Tuning Curvature in Quadratic Regression via Caputo Fractional Derivatives: Theory and Applications

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INFORMATION

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Tuning Curvature in Quadratic Regression via Caputo Fractional Derivatives: Theory and Applications

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ABSTRACT

Classical regression can only examine the relation between response and predictor variables based on integer order calculus theory. What happens when non integer order calculus is considered is a field where a vast spectrum of studies can be undertaken. The purpose of this study introduces a novel fractional-order quadratic regression model grounded in the Caputo derivative framework, addressing the limitation and the rigidity of classical polynomial regression in adapting to the intrinsic curvature of data. The core innovation is the use of the fractional order ν as a tunable parameter for curvature-sensitive optimization. Our main contributions are fourfold: First, we establish a fundamental theoretical pillar by proving that the second-order Caputo derivative preserves the curvature direction of quadratic functions, enabling a principled optimization framework. Second, we rigorously demonstrate the model's robustness by proving the existence and uniqueness of solutions via Banach's fixed point theorem and establishing stability bounds through a fractional Grönwall inequality. Third, we develop a practical methodology to identify an optimal fractional order ν that minimizes the error-to-explained-variation ratio (SSE/SSR). Finally, we validate the framework on four diverse real-world datasets from air quality, soil science, education, and meteorology. The proposed model consistently outperforms classical quadratic regression, achieving a reduction in the SSE/SSR ratio by up to 21% in specific cases. The proposed method yields more efficient models with either lower estimation error or higher correlation coefficients, positioning Caputo fractional quadratic regression as a powerful and theoretically sound alternative for modeling cases where quadratic regression is considered appropriate.

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Abbreviations

SSE	Sum of Squared Errors
SSR	Sum of Squares due to Regression
SST	Total Sum of Squares

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MSE	Mean Squared Error
MAE	Mean Absolute Error
R	Correlation Coefficient
R^2	Coefficient of Determination
NMHC	Non-Methane Hydrocarbons
NO_2	Nitrogen Dioxide
CO	Carbon Monoxide
C6H6	Benzene
pH	Potential of Hydrogen (acidity/basicity measure)
Zn	Zinc
Sal	Salinity
HT	Highest Temperature
LT	Lowest Temperature

1 Introduction

Fractional calculus, a field of mathematical analysis that extends differentiation and integration to non-integer orders, has evolved from an early theoretical curiosity to a powerful framework for modeling complex systems. Originating from the historical correspondence between Leibniz and L'Hôpital on the concept of a "half derivative", fractional calculus was long regarded as abstract. In recent decades, however, it has gained significant prominence due to its ability to capture memory effects and long-range dependencies inherent in natural and engineered systems [1,2]. Unlike classical derivatives, fractional operators are intrinsically non-local, allowing them to incorporate the historical behavior of processes into their present dynamics. This property has proven essential in diverse domains, including biology, finance, engineering, epidemiology [3–7], pandemic studies [8], and atmospheric pollutant dispersion [9]. Among the various definitions, the Caputo fractional derivative has received particular attention because it permits the formulation of physically meaningful initial conditions in a way comparable to classical differential equations [10].

To date, the majority of fractional calculus applications have focused on solving differential equations, with successful implementations in areas such as anomalous diffusion, viscoelasticity, signal processing [4–6], initial value problems [11], plant disease modelling [12], thermodynamics [13], and controlling synchronization in fractional-order complex networks [14]. By contrast, its integration into statistical modeling, and particularly regression analysis, has remained limited. Regression is one of the most fundamental tools for describing relationships between variables, yet classical polynomial regression, constrained by its integer-order formulation, often struggles to adapt to the multi-scale and nonlinear patterns present in real-world data [15]. Bridging this methodological gap—by combining the flexibility of fractional calculus with the power of regression models—represents a promising but underexplored research frontier.

Recent pioneering efforts have begun to highlight this potential. Torres-Hernandez et al. [16] proposed incorporating fractional operators into polynomial regression to reduce overfitting, while Ramalho [17] presented fractional regression as a natural framework for proportional data analysis. A comprehensive review of the geometric and physical interpretation of fractional operators, are discussed in [18], Still, these studies lack a systematic theoretical foundation that rigorously addresses curvature sensitivity, existence and uniqueness of solutions, and stability analysis. Moreover, empirical validations across diverse datasets are still scarce.

This study seeks to fill this gap by developing a fractional quadratic regression model based on the Caputo derivative. In this framework, the fractional order (ν) is introduced as a novel, tunable parameter that enables curvature-sensitive optimization. The classical quadratic regression model ($\nu = 1$) naturally emerges as a special case, serving both as a benchmark and a reference point against which fractional models can be compared. This approach is particularly advantageous for processes characterized by uncertainty, self-similarity, or memory effects, offering a more flexible and realistic modeling alternative.

The main contributions of this work can be summarized as follows:

1. We prove that the second-order Caputo derivative preserves the direction of curvature, laying the foundation for a curvature-sensitive regression optimization framework.
2. We establish the theoretical robustness of the model by demonstrating the existence, uniqueness, and stability of its solutions via Banach's Fixed Point Theorem [19,20] and the fractional Grönwall inequality [21,22].
3. We propose a practical methodology for optimizing the fractional order ν and link it directly to key regression metrics such as regression error SSE, correlation coefficient R, and the ratio of error to explained part of regression SSE/SSR.
4. We validate the proposed approach using four diverse real-world datasets from air quality, soil science, education, and meteorology, showing that the fractional model can outperform the classical model by either reducing the ratio of error to explained part of regression SSE/SSR or increasing correlation.

This extended framework not only bridges the gap between fractional calculus and regression analysis but also provides scientists and data analysts with a more flexible, interpretable, and powerful tool for uncovering complex relationships in data.

Following the Introduction, the paper is organized as follows: Section 2 reviews essential mathematical preliminaries. Section 3 introduces the formulation of the fractional quadratic regression model. Section 4 presents the core curvature analysis and its geometric interpretation. Sections 5 and 6 are dedicated to the proofs of existence, uniqueness, and stability. The application to real world data is in Section 7, while the discussion of the results is provided in Section 8. Finally, the conclusions and avenues for future work are presented in Section 9.

2 Preliminaries and Mathematical Background

In this section, some fundamental concepts that are relevant to this study are given.

Definition 1 [3]: *The Riemann-Liouville fractional integral of order $q > 0$ for a function $g: [0, +\infty] \to \mathbb{R}$ is defined as*

$$\left({}_{RL}I^{q}_{0+}g\right)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s)\, ds. \tag{1}$$

Provided that the right-hand side of the integral is point-wise defined on $(0, +\infty)$ and Γ is the gamma function [23]:

$$\Gamma(\upsilon) = \int_0^\infty e^{-t} t^{\upsilon-1} dt, \forall \upsilon > 0. \tag{2}$$

Definition 2 [3,11]: *The Caputo derivative of order $q > 0$ for a function $g\colon [0, +\infty] \to \mathbb{R}$ is defined as*

$$\left({}_{C}D^{q}_{0+}g\right)(t) = \begin{cases} \int_0^t \frac{(t-s)^{n-q-1} g^{(n)}(s)}{\Gamma(n-q)} ds & , n-1 < q < n, q \in \mathbb{R}, \\ g^{(n)}(t) & , q \in N, \end{cases} \tag{3}$$

where $n = [q] + 1, [q]$ is the integer part of q.

Definition 3 [3,11]: *The two parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined as*

$$E_{\alpha,\beta}(z) : = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ z \in \mathbb{C}; \ \Re(\alpha) > 0 \tag{4}$$

If $\beta = 1$ it becomes the one parameter Mittag-Leffler function

$$E_{\alpha}(z) : = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \ z \in \mathbb{C}; \ \Re(\alpha) > 0 \tag{5}$$

It is an entire function of z with order $[\Re(\alpha)]^{-1}$.

Definition 4 [23,24]: *Asymptotic behavior of the Gamma function:*

Given the Gamma function, let $\varepsilon \in \mathbb{R}$ be such that $\varepsilon \to 0^+$ the following asymptotic expansion holds near zero: $\Gamma(\varepsilon) \sim \frac{1}{\varepsilon} - \gamma + O(\varepsilon)$, also simply written as $\Gamma(\varepsilon) \sim \frac{1}{\varepsilon}$, where $\gamma = 0.5772156\ldots$ is the Euler–Mascheroni constant.

Definition 5 [24]: *Under the classical quadratic regression model $\hat{y} = b_2 x^2 + b_1 x + b_0$ based on a data set of n tuples $(x_i, y_i)_{i=1}^n$, the following parameters are frequently used to assess the validity of a fitted regression model.*

i. the size of the error committed, also called the variation unexplained by the fitted regression model and defined as

$$SSE = \sum_{i=1}^{n} \left(y_i - \hat{y}_i\right)^2 \tag{6}$$

Eq. (6) represents the sum of the squares of the difference between the observed values and the corresponding estimated values.

ii. variation from the mean of the observed data in the response variable explained by the fitted model given as

$$SSR = \sum_{i=1}^{n} \left(\hat{y}_i - \bar{y}\right)^2 \tag{7}$$

The sum of the squares of the difference between the estimated values and the mean of the observed data is represented by Eq. (7).

iii. Total variation around the mean of the observed response variable values is given as

$$SST = \sum_{i=1}^{n} (y_i - \bar{y})^2 = SSE + SSR$$

iv. Correlation coefficient between the response and a predictor variables $R = \sqrt{SSR/SST}$.

3 Fractional Quadratic Regression Model

The Classical quadratic regression model is based on the idea of fitting a theoretical quadratic model $Y = b_2x^2 + b_1x + b_0$ for a given data set consisting of n tuples $(x_i, y_i)_{i=1}^{n}$. Due to the random nature of data the coefficients B_0, B_1, B_2 to be computed will be the estimators of true but unknown coefficients b_0, b_1, b_2. Then the estimated or fitted quadratic regression model will be $\hat{Y} = B_2x^2 + B_1x + B_0$, yielding the corresponding estimated values $\left(x_i, \hat{Y}_i\right)_{i=1}^{n}$. The process of finding the optimal fitted model is based on the minimization of the sum of the squares of the errors (SSE) (6).

Through the minimization process the optimal values for the coefficients B_0, B_1, B_2 are obtained from the following system.

$$\begin{bmatrix} \sum_{j=1}^{n} x_j^4 & \sum_{j=1}^{n} x_j^3 & \sum_{j=1}^{n} x_j^2 \\ \sum_{j=1}^{n} x_j^3 & \sum_{j=1}^{n} x_j^2 & \sum_{j=1}^{n} x_j \\ \sum_{j=1}^{n} x_j^2 & \sum_{j=1}^{n} x_j & n \end{bmatrix} \begin{bmatrix} B_2 \\ B_1 \\ B_0 \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} x_j^2 y_j \\ \sum_{j=1}^{n} x_j y_j \\ \sum_{j=1}^{n} y_j \end{bmatrix} \tag{8}$$

In a recent work [15], building on the classical approach of minimizing the sum of squared errors (SSE), the derivation of quadratic regression coefficients was undertaken within the framework of fractional calculus. The system given in Eq. (9) is developed.

$$\begin{bmatrix} \frac{1}{(2-v)} \sum_{j=1}^{n} x_j^4 & \sum_{j=1}^{n} x_j^3 & \sum_{j=1}^{n} x_j^2 \\ \sum_{j=1}^{n} x_j^3 & \frac{1}{(2-v)} \sum_{j=1}^{n} x_j^2 & \sum_{j=1}^{n} x_j \\ \sum_{j=1}^{n} x_j^2 & \sum_{j=1}^{n} x_j & \frac{1}{(2-v)} n \end{bmatrix} \begin{bmatrix} B_2 \\ B_1 \\ B_0 \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} x_j^2 y_j \\ \sum_{j=1}^{n} x_j y_j \\ \sum_{j=1}^{n} y_j \end{bmatrix} \tag{9}$$

Eq. (9) enables the computation of as many quadratic models as required by simply changing the value of the order of differentiation (v). In this study, we identified the metrics SSE/SSR and R as the most critical for assessing and comparing the performance of the fractional quadratic regression model against the classical counterpart. As detailed in Section 7.2, these metrics effectively capture the model's explanatory power and residual behavior, offering a robust basis for evaluating the advantages of the fractional approach.

Understanding the curvature sensitivity of the fractional model emerges as a crucial aspect to be investigated. These underlying patterns in the data are initially explored through the application of the Caputo derivative of non-integer order, as presented in Section 4 of the manuscript. This approach enables the detection of subtle geometric variations and long-range dependencies that are not captured by classical models.

4 Curvature Analysis of the Fractional Model

When transitioning from classical to fractional analysis of a quadratic regression model, it becomes evident that, in theory, infinitely many models can be fitted to a dataset exhibiting quadratic behavior. Among the many quadratic models that can be generated based on varying orders of differentiation—as indicated by Eq. (9)—it becomes essential to identify which model most accurately represents the data. To address this, a curvature analysis of the quadratic regression equation is performed using the Caputo derivative. The Caputo method of fractional differentiation is chosen because it enables the observation of real-world phenomena through the solutions of the corresponding fractional Caputo differential equations.

4.1 Sign of the Second Caputo Derivative

Theorem 1: *Let* $\hat{Y}(x) = B_2x^2 + B_1x + B_0$ *be a quadratic regression model fitted to a data set on an interval* $[a, b] \in R$.

(a) With $B_2 > 0$ and for any fractional order $v \in (1, 2)$, the Caputo fractional derivative of order v denoted as ${}^cD_a^v\hat{Y}(x)$ satisfies

$$ {}^cD_a^v\hat{Y}(x) > 0, \ \forall x \in (a,\ b] \quad (10) $$

(b) For any fractional order $v \in (1, 2)$, the Caputo fractional derivative of order v satisfies

$$ {}^cD_a^v\hat{Y}(x) < 0, \ \forall x \in (a,\ b] \quad (11) $$

when $B_2 < 0$.

Proof of part a: Consider the quadratic model $\hat{Y}(x) = B_2x^2 + B_1x + B_0$.

When $B_2 > 0$, the model is concave up and $\hat{Y}''(x) = 2B_2$, hence $\hat{Y}''(x)$ is constant.

The Caputo fractional derivative of order $v \in (1, 2)$ is given by

$$ {}^cD_a^v\hat{Y}(x) = \frac{1}{\Gamma(2-v)}\int_a^x (x-t)^{1-v}\,\hat{Y}''(t)\,dt \quad (12) $$

As $\hat{Y}''(x) = \hat{Y}''(t) = 2B_2$ leads to

$$ {}^cD_a^v\hat{Y}(x) = \frac{2B_2}{\Gamma(2-v)}\int_a^x (x-t)^{1-v}\,dt \quad (13) $$

Evaluating the integral $\int_a^x (x-t)^{1-v}\,dt = \frac{(x-a)^{2-v}}{2-v}$.

Thus, the Caputo fractional derivative becomes

$$ {}^cD_a^v\hat{Y}(x) = \frac{2B_2}{\Gamma(2-v)}\frac{(x-a)^{2-v}}{2-v} \quad (14) $$

Since $B_2 > 0$ is given and

$\Gamma(2-v) > 0,\ \forall v \in (1,\ 2)$

$2 - v > 0,$
$(x-a)^{2-v} > 0, \forall x > a$

it follows that ${}^cD_a^v\hat{Y}(x) > 0, \forall x \in (a, b]$. □

Proof of part b follows the same logic leading to ${}^cD_a^v\hat{Y}(x) < 0, \forall x \in (a, b]$.

4.2 Geometric Interpretation: Concave up/down Behavior

Based on Theorem 1 the following analytical interpretations can be made.

Case I. For concave-up models ($B_2 > 0$)

➢ Caputo derivative is positive, increasing with $(x-a)$, but the exponent $2-\nu$ governs the speed of growth:
- If $\nu < 1$ exponent $2-\nu > 1$: indicates faster increase of the fractional derivative function.
- If $\nu > 1 \rightarrow$ exponent $2-\nu < 1$: slower increase the fractional derivative function is observed.

Case II. For concave-down models ($B_2 < 0$)

➢ Caputo derivative is negative, and governed by the exponent $2-\nu$.

➢ In this case:

➢ For $\nu < 1$, $2-\nu > 1$, fast decline in the fractional derivative

➢ For $\nu > 1$, $2-\nu < 1$, slow decline in the derivative

However, before deciding whether a fractional quadratic model offers any advantage over the classical, one has to carefully compare the measures such as Sum of Squares of the Errors (SSE), the sum of the squares of the explained part of variation around the mean $\overline{Y}$ (SSR), Coefficient of determination ($R^2 = SSR/(SSE+SSR)$), and correlation coefficient $\left(R=\sqrt{R^2}\right)$ values between the fractional and classical methods. The ratio SSE/SSR becomes an important indicator in deciding whether the fractional regression model performs better than the classical one. This is explored under Section 7 in more detail.

4.3 Connection to Curvature-Sensitive Optimization

Theorem 2: *Let* $\hat{Y}(x) = B_2x^2 + B_1x + B_0$ *be a quadratic regression model defined on* $[a, T]$, *and let* ${}^{C}D_x^{2\nu}$ *denote the Caputo fractional derivative of order* $2\nu \in (1, 2)$. *Then,*

$$\lim_{\nu\to 1} {}^{c}D_x^{2\nu}\hat{Y}(x) = \frac{d^2\hat{Y}(x)}{dx^2}$$

That is, the Caputo fractional derivative of order 2ν converges to the classical second derivative as $\nu \rightarrow 1$.

Proof: Given $\hat{Y}(x) = B_2x^2 + B_1x + B_0$

$$\Rightarrow \frac{d^2\hat{Y}(x)}{dx^2} = 2B_2 \text{ (a constant).}$$

Now apply the Caputo defination for $1 < 2\nu < 2$. We obtain

$$\begin{aligned} {}^{C}D_x^{2\nu}\hat{Y}(x) &= \frac{1}{\Gamma(2-2\nu)}\int_0^x (x-t)^{1-2\nu}\frac{d^2\hat{Y}(t)}{dt^2}dt \\ &= \frac{1}{\Gamma(2-2\nu)}\int_0^x (x-t)^{1-2\nu}\,2B_2 dt \\ &= \frac{2B_2}{\Gamma(2-2\nu)}\int_0^x (x-t)^{1-2\nu}\,dt \\ &= \frac{2B_2}{\Gamma(2-2\nu)}\frac{x^{2-2\nu}}{2-2\nu} \end{aligned} \quad (15)$$

So we obtain

$$^{C}D_x^{2\nu}\hat{Y}(x) = \frac{2B_2}{(2-2\nu)\,\Gamma(2-2\nu)}x^{2-2\nu} \tag{16}$$

Now consider the limit as $\nu \to 1$

Let $\varepsilon = 2 - 2\nu \Rightarrow \varepsilon \to 0^+$

Keeping in mind the asymptotic expansion of $\Gamma(\varepsilon)$ as $\varepsilon \to 0^+$ and using the identity:

$$\Gamma(\varepsilon) \sim \frac{1}{\varepsilon} \text{ as } \varepsilon \to 0^+$$

$$\Rightarrow \varepsilon \cdot \Gamma(\varepsilon) \to 1$$

so that $(2-2\nu)\,\Gamma(2-2\nu) = \varepsilon\Gamma(\varepsilon) \to 1$ and $x^{2-2\nu} \to x^0 = 1$

Hence

$$\lim_{\nu\to 1} {}^{c}D_x^{2\nu}\hat{Y}(x) = 2B_2 = \frac{d^2\hat{Y}(x)}{dx^2}. \square$$

5 Existence and Uniqueness of Solutions

Existence and uniqueness of the fractional quadratic regression model is shown using the Caputo fractional derivative. This is necessary to ensure that the developed model converges to a unique solution and therefore it is reliable for implementation. The proof is developed using Banach's Fixed Point Theorem within an appropriate function space, under standard continuity [19,20] and Lipschitz conditions on the regression kernel.

The initial value problem for a nonlinear Caputo fractional differential equation of order $\nu \in (0,1)$

$$^{C}D_x^{\nu}\hat{Y}(x) = f\left(x, \hat{Y}(x)\right), \ \hat{Y}(0) = \hat{Y}_0, 0 < \nu < 1 \tag{17}$$

is considered. Here $^{C}D_x^{\nu}$ denotes the Caputo fractional derivative of order ν, and f is a nonlinear function representing the structure of the fractional regression model. Our goal is to show that the model admits a unique continuous solution on a finite interval $[0, X]$, under suitable assumptions on f.

Theorem 3: *Let $f: [0, X] \times \mathbb{R} \to \mathbb{R}$ be continuous function that satisfies a Lipschitz condition. That is, there exists a constant $L > 0$ such that*

$$|f(x, y_1) - f(x, y_2)| \leq L\,|y_1 - y_2|\,, \forall x \in [0, X]\,, y_1, y_2 \in \mathbb{R}$$

Then the initial value problem from Eq. (17)

$$^{C}D_x^{\nu}\hat{Y}(x) = f\left(x, \hat{Y}(x)\right), \ \hat{Y}(0) = \hat{Y}_0, \ 0 < \nu < 1$$

has a unique solution on $[0, X]$, provided that

$$\frac{LX^{\nu}}{\Gamma(\nu+1)} < 1$$

Proof: Beginning with the Caputo fractional differential Eq. (17) in its equivalent Volterra integral form

$$\hat{Y}(x) = \hat{Y}_0 + \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f\left(t, \hat{Y}(t)\right) dt \tag{18}$$

Define the operator T on the Banach Space $C[0, X]$ of continuous functions by

$$\left(T\hat{Y}\right)(x) := \hat{Y}_0 + \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f\left(t, \hat{Y}(t)\right) dt \tag{19}$$

By applying Banach's fixed point theorem [19,20]:

Let $M > 0$, and define the closed ball centered at $\hat{Y}_0$ by

$$B_M := \left\{\hat{Y} \in C[0, X] : \left\|\hat{Y} - \hat{Y}_0\right\|_\infty \le M\right\} \tag{20}$$

where $\hat{Y}_1, \hat{Y}_2 \in B_M$, then

$$\begin{aligned} \left\|T\hat{Y}_1 - T\hat{Y}_2\right\|_\infty &\le \frac{L}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} \left\|\hat{Y}_1 - \hat{Y}_2\right\|_\infty dt \\ &= \frac{L\left\|\hat{Y}_1 - \hat{Y}_2\right\|}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} dt \\ &= \frac{Lx^\nu}{\Gamma(\nu+1)} \left\|\hat{Y}_1 - \hat{Y}_2\right\|_\infty \end{aligned} \tag{21}$$

But it is a assumed that $\frac{Lx^\nu}{\Gamma(\nu+1)} < 1$, implying $\left\|T\hat{Y}_1 - T\hat{Y}_2\right\|_\infty \le \left\|\hat{Y}_1 - \hat{Y}_2\right\|_\infty$. That is the operator T is a contraction mapping on B_M. By Banach's fixed point theorem, T has a unique fixed point in B_M, which is a unique continuous function $\hat{Y}(x)$ satisfying the integral equation, and thus also the orginal differential Eq. (17). □

6 Stability Analysis

In a developed model it is desired that for minor variations in initial conditions, input data and/or model parmeters does not cause large or unacceptable deviations in the output. Therefore, the stability of the developed model is essential [25]. Through the stability analysis of the quadratic regression model, its robustness and reliability of the solutions is ensured. This, in turn, provides theoretical confidence in the constructed model.

In this section, we investigate the stability properties of the Caputo fractional regression model. Stability, in this context, refers to the continuous dependence of the solution on the initial condition. This is an essential theoretical property, ensuring that the regression model remains robust under small perturbations in data.

We consider the fractional differential equation:

$${}^{C}D_x^\nu \hat{Y}(x) = f\left(x, \hat{Y}(x)\right), \hat{Y}(0) = \hat{Y}_0, 0 < \nu < 1 \tag{22}$$

where ${}^{C}D_x^\nu$ denotes the Caputo fractional derivative of order ν, and f is a nonlinear function representing the structure of the fractional regression model.

The following theorem establishes a stability estimate for solutions of the above equation under standard Lipschitz assumptions.

Theorem 4: *Stability Estimate for Caputo Fractional Regression.*

Let $\hat{Y}(x)$ *and* $\tilde{Y}(x)$ *be two solutions of the fractional regression problem*

$$ {}^{C}D_{x}^{\nu}\hat{Y}(x) = f\left(x, \hat{Y}(x)\right), \text{ and } {}^{C}D_{x}^{\nu}\tilde{Y}(x) = f\left(x, \tilde{Y}(x)\right) \tag{23}$$

with different initial conditions

$\hat{Y}(0) = \hat{Y}_0$ and $\tilde{Y}(0) = \tilde{Y}_0$

Assume,

1. f is a continuous on $[0, X] \times \mathbb{R}$
2. f is Lipschitz the second variable

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|, \forall x \in [0, X], \forall y_1, y_2 \in \mathbb{R}$$

Then, the difference $E(x) := \left|\hat{Y}(x) - \tilde{Y}(x)\right|$ *satisfies the stability estimate:*

$$\left|\hat{Y}(x) - \tilde{Y}(x)\right| \leq \left|\hat{Y}_0 - \tilde{Y}_0\right|, E_{\nu}(Lx^{\nu}), \forall x \in [0, X] \tag{24}$$

where $E_{\nu}(z)$ *is the Mittag Leffler function [26] defined as* $E_{\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu k + 1)}$.

Proof: Since $\hat{Y}(x), \tilde{Y}(x)$ satisfies the Caputo fractional differantial equation, we can write

$$\hat{Y}(x) = \hat{Y}_0 + \frac{1}{\Gamma(\nu)} \int_0^x (x - t)^{\nu - 1} f\left(t, \hat{Y}(t)\right) dt \tag{25}$$

$$\tilde{Y}(x) = \tilde{Y}_0 + \frac{1}{\Gamma(\nu)} \int_0^x (x - t)^{\nu - 1} f\left(t, \tilde{Y}(t)\right) dt \tag{26}$$

subtracting Eqs. (25) from (26)

$$\hat{Y}(x) - \tilde{Y}(x) = \hat{Y}_0 - \tilde{Y}_0 + \frac{1}{\Gamma(\nu)} \int_0^x (x - t)^{\nu - 1} \left((f\left(t, \hat{Y}(t)\right) - f\left(t, \tilde{Y}(t)\right)\right) dt$$

$$\left|\hat{Y}(x) - \tilde{Y}(x)\right| \leq \left|\hat{Y}_0 - \tilde{Y}_0\right| + \frac{1}{\Gamma(\nu)} \int_0^x (x - t)^{\nu - 1} dt$$

$$\leq \left|\hat{Y}_0 - \tilde{Y}_0\right| + \frac{1}{\Gamma(\nu)} \int_0^x (x - t)^{\nu - 1} \left|\hat{Y}(t) - \tilde{Y}(t)\right| dt \tag{27}$$

Define.

$$E(x) := \left|\hat{Y}(x) - \tilde{Y}(x)\right| \tag{28}$$

Then,

$$E(x) \leq \left|\hat{Y}_0 - \tilde{Y}_0\right| + \frac{1}{\Gamma(\nu)} \int_0^x (x - t)^{\nu - 1} E(t)\, dt \tag{29}$$

This is the standard fractional Grönwall inequality. According to fractional Grönwall lemma [21,22]. □

$$E(x) \leq \left| \hat{Y}_0 - \tilde{Y}_0 \right| \cdot E_\nu (Lx^\nu) \tag{30}$$

7 Application to Different Data Sets

7.1 Data Description and Descriptive Statistics

Prior to model fitting, a descriptive analysis of each dataset was conducted to understand the scale, central tendency, and variability of the variables under investigation. This foundational step provides essential context for interpreting the regression results. The key descriptive statistics—including number of observations (n), mean, standard deviation (Std. Dev.), minimum (Min), and maximum (Max)—for the predictor (X) and response (Y) variables in each studied pair are summarized in Table 1.

Table 1: Descriptive statistics for the variables used in the fractional quadratic regression analysis

Dataset	Variable pair	Variable	Role	n	Mean	Std. Dev.	Min	Max	$\frac{Std}{Avr}$
1: Air quality [27]	NO_2 on CO	CO (mg/m^3)	Predictor (X)	33	3.31	1.64	0.40	6.40	0.49
	C_6H_6 on NO_2	NO_2 ($\mu g/m^3$)	Response (Y)	33	113.85	29.10	42.00	159.00	0.25
		NO_2 ($\mu g/m^3$)	Predictor (X)	60	171.39	53.78	48.60	264.60	0.31
	NMHC on CO	C_6H_6 ($\mu g/m^3$)	Response (Y)	60	11.31	6.19	1.53	26.81	0.55
		CO (mg/m^3)	Predictor (X)	169	2.42	1.29	0.3	6.3	0.53
2: Soil science [28]	pH on Zn	NMHC (mg/m^3)	Response (Y)	169	295.28	239.02	23	1084	0.81
		Zn (ppm)	Predictor (X)	18	20.37	6.07	9.60	31.29	0.3
	Sal on Zn	pH	Response (Y)	18	4.70	0.71	3.25	5.60	0.15
		Zn (ppm)	Predictor (X)	15	19.99	4.44	13.68	28.59	0.22
3: Education [29]	A on M	Sal (gr/kg)	Response (Y)	15	31.00	4.47	24.00	38.00	0.14
		Multiplication (M)	Predictor (X)	20	72.05	11.93	45.00	90.00	0.17
4: Meteorology [30]	HT on LT	Addition (A)	Response (Y)	20	81.85	13.63	47.00	100.00	0.17
		LT (°C)	Predictor (X)	15	19.00	5.90	8.00	29.00	0.31

Note: mg/m^3 = milligrams per cubic meter; $\mu g/m^3$ = micrograms per cubic meter; ppm = parts per million; gr/kg grams per kilogram.

The analysis reveals the diverse nature of the datasets. For instance, the Air Quality dataset (Dataset 1) exhibits a high degree of variability in pollutant concentrations, as indicated by the large standard deviations relative to the means except NO_2. In contrast, Data sets 2, 3, and 4 do not exhibit large variabilities around their means. The Soil Science data (Dataset 2), with a smaller sample size (n = 15 and 18), does not shows high variability indicating less variability. The Education data (Dataset 3) points towards a higher performance in addition compared with multiplication. Finally, the Meteorological data (Dataset 4) shows the expected strong co-movement between daily high and low temperatures. This descriptive overview confirms that the subsequent regression modeling is performed on datasets with varying scales and statistical properties, which helps in demonstrating the general applicability of the proposed fractional approach.

7.2 Criteria to Be Used Is Assessing Fitted Regression Models

Using the Caputo fractional quadratic regression approach, models obtained as a function of the order of derivative (v) will result in different values for the parameters given in Definition 5. Clearly SSE is of prime importance and will always be a minimum in classical quadratic regression. The correlation coefficient (R) between the independent and the response variables depends on both the SSE and the SSR values as

$$R^2 = 1 - SSE/SST = SSR/SST \rightarrow R = \sqrt{R^2} \tag{31}$$

where R^2 is the coefficient of determination.

Therefore, the ratio$(SSE/SSR)_{v\neq1}$ will act as an indicator for the comparison of different models generated by the fractional approach with that obtained from the classical approach $(SSE/SSR)_{v=1}$. The following becomes evident

- If

$$(SSE/SSR)_{v\neq1} < (SSE/SSR)_{v=1} \rightarrow R_{v\neq1} > R_{v=1} \tag{32}$$

- If

$$(SSE/SSR)_{v\neq1} > (SSE/SSR)_{v=1} \rightarrow R_{v\neq1} < R_{v=1} \tag{33}$$

Obtaining an $R_{v\neq1} > R_{v=1}$ at any $v < 1$ or $v > 1$ does not always mean a better fit than the classical model. The model allows for three possible scenarios

$$\begin{Bmatrix} i.\ MinSSE \\ ii.\ MaxR \\ iii.\ (SSE, R)_{Opt} \end{Bmatrix} \tag{34}$$

i) **Minimum Error Model:** This model yields the lowest Sum of Squared Errors (SSE) and is preferable when minimizing prediction error is the primary objective.

ii) **Maximum Correlation Model:** Characterized by the highest correlation coefficient, this model is ideal when maximizing the strength of association between variables is the goal.

iii) **Optimum Fractional Model:** Identified as the optimal choice, this fractional model achieves the lowest SSE while also surpassing the classical model in terms of correlation (R value).

It is important to note that the application of the Caputo quadratic regression method does not always yield all three possible cases outlined in Eq. (34). In some instances, the classical model alone may prove to be the optimal solution. Nonetheless, other combinations of the three cases may also arise, depending on the case under study.

It is therefore essential to evaluate the fractional models $\hat{Y}_{v\neq1}$ in comparison with the classical model $\hat{Y}_{v=1}$. Table 2 presents a summary of the four cases and seven models analyzed. The values in the final column are determined with reference to Eq. (34).

Table 2: Summary of fractional quadratic regression analysis conducted for the 4 different cases and 7 distinct models

Data set	Cases	Regression variables (Y on X)	Model concavity	Min SSE When $v = 1$	Max R at v	Optimum SSE & R at v
Data set 1	Case 1	NO_2 on CO	Con. down	6967.05121,	0.8620384, $v =$ 0.99999	6967.0611, 0.8620384, $v =$ 0.99999
	Case 2	C_6H_6 on NO_2	Con. up	483.106	0.905 $v = 0.94$	556.161, 0.904 $v = 0.96$
	Case 3	$NMHC$ on CO	Con. up	516975.9261	0.8619695 $v = 1$	516975.93 0.8619695 $v = 1$
Data set 2	Case 1	pH on Zn	Con. down	2.46429	0.87123 $v = 0.9995$	3.70660, 0.87123 $v = 0.9995$
	Case 2	Sal on Zn	Con. up	120.6071	0.78938 $v = 0.98$	300.837, 0.78938 $v = 0.98$
Data set 3	Case 1	Add on $Mult$	Con. down	1020.1389	0.856858 $v = 1$	1062.763, 0.856858 $v = 1$
Data set 4	Case 1	HT on LT	Con. down	34.53487	0.973399 $v = 0.99999$	34.53485, 0.973376, $v = 1.000001$

The operational methodology for applying the Caputo fractional quadratic regression model involves a systematic search for the optimal fractional order v. The process involves the following steps:

(1) For a given dataset, a range of v values centered around the classical case ($v = 1$) is established.
(2) For each candidate value of v, the system of fractional normal equations, Eq. (9) is solved to obtain the regression coefficients $b_0(v)$, $b_1(v)$, and $b_2(v)$.
(3) The resulting model $\hat{Y}(v) = b_0(v) + b_1(v)X + b_2(v)X^2$ is used to calculate the performance metrics (SSE, R, SSE/SSR).
(4) The values of these metrics are then compared across all v to identify the optimal model according to the criteria in Eq. (34).

This process demonstrates how the model operates under different 'working conditions' defined by the fractional order v. The following case studies illustrate this process in detail.

7.3 Case Studies

Out of the four datasets from which seven cases were analyzed, as summarized in Table 3, Case 2 from Dataset 1 and Case 1 from Dataset 4—representing concave-up and concave-down behavior, respectively—are discussed in detail to highlight the application of fractional quadratic regression.

Dataset 1 Case 2: This case involves applying the fractional quadratic regression model, as introduced in Section 4, to urban air pollution data. The dataset comprises 15 variables associated with traffic emissions in a specific Italian town [27]. From this, a subset of 60 data pairs focusing on benzene C_6H_6 and nitrogen dioxide NO_2 is selected, revealing a correlation coefficient of 0.887 between the two. Based on the scatter plot—where NO_2 serves as the predictor and C_6H_6 as the response variable—a classical quadratic regression model, expressed in Eq. (35) is fitted. (Fig. 1)

$$\hat{y} = 0.0004x^2 - 0.00224x + 3.01316 \tag{35}$$

Table 3: Important parameters for the fractional models where $R_{\nu\neq1} > R_{\nu=1}$

ν	SSE	SSR	SST	R^2	R	SSE/SSR
0.92	1130.969	4508.953	5639.922	0.799	0.894	0.251
0.94	703.604	3190.981	3894.585	0.819	0.905	0.220
0.96	556.161	2490.040	3046.201	0.817	0.904	0.223
0.98	507.602	2067.781	2575.383	0.803	0.896	0.245
1.00	483.106	1778.750	2261.856	0.786	0.887	0.272

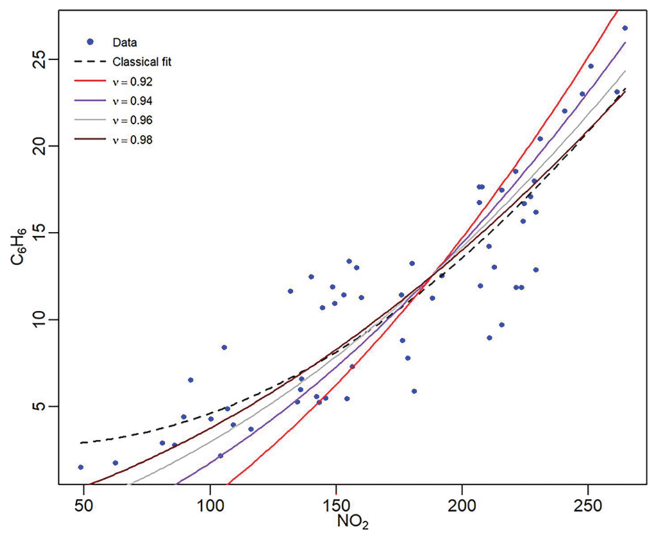

Figure 1: Comparison of selected fractional and classical regression models for C_6H_6 on NO_2

To implement the fractional regression approach, a grid search was conducted over the fractional order ν within the interval (0.9, 1.1) using a step size of 0.02. This resulted in 11 distinct evaluation points. For each value of ν the corresponding system of fractional normal equations, as defined in Eq. (9), was solved to obtain the regression coefficients. The performance of each resulting model was then assessed based on key metrics SSE, SSE/SSR, and R. Fig. 1 displays the fractional regression models identified through our grid search that meet the criteria outlined in Eq. (32), alongside the original data points and the classical case ($\nu = 1$). Notably, the model at $\nu = 0.98$ is visually very close to the classical model yet provides a statistically superior fit with a higher correlation coefficient, as quantified in Table 3.

The curvatures of selected fractional regression models, conform to the geometric interpretation described in Section 4.2. Representative curvature plots illustrating this behavior are presented in Fig. 2. This figure visualizes the curvature, as defined by the sign of the second-order Caputo derivative, for selected fractional models. It empirically validates Theorem 1, showing that all models maintain a consistent concave-up shape regardless of the fractional order ν. Furthermore, the curvature for $v > 1$ increases at a slower rate compared to the curvature at $v = 1$, whereas for $v < 1$, the curvature increases more rapidly. This behavior highlights how the sharpness of the curvature is modulated by the parameter v, forming the foundation of our curvature-sensitive optimization framework.

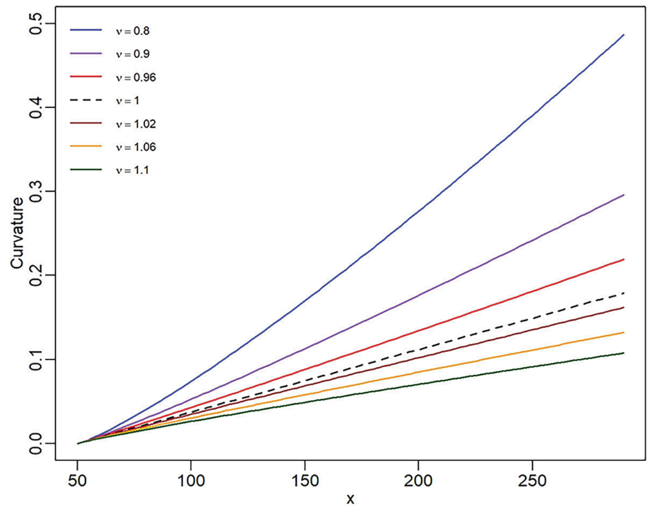

Figure 2: Curvature profiles of selected fractional regression models for the regression of C_6H_6 on NO_2

The visualization demonstrates how tuning the fractional order v adjusts the curvature of the quadratic model to better fit the data distribution. Notably, the model at $v = 0.98$ is visually very close to the classical model yet provides a statistically superior fit with a higher correlation coefficient, as quantified in Table 3.

Key parameters relevant to interpreting models that satisfy Eq. (32) are summarized in Table 3.

As shown in Table 3, the minimum SSE occurs at $v = 1$, while the highest correlation coefficient (R) is observed at $v = 0.94$. Among the fractional models where $v \neq 1$, the lowest SSE is achieved at $v = 0.98$. Therefore, based on the criteria outlined in Eq. (34), the model at $v = 0.98$ is considered optimal, as it combines a lower error amongst the $v \neq 1$ cases and with a higher R value than the classical case at $v = 1$.

Dataset 4 Case 1: Between January and July 2025, a dataset comprising 15 randomly selected daily temperature pairs, representing the highest and lowest temperatures in degrees Celsius °C, was collected from the MSN Weather Forecast page [30] for Monarga (Bogaztepe), Cyprus. Analysis revealed a strong correlation of $R = 0.9734$ between the two variables. A classical quadratic regression model was fitted to the scatter plot, with the highest temperature (HT) serving as the response variable (Y) and the lowest temperature (LT) as the predictor (X), resulting in the regression Eq. (36).

$$\hat{y} = -0.0191x^2 + 1.8283x - 0.5526 \tag{36}$$

The curvature of the Caputo fractional quadratic regression models was examined in accordance with Theorem 1, which addresses the concave-down case. Fig. 3 illustrates the graphical behavior of selected models for various v values near $v = 1$, including the classical case. All models conform to the geometric interpretation outlined in Section 4.2 for concave-down curvature. Specifically, relative to the curvature observed at $v = 1$, the curvature becomes steeper when $v < 1$, and more gradual when $v > 1$.

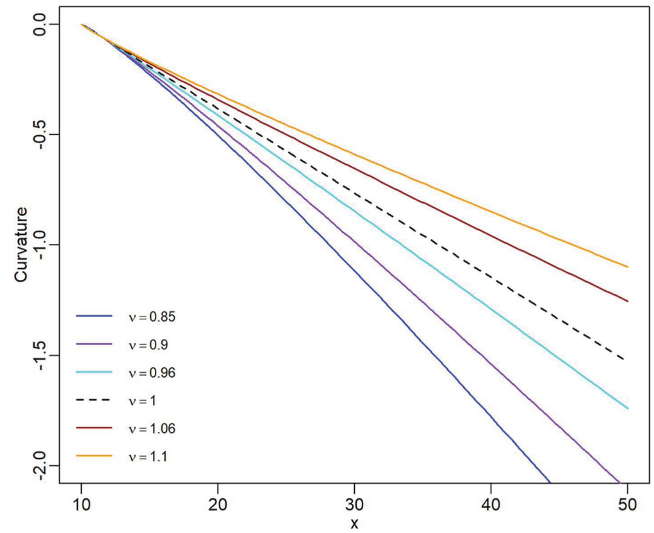

Figure 3: Curvature analysis results of selected fractional regression models for the regression of HT on LT

Fractional regression models based on the Caputo approach that satisfy the criteria outlined in Eq. (32) are presented alongside the original data points and the classical model $\nu = 1$ in Fig. 4.

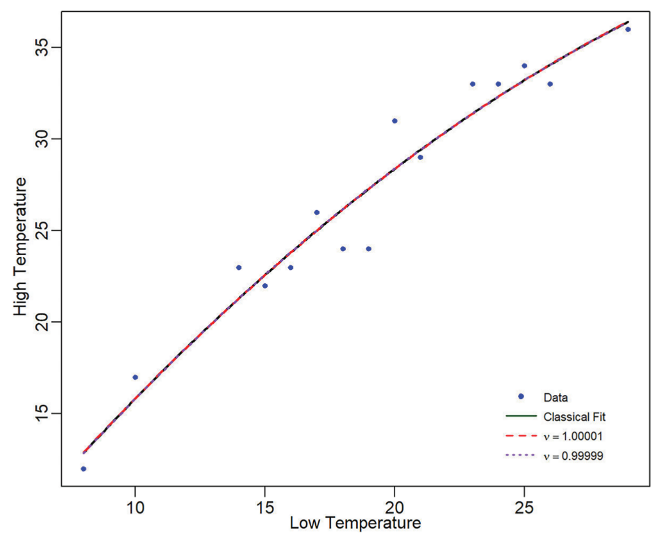

Figure 4: High-resolution comparison of models near the classical limit for HT on LT

This figure provides a detailed view of the fractional and classical models in the immediate vicinity of $\nu = 1$. Due to the scale, the curves for $\nu = 0.99999$, 1.00, and 1.000001 are nearly indistinguishable visually, yet their statistical metrics (SSE, SSR, and SSE/SSR) differ as shown in Table 4. This underscores the sensitivity of our optimization process and the fact that the absolute best-fit model can be a fractional one, even when it is extremely close to the classical solution.

Table 4 indicates that the regression model corresponding to $\nu = 1$ yields the optimal performance, satisfying the criteria for minimum error and improved correlation as outlined in Eq. (34).

Table 4: Important parameters for the fractional models when $R_{\nu\neq1} > R_{\nu=1}$, as $\nu \to 1$

ν	SSE	SSR	SST	R^2	R	SSE/SSR
0.99999	34.53596	623.3801	657.9161	0.947507	0.9734	0.055401
1	34.53487	622.6787	657.2136	0.947453	0.973372	0.055462
1.000001	34.53485	622.7735	657.3083	0.94746	0.973376	0.055453

8 Discussion

This study successfully developed and validated a novel framework for quadratic regression by integrating the principles of Caputo fractional calculus. The results presented in Section 7 demonstrate that the proposed fractional-order model is not merely a theoretical construct but a practical tool that can, under specific conditions, outperform the classical integer-order approach. The discussion that follows interprets these findings, explores their implications, and positions this work within the broader scientific landscape.

8.1 Interpretation of Key Findings and the Curvature-Sensitivity Paradigm

The core theoretical insight of this work, established in Theorem 1, is that the second-order Caputo derivative preserves the concavity of a quadratic function. This property is the cornerstone of our curvature-sensitive optimization framework. It provides a rigorous mathematical justification for using the fractional order ν as a tunable parameter to tailor the regression model to the inherent geometric structure of the data. The empirical results confirm this theory: for a given dataset, varying ν produces a family of models whose performance metrics (SSE, SSE/SSR, and R) change in a predictable and continuous manner, allowing for the selection of a model that is optimal for a specific criterion (e.g., lowest error or highest correlation).

The performance of the fractional model relative to the classical benchmark ($\nu = 1$) can be interpreted through the lens of the SSE/SSR ratio. A fractional model that achieves a lower SSE/SSR ratio for a similar R, or a higher R for a similar SSE/SSR, indicates a more efficient explanation of the variance within the data. For instance, in the C_6H_6 on NO_2 regression (Dataset 1-case 2), the optimum model at $\nu = 0.98$ achieved a significantly lower SSE/SSR ratio (0.245) compared to the classical model (0.272), suggesting that the fractional approach captured the underlying relationship with less error relative to the explained variation. This finding is critical as it moves beyond a simple goodness-of-fit metric and provides a relative measure of model efficiency.

The success of the fractional model in certain cases can be attributed to the inherent properties of fractional derivatives. Unlike integer-order derivatives, which are local operators, fractional derivatives are non-local and capture memory effects and long-range dependencies within the data. For instance, the better performance of the Caputo fractional model for C_6H_6 on NO_2 regression could be interpreted as the system having a “memory”—where past levels of NO_2 have a lingering, non-instantaneous effect on benzene formation, a phenomenon that the classical, memoryless model cannot capture. Conversely, in cases where the classical model was optimal (e.g., NMHC on CO), the relationships appear to be more immediate and local, well-described by standard calculus. This provides a physical and intuitive explanation for when and why the fractional model offers an advantage.

8.2 Comparative Analysis with Advanced Modeling Techniques

The proposed fractional regression framework offers a distinct advantage when compared to other advanced modelling techniques for handling complexity, such as deep neural networks (DNNs) [31] or fractal fractional operators [32]. While DNNs are powerful black-box tools capable of modelling extremely non-linear and high dimensional relationships, they often require large data sets and significant computational resources, while their internal mechanisms can be opaque. In contrast our fractional quadratic model is parsimonious and interpretable. The fractional order v provides a single, intuitive parameter whose adjustment has a clear geometric meaning (curvature tuning), making the model's behavior easier to understand and justify for physical or natural systems where interpretability is key.

Similarly, while fractal-fractional operators [32] introduce even more degrees of freedom for capturing complex dynamics, our use of the standard Caputo derivative demonstrates that substantial improvements can be achieved within a simpler and well established mathematical framework. This work shows that significant value lies in enhancing foundational statistical models (like quadratic regression) with fractional calculus, providing a middle ground between classical models and highly complex modern techniques.

8.3 Implications and Applicability

The ability of the fractional model to outperform the classical one in specific cases (e.g., Dataset 1 Case 2 and Dataset 4 Case 1) has profound implications. It suggests that many natural and engineered systems traditionally modeled with integer-order calculus may exhibit hidden fractional-order dynamics. The improvement in fit could be attributed to the fractional derivative's ability to better capture memory effects, long-range dependencies, or anomalous diffusion processes that are not accounted for in classical regression. This makes the framework particularly relevant for fields like environmental science (e.g., pollutant dispersion), thermodynamics (e.g., temperature equilibration), and physiology (e.g., growth patterns), where such phenomena are common.

Furthermore, the theoretical guarantees provided by Theorems 3 and 4 (existence, uniqueness and stability) are not merely mathematical formalities. They ensure that the fractional regression model is robust, reliable, and that its solutions are continuous and depend predictably on the initial data. This mathematical rigor is essential for building confidence in the model's predictions and for its future application in sensitive domains.

8.4 On the Optimality of the Classical Model

It is important to contextualize the results where the classical model remained optimal (e.g., NMHC on CO in Dataset 1, Add on Mult in Dataset 3). This outcome is not a failure of the fractional approach but rather an important result itself. It indicates that for those specific variable pairs, the integer-order calculus sufficiently describes the relationship. The fractional framework gracefully includes the classical model as a special case ($\nu = 1$) and provides a systematic method to test whether a more complex fractional description is warranted. The choice between models can thus be guided by objective criteria (e.g., SSE/SSR, R) within the proposed framework, moving the selection process from *ad hoc* to principled.

In conclusion, the Caputo fractional quadratic regression model presented herein represents a meaningful advancement in regression analysis. It combines a solid theoretical foundation with practical utility, offering a new, interpretable, and powerful tool for data analysts and scientists seeking to extract the most accurate and meaningful relationships from their data.

9 Conclusion and Future Work

9.1 Conclusion

This study has successfully established a comprehensive and theoretically robust framework for Caputo fractional quadratic regression. We have rigorously proven that the model preserves the intrinsic curvature direction of data, providing a geometrically interpretable foundation. Furthermore, we furnished essential theoretical guarantees by demonstrating the existence, uniqueness, and stability of the solutions, ensuring the model's reliability. The central innovation of this work is the introduction of the fractional order ν as a tunable parameter for curvature-sensitive optimization, moving beyond the rigid structure of classical integer-order calculus.

The practical efficacy of the proposed framework was empirically validated across four distinct real-world datasets. The key results, summarizing the performance of the classical model against the best-performing fractional model for each variable pair, are presented in Table 5 below. The percentage improvement in the critical SSE/SSR ratio is used as the primary metric for comparison.

Table 5: Summary of key results: classical vs. optimal fractional model performance

Dataset	Variable pair (Y on X)	Classical model ($\nu = 1$)	Optimal fractional model	% Improvement in SSE/SSR	Best case scenario
Air quality	NO_2 on CO	SSE/SSR = 0.3459	$\nu = 0.99999$, SSE/SSR = 0.3457	0.0006%	Opt. fract. model marginally better
Air quality	C_6H_6 on NO_2	SSE/SSR = 0.272	$\nu = 0.98$, SSE/SSR = 0.245	9.9%	Opt. fract. model better
Air quality	NMHC on CO	SSE/SSR = 0.0569	$\nu = 1.00$, SSE/SSR = 0.0569	0.0%	Classical Model is Optimal
Soil science	pH on Zn	SSE/SSR = 0.40196	$\nu = 0.9995$, SSE/SSR = 0.31744	21%	Opt. fract. case better
Soil science	Sal on Zn	SSE/SSR = 0.7567	$\nu = 0.98$, SSE/SSR = 0.6048	20%	Opt. fract. case better
Education	Add on Mult	SSE/SSR = 0.362	$\nu = 1.00$, SSE/SSR = 0.362	0.0%	Classical Model is Optimal
Meteorology	HT on LT	SSE/SSR = 0.05546	$\nu = 1.000001$, SSE/SSR = 0.05540	0.001%	Opt. fract. model marginally better

The results presented in Table 5 underscore the improvements the fractional framework can offer. Notably, the pH on Zn regression exhibited a 21% reduction in the SSE/SSR ratio when $\nu = 0.98$, indicating a model with a stronger correlation coefficient and improved fit. Among the seven regression models evaluated, only two cases—NMHC on CO and Add on Mult—showed no improvement, correctly identifying the classical model as optimal with 0% gain in SSE/SSR.

These results conclusively demonstrate that the Caputo fractional quadratic regression model is a powerful, interpretable, and theoretically sound alternative for data analysis, capable of uncovering more efficient representations of complex relationships where classical models do not offer this facility.

9.2 Limitations and Future Work

Despite the promising results, this work has certain limitations that pave the way for future research.

9.2.1 Computational Search

The current method for finding the optimal fractional order ν involves a grid search, which can be computationally expensive for very large datasets. Future work will focus on developing efficient optimization algorithms, such as gradient-based or metaheuristic methods, to automate and accelerate this process.

9.2.2 Model Scope

The present framework is confined to bivariate quadratic regression. A significant and logical extension is to develop a multivariate fractional polynomial regression model. This would involve formulating and solving systems of fractional normal equations for multiple predictor variables.

9.2.3 Operator Exploration

This study focused on the Caputo derivative due to its advantageous properties for initial value problems. Future investigations could explore the performance and theoretical implications of using other fractional operators, such as the Riemann–Liouville, Atangana–Baleanu, or tempered fractional derivatives, within this regression framework.

9.2.4 Theoretical Extensions

Further theoretical work could explore the asymptotic properties of the fractional regression coefficients and develop hypothesis tests for the significance of the fractional order ν.

By addressing these limitations, the potential of fractional calculus in regression analysis can be further unlocked, leading to more sophisticated and accurate tools for statistical modeling and data analysis.

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Availability of Data and Materials: The datasets analyzed during the current study are publicly available and cited within the article as references [27–30].

Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest to report regarding the present study.

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