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The braid of numbers

E. Oñate

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1 THE AURA OF NUMBERS

Looking back towards the origins of mankind it is surprising how humans have evolved in parallel with their wish to quantify the phenomena of nature. Indeed, it is now clear that the progress of civilization has been lead by the capacity of men to express numerically the solution to their everyday's problems.

The relationship between man and numbers is interwined with the development of all sciences, and mainly with mathematics and philosophy. Engineers can perhaps realize clearer than others that numbers form an intrinsic part of our professional life. In our daily tasks we look for quantitative answers to questions such as which dimensions must have a mechanical part or a reinforced concrete structure, when will fracture occur in certain material or which will the manufacturing cost of a new product be. As years go by I have gradually and imperceptibly noticed that, without reaching the Pitagorean creed that "number is everything", it is true that numbers irradiate their own light which lights up our lives, and that the success of man in the fight for improving his existence lays in knowing first and influencing next, the circumstances which surround him which, in addition to any other subjective aspect, have a distinct numerical value.

The joint path followed by man and numbers through history has distinct features. The numerological conception of the universe initiated by the school of Pitagoras, evolved for over twenty three centuries up to the discovery of infinitesimal calculus by Newton and Leibniz. Nowadays, once the exact (analytical) solution of these equations has proven to be impossible in most cases, numbers return as the protagonists of the story. The come back of numbers has happened throughout the numerical solution of the differential equations, that is by providing quantitative values to the parameters which govern the mathematical equations of a problem in science or engineering.

There is nothing new in above ideas. The role of numbers made the great Bertran Russel say that "what is more astonishing in modern science is its return to Pitagorism". These words spoken more than fifty years ago are even truer nowadays. The recent spectacular progresses in science and technology have been driven by the advances in the so-called numerical or computational methods, aiming to extract quantitative answers from the differential equations deduced by mathematicians in the past two centuries.

The triad numbers – differential equations-numbers has many examples of its admirable symmetry. The initial conception of numbers since several millennia before our era reaches its peak with Pitagoras when numbers took up the center role in universe. The subsequent numberization of sciences allowed the development of the philosophy of Plato and Aristoteles, the geometry and algebra of Euclides and the physics and quantitative methods or Archimedes which have influenced for centuries (even nowadays) the development of mathematics, science and technology. From those inspiring times to the formulation of everything in nature in terms of differential equations took almost twenty centuries.

After the discovery of infinitesimal calculus, the inconditionals of Newton and Leibniz probably thought that "differential equations are everything". Nowadays the loop has been closed and any computational method is good only in terms of its capacity to provide "acceptable" numerical results. Numbers are back, two thousand five hundred years after they were taken to the zenith by the Pitagoreans.

The following lines aim tries to present an overview on some of the fundamental steps in the magic braid of numbers. I will refer to the initial conception of numbers, centered around the golden era in ancient Greece, the evolution of “quantitative methods” up to the discovery of infinitesimal calculus in the XVII century and, finally, the resurrection of numbers with the help of numerical methods and computers.

This written journey is obviously equivalent to describe a great part of the history of science and technology and obviously it can not be undertaken in a few pages. Many important achievements and facts will therefore be omitted with the hope that the main argument of the story remains. That is, to show the invisible line which links the first perceptions of numbers some three thousand years ago with the generalized applications of numerical methods nowadays, which even allow to translate into feelings, through virtual reality machines, the output of lengthy computations.

2 THE PERCEPTION OF NUMBERS

The idea that the concept of number is innate to man has many supporters. The natural perception of numbers reveals itself in human beings, and even in some animals, by the fact that they can detect the presence of small quantities.

Does this necessarily mean that animals can count? The answers is no. For that purpose they should be able to enumerate any series of numbers. Up to now, there is no creature known to be capable of such an achievement. Therefore, men are still the only living species who can argue: “I count therefore I am”.

There are no well defined references of when and by whom numbers were invented. It is however accepted that their origins precede in some thousand years to the Egyptian warriors who counted their captures. Nor even the inquisitive Greeks asked themselves explicitly what the numbers were, even though Pitagoras and his followers spoke of numbers constantly as if they were alive.

The question of who invented numbers is probably ill posed. It is possible that numbers were never deliberately invented by a single man or a group of men. Most probably the concept of number evolved almost imperceptibly through the centuries. Somewhere, some time, humans started to get used to use numbers without precisely realizing what they were doing. However, numbers 1,2,3 exhibit the trademark of a conscious inspiration and invention. Although it may be a mere speculation, we can imagine an unknown genius suddenly perceiving that a man and a woman, a dog and a cat, a dawn and a sun set and, in fact, every pair of things have all something in common they are “two”. From that instant to the conception of the number two it many have been a big step forward, but, surely, someone took it well before the Egyptian soldiers counted their captures.

By admitting that numbers were invented one takes a position in front of many eminent philosophers, Plato amongst others, and relevant mathematicians from the XIX and XX centuries who defend other option. If numbers were not invented by humans they can –not necessarily must- have been “discovered”. This is the cross point where knowledge ends and opinion starts. It is the old divergence between those who believe that mathematics originate from within our mids and those who defend that they come from outside us. The formers are convinced that we invent mathematics as a useful tool for describing the surrounding world.

The latter maintain that we discover mathematics, that they are “out there” and they will still be there even if there were not mathematicians.

I do not intend to take part in this old controversy. My only concern here is to point out the analogy between the influence that the first perception of numbers had in the origins of our civilization and the impact of numbers nowadays. It the question on the origin of numbers falls within the class of the “non decideables” is of little relevance in front of the indisputable fact that numbers have influenced as any other concept the development of man’s life. Moreover, it all indicates that the presence and company of numbers among us will increase to help us in the design of our future.

3 THE DAYBREAK OF NUMBERS

It is will known that the Greek scholars had renowned predecessors who were also interested in aritmetics and algebra. The oldest archeological evidences show that towards the end of the Neolithic times (3300—3100 B.C.) when the Greeks were little more than a group of nomadic tribes moving around Minor Asia, wise men in Mesopotamia and Egypt had already independently elaborated a numbering system.

Babylonians, differently from Egyptians, were not only good mathematicians and algebrists. The erudites of Babylon developed the sexagesimal numbering system. Moreover they are also credited for having discovered the positional principle, which is the basis of the current representation of numbers.

However, if ever the Egyptian and Babylonian scholars made a mistake that was the inability to transmit their achievements by creating schools of thought aiming to ensure the survival of their ideas. Overcoming this shortage was, undoubtedly, one the greatest merits of Greek civilization whose benefits have been collected throughout the subsequent centuries up to nowadays.

4 THE DECISIVE CENTURY

If we believe in points of inflexion in history, the VI century B.C. was a clear example. In that century two Greeks, Thales and Pitagoras, probably the first immortals of exact sciences, produced so decisive advances in mathematics and, in science in general, that made possible the work of Galileo and Newton may centuries later. In all aspects, either scientific, mathematic or religious, the VI century B.C. was a memorable time for western civilization.

In this respect, I can only agree with the american mathematician Eric. T. Bell [1] that in that critical period our civilization shifted its evolution from East to West. It is indeed difficult to imagine what would nowadays be the state of the world should this shift never had occurred.

Of Thales of Mileto there are so many stories, as evidences of its deep interest and knowledge of geometry and calculus. The most famous anecdote is perhaps the prediction of a sun eclipse during one the Medic wars. Coetaneous of Creso, the king famous for this addiction to gold and wealth, Thales used to answer to his thankful citizens when they asked him the prize for his services that the just wished “recognition for his discoveries”. In this way, Thales became probably the first man who clearly noticed that the intangibles of fame are far superior than material riches.

4.1 The revelation of Pithagoras

The definitive turn towards civilization was, however, given a few years later by Pithagoras.

Two important consequences for the future of science and philosophy emerged from the fascinating contributions from Pithagoras. The first was the belief that the physical universe can consistently be described in terms of numbers. The second was his deep conviction that conclusions reached by means of mathematical reasoning are of greater certitude than those obtained by any other mean. Both opinions have been frequently questioned. Both have been modified once and again to accommodate the advances of knowledge. However, the essence of these two asseverations remains basically unchanged and, nowadays, they can be considered complementary postulates of a single thesis not yet verified: the rational comprehension of the world is possible and, once this occurs, it will agree with the experiences of the senses and will allow man to predict the course of nature.

This is the dream that nowadays despite of the every new discovery seems a little more distant, although still reachable for many. Pithagoras believed he have found the magic formula in his idea that “everything is number”. In the most primitive version, the numerology of Pithagoras covered literally “everything”, from heaven to the musical armonies and the intimate human emotions. As the knowledge of universe increased, the “everything” was progressively reduced to more modest proportions. Thus, towards the first half of the XIX century, “everything” covered just the astronomical and physical sciences. The recent evidence that practically all phenomena in nature are expressable by mathematical equations which solutions can be found in terms of numbers, brings back the possibility that “everything” can be explained by numbers. The Pithagorean creed remains thus unchanged, supporting those who believe that man and number form an undissociable binomial from the origin of times.

Before going any further, it is necessary to analyze a doubt which worried Pithagoras during his last days of existence. Indeed the same doubt has returned twenty-five centuries later to disturb modern Pithagoreans.

The basic assumption supporting all applications of numbers into science is that nature laws are rational, that is, they are accessible by a healthy mind. This might however not be true.

Concepts such as that of infinity, or the evidence that it was impossible to measure the diagonal of a square which sides were rational numbers, or the fact that the length of a circumference which diameter was a rational number was not computable, brought a serious shade of doubt into Pithagoras assumption that the universe could be expressed in terms of known numbers. Something seemed to indicate that a part of the universe is beyond the understanding and control of man. In the more general sense, the universe suddenly appeared to Pithagoras as numerically “irrational”. From our own perspective twenty-five centuries later we see clearly that the word irrational should not be interpreted here as “contrary to reason”, but rather contrary to the axioms on numbers accepted in those times.

The evidence that the mathematical truth is beyond all axioms and rules has been defended by many scientists in all ages. In Pithagoras times, Zenon of Enea invented his famous and controversial paradoxes on movement which helped to demolish the classical concepts of space and time and which have influenced many later developments in science and philosophy.

Many centuries after Zenon, in 1931, the Viennese mathematician Kurt Gödel explained in his own way that in order to fully understand nature through mathematics one should go out from mathematics. Gödel's proof on the unavoidness of undecideability has been the incentive for many applications to other areas of knowledge. In particular, much has been discussed on its consequences for any comprehensive understanding of the universe by means of mathematical methods. It has been said that as we can "see" the evidence of Gödel's statement, this necessarily means that the human mind can not be a formal system and, therefore, the sophisticated attempts of the so called artificial intelligence methods, based in reducing the behavior of the mind to a finite set of algorithms, can never be successful.

Accepting the conclusions of Gödel's theorem, we have to admit humbly that our knowledge of nature is only possible in an "irrational" sense. Thus, regardless of the sophistication of any new theory contributed to the existing ones, new problems will always appear which solution will be undecideable using the known methods. Once more, the persevering man will advance in his discoveries until he reaches a new crossroads in science and so on. This process is analogous to the search for all figures of some irrational numbers, such as the number p of which in thirty centuries we have passed from knowing 15 figures to some millions, remaining still the whole infinity of figures to be found.

5 THREE GLORIOUS CENTURIES

One of the most faithful inheritants of the Pithagorean culture was Plato. Accepting the philosophy of numbers, Plato codified and amplified it by providing a rational basis for the "everything is number" of this mystical predecessor.

In this Republic, Plato ordered an intensive education in mathematics for the guardians of his ideal city, since, as he used to say, "all arts and sciences involve numbers and computations". Plato took this obsession into practice in the every day's life of his Academy where over the entrance he posted the ban: "Let no ignorant of geometry enter my doors".

Plato, differently from his disciple and ideological rival Aristoteles, was perhaps the clearest example after Pithagoras of a life devoted to the interpretation of the world through mathematics and also, in a lesser sense, to trying to capture the essence of the Ideas, the mathematical truths, through the experiences of senses. His conclusion, that "mathematical reality lies outside us" was shared by many subsequent philosophers and it links with the belief in a mathematical universe that man can progressively discover and explore.

5.1 Euclides of Alexandria

During his expedition to Egypt in 332 B.C. Alexander the Great founded the city of Alexandria where it flourished a cosmopolitan community of mathematicians.

Among the wise man who came to Alexandria from Greece there was Euclides.

Despite his many contributions, Euclides fame has reached us mainly for being the author of the famous book the *Elements*, probably the more re-published book in history.

The influence of the *Elements* in the development of all branches of geometry was enormeous. From a computational point of view geometrical methods were the basis for the evaluation of areas and volumes, which helped to conceive the principles of infinitesimal calculus. Nowadays, geometry plays an essential role in the development of numerical methods for the solution of differential equations over a domain using discretization techniques. These techniques are based on the division of the domain shape into simple geometrical elements, such as triangles and quadrilaterals in the plane, or tetrahedral and hexaedra in space. Indeed geometry is also essential for developing graphic representation methods for visualizing the results of computations.

5.2 Arquimedes of Siracusa

Arquimedes was the best disciple of Euclides and probably the last representative of the school of thought in ancient Greece.

Arquimedes is seen by many as the father of physics as a science and also as the first engineer scientist: the man in search of general principles for application into specific problems. In this respect Arquimedes can be considered as the clear predecessor of modern computational methods.

Most developments of Arquimedes were motivated by the need of finding the solution to practical problems, generally of military nature. It is remarkable how he managed to combine and extend concepts in mathematics and physics, reaching, in many occasions, to general conclusions such as mathematical formulae of universal validity and, in other, to the numerical solution of everyday’s problems. As such, Arquimedes can be considered as the father of modern applied mathematics and, more specifically, of numerical methods.

Arquimedes was also the first in proposing a method for computing the number p with any precision. The technique, a predecessor of discretization methods, is based in the fact that the perimeter of a polygon inscribed or circumscribed in a circle approximates the length of the circumference.

The approximate value of p is computed by evaluating the perimeter of the polygon and dividing this by the diameter of the circle. Obviously, as the number of sides increases, so it does the approximation of p . It is indeed fascinating the analogy of this discretization technique with the approach followed by many numerical methods used nowadays to solve more complex problems with the help of computers.

6 THE LONG DARK JOURNEY

A number of prestigious Greek mathematicians remained in Alexandria during the Roman Empire. Important names were Apolonio, Erastotenes, who estimated quite accurately the length of the earth meridian, the astronomers Ptolomeo, Heron, Pappus, Diophantus and others.

Despite the efforts of these eminent scientists, Greek mathematics, as many other sciences, elapsed slowly under the power of Rome. It was the prelude of a long period of darkness in Europe, which lasted for many subsequent centuries.

6.1 The shinning from other cultures of number

From the fall of the Roman Empire up to the end of the IX century Western Europe devastated by epidemies, hunger and wars, sank in the deepest political chaos, the economy of the mere subsistence and the complete medieval darkness.

As a mere example, arithmetics in those times was simply based on the old Roman numbering system, and the use of pebbles or chips on the abacus inherited from the Romans, including the method of counting with fingers taught by Saint Isidorus of Seville.

Fortunately for Europe, other men in far away countries like China and India, continued the inexorable progress in mathematics and physics. Indeed Europe was saved by these remote cultures and, paradoxically enough, through the biggest enemy of Christianity in the Middle Ages.

The Hindu civilization deserves an special remark as it produced the master piece of our actual numbering system based on the three following great ideas:

- The description of figures by graphic signs which do not evoke the number of units they represent.
- The use of a decimal positional system.
- And, last, but not least, the invention of number zero.

These fundamental contributions modified completely the existence of human beings. From that point in history onwards, any arithmetic operation was possible without difficulty and this opened the door to the development of mathematics, science and technology.

Unfortunately, the revolutionary contributions of the Hindu mathematical culture, already developed by the mid V century A.D., took almost eight centuries to be fully accepted by the western world. In fact, the influence of Hindu civilization did not reach Europe directly. This task was the credit of the Arab scholars, who transmitted the science from the Hindus, playing, among many other roles, that of intermediaries between the East and the West worlds.

Among the many Arab scientists who contributed to the dissemination of the Hindu numbers and arithmetics stands out the mathematician Al-Khuwarizmi.

The scientific contributions of Al-Khuwarizmi are collected on two fundamental books. The first, entitled *Al jabr wa'l muqabala* gained such popularity in his days that name of the book was the origin of the word *algebra*. The second book of Al-Khuwarizmi was the *Book of Addition and Sustraction following the Arithmetic of Hindus*. This book, as the former, became so popular and prestigious that the name of his author was used as generic reference to the new arithmetic system. In this way, the name Al-Khuwarizmi, once latinized, become *Algorithm*, which initially denoted the system formed by the number zero, the nine significant figures and the calculation methods originated in India. As years went by the name *algorithm* progressively took the more general and abstract meaning given nowadays.

Thanks to the scientific and technical contributions of the Islam scholars, generally unknown for the majority of western world, Europe was able to initiate, around the XI century, its intellectual renovation .

One of the predecessors of the introduction of Arabic science in Western Europe was the French monk Gerbert d'Aurillac who in 999 became Pope under the name of Silvester II.

During his many travels through the Muslim Spain, Gerbert d'Aurillac became acquainted with Arab numbers which then he introduced in other European countries. Unfortunately his contribution was limited to the first nine numbers, excluding number zero and the calculation methods from the Hindus. The explanation of this odd circumstance which delayed considerably the progress of European science was the resistance and conservatism of some officers of the Christian community hooked to the culture of the Roman numbering system.

Gerbert d'Aurillac could not escape to the retrograde spirit of those times. Some said that he practiced alchemy and witchcraft and many thought that having tasted the science of the infidel Sarracens, he surely must have sold his soul to Lucifer.

However, the efforts of the Pope Silvester were not in vain. The first lights of dawn in the medieval night brought up by his work lit the path to many in the slow come back of Europe to scientific activity. This rising of the western spirit, in times of Richard Lion Heart, occurred, paradoxically enough, under the sign of the Cross, precisely the same which indirectly has kept Europe in the darkness.

Thus, through the many trips of the crusaders, from the XII century onwards Europe started to be acquainted with the work of the Arabs and also with that of Greeks and Hindus from books translated into Arabic language. The cultural contacts between both worlds increased and so was the number of Europeans wishing to get instruction in arithmetics, mathematics, astronomy, natural sciences and philosophy.

In summary, the Crusades allowed the big step forward which neither the science or the willingness of the Pope Silvester has achieved: to impose the number zero and the new calculation methods to western Europeans.

The subsequent invention of the printing and the necessity of greater technical knowledges derived from a time of many geographical expeditions, contributed to the dissemination of Arab numbers and the written calculation techniques. These, as a new religion, were finally accepted by the majority of people, therefore contributing to the democratization of calculation methods in Europe.

Unfortunately the guards of medieval orthodoxy resisted hardly to accept the new winds which blowed in all directions with progressive strength. The transition from the Middle Ages to the Renaissance in the scientific-mathematic world was long and painful and indeed had its victims. As far from the official end of Middle Ages, in 1600, Giordano Bruno, was sent to the fire in Rome for defending Copernicus astronomy. Few years later, in 1633, Galileo at the age of 70, had to retract his ideas on the earch moving around the sun.

The effort of these distinguished scientists was however not in vain. Soon before the end of the year of Galileo's death, in Lincolnshire (England) a child was born who was to take and extend Galileo's legacy to such a high levels that both their names will remain as a necessary reference in the history of progress of men. The name of the boy was Isaac Newton.

7 NUMBERS LIGHT UP EVERYTHING

It can be said that the consequences of the discovery of infinitesimal calculus by Newton and Leibniz for science and technology are comparable to those of the finding of fire for primitive men, or electricity for the industrial revolution.

This statement is, by no means, exaggerated. Before the days of Newton and Leibniz, there was no general procedure for describing in terms of mathematic equations the behavior of a specific problem in physics, such as the transmition of heat in a body, the flow of a fluid or the deformation of an elastic solid. Obviously, as the problem could not be posed in mathematical form its solution was impossible. After the contributions of Newton and Leibniz it was then possible to describe the behavior of any physical system of either solid, liquid or gas form by differential and integral equations. Moreover, new techniques were made available to solve these equations in many cases which, despite they were simplifications of the general problem, allowed important advances in scientific and technical knowledge. The disciples of Newton and Leibniz were able to say with full conviction that “mathematics are everything”, meaning that from the end of the XVII century onwards any problem could be expressed in mathematical form using the tools provided by the new calculus.

The optimism injected to the scientific community by the first successes of infinitesimal calculus was soon shaded by an unpleasant evidence. It was indeed true that any problem in nature could be posed in mathematical form by means of differential equations. However, the “exact” solution of these equations it was only possible for a few particular cases, which frequently represented coarse simplifications of reality. The difficulties for finding the universal mathematical formula dreamt by Pithagoras which would solve practical problems in science and technology forced the need for deriving alternative methods for solving the new differential equations.

In this way, at the turn of the XX century a number of scientists and engineers observed that approximate numerical values of the unknown parameters of a problem could be obtained by “discretizing” the governing differential equations, using analogous techniques to those used by Archimedes to estimate the value of number p . Numerical methods were then born.

The strategy followed by most numerical methods is to transform the differential equations governing a problem into a set of algebraic equations which depend on a finite number of unknown parameters. This number is in most practical cases of many thousands (and even millions) unknowns and therefore the final system of equations can only be solved with the help of computers. This explains why even though many numerical methods were known since the XVIII and XIX centuries, their development and popularity has occurred parallel to the progress of modern computers in the XX century.

Numerical methods represent, in fact, the return of numbers as the true protagonists in the solution of a problem. The loop initiated by Pithagoras 25 centuries ago has been closed in last few decades with the evidence that with the help of numerical methods we can find precise answers to the problems of universe.

7.1 Perspectives of numerical methods

History tell us clearly that the progress in science and technology has runned in parallel with the increased knowledge by men of the phenomena of nature and the impact of human interventions in those. The unavoidable need for "quantifying" the solution of a problem such as, for instance, the design and construction of a building, the prediction of life in a cell or the economic production of food cans has even increased nowadays. The aura of number which has fascinated man since the origins of times, finds out its raison d'être through the extended use of numerical methods fostering developments in all branches in science and engineering.

We can not forget that numerical methods are inseparable from mathematics, material modeling and computer science. Nowadays it is unthinkable to attempt the development of a new numerical method for solution of a problem in science or engineering without referring to those disciplines. As an example, any new numerical method for solving large scale problems has to take into account the future hardware environment (most probably using parallel computing facilities). Also a modern computer program should be able to incorporate easily the continuous advances in the modeling of new materials.

The word which perhaps synthesizes best the immediate future of numerical methods is "computational multiphysics". The solution of problems will not be attempted from the perspective of a single physical medium and it will incorporate all the couplings which characterize the complexity of reality. For instance, the design of a structural component for a vehicle (an automotive, an airplane, etc.) will take into account the manufacturing process and the function which the component will play through its practical life time. Structures in civil engineering will be studied taking into account the surrounding environment (soil ground, water, air). Similar examples can be found in mechanical, naval and aeronautic engineering, among others, as well as in bio-engineering and indeed in practically all branches of science. Accounting for the non deterministic character of data will be essential for estimating the probability that the new products and processes conceived by men behave as planned. The huge computational needs resulting from a "stochastic multiphisic" perspective will demand next century better numerical methods, new material models and, indeed, faster computers.

It is only from the perspective of a narrow cooperation between all sides of the triangle formed by the deep knowledge of the physical and mathematical basis of a problem, numerical methods and informatics, that effective solutions will be found for the mega problems of next century. This cooperation should be also reflected in a greater emphasis for optimizing the human and material resources required for confronting with enough confidence the change of scale in the solution of future problems. Last but not least, innovative training schemes will be needed to accordingly educate the new generations who, with the help of numbers, will successfully solve multi-disciplinary problems.

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