

Bifurcation Analysis and Dynamical Investigation of Nonlinear Diffusion-Reaction Equations with Nonlinear Convective Flux Term: Stability and Applications

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ABSTRACT

The aim of this work is to examine the rich dynamics of quadratic and quartic nonlinear diffusion-reaction (DR) equations with a nonlinear convective flux term. These equations are crucial for simulating a variety of biological and physical processes, such as the dynamics of species populations. The main goal is to use the modified extended simple equation method (mESEM), a generalization of the standard simple equation method that hasn't been used in this situation before, to extend the analytical treatment of such equations. We obtain a variety of new exact solutions using this method, such as breathers, kink and anti-kink waves, multi-peak solitons, bright-dark solitons, periodic waves, and waveforms represented by hyperbolic, trigonometric, and rational functions. Analyzing the stability and physical relevance of these solutions is another major goal of this work. Modulational instability analysis verifies the robustness of the obtained waveforms, while bifurcation analysis reveals qualitative changes in system behavior under parameter variations. The various wave structures and their dynamical characteristics are further highlighted with graphic illustrations. In general, the study highlights the potential of mESEM to reveal rich wave phenomena with applications spanning fluid dynamics, plasma physics, chemical reaction processes, population biology, neuroscience, and optical fiber communication, in addition to showcasing its effectiveness and versatility in solving nonlinear DR equations.

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1 Introduction

Nonlinear wave equations hold considerable importance across various disciplines in both pure and applied sciences, such as solid-state physics, fluid dynamics, plasma physics, population modeling, chemical reactions, nonlinear optics, and the transmission of fluxons in Josephson junctions, among others [1–4]. In scientific investigations, complex systems are typically described using mathematical models, which are frequently nonlinear in nature [5–8]. Determining precise analytical solutions to these nonlinear models is a crucial next step that helps researchers better comprehend, regulate, and measure the fundamental behaviors of the systems [9–12].

In recent decades, significant advancements have been achieved in developing a wide range of effective techniques for finding exact analytical solutions to nonlinear evolution equations (NLEEs) [13,14]. To derive exact results of NLEEs, a variety of analytical schemes have been proposed and documented in the literature [15–17]. The Kudryashov approaches [18], the homogeneous balance scheme [19], the tanh-coth method [20], quasi-linear and monotone iterative techniques [21] and the auxiliary equation method [22] are a few of these. Other noteworthy techniques include the extended (G'/G) -expansion method [23], the variational scheme [24], the unified approach [25] and other techniques in [26,27]. Furthermore, the bilinear neural network scheme [28], the enhanced expansion technique [29], the Reduced differential transform technique [30], and the Jacobian elliptic function method [31] have also been investigated by researchers. The enhanced Adomian decomposition method [32], the homotopy analysis method [33], the Painlevé analysis method [34], and a number of others [35,36] are examples of additional advancements in the construction of approximate solutions. The SEM is a relatively new and improved technique for effectively solving nonlinear wave equations [37]. Since then, many researchers have embraced this approach to investigate a variety of nonlinear phenomena [38]. Moreover, researchers have also investigated other related techniques [39–42].

Although reaction-diffusion systems are naturally found in chemistry, they are also frequently used to characterize dynamical processes that are not chemical in nature. Applications can be found in ecology, physics, geology, biology, and sociology. These systems are usually represented mathematically as semi-linear parabolic partial differential equations, which model different spatial and spatiotemporal patterns in domains like morphogenesis, chemical reactions, predator-prey interactions, ecological systems, nerve fiber dynamics, and flame propagation [43,44]. The DR model in its most basic form [45] is provided by

$$\mathcal{V}_t = r\mathcal{V} \left(1 - \frac{\mathcal{V}}{\kappa} \right) + D\mathcal{V}_{xx}, \quad (1)$$

where D is the diffusion coefficient, κ is the environment's carrying capacity, and r is the linear growth rate. With the addition of linear diffusion for population dispersal, this equation is a logical progression of the logistic growth model. By changing the parameters to $r = \alpha$ and $\frac{r}{\kappa} = \beta$, above equation reduces to

$$\mathcal{V}_t = D\mathcal{V}_{xx} + \alpha\mathcal{V} - \beta^2\mathcal{V}. \quad (2)$$

We will go into more detail about the mathematical and physical aspects of this equation in the next section because of its broad applicability in natural phenomena. Whether a linear or nonlinear form of the equation is appropriate depends on the particular phenomenon being studied in one or more dimensions.

This study focuses on nonlinear diffusion-reaction (DR) equations, which are commonly used to model phenomena in physical, chemical, and biological systems. DR equations are the generalized

form of Eq. (2). Over time, many simplified forms of these nonlinear DR equations have been explored. For instance, Triki et al. [46] analyzed three versions of nonlinear DR equations featuring both derivative-based and algebraic non-linearities, along with long and short range diffusion components in inhomogeneous environments, employing the auxiliary scheme. The researchers in [47] derived various types of solutions such as periodic, double-kink, bell-shaped, and anti-kink for the cubic-quintic nonlinear DR equation having with coefficients of variable convection. In a more recent work [48], the authors investigated a specific case of the DR equation where the diffusion coefficient D was assumed to be independent of concentration or density. However, in many real-world scenarios, such as models describing insect movement or the dispersal of small rodents, the diffusion coefficient D becomes dependent on population density [2,49]. In such cases, an increase in species concentration leads to a corresponding rise in the diffusion rate.

Bifurcation analysis [50,51] in the context of reaction-diffusion equations is a powerful mathematical tool used to investigate how the qualitative behavior of solutions changes as system parameters vary. This analysis helps identify critical points at which a small change in a parameter can lead to a sudden shift in the system's dynamics, such as the emergence of new steady states or periodic solutions. In reaction-diffusion systems, bifurcation phenomena often explain the formation of spatial patterns or temporal oscillations observed in biological, chemical, or ecological processes. Techniques such as linear stability analysis and numerical continuation are commonly employed to trace these bifurcations and understand the transitions between different dynamical regimes [52].

This study aims to improve the precision of soliton solutions for nonlinear DR equations by utilizing a novel extended technique. The method, known as the modified extended Simple Equation Method (mESEM), is relatively new and has not yet been applied to DR equations that incorporate nonlinear convective flux terms. Nevertheless, mESEM has proven to be reliability and efficient in resolving a variety of nonlinear wave equations in disciplines like applied mathematics, physics, and engineering. Rouge waves, complex solitary waves, singular waves, kink and anti-kink profiles, and periodic-singular solutions are among the wave structures that are revealed by this method. Furthermore, rational solutions emerge naturally throughout the analytical procedure. While modulation instability analysis is used to evaluate the model's stability, bifurcation analysis is used to investigate the qualitative behavior of the DR equations in order to better understand their dynamics. The outcomes obtained from this research have potential applications in diverse scientific and engineering domains such as chemical kinetics, population dynamics, biological pattern formation, heat and mass transfer, and fluid mechanics.

The structure of this article is organized as follows: Section 2 offers an overview of the mESEM, which is employed to obtain traveling wave results for NLEEs. Section 3 introduces the governing mathematical models. In Section 4, the DR equations with quadratic and cubic nonlinear terms are solved using the proposed approach. Section 5 is dedicated to the bifurcation analysis, whereas Section 6 examines the stability characteristics of the models. A detailed analysis and discussion of the obtained results are presented in Section 7. Lastly, Section 8 provides a summary of the main findings.

2 Methodology

In this part, we describe the mESEM to find exact solutions nonlinear evolution equations. Considering the higher order NLEE below:

$$P(\mathcal{V}, \mathcal{V}_t, \mathcal{V}_x, \mathcal{V}_{tt}, \mathcal{V}_{tx}, \mathcal{V}_{xx}, \dots) = 0, \quad (3)$$

where P denotes a polynomial in $\mathcal{V}(x, t)$ and its partial derivatives with respect to x and t . The key stages of this method are as

Step-I: Considering the traveling wave transformation

$$\mathcal{V}(x, t) = \mathcal{F}(\xi), \quad \xi = kx - vt, \quad (4)$$

where k and v are wave number and frequency of waves. This transformation converts Eq. (3) into a nonlinear ODE represented as follows:

$$H(\mathcal{F}, -v\mathcal{F}', k\mathcal{F}', v^2\mathcal{F}'', -kv\mathcal{F}'', k^2\mathcal{F}'', \dots) = 0. \quad (5)$$

Step-II: Assuming that the solution to Eq. (5) takes the following form:

$$\mathcal{F}(\xi) = \sum_{i=-M}^M A_i \Phi^i(\xi), \quad (6)$$

here A_i are arbitrary constants and M is a positive integer determined by applying the balancing principle to Eq. (5), and $\Phi(\xi)$ adheres to the structure of the newly introduced ansatz equation

$$\Phi'(\xi) = b_0 + b_1\Phi + b_2\Phi^2, \quad (7)$$

where the arbitrary constants are b_0, b_1 and b_2 .

Step-III: Merging Eqs. (6) and (7) into Eq. (5) and setting the coefficients of powers of Φ^i to zero, yields an algebraic system of equations with parameters A_i, b_0, b_1, b_2, k and v . Mathematica solves the algebraic system and the values of the parameters to be determined.

Step-IV: By replacing the values of the parameters obtained in Stage 3rd and $\Phi(\xi)$ into Eq. (7), one can derive the solution to Eq. (3).

3 Governing Models

The nonlinear DR equations [10,53] include a convective flux term with quadratic and quartic non-linearities, given as follows:

$$\mathcal{V}_t + \alpha \mathcal{V} \mathcal{V}_x = D \mathcal{V}_{xx} + \beta \mathcal{V} - \gamma \mathcal{V}^2, \quad (8)$$

$$\mathcal{V}_t + \alpha \mathcal{V}^2 \mathcal{V}_x = D \mathcal{V}_{xx} + \beta \mathcal{V} - \gamma \mathcal{V}^4, \quad (9)$$

where α, β and γ are physical parameters to be identified, D denotes the diffusion coefficient, and $\mathcal{V}(x, t)$ varies in meaning depending on the specific phenomenon being studied. Each parameter's efficiency [10,54] can be summed up as follows: greater α reinforces nonlinear convection, resulting in sharper and more asymmetric wave profiles; larger D improves diffusion, smoothing gradients and lowering peak values. The saturation γ regulates nonlinear damping, whereas the growth rate β amplifies the solution and raises the equilibrium level. While the steady state for the quartic reaction $\beta \mathcal{V} - \gamma \mathcal{V}^4$ has $\mathcal{V}^* = (\beta/\gamma)^{1/3}$, indicating less sensitivity to parameter changes, the steady state for the quadratic reaction $\beta \mathcal{V} - \gamma \mathcal{V}^2$ is $\mathcal{V}^* = \beta/\gamma$. In these equations, quadratic nonlinearities are associated to reaction-diffusion biology [2], while quartic terms arise in nonlinear transport and chemical kinetics [48,49]. These equations also model transport processes in which both diffusion and convection play significant roles, assuming that nonlinear diffusion behaves similarly to nonlinear convection. By employing the transformation $\mathcal{V}(x, t) = \mathcal{F}(\xi)$, $\xi = kx - vt$ (given in Eq.(4)) on Eqs. (8) and (9), we obtain the ODEs in following form as

$$Dk^2 \mathcal{F}'' + (v - \alpha k \mathcal{F}) \mathcal{F}' + \beta \mathcal{F} - \gamma \mathcal{F}^2 = 0. \quad (10)$$

$$Dk^2 \mathcal{F}'' + (v - \alpha k \mathcal{F}^2) \mathcal{F}' + \beta \mathcal{F} - \gamma \mathcal{F}^4 = 0. \quad (11)$$

4 Mathematical Formulation of Results

In this section, we construct the solitons and other wave outcomes of Eqs. (10) and (11) using modified extended simple equation mathematical technique.

4.1 Quadratic Dynamical Model

By employing the balance principle on Eq. (10), obtain $N = 2$. Thus, the solution of Eq. (10) is the form as

$$\mathcal{F}(\xi) = \frac{A_{-2}}{\Phi^2} + \frac{A_{-1}}{\Phi} + A_0 + A_1 \Phi + A_2 \Phi^2. \quad (12)$$

By substituting Eq. (12) and Eq. (7) into Eq. (10) and setting the coefficients of Φ^i to zero, obtain a system of algebraic equations involving $A_i, b_0, b_1, b_2, k, \alpha, \beta, \gamma$ and v . Solving these equations yields the following cases as:

Case-I: As shown in Fig. 1. If $b_1 = 0$

$$A_{-2} = 0, A_{-1} = \frac{\alpha \beta^2}{8b_2 \gamma^2 dk}, A_0 = \frac{\beta}{2\gamma}, A_1 = 0, A_2 = 0, b_0 = -\frac{\alpha^2 \beta^2}{16b_2 \gamma^2 d^2 k^2}, v = \frac{k(\alpha^2 \beta + 4\gamma^2 d)}{2\alpha \gamma}. \quad (13)$$

$$A_{-2} = 0, A_{-1} = 0, A_0 = \frac{\beta}{2\gamma}, A_1 = \frac{2b_2 dk}{\alpha}, A_2 = 0, b_0 = -\frac{\alpha^2 \beta^2}{16b_2 \gamma^2 d^2 k^2}, v = k \left(\frac{\alpha \beta}{2\gamma} + \frac{2\gamma d}{\alpha} \right). \quad (14)$$

$$A_{-2} = 0, A_{-1} = \frac{\alpha \beta^2}{32b_2 \gamma^2 dk}, A_0 = \frac{\beta}{2\gamma}, A_1 = \frac{2b_2 dk}{\alpha}, A_2 = 0, b_0 = -\frac{\alpha^2 \beta^2}{64b_2 \gamma^2 d^2 k^2},$$

$$v = k \left(\frac{\alpha \beta}{2\gamma} + \frac{2\gamma d}{\alpha} \right). \quad (15)$$

By substituting the solution sets (13), (14) and (15), together with the solution of Eq. (7) into Eq. (12), the wave solutions of Eq. (8) can be obtained in the form as

$$\mathcal{V}_{1,1}^1(x, t) = \frac{\beta}{8\gamma^2} \left(\frac{\alpha \beta \cot(\sqrt{b_0 b_2}(\xi + \xi_0))}{\sqrt{b_0 b_2} dk} + 4\gamma \right), \quad b_0 b_2 > 0. \quad (16)$$

$$\mathcal{V}_{1,2}^1(x, t) = \frac{\beta}{8\gamma^2} \left(4\gamma - \frac{\alpha \beta \coth\left(\sqrt{-b_0 b_2} \xi + \frac{\rho \log(\xi_0)}{2}\right)}{\sqrt{-b_0 b_2} dk} \right), \quad \xi_0 > 0, \rho = \pm 1, b_0 b_2 < 0. \quad (17)$$

$$\mathcal{V}_{1,3}^1(x, t) = \frac{\beta}{2\gamma} + \frac{2\sqrt{b_0 b_2} dk \tan(\sqrt{b_0 b_2}(\xi + xi_0))}{\alpha}, \quad b_0 b_2 > 0. \quad (18)$$

$$\mathcal{V}_{1,4}^1(x, t) = \frac{\beta}{2\gamma} - \frac{2\sqrt{-b_0 b_2} dk \tanh\left(\sqrt{-b_0 b_2} \xi + \frac{\rho \log(\xi_0)}{2}\right)}{\alpha}, \quad \xi_0 > 0, \rho = \pm 1, b_0 b_2 < 0. \quad (19)$$

$$\mathcal{V}_{1,5}^1(x, t) = \frac{\tan(\sqrt{b_0 b_2}(\xi + \xi_0)) (\alpha \beta \cot(\sqrt{b_0 b_2}(\xi + \xi_0)) + 8\sqrt{b_0 b_2} \gamma dk)^2}{32\alpha \sqrt{b_0 b_2} \gamma^2 dk}, \quad b_0 b_2 > 0. \quad (20)$$

$$\mathcal{V}_{1,6}^1(x, t) = -\frac{\tanh(\sqrt{-b_0 b_2} \xi + \frac{\rho \log(\xi_0)}{2}) (\alpha \beta \coth(\sqrt{-b_0 b_2} \xi + \frac{\rho \log(\xi_0)}{2}) - 8\sqrt{-b_0 b_2} \gamma dk)^2}{32\alpha \sqrt{-b_0 b_2} \gamma^2 dk},$$

$$\xi_0 > 0, \rho = \pm 1, b_0 b_2 < 0. \quad (21)$$

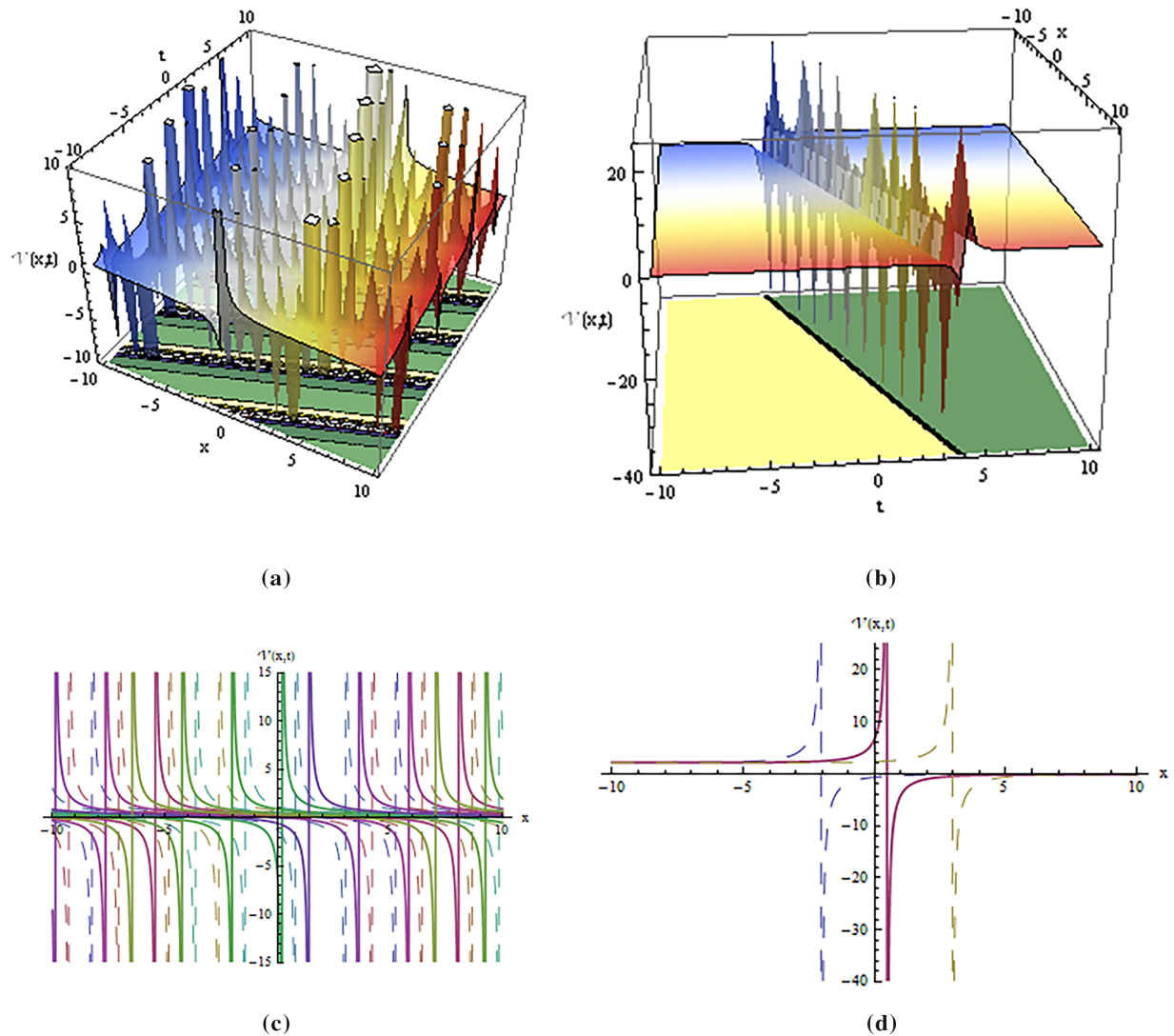


Figure 1: The solutions (16) and (17) are depicted using appropriately chosen parameters as follows: (a) shows a singular breather types solitary waves having diverse amplitude with CP and its 2D representation in (c), (b) depicts a combine bright-dark multi-peak solitary waves with CP, and its 2D representation is provided in (d), respectively

Case-II: As shown in Figs. 2 and 3. If we take $b_0 = 0$

$$A_{-2} = 0, A_0 = \frac{A_{-1}b_2}{b_1} + \frac{\beta}{\gamma}, A_1 = 0, A_2 = 0, k = \frac{\gamma}{\alpha b_1}, v = \frac{\gamma^2 d}{\alpha^2 b_1}. \quad (22)$$

$$A_{-2} = 0, A_{-1} = 0, A_0 = \frac{\beta}{\gamma}, A_1 = \frac{\beta b_2}{b_1 \gamma}, A_2 = 0, k = \frac{\alpha \beta}{2b_1 \gamma d}, v = \frac{\beta (\alpha^2 \beta + 4\gamma^2 d)}{4b_1 \gamma^2 d}. \quad (23)$$

$$A_{-2} = \frac{A_{-1}b_1}{2b_2}, A_0 = \frac{A_{-1}b_2}{2b_1}, A_1 = 0, A_2 = 0, k = \frac{\gamma}{2\alpha b_1}, v = \frac{\alpha^2 \beta + \gamma^2 d}{2\alpha^2 b_1}. \quad (24)$$

$$A_{-2} = 0, A_{-1} = 0, A_0 = 0, A_1 = -\frac{\beta b_2}{b_1 \gamma}, A_2 = 0, k = -\frac{\alpha \beta}{2b_1 \gamma d}, v = -\frac{\beta (\alpha^2 \beta + 4\gamma^2 d)}{4b_1 \gamma^2 d}. \quad (25)$$

$$A_{-2} = 0, A_0 = \frac{A_{-1}b_2}{b_1}, A_1 = 0, A_2 = 0, k = \frac{\gamma}{\alpha b_1}, v = \frac{\alpha^2 \beta + \gamma^2 d}{\alpha^2 b_1}. \quad (26)$$

$$A_{-2} = \frac{A_{-1}b_1}{2b_2}, A_0 = \frac{A_{-1}b_2}{2b_1} + \frac{\beta}{\gamma}, A_1 = 0, A_2 = 0, k = \frac{\gamma}{2\alpha b_1}, v = \frac{\gamma^2 d}{2\alpha^2 b_1}. \quad (27)$$

By substituting the solution sets (22), (23) and (24), together with the solution of Eq. (7) into Eq. (12), the solitary wave results of Eq. (8) can be obtained in the form as

$$\mathcal{V}_{2,1}^1(x, t) = \frac{\beta}{\gamma} + \frac{A_{-1}e^{-b_1(\xi + \xi_0)}}{b_1}, \quad b_1 > 0. \quad (28)$$

$$\mathcal{V}_{2,2}^1(x, t) = \frac{\beta}{\gamma} - \frac{A_{-1}e^{-b_1(\xi + \xi_0)}}{b_1}, \quad b_1 < 0. \quad (29)$$

$$\mathcal{V}_{2,3}^1(x, t) = \frac{\beta}{\gamma - b_2 \gamma e^{b_1(\xi + \xi_0)}}, \quad b_1 > 0. \quad (30)$$

$$\mathcal{V}_{2,4}^1(x, t) = \frac{\beta}{b_2 \gamma e^{b_1(\xi + \xi_0)} + \gamma}, \quad b_1 < 0. \quad (31)$$

$$\mathcal{V}_{2,5}^1(x, t) = \frac{A_{-1}e^{-2b_1(\xi + \xi_0)}}{2b_1 b_2}, \quad b_1 > 0. \quad (32)$$

$$\mathcal{V}_{2,6}^1(x, t) = A_{-1} \left(\frac{b_2}{2b_1} + \frac{e^{-2b_1(\xi + \xi_0)} - b_2^2}{2b_1 b_2} \right), \quad b_1 < 0. \quad (33)$$

One can construct more results in more generalized form of Eq. (8) from solution sets (25), (26) and (27).

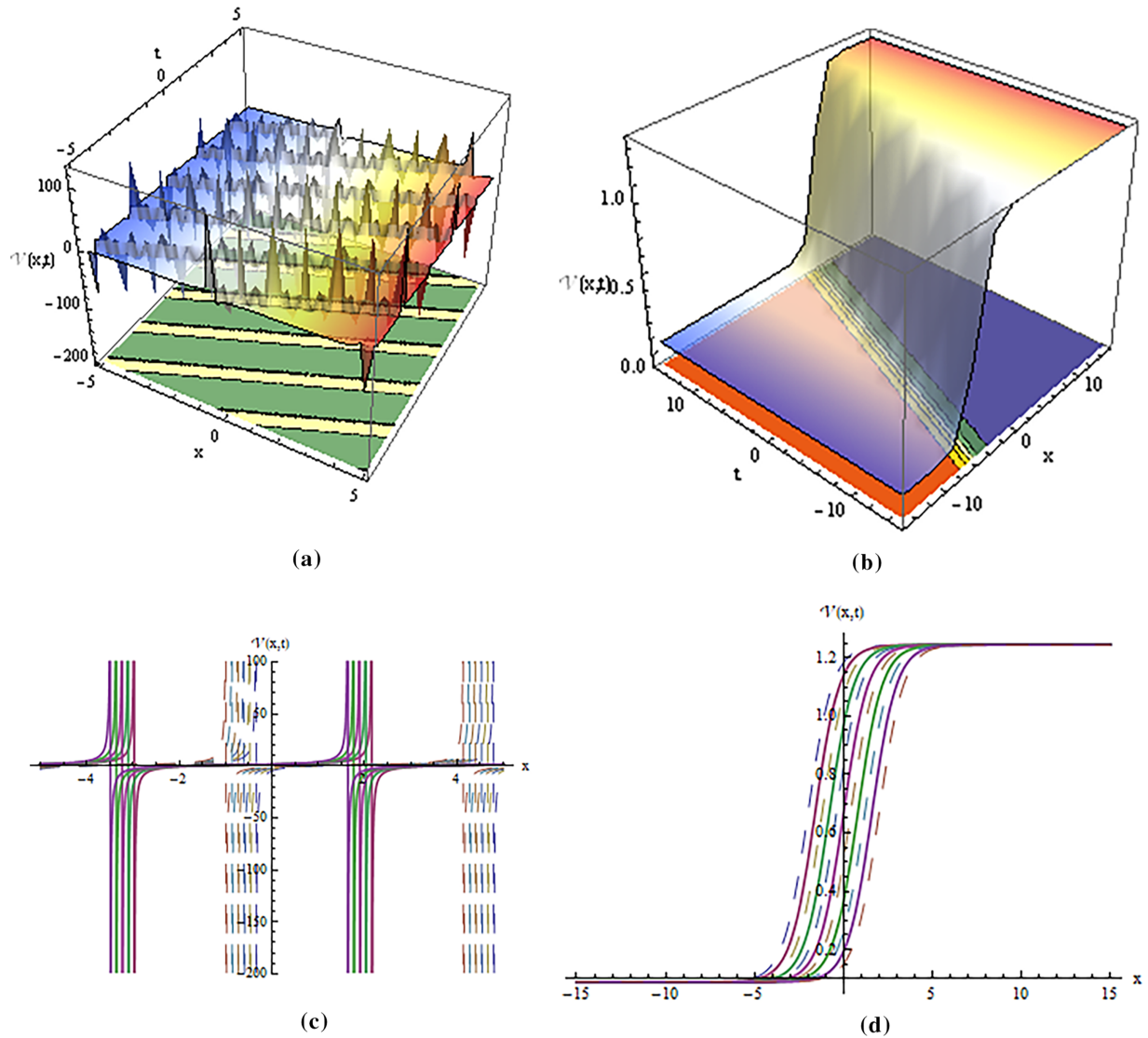


Figure 2: The solutions (18) and (19) are depicted by assigning suitable parameter values as follows: (a) shows periodic multi-peak solitary waves having diverse amplitudes with CP and its 2D representation in (c), (b) depicts a Kink Soliton with CP, and its 2D representation is provided in (d), respectively

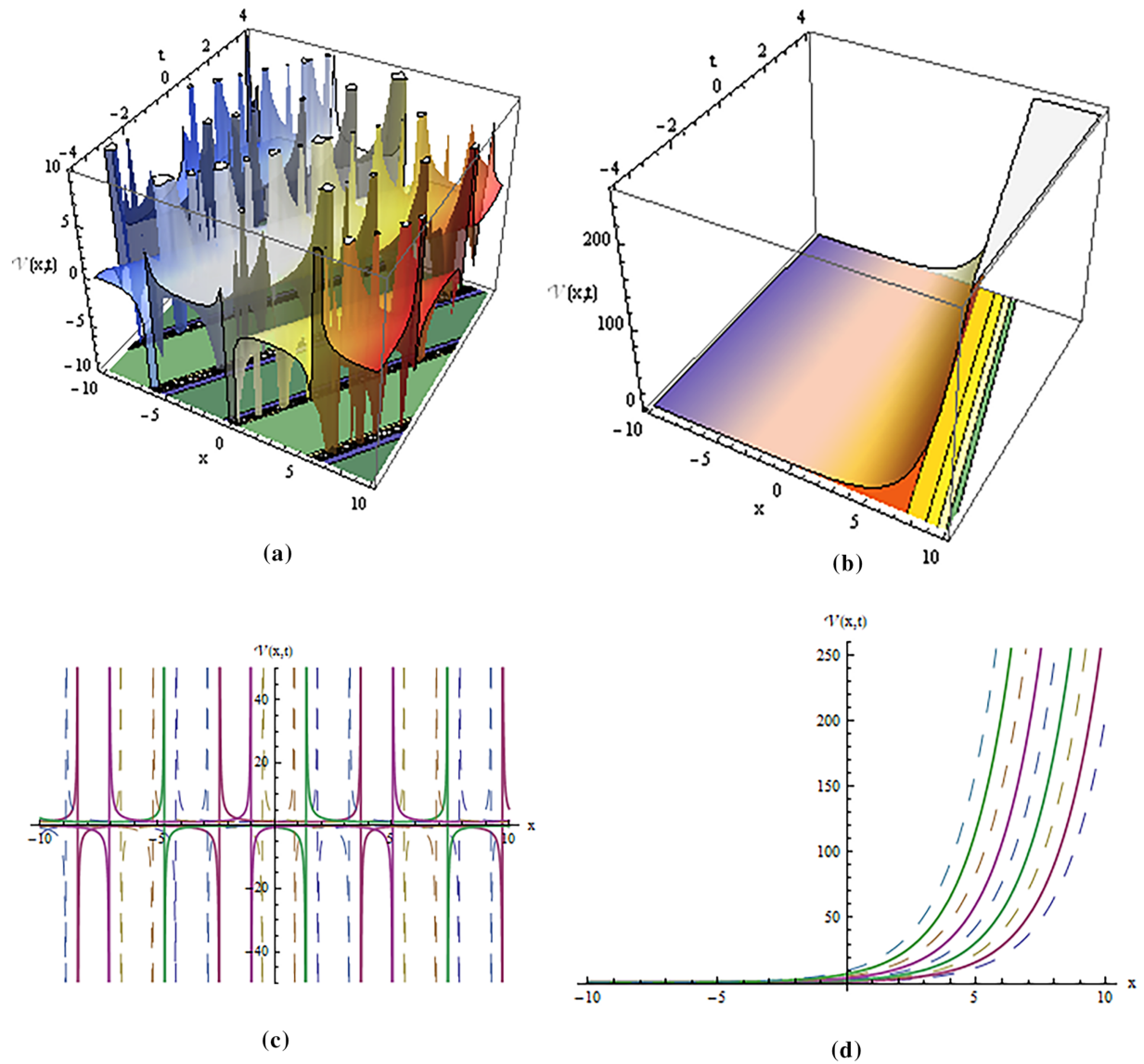


Figure 3: The solutions (20) and (28) are depicted using appropriately chosen parameters as follows: (a) shows periodic breathers type solitary waves having diverse amplitudes with CP and its 2D representation in (c), (b) depicts solitary wave with CP, and its 2D representation is provided in (d), respectively

Case-III: As shown in Fig. 4.

$$A_{-2} = 0, A_{-1} = 0, A_1 = \frac{2b_2 dk}{\alpha}, A_{-2} = 0, \beta = 2\gamma \left(A_0 - \frac{b_1 dk}{\alpha} \right), b_0 = \frac{\alpha A_0 (2b_1 dk - \alpha A_0)}{4b_2 d^2 k^2},$$

$$v = k \left(\alpha A_0 - b_1 dk + \frac{2\gamma d}{\alpha} \right). \quad (34)$$

$$A_{-2} = 0, A_{-1} = -\frac{2b_0dk}{\alpha}, A_1 = 0, A_2 = 0, \beta = \frac{2\gamma(\alpha A_0 + b_1dk)}{\alpha}, b_2 = -\frac{\alpha A_0(\alpha A_0 + 2b_1dk)}{4b_0d^2k^2},$$

$$v = k \left(\alpha A_0 + b_1dk + \frac{2\gamma d}{\alpha} \right). \quad (35)$$

$$A_{-2} = 0, A_{-1} = 0, A_0 = \frac{\beta}{2\gamma} \left(\frac{b_1}{\sqrt{b_1^2 - 4b_0b_2}} + 1 \right), A_1 = \frac{\beta b_2}{\sqrt{b_1^2 - 4b_0b_2}\gamma}, A_2 = 0,$$

$$k = \frac{\alpha\beta}{2\sqrt{b_1^2 - 4b_0b_2}\gamma d}, v = \frac{\beta(\alpha^2\beta + 4\gamma^2d)}{4\sqrt{b_1^2 - 4b_0b_2}\gamma^2d}. \quad (36)$$

$$A_{-2} = 0, A_{-1} = -\frac{\beta b_0}{\sqrt{b_1^2 - 4b_0b_2}\gamma}, A_0 = \frac{1}{2\gamma} \left(\beta - \frac{\beta b_1}{\sqrt{b_1^2 - 4b_0b_2}} \right), A_1 = 0, A_2 = 0,$$

$$k = \frac{\alpha\beta}{2\sqrt{b_1^2 - 4b_0b_2}\gamma d}, v = \frac{\beta(\alpha^2\beta + 4\gamma^2d)}{4\sqrt{b_1^2 - 4b_0b_2}\gamma^2d}. \quad (37)$$

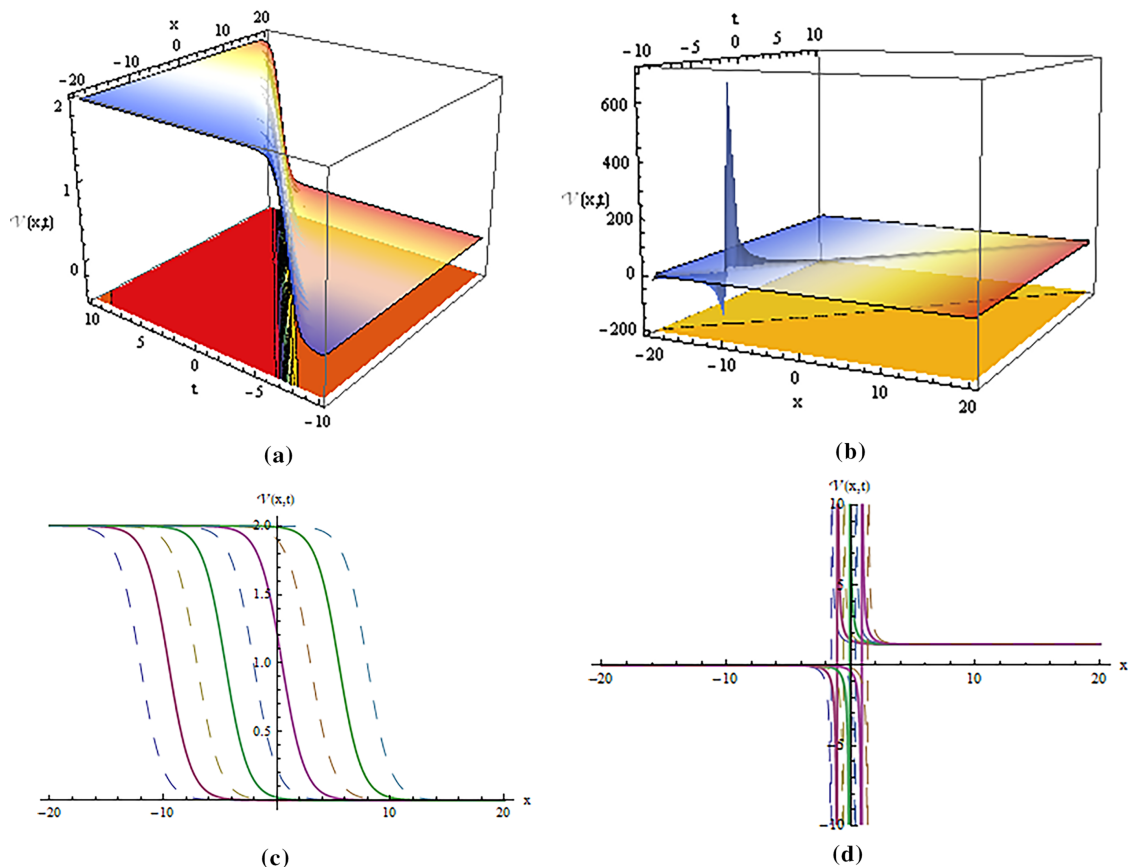


Figure 4: The solutions (30) and (31) are defined using appropriately chosen parameters as follows: (a) shows a anti-Kink soliton with CP and its 2D representation in (c), and (b) shows a combine bright dark solitary wave with CP and its 2D representation in (d), respectively

By substituting the solution set (34), together with the solution of Eq. (7) into Eq. (12), the wave solutions of Eq. (8) can be obtained in the form as

$$\mathcal{V}_{3,1}^1(x, t) = A_0 + \frac{dk}{\alpha} \left(\sqrt{4b_0b_2 - b_1^2} \tan \left(\frac{\sqrt{4b_0b_2 - b_1^2}}{2} (\xi + \xi_0) \right) - b_1 \right), \quad 4b_0b_2 > b_1^2, \quad b_1 > 0. \quad (38)$$

$$\mathcal{V}_{3,2}^1(x, t) = A_0 + \frac{dk \left(\sqrt{4b_0b_2 - b_1^2} \tan \left(\frac{\sqrt{4b_0b_2 - b_1^2}}{2} (\xi + \chi) \right) + b_1 \right)}{\alpha}, \quad 4b_0b_2 > b_1^2, \quad b_1 < 0. \quad (39)$$

One can construct more results in more generalized form of Eq. (8) from solution sets (35), (36) and (37).

4.2 Quartic Dynamical Model

By employing the balance principle on Eq. (11), obtain $N = 1$. Thus, the solution of Eq. (11) is the form as

$$\mathcal{F}(\xi) = \frac{A_{-1}}{\Phi(\xi)} + A_0 + A_1\Phi(\xi). \quad (40)$$

By substituting Eq. (40) and Eq. (7) into Eq. (11) and setting the coefficients of Φ^i to zero, obtain a system of algebraic equations involving $A_{-1}, A_0, A_1, b_0, b_1, b_2, k, \alpha, \beta, \gamma$ and ν . Solving these equations yields the following cases as:

Case-I: As shown in Fig. 5. If $b_1 = 0$

$$A_0 = \frac{\gamma d}{\alpha^2}, A_1 = 0, \beta = \mp \frac{8\gamma^4 d^3}{\alpha^6}, b_0 = \pm \frac{\alpha^4 A_{-1}^2 b_2}{3\gamma^2 d^2}, k = \pm \frac{3\gamma^3 d^2}{\alpha^5 A_{-1} b_2}, \nu = \mp \frac{6\gamma^5 d^4}{\alpha^8 A_{-1} b_2}. \quad (41)$$

$$A_{-1} = 0, A_0 = \frac{\gamma d}{\alpha^2}, \beta = \pm \frac{8\gamma^4 d^3}{\alpha^6}, b_0 = \mp \frac{b_2 \gamma^2 d^2}{\alpha^4 A_1^2}, k = -\frac{A_1 \gamma}{\alpha b_2}, \nu = \frac{6A_1 \gamma^3 d^2}{\alpha^4 b_2}. \quad (42)$$

$$A_{-1} = \pm \frac{3\gamma^3 d^2}{4\alpha^5 b_2 k}, A_0 = \frac{\gamma d}{\alpha^2}, A_1 = -\frac{\alpha b_2 k}{\gamma}, \beta = \mp \frac{8\gamma^4 d^3}{\alpha^6}, b_0 = \pm \frac{3\gamma^4 d^2}{4\alpha^6 b_2 k^2}, \nu = -\frac{2\gamma^2 d^2 k}{\alpha^3}. \quad (43)$$

By substituting the solution sets (41), (42) and (43), together with the solution of Eq. (7) into Eq. (40), the wave solutions of Eq. (9) can be obtained in the form as

$$\mathcal{V}_{1,1}^2(x, t) = \frac{\gamma d}{\alpha^2} + \frac{A_{-1} b_2 \cot \left(\sqrt{b_0 b_2} (\xi + \xi_0) \right)}{\sqrt{b_0 b_2}}, \quad b_0 b_2 > 0. \quad (44)$$

$$\mathcal{V}_{1,2}^2(x, t) = \frac{A_{-1} \sqrt{-b_0 b_2} \coth \left(\sqrt{-b_0 b_2} \xi + \frac{\rho \log(\xi_0)}{2} \right)}{b_0} + \frac{\gamma d}{\alpha^2}, \quad \xi_0 > 0, \quad \rho = \pm 1, \quad b_0 b_2 < 0. \quad (45)$$

$$\mathcal{V}_{1,3}^2(x, t) = \frac{d\gamma}{\alpha^2} + \frac{A_1 b_0 \tan \left(\sqrt{b_0 b_2} (\xi + \xi_0) \right)}{\sqrt{b_0 b_2}}, \quad b_0 b_2 > 0. \quad (46)$$

$$\mathcal{V}_{1,4}^2(x, t) = \frac{\gamma d}{\alpha^2} + \frac{A_1 b_0 \tanh \left(\sqrt{-b_0 b_2} \xi + \frac{\rho \log(\xi_0)}{2} \right)}{\sqrt{-b_0 b_2}}, \quad \xi_0 > 0, \quad \rho = \pm 1, \quad b_0 b_2 < 0. \quad (47)$$

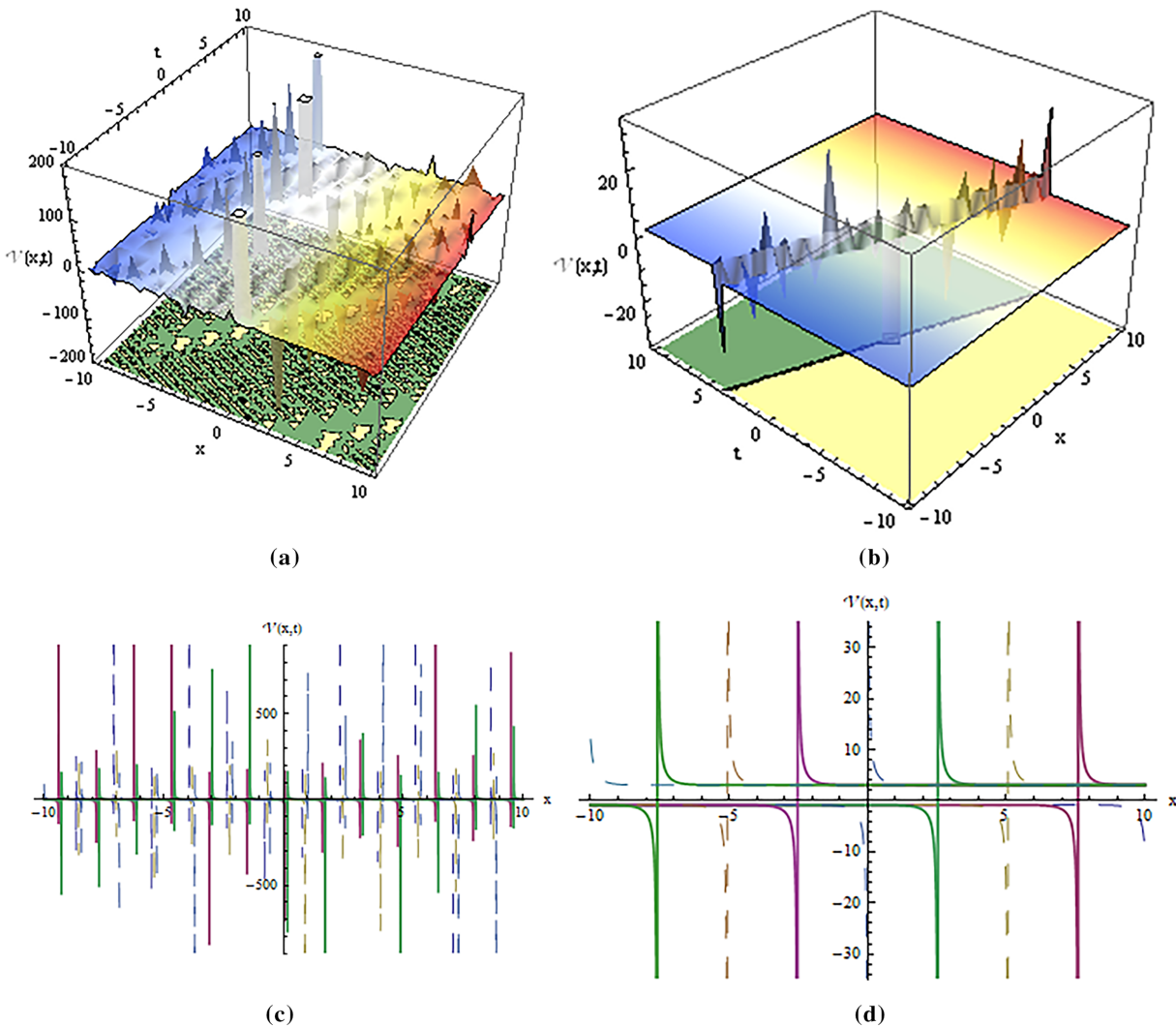


Figure 5: The solutions (44) and (45) are depicted by assigning suitable parameter values as follows: (a) shows a breather types solitary waves having diverse amplitude with CP and its 2D representation in (c), (b) depicts a breather type multi-peak solitary waves with CP, and its 2D representation is provided in (d), respectively

Similarly, one can construct more results in more generalized form of Eq. (9) from solution set (43).

Case-II: As shown in Figs. 6 and 7. If we take $b_0 = 0$

$$A_{-1} = 0, A_0 = 0, A_1 = -\frac{2b_2\gamma d}{\alpha^2 b_1}, \beta = \frac{8\gamma^4 d^3}{\alpha^6}, k = \frac{2\gamma^2 d}{\alpha^3 b_1}, v = -\frac{12\gamma^4 d^3}{\alpha^6 b_1}. \quad (48)$$

$$A_{-1} = 0, A_0 = \frac{2\gamma d}{\alpha^2}, A_1 = \frac{2b_2\gamma d}{\alpha^2 b_1}, \beta = \frac{8\gamma^4 d^3}{\alpha^6}, k = -\frac{2\gamma^2 d}{\alpha^3 b_1}, v = \frac{12\gamma^4 d^3}{\alpha^6 b_1}. \quad (49)$$

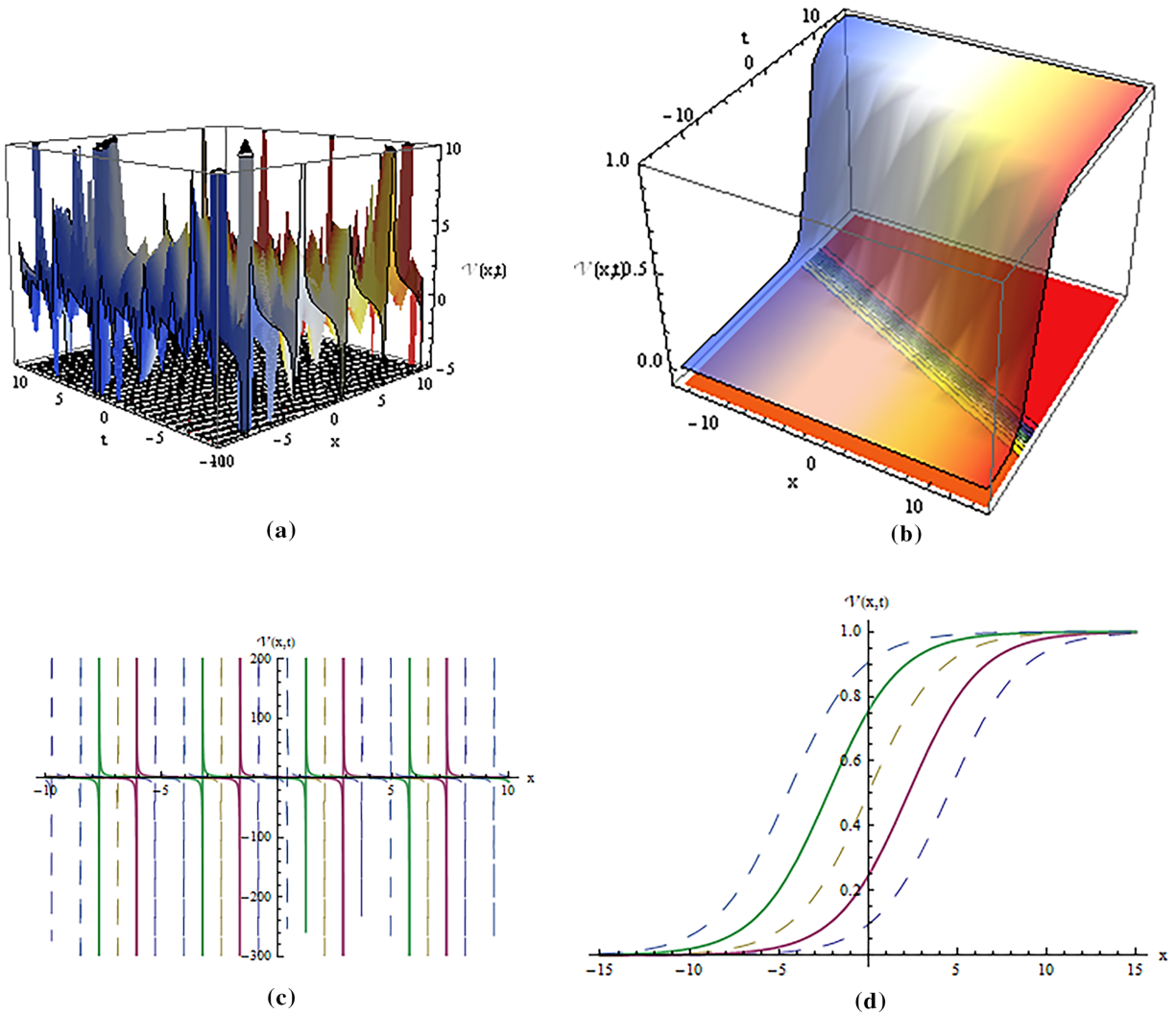


Figure 6: The solutions (46) and (47) are depicted by assigning suitable parameter values as follows: (a) shows periodic solitary waves having diverse amplitudes with CP and its 2D representation in (c), (b) depicts a Kink Soliton with CP, and its 2D representation is provided in (d), respectively

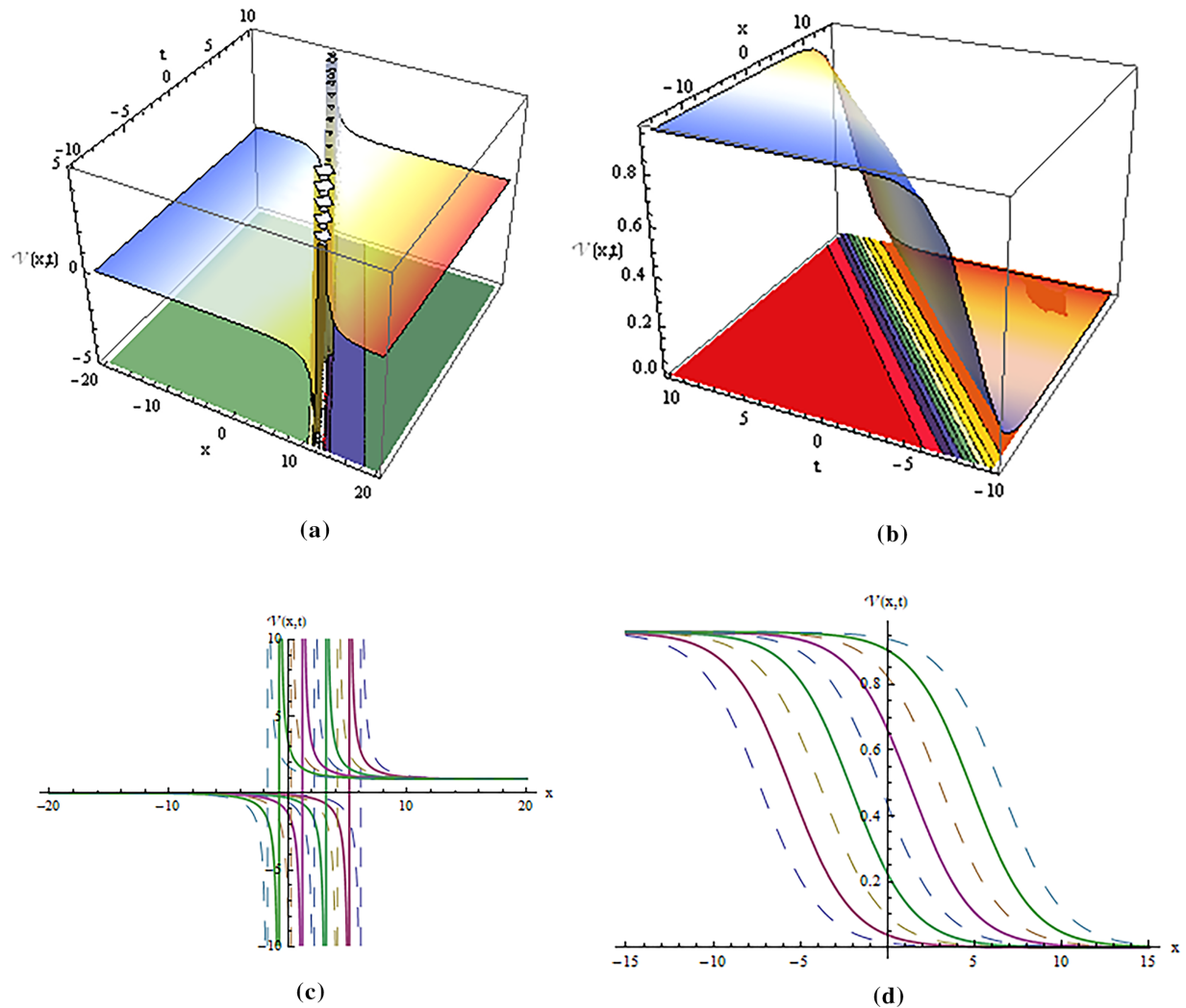


Figure 7: The solutions (50) and (51) are defined using appropriately chosen parameters as follows: (a) shows a singular combine dark-bright solitary wave with CP and its 2D representation in (c), and (b) shows a anti-Kink solitary wave with CP and its 2D representation in (d), respectively

By substituting the solution sets (48) and (49), together with the solution of Eq. (7) into Eq. (40), the solitary wave results of Eq. (9) can be obtained in the form as

$$\mathcal{V}_{2,1}^2(x, t) = \frac{2b_2\gamma de^{b_1(\xi+\xi_0)}}{\alpha^2 (b_2e^{b_1(\xi+\xi_0)} - 1)}, \quad b_1 > 0. \quad (50)$$

$$\mathcal{V}_{2,2}^2(x, t) = \frac{2b_2\gamma de^{b_1(\xi+\xi_0)}}{\alpha^2 (b_2e^{b_1(\xi+\xi_0)} + 1)}, \quad b_1 < 0. \quad (51)$$

$$\mathcal{V}_{2,3}^2(x, t) = -\frac{2\gamma d}{\alpha^2 (b_2e^{b_1(\xi+\xi_0)} - 1)}, \quad b_1 > 0. \quad (52)$$

$$\mathcal{V}_{2,4}^2(x, t) = \frac{2\gamma d}{\alpha^2 (b_2e^{b_1(\xi+\xi_0)} + 1)}, \quad b_1 < 0. \quad (53)$$

Case-III:

$$A_{-1} = 0, A_0 = \frac{\gamma d}{\alpha^2} \left(1 + \frac{\sqrt{3}b_1}{\sqrt{4b_0b_2 - b_1^2}} \right), A_1 = \frac{2\sqrt{3}b_2\gamma d}{\alpha^2\sqrt{4b_0b_2 - b_1^2}}, \beta = -\frac{8\gamma^4d^3}{\alpha^6},$$

$$k = -\frac{2\sqrt{3}\gamma^2d}{\alpha^3\sqrt{4b_0b_2 - b_1^2}}, \nu = \frac{4\sqrt{3}\gamma^4d^3}{\alpha^6\sqrt{4b_0b_2 - b_1^2}}. \quad (54)$$

$$A_{-1} = \frac{2\sqrt{3}b_0\gamma d}{\alpha^2\sqrt{4b_0b_2 - b_1^2}}, A_0 = \frac{\gamma d}{\alpha^2} \left(1 + \frac{\sqrt{3}b_1}{\sqrt{4b_0b_2 - b_1^2}} \right), A_1 = 0, \beta = -\frac{8\gamma^4d^3}{\alpha^6},$$

$$k = \frac{2\sqrt{3}\gamma^2d}{\alpha^3\sqrt{4b_0b_2 - b_1^2}}, \nu = -\frac{4\sqrt{3}\gamma^4d^3}{\alpha^6\sqrt{4b_0b_2 - b_1^2}}. \quad (55)$$

$$A_{-1} = 0, A_0 = \frac{\gamma d}{\alpha^2} \left(1 - \frac{\sqrt{3}b_1}{\sqrt{4b_0b_2 - b_1^2}} \right), A_1 = -\frac{2\sqrt{3}b_2\gamma d}{\alpha^2\sqrt{4b_0b_2 - b_1^2}}, \beta = -\frac{8\gamma^4d^3}{\alpha^6},$$

$$k = \frac{2\sqrt{3}\gamma^2d}{\alpha^3\sqrt{4b_0b_2 - b_1^2}}, \nu = -\frac{4\sqrt{3}\gamma^4d^3}{\alpha^6\sqrt{4b_0b_2 - b_1^2}}. \quad (56)$$

$$A_{-1} = -\frac{2\sqrt{3}b_0\gamma d}{\alpha^2\sqrt{4b_0b_2 - b_1^2}}, A_0 = \frac{\gamma d}{\alpha^2} \left(1 - \frac{\sqrt{3}b_1}{\sqrt{4b_0b_2 - b_1^2}} \right), A_1 = 0, \beta = -\frac{8\gamma^4d^3}{\alpha^6},$$

$$k = -\frac{2\sqrt{3}\gamma^2d}{\alpha^3\sqrt{4b_0b_2 - b_1^2}}, \nu = \frac{4\sqrt{3}\gamma^4d^3}{\alpha^6\sqrt{4b_0b_2 - b_1^2}}. \quad (57)$$

By substituting the solution sets (54) and (55), together with the solution of Eq. (7) into Eq. (40), the wave solutions of Eq. (9) can be obtained in the form as

$$\mathcal{V}_{3,1}^2(x, t) = \frac{\gamma d}{\alpha^2} \left(1 + \sqrt{3} \tan \left(\frac{\sqrt{4b_0b_2 - b_1^2}}{2} (\xi + \xi_0) \right) + 1 \right), \quad 4b_0b_2 > b_1^2, \quad b_1 > 0. \quad (58)$$

$$\mathcal{V}_{3,2}^2(x, t) = \frac{\gamma d}{\alpha^2} \left(1 + \frac{2\sqrt{3}b_1}{\sqrt{4b_0b_2 - b_1^2}} + \sqrt{3} \tan \left(\frac{\sqrt{4b_0b_2 - b_1^2}}{2} (\xi + \xi_0) \right) \right), \quad 4b_0b_2 > b_1^2, \quad b_1 < 0. \quad (59)$$

$$\mathcal{V}_{3,3}^2(x, t) = \frac{\gamma d}{\alpha^2} \left(1 + \frac{\sqrt{3}b_1}{\sqrt{4b_0b_2 - b_1^2}} - \frac{4\sqrt{3}b_0b_2}{\sqrt{4b_0b_2 - b_1^2} \left(b_1 - \sqrt{4b_0b_2 - b_1^2} \tan \left(\frac{\sqrt{4b_0b_2 - b_1^2}}{2} (\xi + \xi_0) \right) \right)} \right),$$

$$4b_0b_2 > b_1^2, \quad b_1 > 0. \quad (60)$$

$$\mathcal{V}_{3,4}^2(x, t) = \frac{\gamma d}{\alpha^2} \left(1 + \frac{\sqrt{3}b_1}{\sqrt{4b_0b_2 - b_1^2}} + \frac{4\sqrt{3}b_0b_2}{\sqrt{4b_0b_2 - b_1^2} \left(\sqrt{4b_0b_2 - b_1^2} \tan \left(\frac{1}{2} \sqrt{4b_0b_2 - b_1^2} (\xi + \chi) \right) + b_1 \right)} \right),$$

$$4b_0b_2 > b_1^2, \quad b_1 < 0. \quad (61)$$

One can construct more results in more generalized form of Eq. (9) from solution sets (56) and (57).

5 Bifurcation Analysis

The bifurcation analysis of the nonlinear DR equations is examined in this section. The study of bifurcation theory looks at how minor adjustments to system parameters can result in significant shifts in how solutions behave. We find critical points where the dynamics of the system change, resulting in phenomena like the appearance of new equilibrium states or periodic solutions, by methodically altering important parameters. Our comprehension of the intricate behaviors displayed by nonlinear DR systems is improved by this analysis, which sheds light on the stability and multiplicity of solutions.

5.1 Quadratic Dynamical Model

Through the Galilean transformation on Eq. (10), the equation can be rewritten as the following planer dynamical system:

$$\begin{aligned}\mathcal{F}' &= R, \\ R' &= a\mathcal{F}^2 - b\mathcal{F} + c\mathcal{F}R - eR.\end{aligned}\quad (62)$$

where $a = \frac{\gamma}{Dk^2}$, $b = \frac{\beta}{Dk^2}$, $c = \frac{\alpha}{Dk}$ and $e = \frac{\nu}{Dk^2}$. Next, we will discuss the bifurcations of the phase profiles in the parameters paces a, b, c and e of system (62). According to the bifurcation theory applied to two-dimensional nonlinear systems, the nature of the equilibrium point $(\mathcal{F}, R)$ depends on the sign of the Jacobian determinant $J(\mathcal{F}, R)$. Here, $J(\mathcal{F}, R)$ represents the determinant of the system's Jacobian matrix. The system in (62) possesses two equilibrium points: $P = (0, 0)$ and $Q = (\frac{b}{a}, 0)$.

Determinant of the Jacobian matrix of the system (62) as

$$J(\mathcal{F}, R) = b - 2a\mathcal{F} - cR. \quad (63)$$

The nature of each equilibrium point is determined by evaluating the Jacobian determinant at that point. The trace of the Jacobian

$$\tau = c\mathcal{F} - e,$$

is also be important for stability. Now analyze the two points for different values of parameter b . (Table 1, Figs. 8 and 9)

Table 1: Classification of fixed points and bifurcation type depending on parameter b

Condition	Fixed points	Type of fixed point P (0, 0)	Type of fixed point Q ($b/a, 0$)	Bifurcation type
$b > 0$	2	Stable (Spiral/Node)	Saddle (Unstable)	No bifurcation
$b = 0$	1 (P and Q coincide)	Linearization fails	Linearization fails	Transcritical bifurcation
$b < 0$	2	Saddle (Unstable)	Stable (Spiral/Node)	No bifurcation

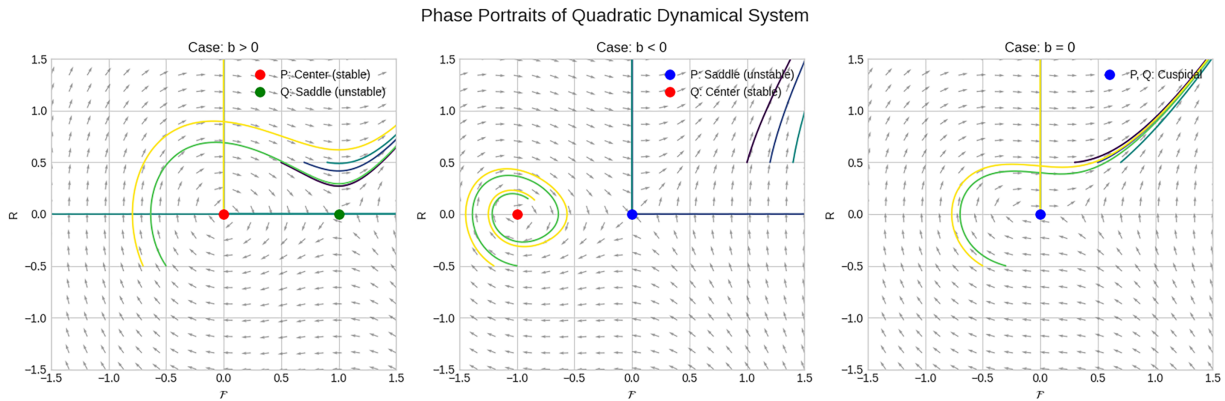


Figure 8: Phase portraits of system (62) for different values of parameter b

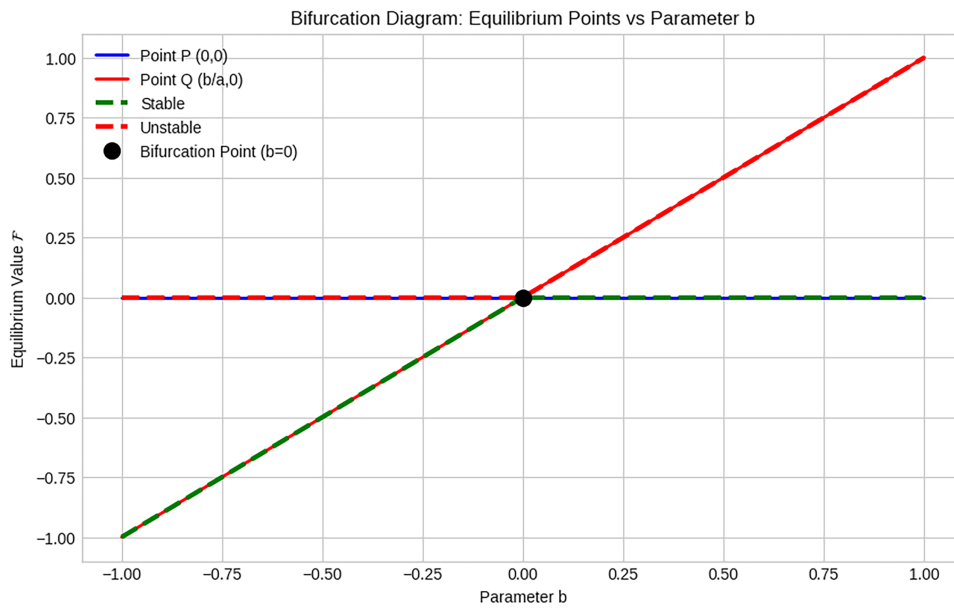


Figure 9: Bifurcation diagram showing equilibrium points as a function of parameter b

Case-I: $b > 0$

At Point $P(0, 0)$:

- Determinant: $J(P) = b > 0$.
- Trace: $\tau(P) = c \cdot 0 - e = -e$. If we assume small or positive damping ($e \geq 0$), the trace is ≤ 0 .
- Conclusion: Since $J > 0$ and $\tau \leq 0$, P is a stable equilibrium. It is a stable spiral if $\tau^2 < 4J$, or a stable node if $\tau^2 > 4J$. If $e = 0$, it is a center.

At Point $Q\left(\frac{b}{a}, 0\right)$:

- Determinant: $J(Q) = -b < 0$.
- Conclusion: Since $J < 0$, Q is a saddle point, which is always unstable.

Case-II: $b < 0$

At Point $P(0, 0)$:

- Determinant: $J(P) = b < 0$.
- Conclusion: Since $J < 0$, P is a saddle point and is unstable.

At Point $Q\left(\frac{b}{a}, 0\right)$:

- Note: Since $b < 0$, $\frac{b}{a}$ must be positive for Q to be physically relevant (assuming $a > 0$). This requires $a < 0$. Let's assume $a > 0$, making Q a non-physical point in this parameter region. The analysis would need to be adjusted if a can change sign. For the standard case ($a > 0, b < 0$), Point Q does not exist in the real plane.

Case-III: $b = 00$ (Bifurcation Point)

At Point $P(0, 0)$:

- Determinant: $J(P) = 0$.
- The linearization fails. This is a non-hyperbolic point, indicating a potential bifurcation.
- The system undergoes a transcritical bifurcation at $b = 0$. In this bifurcation, the stability of the two fixed points P and Q is exchanged as parameter b passes through zero.

5.2 Quartic Dynamical Model

Through the Galilean transformation in Eq. (11), the equation can be rewritten as the following planer dynamical system:

$$\mathcal{F}' = R,$$

$$R' = a\mathcal{F}^4 - b\mathcal{F} + c\mathcal{F}^2R - eR. \quad (64)$$

where $a = \frac{\gamma}{Dk^2}$, $b = \frac{\beta}{Dk^2}$, $c = \frac{\alpha}{Dk}$ and $e = \frac{\nu}{Dk^2}$. Next, we will discuss the bifurcations of the phase profiles in the parameters paces a, b, c and e of system (64). The system in (64) possesses two equilibrium points: $P_1 = (0, 0)$ and $Q_1 = \left(\left(\frac{b}{a}\right)^{\frac{1}{3}}, 0\right)$.

Determinant of the Jacobian matrix of the system (62) as

$$J(\mathcal{F}, R) = b - 4a\mathcal{F}^3 - 2c\mathcal{F}R. \quad (65)$$

At the equilibrium points (where $R = 0$), the determinant simplifies to:

$$J(\mathcal{F}, 0) = -(4a\mathcal{F}^3 - b).$$

Now, analyze the stability for different parameter regimes (Table 2 and Figs. 10 and 11):

Case-I: For $b > 0$,

- Point $P(0, 0)$: $J(0, 0) = b > 0$ – Stable center
- Point $Q\left(\sqrt[3]{\frac{b}{a}}, 0\right)$: $J(Q) = -(4b - b) = -3b < 0$ – Unstable saddle point

Case-II: $b < 0$,

- Point $P(0, 0)$: $J(0, 0) = b < 0$ – Unstable saddle point
- Point $Q\left(\sqrt[3]{\frac{b}{a}}, 0\right)$: $J(Q) = -3b > 0$ – Stable center

Case-III: $b = 00$,

- Point P (0,0): $J(0,0) = 0$ – Degenerate case (linearization fails)
- The system undergoes a pitchfork bifurcation at $b = 0$

Table 2: Stability and bifurcation analysis of the quartic dynamical system

Condition	Fixed points	Type of point P (0, 0)	Type of point Q ($\sqrt{b/a}$, 0)	Bifurcation type
$b > 0$	2	Center (stable)	Saddle (unstable)	No bifurcation
$b = 0$	1	Degenerate	N/A	Pitchfork bifurcation
$b < 0$	2	Saddle (unstable)	Center (stable)	No bifurcation

Stability Analysis of Quartic Dynamical System

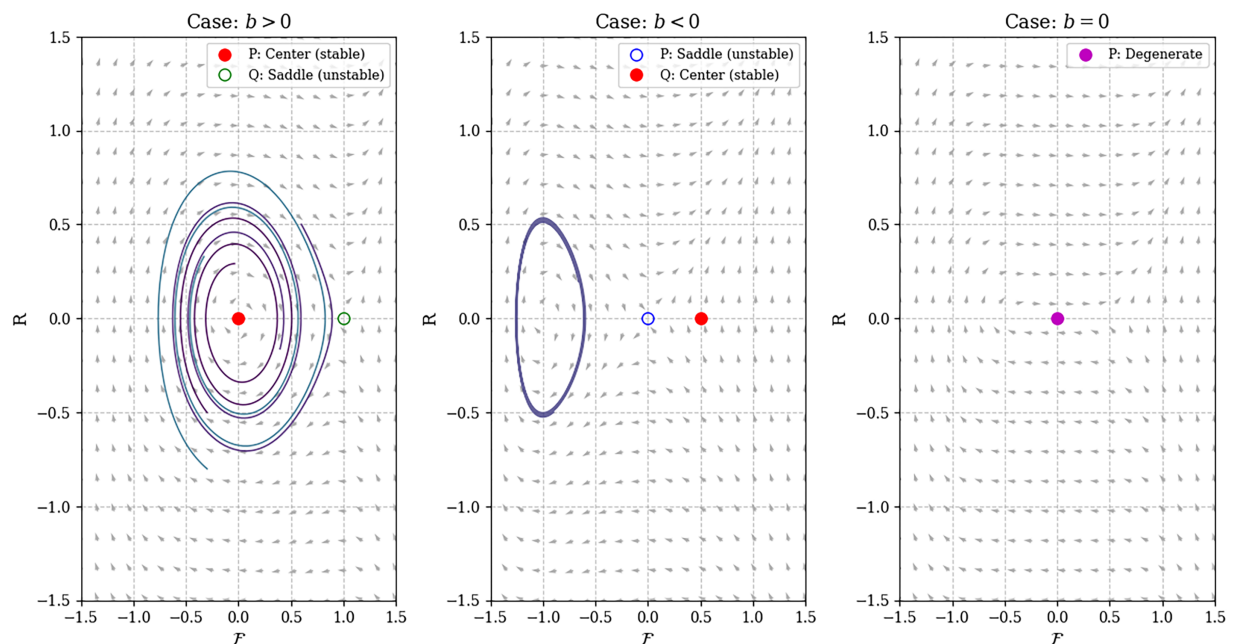


Figure 10: Phase portraits of system (64) for different values of parameter b

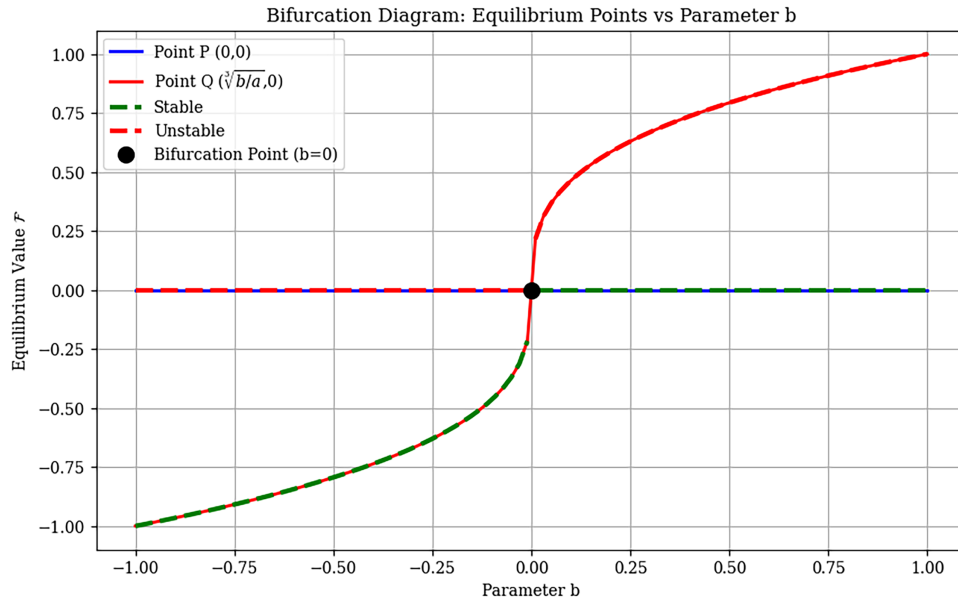


Figure 11: Bifurcation diagram showing equilibrium points as a function of parameter b

6 Modulational Instability Analysis

Modulational Instability (MI) Analysis is a fundamental technique used in nonlinear wave theory, especially in optics, fluid dynamics and Bose-Einstein condensates. It describes the process where a uniform (or plane wave) solution of a nonlinear dispersive equation becomes unstable under the influence of small perturbations, leading to the growth of modulated waveforms often resulting in localized structures like solitons or rogue waves. For diffusion-reaction equations, MI can arise when the system exhibits certain nonlinearities, such as in reaction-diffusion systems that describe the spatiotemporal evolution of concentrations of chemical species. Using a standard linear stability analysis [1,52], we develop the analysis of MI for the dynamical models (8) and (9).

6.1 Quadratic Dynamical Model

The steady-state solution (S-SS) of dynamical model (8) takes the form as

$$\mathcal{V}(x, t) = \left(\sqrt{S} + \Psi(x, t) \right) e^{\Omega(t)}, \quad \Omega(t) = S\lambda\epsilon t, \quad (66)$$

here, S denotes the normalized optical power. To ensure that $\Psi \ll \sqrt{S}$, a small perturbation $\Psi(x, t)$ is incorporated. Substituting Eq. (66) into Eq. (8) and performing a linearization yields as

$$\frac{\partial \Psi}{\partial t} - D \frac{\partial^2 \Psi}{\partial x^2} + (S\lambda\epsilon - \beta) \Psi = 0. \quad (67)$$

Considering the solution of Eq. (67) has as

$$\Psi(x, t) = \eta e^{\sigma x - \omega t}, \quad (68)$$

where σ and ω are the wave numbers and frequency of Ψ , respectively. When Eq. (68) is substituting Eqs. (68) into (67), the relation is obtained as

$$\omega = \lambda S\epsilon - \beta - D\sigma^2. \quad (69)$$

The dispersion relation in (69) show that the wave number, self-phase modulation, and Stimulated Raman scattering influences the stability of the steady-state condition. For all σ , the ω in (69) remains real, indicating that the steady state is stable under small perturbations (Fig. 12).

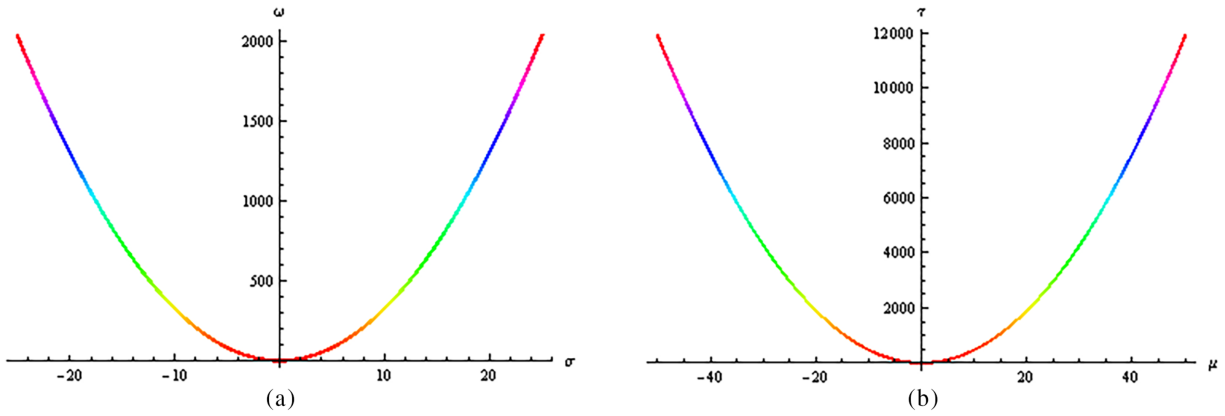


Figure 12: The relations in (69) and (73) are shown in (a) and (b), respectively

6.2 Quartic Dynamical Model

The S-SS of dynamical model (9) takes the form as

$$\mathcal{V}(x, t) = \left(\sqrt{R} + \Psi(x, t) \right) e^{\Omega(t)}, \quad \Omega(t) = R\lambda\epsilon t, \quad (70)$$

here, R denotes the normalized optical power. To ensure that $\Psi \ll \sqrt{R}$, a small perturbation $\Psi(x, t)$ is incorporated. Substituting Eq. (70) into Eq. (9) and performing a linearization yields as

$$\frac{\partial \Psi}{\partial t} - D \frac{\partial^2 \Psi}{\partial x^2} + (R\lambda\epsilon - \beta) \Psi = 0. \quad (71)$$

Considering the solution of Eq. (71) has as

$$\Psi(x, t) = \eta e^{\mu x - \tau t}, \quad (72)$$

where μ and τ are the wave numbers and frequency of Ψ , respectively. When Eq. (72) is substituting into Eq. (71), the relation is obtained as

$$\tau = \lambda R\epsilon - \beta - D\mu^2. \quad (73)$$

The dispersion relation in (73) show that the wave number, self-phase modulation, and Stimulated Raman scattering influences the stability of the steady-state condition. For all μ , the τ in (73) remains real, indicating that the steady state is stable under small perturbations.

7 Discussion and Physical Interpretation

The dynamics of nonlinear DR systems with quadratic and quartic non-linearities in the presence of nonlinear convective flux terms are the main focus of the study. These equations interplay of quartic and quadratic nonlinear terms enriches the solution space by enabling a broad range of waveforms and transition patterns. Analytical soliton solutions for these equations are obtained using the mESEM. These kinds of NLEEs are essential for simulating physics, biology, and chemistry phenomena like

population migration, chemical kinetics, and the formation of biological patterns. This method yields a large class of exact solitary wave solutions, such as breather-type waves, periodic and rational solutions, kink and anti-kink solitons, and combined bright-dark solitary structures. These solutions are relevant to physical and biological phenomena such as species dispersal, chemical transport, and pattern formation in ecological systems. Through bifurcation and modulational instability analyses, the study identifies critical parameter thresholds that determine the emergence and stability of various waveforms. The results show that small variations in diffusion coefficients, reaction rates, or convective terms can significantly influence the nature, amplitude, and speed of the wave propagation, offering valuable insight into the qualitative behavior of real-world systems. Our findings offer new wave structures that are very different from those found in earlier research, and they constitute an enhanced and more varied version of the current findings. These novel structures demonstrate how well our method captures intricate dynamical behaviors while also expanding the pool of potential solutions. Different authors applied traditional methods to construct the wave solutions of these models for constructing the wave solutions. The authors in [10] applied three analytical methods namely the (G'/G) -expansion scheme, its generalized version and Kudryashov scheme for constructing the traveling wave solutions. The researchers in [53] used direct algebraic scheme for obtaining the solitary wave solutions of these models. In [55], the authors obtained kink and anti-kink wave solutions of Eq. (8). In [48], the researchers investigated a specific case of the DR equation where the diffusion coefficient D was assumed to be independent of concentration or density. Hence, the findings of this work represent original contributions that have not been addressed in prior studies.

The figures illustrate various types of soliton structures derived from the obtained solutions by selecting appropriate parameter values for the corresponding equations. In Fig. 1, solutions (16) and (17) are visualized: (a) shows singular breather-type solitary waves with CP and diverse amplitudes; (b) illustrates combined bright-dark multi-peak solitary waves with CP; (c)–(d) provide their corresponding 2D views. Fig. 2 shows solutions (18) and (19): (a) depicts periodic multi-peak solitary waves with CP; (b) presents a kink soliton with CP; with (c)–(d) as their 2D counterparts. In Fig. 3, solutions (20) and (28) include (a) periodic breather-type waves with CP and (b) a solitary wave with CP, along with their 2D views in (c) and (d).

Fig. 4 illustrates solutions (30) and (31): (a) shows an anti-kink soliton with CP; (b) displays a combined bright-dark solitary wave with CP, both with 2D versions in (c)–(d). Figs. 5 and 6 both visualize solutions (46) and (47): (a) shows periodic solitary waves; (b) shows a kink soliton, with respective 2D representations. In Fig. 7, solutions (50) and (51) depict: (a) a singular combined dark-bright soliton; (b) an anti-kink soliton, with 2D plots in (c) and (d). Figs. 8–11 show phase portraits of systems (62) and (64) for varying parameters. In Fig. 12, the relations given in (69) and (73) are illustrated in subfigures (a) and (b), respectively.

8 Conclusion

In this paper, the mESEM has been effectively applied to nonlinear diffusion-reaction equations incorporating both quadratic and quartic non-linearities having non-linear convective flux terms. These types of equations frequently arise in various physical and biological contexts, such as in modeling species population dynamics. A wide range of novel travelling and solitary wave solutions is obtained, including multi-peak breather-type waves, kink and anti-kink structures, as well as solutions expressed through hyperbolic, trigonometric, and rational functions. These solutions provide rich insight into the complex wave structures that can emerge in systems governed by diffusion, convection, and nonlinear reaction dynamics. These solutions also have applications in ecology, fluid mechanics,

and chemical transport. Bifurcation analysis is employed to explore the qualitative behavior of the solutions under varying parameters. Additionally, modulational instability analysis is conducted to assess the stability of the obtained solutions, confirming that they are both exact and stable. The graphical representation of several solutions illustrates their physical significance and demonstrates the wave-like nature of the solitons in realistic settings. The results confirm that mSEM is a powerful and flexible tool for solving higher-order NLEEs and can contribute significantly to the modeling of various physical, biological, and engineering phenomena involving nonlinear wave propagation. Future investigations will focus on multi-dimensional generalizations to examine more intricate spatial interactions and wave structures, as well as fractional-order extensions to capture memory and hereditary effects.

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Availability of Data and Materials: The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest to report regarding the present study.

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