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ADAPTIVE REFINEMENT CRITERION FOR ELLIPTIC PROBLEMS DISCRETIZED BY FEM

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SUMMARY

In a recent paper[1] we presented a data structure to be used with multigrid techniques on non-homogeneously refined FEM meshes. This paper focuses on the adaptive refinement techniques used there. The error estimate is obtained from standard Taylor series. For each element we compute its efficiency in terms of the size, the norm of the second derivatives of the unknown and the parameter p, where L_p is the chosen norm. The way the norm influences the optimal mesh is studied. The number of elements to be refined at each step is such to produce a fast convergence to the optimal mesh, followed by successive homogeneous refinements. We hope that the analysis of these two subjects could be of value for people working with other (perhaps very dissimilar) adaptive refinement techniques (error estimate and data structure, for instance).

1. INTRODUCTION

In previous years, more efficient solutions of discretized continuum problems were searched in two ways: optimal meshing and more efficient solution of the resulting discrete equations. In our work optimal finite-element meshing is achieved through a process of adaptive refinement, and the solution is obtained at each step with an extension of the multigrid method for non-structured meshes. This subject has been dealt with in depth in a recent paper.[1]

With reference to the adaptive refinement strategy, much effort was devoted to two of the most important topics in this area, i.e. the error estimator and the type of adaptive improvement for use. To begin with the error estimator, we could make a brief list of the various kinds of estimators that exist in the literature. These are based on:

(a) resolution of local problems. The estimators within this class produce the estimates of local errors over individual elements or over a patch of elements surrounding each of the elements by defining an indicator of how much the approximation fails to satisfy the governing differential equations and boundary conditions[2–5]

(b) interpolations methods and superconvergence. These methods use the interpolation theory of finite elements in Sobolev norms to produce the estimates of the local errors over individual elements[6–10]

(c) complementary variational principle. These ones, valid for self-adjoint elliptic problems, use the duality theory of convex optimization to derive upper and lower bounds of the element errors[11]

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(d) estimates of derivatives by incremental quotients of u_h[12]
(e) residual error estimates. The value of the residual over each element is taken as a measure of the error size and is used as a refinement criterion.[13-17]

For our work we adopt an estimator that falls inside item (b). Our algorithm uses an error indicator based on the truncation error obtained by Taylor series expansions. It has several attractive features, as follows:

(1) The interpolation error is a local quantity defined for individual elements. Any local changes in mesh affect the values of the interpolation error on the elements where the changes have been made.
(2) It is very easy to implement in a finite-element code, and has low computational cost.
(3) In relation to our applications, the fact that the error indicators would be sensible to the high first derivative of the solutions is a good feature because the problems to be solved have this type of discontinuity.

On the other hand, extensions to non-elliptic problems do not produce the same results. The error indicator and element size serve to compute a parameter that plays the role of an element computational efficiency. This point will be dealt with in depth in Section 2.

Continuing with the other topics, we come to discuss the refinement type to be used. There are four principal types:

(a) h-type versions
(b) p-type versions
(c) h–p type versions
(d) node relocation or R-type.

In the first strategy, the order of polynomials used to interpolate the solution is kept constant but the mesh size varies. In the second strategy the order of polynomial approximation inside the element is variable and the element size fixed. The third strategy is based on a mixture of the two strategies above. In the fourth strategy, the location of the nodes is changed to improve the finite-element solution, keeping the number of nodes and the degree of the elements fixed.

We do not make any specific reference to the great number of papers published on this topic, but we just mention one state-of-the-art paper,[18] in which almost 200 references can be found. It is a common problem of all types of strategies to find an optimal mesh. Our work is based on the h-type strategy as part of a full multigrid method.

In our work, we define an optimal mesh as the mesh, among the class of admissible ones, for which the efficiency for all the elements should be constant and should give a minimum error for a fixed computational effort. The strategy to choose the amount of elements for refinement is discussed later on.

We decide to use h-type refinement over a mesh composed of quadrangular elements. The data structure used is similar to those appearing in References 19 and 20, with some particular aspects due to its application in a multigrid context.[1]

2. THE ERROR ESTIMATOR

We estimate the error as being given mainly by the interpolation one

$$\| u^R - u \| \leqslant \| u^R - u^I \| + \| u^I - u \| \approx \| u^I - u \| \tag{1}$$

This assertion is true in L_2 norms. Sewell used it as valid in the L_∞ context, and in this paper we use it in general L_p norms. Like Sewell, we have no theoretical foundations to do this. The L_p-norm error is calculated element by element and the local error is approximated by a truncated Taylor series:[21]

$$\| u^I - u \|_p = \left(\int_\Omega | u^I - u |^p \, d\Omega \right)^{1/p} = \left(\sum_e \int_{\Omega_e} | u^I - u |^p \, d\Omega \right)^{1/p}$$

$$| u^I(\mathbf{x}) - u(\mathbf{x}) | \approx \left| \sum_{i,j=1}^{n_d} \left(\frac{\partial^2 u}{\partial x_i \, \partial x_j} \right)_{v_k} \Delta x_i \, \Delta x_j \right| \tag{2}$$

we take v_k as the nearest node of the element containing the point $\mathbf{x}$, so that:

$$\Delta x_i \leqslant \frac{h_e}{2} \tag{3}$$

$$\begin{aligned} \left| \sum_{i,j=1}^{n_d} \left(\frac{\partial^2 u}{\partial x_i \, \partial x_j} \right)_{v_k} \Delta x_i \, \Delta x_j \right| &\leqslant \sum_{i,j=1}^{n_d} \left| \left(\frac{\partial^2 u}{\partial x_i \, \partial x_j} \right)_{v_k} \Delta x_i \, \Delta x_j \right| \\ &\leqslant \sum_{i,j=1}^{n_d} \left| \left(\frac{\partial^2 u}{\partial x_i \, \partial x_j} \right)_{v_k} \right| \frac{h^2}{4} \leqslant \tilde{\Phi}_e \frac{h^2}{4} \end{aligned} \tag{4}$$

where

$$\tilde{\Phi}_e = \max_{v_k \in \Omega_e} \left\{ \sum_{i,j=1}^{n_d} \left| \left(\frac{\partial^2 u}{\partial x_i \, \partial x_j} \right)_{v_k} \right| \right\} \tag{5}$$

Replacing (4) in (2) we obtain our error estimator:

$$\| u^R - u \|_p \approx \left(\sum_e \Phi_e^p \left(\frac{h_e}{2} \right)^{2p} \Omega_e \right)^{1/p} = E_p \tag{6}$$

where Φ_e is calculated with an expression similar to (5), replacing the derivatives of the exact solution u by

$$\left(\frac{\partial^2 u}{\partial x_i \, \partial x_j} \right)_{v_k} \approx \left(- \int_\Omega \frac{\partial N_k}{\partial x_i} \frac{\partial u^R}{\partial x_j} \, d\Omega \right) \Bigg/ \left(\int_{\Omega_B} N_k \, d\Omega \right) \tag{7}$$

Usually, one of the most important requirements to the error estimator is to detect singularities. Since our estimate is based on the second derivatives, it is evident that it will detect jump singularities (as shocks in compressible fluid dynamics), phase change and general free boundary problems, domain irregularities (as corners in fluid dynamics, elasticity, etc...), boundary data discontinuities, etc.... Even in the case where the second derivative does not exist the error estimator will get very high values in such a situation. Hence, the use of the estimator as a driving force to the feedback procedure is guaranteed.

Now we are going to discuss the use of our estimator as a stopping criterion. Starting with the definition (6) and supposing that $\Phi_{\min} > 0$ is the lower bound for the Φ_es,

$$\left(\frac{4E_p}{\Phi_{\min}} \right)^p \geqslant \sum_e h_e^{2p} \Omega_e \geqslant h_{\min}^{2p} \sum_e \Omega_e = h_{\min}^{2p} \Omega \tag{8}$$

which implies that

$$h_{\min} \leqslant \left(\frac{4E_p}{\Phi_{\min}} \right)^{1/2} \frac{1}{\Omega^{1/2p}} \leqslant C_p E_p^{1/2} \tag{9}$$

so that, if $E_p \to 0$, then $h_{\min} \to 0$ and (if convergence is verified) $u^R \to u$.

3. REFINEMENT STRATEGY

In general, a refinement strategy will give us a set of element parameters $\{\eta_e\}_{e=1}^{N_e}$ so that one refines those elements in which η_e exceeds a certain threshold value. If the refinement criterion is, for instance, to reduce the local error, one can define

$$\eta_e = \Phi_e h_e^2 \approx \max_{\mathbf{x}\in\Omega_e} |u(\mathbf{x}) - u^R(\mathbf{x})| \tag{10}$$

Another criterion could be to reduce the L_p-norm of the error over the element:

$$\eta_e = \Phi_e h_e^{2+n_d/p} \approx 4\left(\int_{\Omega_e} |u - u^R|^p \,\mathrm{d}\Omega\right)^{1/p} = 4\,\|(u - u^R)|_{\Omega_e}\|_p \tag{11}$$

One can generalize all these criteria as $\eta_e = \Phi_e h_e^m$, with $0 < m < \infty$. A small m will give a refinement extremely sensitive to variations in the second derivatives; the opposite is true for the case of a large m, so we will try to find the appropriate value of m. This will be done by optimization of the element error reduction penalized with the increase in computational work that takes place during the refinement of the particular element.[22]

The computational work is given by

$$W \approx wN_e = w\sum_e 1 \approx w\sum_e h^{-n_d}\Omega_e \approx w\int_\Omega h^{-n_d}\,\mathrm{d}\Omega = W[h(\mathbf{x})] \tag{12}$$

where h is taken as a continuous function. The error can be written also as a functional of h as:

$$E[h(\mathbf{x})] = \frac{1}{4}\left(\int_\Omega \tilde{\Phi}^p(\mathbf{x})h^{2p}(\mathbf{x})\,\mathrm{d}\Omega\right)^{1/p} \tag{13}$$

(Note that $\Phi(\mathbf{x})$ has been replaced by $\tilde{\Phi}(\mathbf{x})$). We introduce the computational work in the functional via a Lagrange multiplier λ and pose the Euler equations to find the extrema

$$\delta(E[h(\mathbf{x})] + \lambda W[h(\mathbf{x})]) = 0 \tag{14}$$

which implies

$$\frac{1}{p}\,E^{1-p}\Phi^p 2^p\left(\frac{h}{2}\right)^{2p-1}\frac{1}{2} + \lambda w(-n_d)h^{-n_d-1} = 0 \tag{15}$$

so that

$$\eta_e = \Phi_e h_e^{2+n_d/p} = \left(\frac{\lambda w n_d 2^{2p-1}}{E^{1-p}}\right)^{1/p} = \mathrm{cte} \tag{16}$$

which is the same as the second definition described at the beginning of this section; hence the optimal criterion is to reduce the L_p-norm of the error over the element. η_e can be considered as the inverse of an efficiency:

$$\eta_e^p \propto \lambda = \frac{-\delta E}{\delta W} \tag{17}$$

For the elements with a large η_e, a strong reduction $-\delta E$ of the error can be obtained through a little computational work δW, and the converse is true for small η_e.

The elements to be refined at each refinement step are those which have a low efficiency, i.e. those which satisfy

$$\eta_e > \beta\eta_{\max} \tag{18}$$

where $\eta_{\max}$ is the maximum value of η_e over all the elements and $0 < \beta < 1$.

4. THE NUMBER OF ELEMENTS TO BE REFINED

We will discuss now the choice of β or, in other words, the number of elements that must be refined at each refinement step. As mentioned above, the optimal mesh will be that one with a constant η_e. It implies an optimal mesh size $h_{\text{opt}} \propto \bar{\Phi}^{-1/m}$. Note that the optimal mesh is defined modulo a homogeneous refinement. A value of β close to 1 will produce a very slow convergence to the optimal mesh because only very few elements are refined at each step. Conversely, if β is close to 0 nearly all the elements are refined at each step and the process will not converge at all to the optimal mesh. The optimal β will be that which gives a fast convergence to a mesh with a narrow gap of ηs.

To find the optimal β we will make a simplified analysis based on the following model. Once an element with $\eta_e = \Phi_e h_e^m$ is refined, 2^{n_d} sons are created, with

$$\eta_e' = \Phi_e \left(\frac{h}{2}\right)^m = \eta_e 2^{-m} \tag{19}$$

and the other elements remain with their respective η_e unchanged.

It implicitly assumes that the Φ_es do not change very much when the elements are refined, which is likely to be so, due to the convergence of u^R to u. Let us now consider the $2^{-m} \leqslant \beta < 1$ case (see Figure 1), each curve represents the distribution of elements over η for the G^R-mesh (the mesh obtained after R refinement steps), so that the area below the curve in any interval $[\eta_a, \eta_b]$ is representative of the number of elements which have $\eta_a \leqslant \eta_e \leqslant \eta_b$ (note: the η axis is logarithmic).

In the former steps ($R < R_{\mathrm{C}}$; the expression for R_{C} is shown later), the elements in the interval $[\beta\eta_{\max}^{R-1}, \eta_{\max}^{R-1}]$ (the shadowed area in Figure 1(a)) are moved left a distance $\Delta = m \log 2$ to their position in the shadowed area of Figure 1(b). The net area added below the curve is $(2^{n_d} - 1)$ times the shadowed area in Figure 1(a). Note that $\eta_{\max}$ is moved left a distance $S = -\log \beta$ and $\eta_{\min}$ remains unchanged. This is so in the subsequent refinement steps until at a certain step R_{C} ($\mathrm{G}^{R_{\mathrm{C}}-1} \rightarrow \mathrm{G}^{R_{\mathrm{C}}}$):

$$\eta_{\min}^{R_{\mathrm{C}}} = 2^{-m}\beta\eta_{\max}^{R_{\mathrm{C}}-1} = 2^{-m}\beta^{R_{\mathrm{C}}}\eta_{\max}^{0} \leqslant \eta_{\min}^{0} \tag{20}$$

is satisfied, so that

$$R_{\mathrm{C}} = \left[\frac{1}{S}\log\left(\frac{\eta_{\max}^{0}}{2^m\eta_{\min}^{0}}\right)\right] \tag{21}$$

where [ ] denotes the integer part of its argument. In the R_{C} step the $\eta_{\min}$ value does not remain constant, and for $R > R_{\mathrm{C}}$ the situation is like that described in Figures 1(c, d): the elements in the interval $[\beta\eta_{\max}^{R-1}, \eta_{\max}^{R}]$ are moved into the interval $[2^{-m}\beta\eta_{\max}^{R-1}, 2^{-m}\eta_{\max}^{R}]$ exactly to the left of the interval whose elements remain fixed.

As a consequence, we have for $R > R_{\mathrm{C}}$:

- a constant gap: $\Delta = \log\left(\frac{\eta_{\max}^{R}}{\eta_{\min}^{R}}\right) = m \log 2$
- a constant shift: $S = -\log\left(\frac{\eta_{\max}^{R-1}}{\eta_{\max}^{R}}\right) = -\log\left(\frac{\eta_{\min}^{R+1}}{\eta_{\min}^{R}}\right) = -\log \beta$

We can see that, from all the βs with $2^{-m} \leqslant \beta < 1$, the value $\beta_{\text{opt}} = 2^{-m}$ is the best one because:

- the shift is the highest: $S = m \log 2$

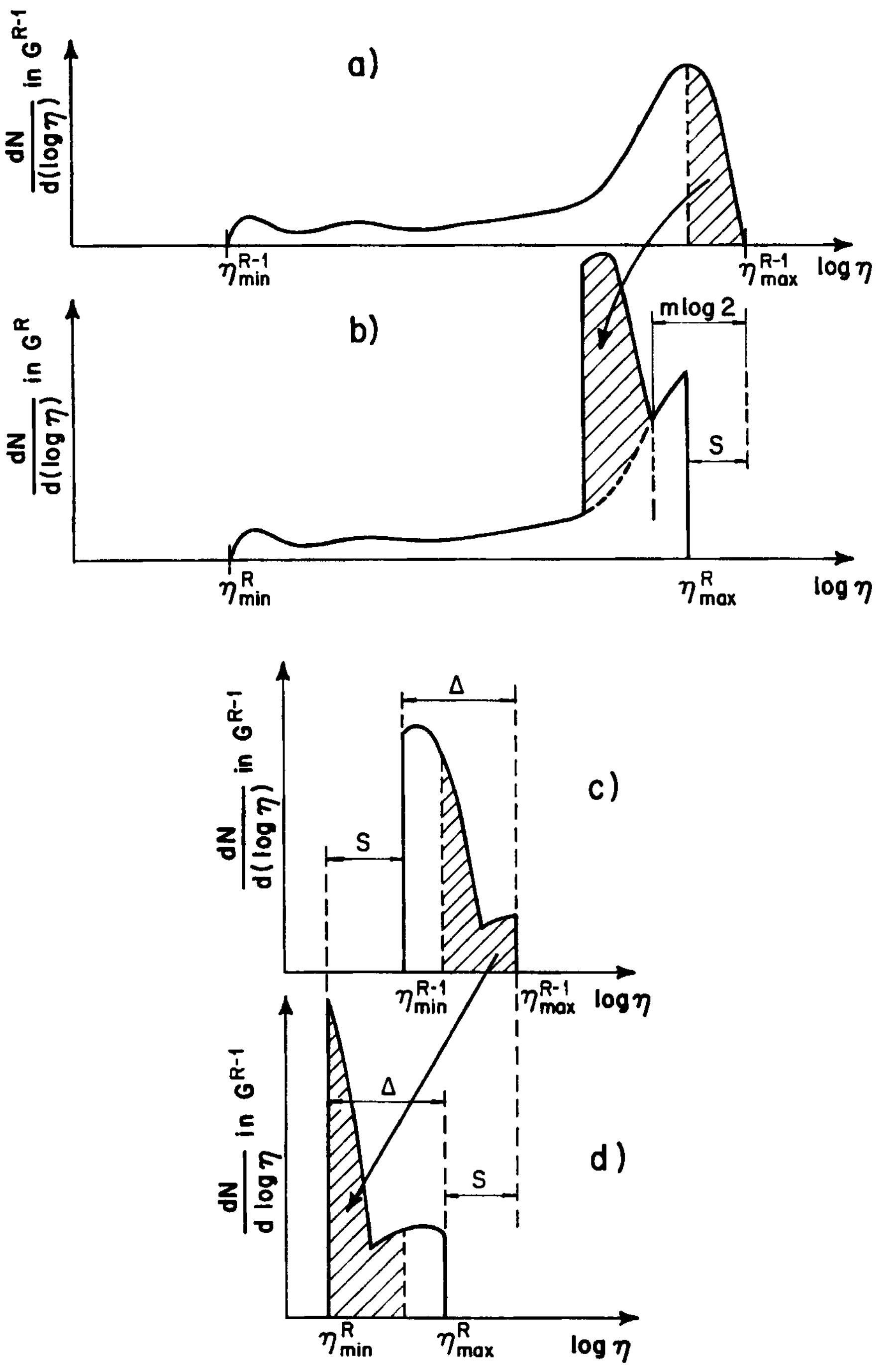


Figure 1. The $2^{-m} \leqslant \beta < 1$ case, $S = -\log \beta$, $\Delta = m \log 2$

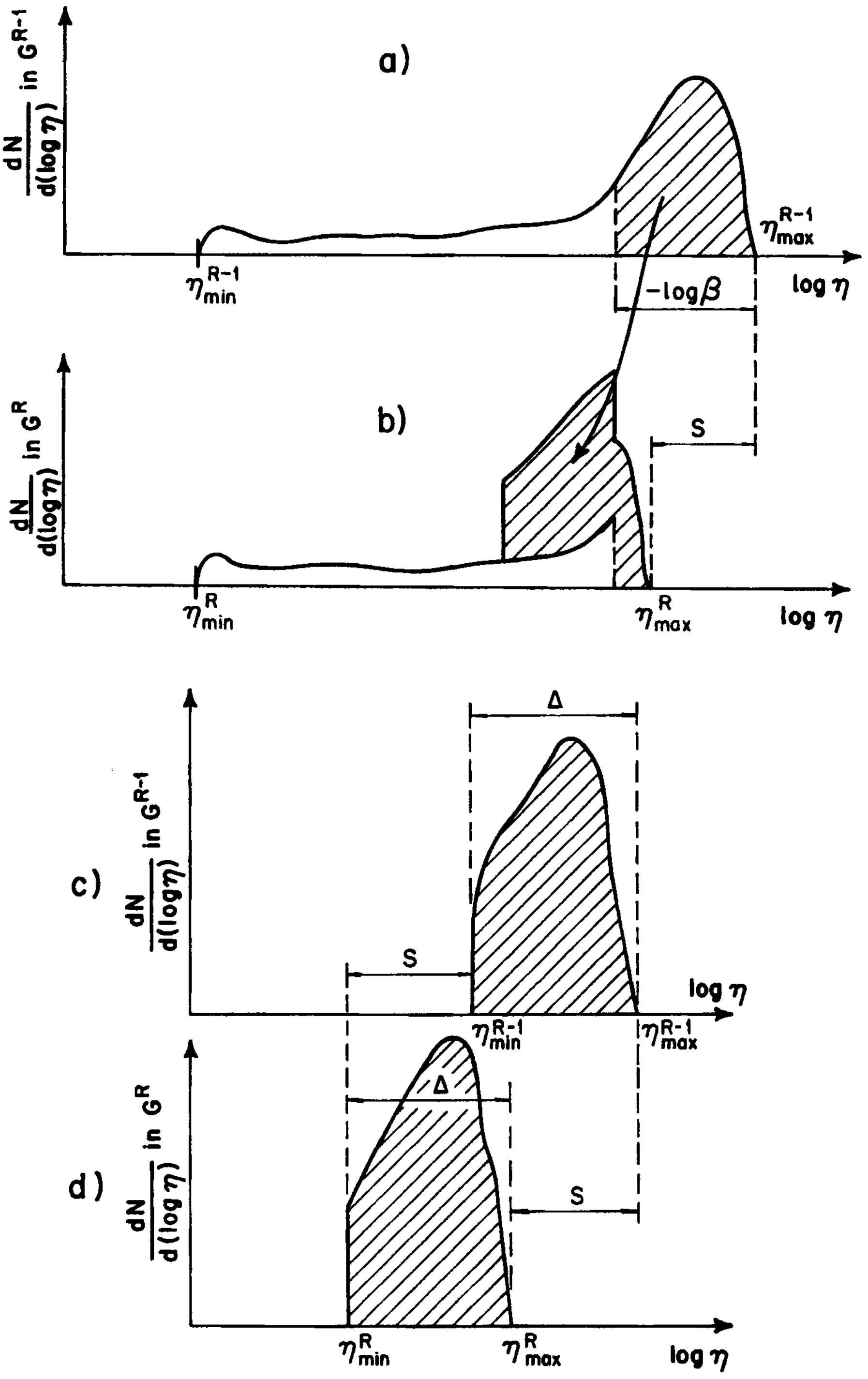


Figure 2. The $0 < \beta \leqslant 2^{-m}$ case, $S = m \log 2$, $\Delta = -\log \beta$

- the number of steps R_C to attain the best mesh is the lowest (see (21))
- the gap is the same as for all the other βs within the interval.

In the same way, it can be shown that for the $0 < \beta \leqslant 2^{-m}$ case we have (see Figure 2):

- a shift $S = m \log 2$
- a gap $\Delta = -\log \beta$ for $R \geqslant R_C$, where R_C is given by the same expression (21).

We have in this interval:

- the shift is the same for all βs
- the number of steps R_C to attain the optimal mesh is the same for all βs
- the gap is the narrowest for $\beta = \beta_{\text{opt}}$.

So the choice $\beta = \beta_{\text{opt}}$ is optimal over all $0 < \beta < 1$.

We have studied this point in detail because we think that it can be applied to other error estimators, as long as the model described above holds. At each circumstance the corresponding m value must be found by a theoretical error analysis or by a heuristic one.

5. NUMERICAL EXAMPLES

We solve the Laplace equation in the unit square with Dirichlet conditions:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = [0,1] \times [0,1] \\ u = \bar{u} & \text{in } \partial\Omega \end{cases} \tag{22}$$

where:

$$\begin{aligned} &\bar{u} = ((x+\varepsilon)^2 + y^2)^{\alpha/2} \cos\left(\alpha \tan^{-1}\left(\frac{y}{x+\varepsilon}\right)\right) = \Re\{\mathrm{f}(z)\} \\ &z = x + \mathrm{i}y, \qquad \mathrm{i} = \sqrt{-1} \\ &\mathrm{f}(z) = (z+\varepsilon)^{\alpha}, \qquad \alpha, \varepsilon \in \mathbb{R} \end{aligned} \tag{23}$$

f is (depending on the choice of α) singular on $z = -\varepsilon$. Hence, by varying α and ε, we control the irregularity of u (over Ω). In fact u is the analytical solution for potential flow around a re-entrant corner (see Figure 3). For $\alpha = -1$ the flow corresponds to that of a dipole. The norm of the derivatives is easily calculated as follows:

$$\begin{aligned} |\nabla u| &= \left|\frac{\mathrm{df}}{\mathrm{dz}}\right| = \alpha r^{\alpha-1} \\ |\nabla\nabla u| &= \sqrt{\left\{\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x\, \partial y}\right)^2\right\}} \\ &= \sqrt{2}\left|\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{dz}^2}\right| = \sqrt{2}\,|\alpha(\alpha-1)|\,r^{\alpha-2} \end{aligned} \tag{24}$$

So, using standard inequalities relating different p-norms on finite-dimension spaces,

$$\bar{\Phi} = \mathrm{C}(x,y)\,|\alpha(\alpha-1)|\,r^{\alpha-2}$$

with $\sqrt{2/3} \leqslant \mathrm{C}(x,y) \leqslant \sqrt{2}$.

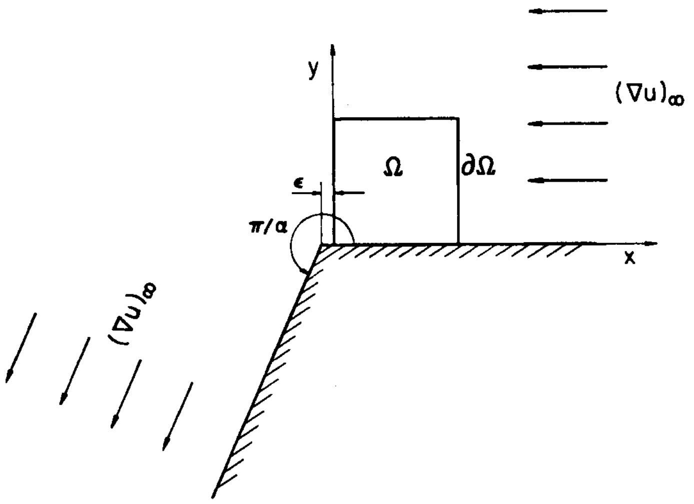


Figure 3. Test problem

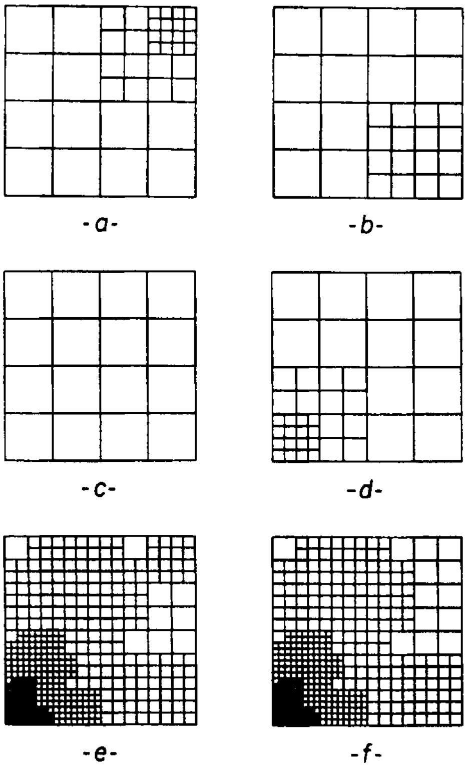


Figure 4. Mesh convergence: (a)–(d) initial meshes; (e) final mesh for mesh (a); (f) final mesh for meshes (b)–(d)

5.1. *Mesh convergence*

The first example deals with the mesh convergence of the refinement process, i.e. we expect that, starting with different initial meshes G^0, we will get similar meshes after several refinement steps.

We have set $\alpha = 0{\cdot}1$ and $\varepsilon = 10^{-3}$ and we have taken as initial meshes those indicated in Figures 4(a–d)). The corresponding final meshes are: 4(e) for 4(a) and 4(f) for 4(b–d). The Figures show a satisfactory convergence to the optimal mesh, regardless of the initial mesh chosen by the user. The final meshes shown are obtained after seven refinement steps for the (a–c) cases and after five steps in the (d) case, which is obviously the best initialization. However, the CPU times are almost the same because the most expensive steps are the final ones, where the mesh configuration is the same.

5.2. β *choice*

The problem parameters are: $\alpha = 0{\cdot}5$, $\varepsilon = 10^{-12}$ and $p = 0{\cdot}25$. In Figure 5(a) we have the initial distribution curve of η_es. The interval shown is always $[\log \eta^R_{\max} - 4, \log \eta^R_{\max}]$. We have

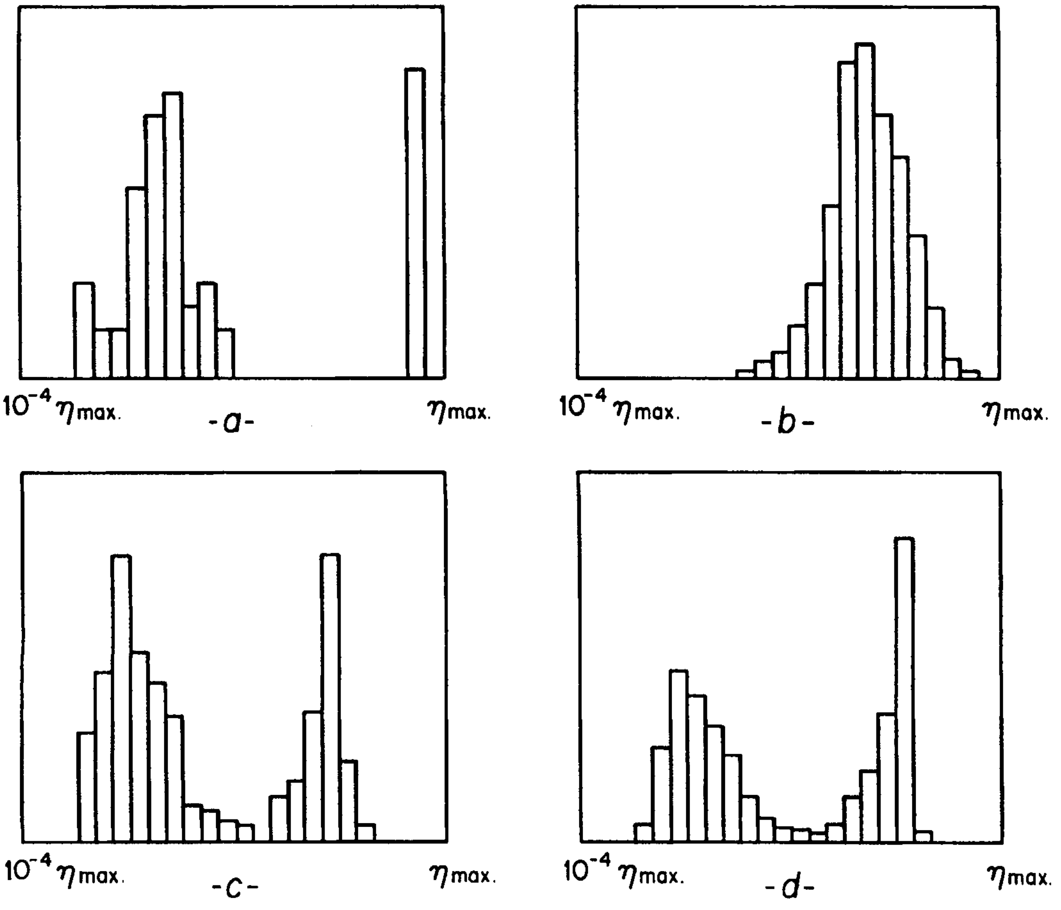


Figure 5. Choice of β parameter

run this problem with:

(i) $\beta = \beta_{\text{opt}} = 2^{-10} \approx 0{\cdot}001$. After three refinement steps we obtain the curve shown in Figure 5(b) with a gap $\Delta \approx 2$ (neglecting the tails of the curve to the right and to the left of the main peak)

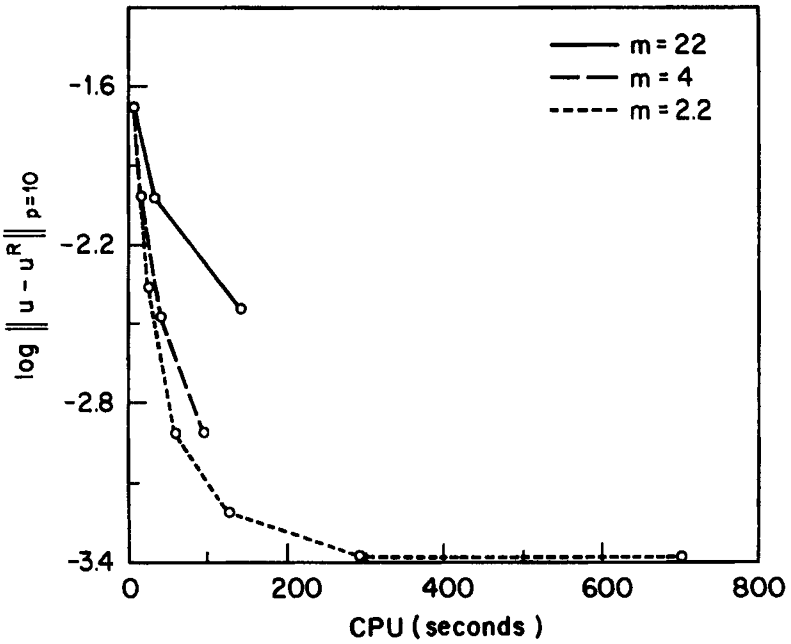
(ii) $\beta = 0{\cdot}1 \gg \beta_{\text{opt}}$. We show the curves for $R = 3$ (Figure 5(c)) and $R = 7$ (Figure 5(d)). Note that the gap is in both curves $\Delta \approx 3$. This high value for Δ is expected since we are in the case $\beta > \beta_{\text{opt}}$ and then we would need a great number of additional steps to reach the optimal mesh.

5.3. *Choice of m and p parameters*

It was shown in Section 5 that the optimal choice of m for a given norm L_p is $m = 2 + 2/p$ (in 2D). It means that if we plot the p-norm of the error $\| u^R - u \|_p$ against the CPU time for several m-values we should find that the choice $m = 2 + 2/p$ gives the fastest reduction of the error. To make this clear we will show a numerical example in which we have made nine runs with all the combinations (m_i, p_j) with:

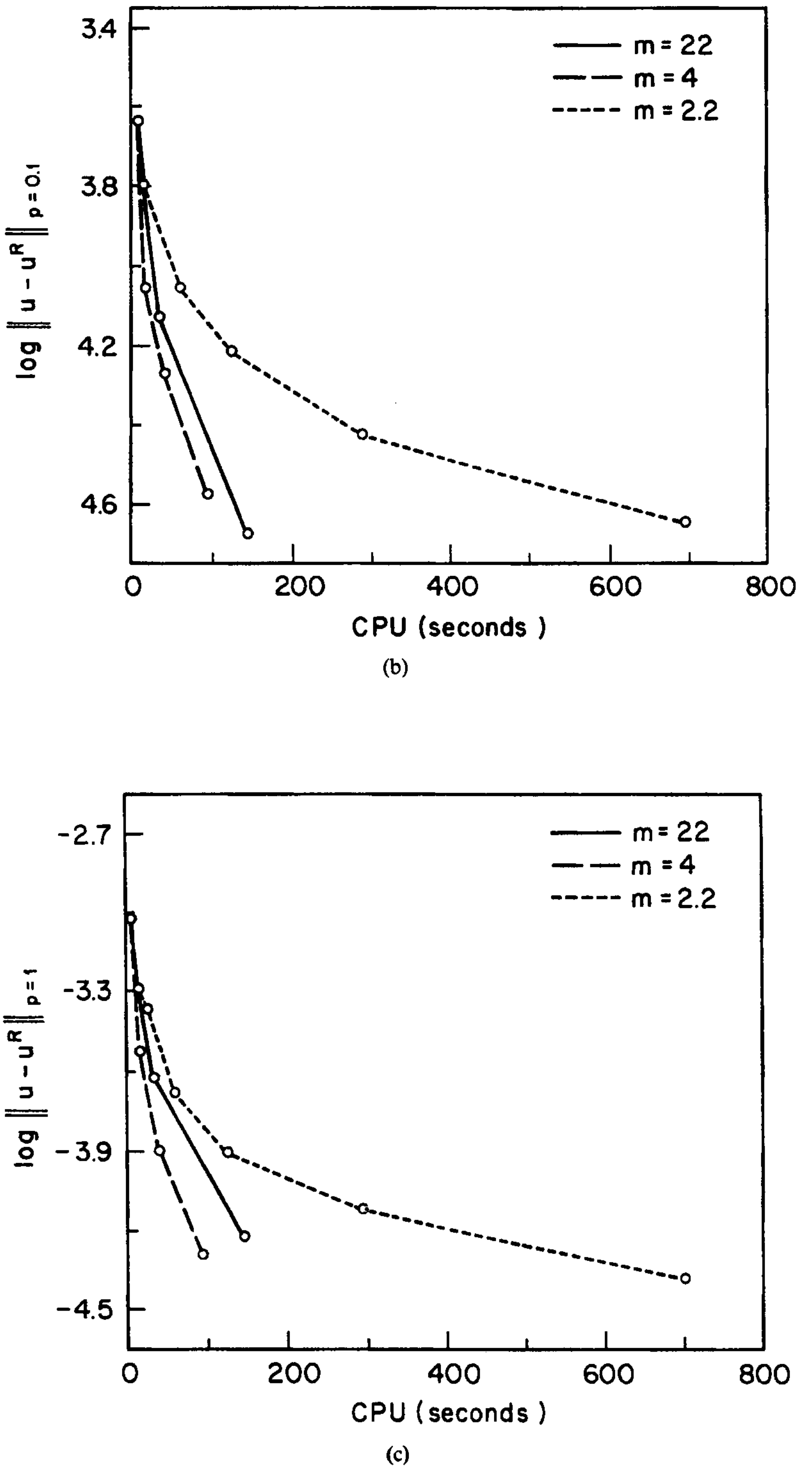
$$m_1 = 2{\cdot}2, \qquad m_2 = 4, \qquad m_3 = 22$$
$$p_1 = 10, \qquad p_2 = 1, \qquad p_3 = 0{\cdot}1$$

In Figures 6(a–c) we see the L_p-norm of the error with the three refinement criteria m_1, m_2 and m_3. The expected behaviour is found for p_1 and p_2. For p_3 (Figure 6(c)) m_2 was the optimum; however m_3 was close to it and the runs should be extended to reach a final conclusion. In



(a)

Figure 6. Choice of m and p: (a) $p_1 = 10$; (b) $p_2 = 1$; (c) $p_3 = 0{\cdot}1$



(b)

(c)

Figure 6. (*Continued*)

practice one is interested in norms L_1 and L_2. This gives m-values 4 and 3, respectively. Note that $m = 4$ was nearly optimal in all the norms $p_i = 0{\cdot}1$, 1 and 10, so that we recommend, in general, a choice of m near these values.

6. CONCLUSIONS

This study is concerned with the already popular adaptive refinement technique and highlights some important points about an optimal use of it. An estimator based on interpolation theory is used and the numerical results give evidence that its behaviour in the presence of different types of discontinuities is correct. A good way to choose the amount of elements to be refined in each step, based on the optimization of a computational efficiency, is developed. Several examples have shown an acceptable accord between the theoretical aspects and the practical ones. Here we include only the Laplacian problem. Probably, future works in this direction would lead to a better utilization of the computational resources.

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APPENDIX

Nomenclature

e = element index
G_l^R = R-step mesh 'unrefined' to level l
h = element size
L_p = norm
l = level number
m = exponent defining the parameters of η
n_d = number of spatial dimensions
R = refinement step index: $G^{R-1} \rightarrow G^R$ during step R
S = logarithmic shift at each refinement step
u = scalar valued functions over Ω
w = average resolution work by element
β = parameter defining the η threshold value for refinement; β_{opt} = optimal value
γ = maximum level
δ = first variation of a functional.
Δ = logarithmic gap, i.e. the width of the distribution curve in logarithmic scale
ε = shift from the corner in the numerical examples
η = element refinement parameters; one refines the elements with high η; η_{max}, η_{min} are the extreme values
Ω = domain.

Super/subscripts

(sub) e = element index
(supra) I = finite-element interpolation
(sub) l = level of 'unrefinement'
(sub/supra) p = parameter in the norm definition
(supra) R = level of refinement.

Special symbols

$\mathbf{x}$ = vectors in bold
$\| x \|$ = norm of a vector
$(\ )_{\mathbf{x}}$ = evaluate ( ) at $\mathbf{x}$
$[x]$ = the integer part of x

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