbf{m}) = \frac{1}{2} dev(\mathbf{m}) : dev(\mathbf{m}) \tag{4.52}$$

Equation (4.50) determines an ellipse in the Mohr's space of center $(\sigma_0, 0)$ and semi-axes A and B (see figure (4.6)).

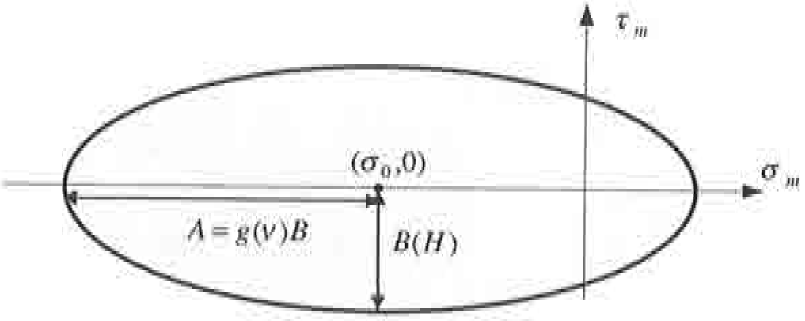


Figure 4-6– Bifurcation condition locus

Remark 4-3

- The center (σ_0) and the shape (A/B) of the ellipse are unaffected by the value of the softening parameter H.
- For a fixed value of $\mathbf{m}$ the size of the ellipse increases with the softening parameter H (parameter B).
- At the elastic regime or for unloading cases $H = \infty \Leftrightarrow \mathbb{E}^{ep} \to \mathbb{E}$ the size of the ellipse is infinite.

For a given value of the stress state $\boldsymbol{\sigma}$ and, consequently, for a fixed $\mathbf{m}(\boldsymbol{\sigma})$ we can consider the possible ellipses of Figure 4-6, containing the solutions of the bifurcation equation (4.49), and the Mohr's circle of Figure 4-5 defining the feasible space (see Figure 4-7).

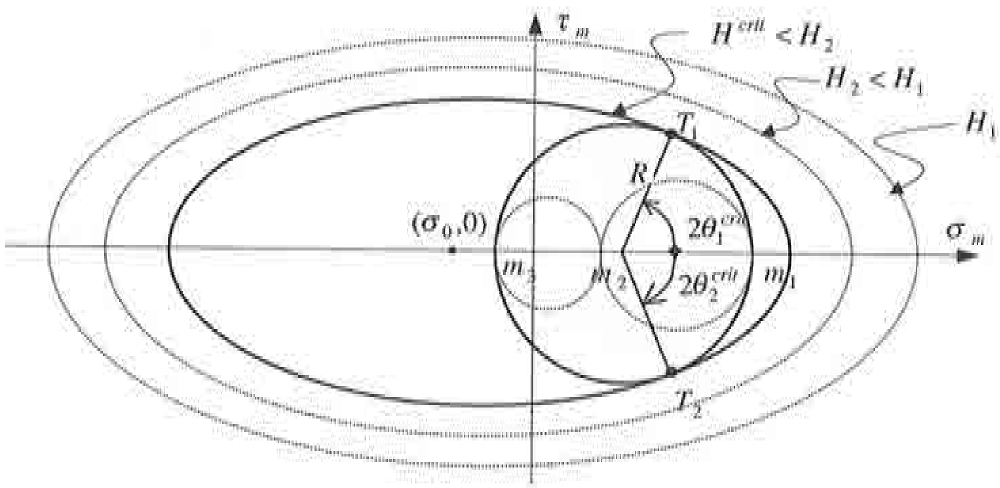


Figure 4-7– Geometrical bifurcation condition

Bifurcation is possible for those values of H whose corresponding ellipse cuts the feasible (shaded) zone of the Mohr's circles. According to equation (4.39) the largest of those is the critical softening parameter H^{crit}, whose corresponding ellipse is tangential to the largest Mohr's circle in points T_1 and T_2 of Figure 4-7. By resorting to the Mohr's circle properties, the associated values of $\mathbf{n}^{crit}$ have the following properties:

1) $\mathbf{n}^{crit}$ is *orthogonal* to the intermediate principal direction of $\mathbf{m}$ $(\rightarrow \mathbf{p}_2)$
2) The angle of $\mathbf{n}^{crit}$ *with the first principal direction* of $\mathbf{m}$ $(\rightarrow \mathbf{p}_1)$ is θ^{crit} given in Figure 4-7.

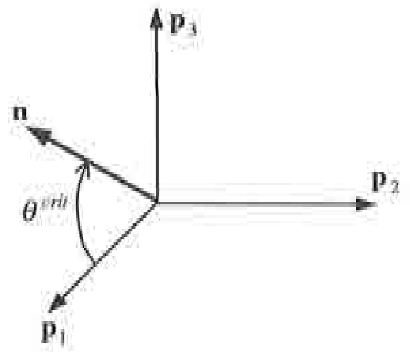


Figure 4-8– Normal $\mathbf{n}$ and the principal directions of the flow tensor $\mathbf{m}$

After some algebraic manipulation the values of H^{crit} and θ^{crit} are determined by:

$$tag^2\theta^{crit} = -\frac{m_3 + \nu\, m_2}{m_1 + \nu\, m_2} \qquad (4.53)$$

$$H^{crit} = -E\, m_2^2 \qquad (4.54)$$

Remark 4-4

Notice in Figure 4-7 that, in general, there will be two possible values of θ^{crit} and, therefore, of $\mathbf{n}^{crit}$ and two possible bifurcations can take place simultaneously. Particular cases are: $2\theta_1^{crit} = 2\theta_2^{crit} = 0 \Rightarrow \mathbf{n}_1^{crit} = \mathbf{n}_2^{crit} = \mathbf{p}_1$ and $2\theta_1^{crit} = 2\theta_2^{crit} = \pi \Rightarrow \mathbf{n}_1^{crit} = \mathbf{n}_2^{crit} = \mathbf{p}_3$.

4.5 Application to several elasto-plastic models

4.5.1 Rankine models

Uniaxial equivalent stress	$\phi(\boldsymbol{\sigma}) = \sigma_1$ $\sigma_1 \geq \sigma_2 \geq \sigma_3 \rightarrow$ principal stresses $\quad \boldsymbol{\sigma} = \sum_1^3 \sigma_i \mathbf{p}_i \otimes \mathbf{p}_i$ $\mathbf{p}_i \; i \in \{1,2,3\} \rightarrow$ eigenvalues	(4.55)
Flow tensor	$\mathbf{m} = \frac{\partial \phi(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = \frac{\partial \sigma_1}{\partial \boldsymbol{\sigma}} = \mathbf{p}_1 \otimes \mathbf{p}_1$ $[\mathbf{m}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	(4.56)
Bifurcation parameters:	$m_1 = 1 \; ; \; m_2 = m_3 = 0 \Rightarrow$ $\boxed{\theta_1^{crit} = \theta_2^{crit} = 0 \Rightarrow \mathbf{n}^{crit} = \mathbf{p}_1 \quad H^{crit} = 0}$	(4.57)

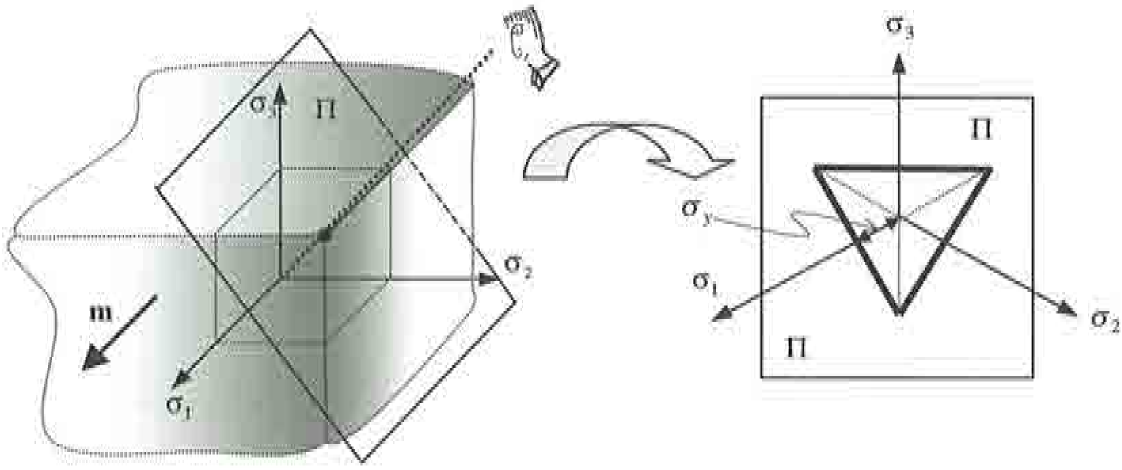


Figure 4-9– Rankine plasticity model

4.5.2 J2 plasticity models

Uniaxial equivalent stress	$\phi(\sigma)=\sqrt{\frac{3}{2}\mathbf{s}:\mathbf{s}}$; $\mathbf{s}=dev(\sigma)$	(4.58)
Flow tensor	$\mathbf{m}=\frac{\partial\phi(\sigma)}{\partial\sigma}=\sqrt{\frac{3}{2}}\frac{\mathbf{s}}{\lVert\mathbf{s}\rVert}$ $[\mathbf{m}]=\sqrt{\frac{3}{2}}\frac{1}{\lVert\mathbf{s}\rVert}\begin{bmatrix}s_1 & 0 & 0\\ 0 & s_2 & 0\\ 0 & 0 & s_3\end{bmatrix}$ $(tr(\mathbf{s})=0)$	(4.59)
Bifurcation parameters:	$m_1+m_2+m_3=0$ $tag^2\theta^{crit}=\frac{s_1+(1-\nu)s_2}{s_1+\nu s_2}$ $H^{crit}=-\frac{3E}{2}\frac{s_2^2}{s_1^2+s_2^2+s_3^2}$	(4.60)

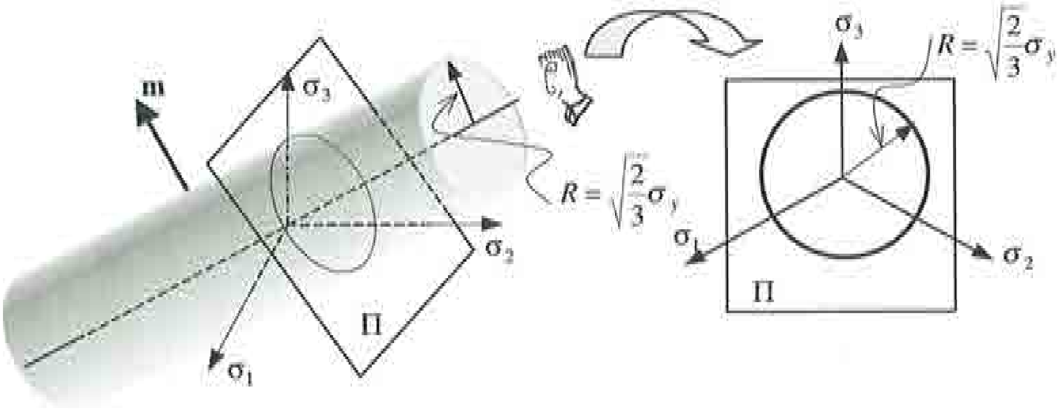


Figure 4-10– J_2 (von Mises) plasticity model

> **Remark 4-5**
> Notice that for the case of pure shear deviatoric stress state $(s_1=-s_3\,;\,s_2=0)$ we obtain $\theta^{crit}=\frac{\pi}{4}$ and $H^{crit}=0$ and one can obtain discontinuous bifurcation with perfect plasticity.

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