14th World Congress on Computational Mechanics (WCCM)
ECCOMAS Congress 2020)
Virtual Congress: 11-15 January 2021
F. Chinesta, R. Abgrall, O. Allix and M. Kaliske (Eds)

SOME REMARKS ON THE A POSTERIORI ERROR ANALYSIS OF THE MIXED LAPLACE EIGENVALUE PROBLEM

FLEURIANNE BERTRAND[1], DANIELE BOFFI[2], JOSCHA GEDICKE[3] AND ARBAZ KHAN[4]

[1] University of Twente
Faculty of Electrical Engineering, Mathematics and Computer Science, Zilverling, P.O. Box 217, 7500 AE Enschede, The Netherlands
f.bertrand@utwente.nl

[2] King Abdullah University of Science and Technology (KAUST) and University of Pavia
Thuwal 23955-6900, Kingdom of Saudi Arabia and Via Ferrata 1, I-27100 Pavia, Italy
daniele.boffi@kaust.edu.sa

[3] University of Bonn
Institute for Numerical Simulation, Friedrich-Hirzebruch-Allee 7, 53115 Bonn, Germany
gedicke@ins.uni-bonn.de

[4] Indian Institute of Technology Roorkee (IITR)
Department of Mathematics, Roorkee, India-247667
arbaz@ma.iitr.ac.in

Key words: eigenvalue problem, adaptivity, a posteriori error estimator, mixed finite element method, postprocessing

Abstract. In this note we consider the a posteriori error analysis of mixed finite element approximations to the Laplace eigenvalue problem based on local postprocessing. The estimator makes use of an improved L^2 approximation for the Raviart-Thomas (RT) and Brezzi-Douglas-Marini (BDM) finite element methods. For the BDM method we also obtain improved eigenvalue convergence for postprocessed eigenvalues. We verify the theoretical results in several numerical examples.

1 INTRODUCTION

The aim of this note is to discuss some remarks about the approximation of the eigensolutions of the mixed Laplace problem: find $\lambda \in \mathbb{R}$ and $u \in L^2(\Omega)$ with $u \neq 0$ such that for some $\sigma \in H(\mathrm{div};\Omega)$ it holds

$$\begin{cases} (\sigma,\tau) + (\mathrm{div}\,\tau, u) = 0 & \forall \tau \in H(\mathrm{div};\Omega) \\ (\mathrm{div}\,\sigma, v) = -\lambda(u,v) & \forall v \in L^2(\Omega). \end{cases} \tag{1}$$

As it is known, this corresponds to the Dirichlet eigenvalue problem for the Laplace, which in strong form reads as follows

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume that the reader is familiar with the characterization of the solution to compact symmetric eigenvalue problems and to the classical Babuška–Osborn theory for their finite element approximation. We refer to [5] for a summary, with particular reference to mixed problems.

We are interested in particular in a posteriori error estimates. Several error estimators for the mixed approximation are known for the corresponding source problem, c.f. [1, 9, 7, 8, 11, 13], and in some cases it has been observed how to generalize them to the eigenvalue problem. Residual a posteriori error estimators for the mixed approximation of the Laplace eigenvalue problem have been studied in [6, 10], and error estimators based on a local postprocessing are given in [3].

In [3] an error estimator was proposed based on the following two steps: first a local postprocessing procedure was performed in the spirit of [14]; then, the postprocessed solution was averaged in order to obtain a continuous reconstruction of the solution. It was proved that this procedure is effective in the case when Raviart–Thomas (RT) elements are used for the approximation of the gradients. The result relies on the estimation of a typical nonlinear term that arises when arguments valid for the source problem are extended to the eigenvalue problem. In [3] it was shown that such nonlinear contribution is a higher order term with respect to the a priori rate of convergence of the quantities of interest.

In [2] it was shown that the same result doesn't apply to the case when Brezzi–Douglas–Marini (BDM) elements are used for the approximation of the gradients. Actually, the nonlinear term has been shown to converge at the same order as the other quantities of interest. This implies that the analysis that was performed in [3] could not be extended to BDM spaces. This remark was the starting point of this research that addresses the a posteriori error analysis for the problem under consideration when the BDM scheme is used.

The eigenvalue problem associated with the Stokes problem was considered in [12] by using a stress-velocity formulation related to linear elasticity. The finite element discretization is based on the Arnold–Winther (AW) space for the approximation of the stresses which can be seen as a generalization of the BDM space. For this reason, the results of [12] prove very useful in this context. More precisely, the results translate almost verbatim to the approximation of the mixed Laplacian based on BDM finite elements. It is then quite natural to compare the obtained results with the discussion in [3] and [2]. The a posteriori estimator of [12] is based on the first step only: a local postprocessing procedure in the spirit of [14]. Moreover, it provides in a natural way a postprocessed solution for the eigenvalues which converges faster than the original one.

In this paper we recall the results of [3] for the RT scheme and we state the results of [12] applied to the BDM scheme.

2 SETTING OF THE PROBLEM

The approximation of (1) consists in choosing finite dimensional subspaces $U_h \subset L^2(\Omega)$ and $\Sigma_h \subset H(\mathrm{div};\Omega)$ and to seek $\lambda_h \in \mathbb{R}$ and $u_h \in U_h$, with $u_h \neq 0$ such that for some $\sigma_h \in \Sigma_h$ it holds

$$\begin{cases} (\sigma_h, \tau) + (\mathrm{div}\,\tau, u_h) = 0 & \forall \tau \in \Sigma_h \\ \qquad (\mathrm{div}\,\sigma_h, v) = -\lambda_h(u_h, v) & \forall v \in U_h. \end{cases} \tag{2}$$

We are going to consider two possible schemes: the RT scheme and the BDM scheme.

We consider a polygonal or polyhedral Lipschitz domain Ω in $\mathbb{R}^d$ $(d = 2,3)$ and a conforming triangulation $\mathcal{T}_h$ of shape regular triangles or tetrahedra. For $k \geq 0$, in both cases the solution space will be

$$U_h = \{v \in U : v|_T \in P_k(T)\ \forall T \in \mathcal{T}_h\},$$

where $P_k(T)$ denotes the space of polynomials of degree not exceeding k.

In the case of the RT scheme we have

$$\Sigma_h = RT_k = \{\tau \in H(\mathrm{div};\Omega) : \tau|_T \in (P_k(T))^d \oplus \mathbf{x}\tilde{P}_k(T)\ \forall T \in \mathcal{T}_h\},$$

where $\tilde{P}_k(T)$ is the space of homogeneous polynomials of degree k.

For the BDM scheme we have

$$\Sigma_h = BDM_{k+1} = \{\tau \in \Sigma : \tau|_T \in P_{k+1}(T)\ \forall T \in \mathcal{T}_h\}.$$

For ease of presentation, we consider a simple eigenvalue λ with the corresponding eigenfunction u, and assume that the eigenfunctions are normalized, such that $\|u\|_0 = \|u_h\|_0 = 1$. It is well known that our choice of spaces provides a stable and convergence approximation so that for each h there exists λ_h and u_h converging to λ and u. Moreover, the system (2) gives a unique σ_h corresponding to u_h and it holds that σ_h converges to $\sigma = \nabla u$.

Since Ω is a polytope, standard regularity results guarantee that the eigenfunction u belongs to $H^{1+s}(\Omega)$, for some $s > 1/2$ and σ belongs to $H^s(\Omega) \cap H(\mathrm{div};\Omega)$.

Then, in the case of the RT scheme, the following a priori estimates are satisfied ($k \geq 0$)

$$\begin{aligned} |\lambda - \lambda_h| &\leq Ch^{2r}|u|_{1+r} \\ \|u - u_h\|_0 &\leq Ch^r|u|_{1+r} \\ \|\sigma - \sigma_h\|_0 &\leq Ch^r|u|_{1+r}, \end{aligned}$$

with $r = \min(s, k+1)$.

When the BDM scheme is used, the effect of a larger space Σ_h is that the previous estimates become unbalanced. The convergence for $\sigma - \sigma_h$ is higher than the other terms and the eigenvalue convergence does not take any advantage by the fact that we are using a richer approximation:

$$\begin{aligned} |\lambda - \lambda_h| &\leq Ch^{2r}|u|_{1+r} \\ \|u - u_h\|_0 &\leq Ch^r|u|_{1+r} \\ \|\sigma - \sigma_h\|_0 &\leq Ch^{r'}|u|_{1+r'}, \end{aligned}$$

where $r = \min(s, k+1)$ as before and $r' = \min(s, k+2)$. If the solution is smooth enough, then $r' = r+1$.

3 THE POSTPROCESSING TECHNIQUE

For $k \geq 0$ and $\ell \geq 1$ we consider the space

$$U_h^* = \{v \in L^2(\Omega) : v|_T \in P_{k+\ell}(T)\ \forall T \in \mathcal{T}_h\}.$$

The local postprocessing of the eigenfunction $u_h \in U_h$ can be performed along the lines of [13, 14] as follows: find $u_h^* \in U_h^*$ such that

$$\begin{cases} P_h u_h^* = u_h \\ (\nabla u_h^*, \nabla v)_T = (\sigma_h, \nabla v) \quad \forall v \in (I - P_h) U_h^*|_T\ \forall T \in \mathcal{T}_h, \end{cases}$$

where P_h denotes the $L^2(\Omega)$-projection onto U_h, I is the identity operator, and σ_h is the component of the solution associated with u_h.

A variational formulation of the construction of u_h^* is obtained, for instance, by solving the following local problem for all T in $\mathcal{T}_h$: find $u_h^* \in U_h^*|_T$ and $z_h \in U_h|_T$ such that

$$\begin{cases} (\nabla u_h^*, \nabla v) + (z_h, v) = (\sigma_h, \nabla v) \quad \forall v \in U_h^*|_T \\ (u_h^*, w) = (u_h, w) \quad \forall w \in U_h|_T. \end{cases}$$

The choice of ℓ depends on the scheme that we are using: in the case of the RT scheme we take $\ell = 1$ and in the case of BDM we choose $\ell = 2$. This is compatible with the fact that the postprocessed solution should be mimicking σ_h and in the case of the BDM approximation, as explained before, we have a higher rate of convergence for the variable approximating ∇u.

3.1 Postprocessing of the eigenvalue for the BDM scheme

The motivation for the eigenvalue postprocessing is related to the aforementioned imbalance of the eigenfunction errors. The identity [10, Lemma 3.2] for the eigenvalue and eigenfunction errors

$$\lambda - \lambda_h = \|\sigma - \sigma_h\|_0^2 - \lambda_h \|u - u_h\|_0^2 \tag{3}$$

shows that the convergence of the eigenvalues is limited by the L^2 convergence of the eigenfunctions. Hence, we may improve the eigenvalue convergence by a postprocessing of both the eigenfunction u_h and the eigenvalue λ_h.

It can be easily seen that the eigenvalues λ_h of (2) are related to u_h and σ_h by the following relation which resembles the Rayleigh quotient

$$\lambda_h = -\frac{(\operatorname{div} \sigma_h, u_h)}{\|u_h\|_0^2}.$$

Following [12, Def. 4.2] it is then natural to define a postprocessed eigenvalue by replacing u_h with the postprocessed solution u_h^*

$$\lambda_h^* = -\frac{(\operatorname{div} \sigma_h, u_h^*)}{\|u_h^*\|_0^2}.$$

It turns out that the postprocessed eigenvalue satisfies the following bound

$$|\lambda-\lambda_h^*| \leq C\left(\|\sigma-\sigma_h\|_0^2+\|u-u_h^*\|_0^2+\|\operatorname{div}(\sigma-\sigma_h)\|_0\|u-u_h^*\|_0+|\lambda-\lambda_h^*|^2\right),$$

that can be obtained in the same way as in [12, Thm. 4.3]. This implies the following superconvergence result if the solution is smooth enough.

Theorem 1. *Let u be in $H^{k+2}(\Omega)$ and σ in $H^{k+2}(\Omega)$, then the following estimate is valid*

$$|\lambda-\lambda_h^*| \leq Ch^{2(k+2)}(\|u\|_{k+2}+\|\sigma\|_{k+2}).$$

3.2 Error estimator for the RT scheme

The error estimator in the case of the RT is obtained in [3] by performing a further postprocessing of the function u_h^*. It consists in a standard averaging technique that returns a function u_h^{**} in $H_0^1(\Omega)$. More precisely, using Oswald interpolation, it is possible to define u_h^{**} in the space

$$U_h^{**}=\{v\in H_0^1(\Omega): v|_T\in P_{k+1}(T)\ \forall T\in\mathcal{T}_h\}$$

satisfying

$$\|\nabla u-\nabla u_h^{**}\|_0 \leq Ch^r\|u\|_{1+r}$$

with $r=\min(s,k+1)$.

A natural error estimator can then be defined as

$$\eta_{RT}(T)=\|\nabla u_h^{**}-\sigma_h\|_{0,T},\quad T\in\mathcal{T}_h,$$
$$\eta_{RT}=\|\nabla u_h^{**}-\sigma_h\|_0.$$

The following local efficiency and global reliability results (up to a higher terms) is proved in [3].

Theorem 2. *Let λ be a simple eigenvalue of* (1) *and (u,σ) the associated eigenfunction. Let λ_h be the corresponding discrete eigenvalue of* (1) *and (u_h,σ_h) converging to (u,σ) when the RT scheme is used.*

Then the following local efficiency holds true for all elements T of the triangulation $\mathcal{T}_h$

$$\eta_{RT}(T) \leq \|\nabla u-\nabla u_h^{**}\|_{0,T}+\|\sigma-\sigma_h\|_{0,T}.$$

Moreover, the following global reliability is satisfied

$$\|\nabla u-\nabla u_h^{**}\|_0^2+\|\sigma-\sigma_h\|_0^2 \leq \eta_{RT}^2+hot(h)$$

with

$$hot(h)=2\|u-u_h^{**}\|_0\|\lambda u-\lambda_h u_h\|_0.$$

The previous theorem has the merit of providing an a posteriori analysis with explicit constants equal to one in the efficiency and reliability bounds. It could be extended to the BDM scheme as well, but in this case the term $hot(t)$ is not of higher order. Numerical experiments in [2] show that indeed it is of the same order as the estimator.

By using the identity (3) and the previous theorem, we easily obtain the following reliability result for the error in the eigenvalue when a slightly modified error estimator is used.

Corollary 1 (reliability for the error in the eigenvalue)**.** *Under the same hypotheses of the previous theorem, for h small enough it holds*

$$|\lambda - \lambda_h| \leq \tilde{\eta}_{RT}^2 + hot(h)$$

with

$$\tilde{\eta}_{RT}^2 := \eta_{RT}^2 + 2\lambda_h \|u_h^{**} - u_h\|_0^2, \qquad hot(h) := 2\lambda_h \|u - u_h^{**}\|_0^2 + 2\|u - u_h^{**}\|_0 \|\lambda u - \lambda_h u_h\|_0.$$

In the next subsection we show how the estimator introduced in [12] for the Stokes eigenproblem can be used for the BDM scheme applied to (2).

3.3 Error estimator for the BDM scheme

For the BDM scheme we define the following error indicator which is based on the postprocessed eigenvalue λ_h^* and eigenfunction u_h^* without the need of any additional regularizations

$$\eta_{BDM}^2 = \sum_{T \in \mathcal{T}_h} \|\nabla u_h^* - \sigma_h\|_{0,T}^2 + \sum_{T \in \mathcal{T}_h} h_T^2 \|\lambda_h^* u_h^* + \operatorname{div} \sigma_h\|_{0,T}^2 + \sum_{E \in \mathcal{E}_h} h_E^{-1} \| [\![u_h^*]\!] \|_{0,E}^2.$$

We now state some reliability and efficiency theorems that are obtained analogously to what has been done in [12]. We don't repeat the proofs, but we give reference to the corresponding results in [12] from which the proofs can be obtained with natural modifications.

Theorem 3 (reliability for the error in the eigenfunction: corresponding to [12, Thm. 5.3])**.** *Let λ be a simple eigenvalue of* (1) *and (u, σ) the associated eigenfunction. Let λ_h be the corresponding discrete eigenvalue of* (1) *and (u_h, σ_h) converging to (u, σ) when the BDM scheme is used. Then the following estimate holds true*

$$\|\sigma - \sigma_h\|_0 + \left(\sum_{T \in \mathcal{T}_h} \|\nabla(u - u_h^*)\|_{0,T}^2 \right)^{1/2} \leq C\left(\eta_{BDM} + hot_1(h) + hot_2(h)\right)$$

with

$$hot_1(h) = \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|\lambda u - \lambda_h^* u_h^*\|_{0,T}^2 \right)^{1/2}, \qquad hot_2(h) = \lambda_h \|u - u_h^*\|_0 + |\lambda - \lambda_h|.$$

From [14, Thm 2.2] we have for the postprocessing of the BDM scheme applied to the associated source problem with source term $f = \lambda u$ satisfies the following bound

$$\|u - u_h^*\|_0 \leq \begin{cases} Ch^{1+r'}(|\sigma|_{r'} + |u|_{1+r'}) & \text{for } k \geq 1 \\ Ch^2(|\sigma|_2 + |u|_2) & \text{for } k = 0. \end{cases}$$

One can show that the same estimates hold true also for the eigenvalue problem. Hence, $hot_1(h)$ is of higher order $O(h^{1+r'})$ for all $k \geq 0$. For the second term $hot_2(h)$ one has to distinguish two cases: for $k \geq 1$, $hot_2(h)$ is of higher order $O(h^{1+r'})$, while for $k = 0$ it is of the same order $O(h^2)$. However, η_{BDM} can still be used as a posteriori error estimator for $k = 0$ in practice, as demonstrated in the numerical examples of Section 4.

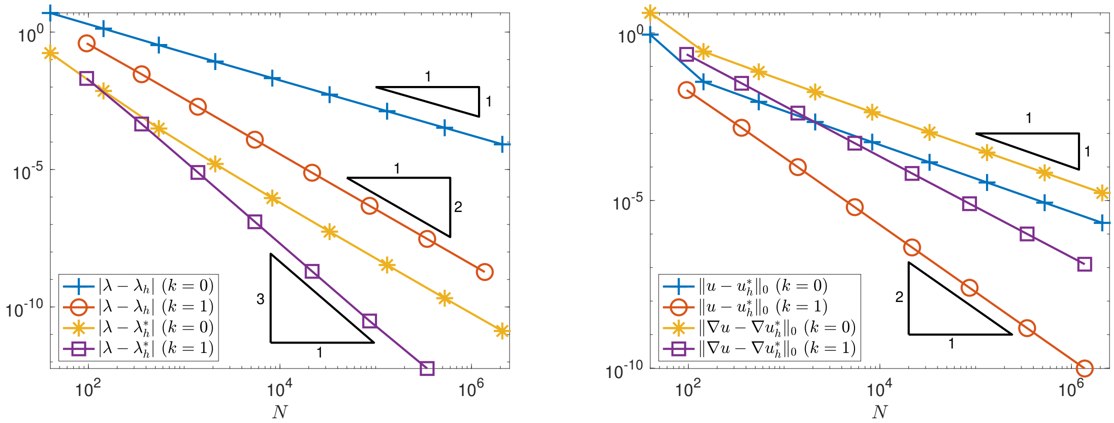

Figure 1: Convergence of the BDM postprocessed eigenvalues (left) and eigenfunctions (right) on the unit square

Theorem 4 (reliability for the error in the eigenvalue: corresponding to [12, Thm. 5.4])**.** *Under the same hypotheses of the previous theorem, for h small enough it holds*

$$|\lambda-\lambda_h^*| \leq C\left(\eta_{BDM}^2 + (hot_1(h)+hot_2(h))^2\right).$$

Theorem 5 (efficiency: corresponding to [12, Thm. 5.5])**.** *Under the same assumptions as in the previous theorem, the following bound is valid*

$$\eta_{BDM} \leq C\Bigg(\|\sigma-\sigma_h\|_0 + \left(\sum_{T\in\mathcal{T}_h} \|\nabla(u-u_h^*)\|_{T,0}^2\right)^{1/2} + h\left(|\lambda-\lambda_h| + \lambda_h\|u-u_h\|_0 + |\lambda-\lambda_h^*| + \lambda_h^*\|u-u_h^*\|_0\right)\Bigg).$$

Note that while reliability holds for any $u_h^* \in U_h^*$, $k \geq 1$, efficiency holds only for convex domains with sufficiently smooth solutions. However, as we will demonstrate in the next section, efficiency also holds numerically in the case of singular eigenfunctions for graded meshes generated by an adaptive algorithm.

4 NUMERICAL EXPERIMENTS

This section is devoted to several numerical experiments using uniform and adaptive h-refinements of triangular meshes for two-dimensional domains. The solution of problem (2) is computed with RT and BDM elements of order $k=0$ and $k=1$.

4.1 Unit square

As first example we consider the unit square $\Omega=(0,1)^2$. The first eigenvalue $\lambda = 2\pi^2$ is known to be simple and the corresponding eigenfunction (L^2-normalized to 1) reads $u(x) = 2\sin(\pi x_1)\sin(\pi x_2)$. In Figure 1 we verify the higher order convergence of the postprocessed eigenvalues λ_h^* and eigenfunctions

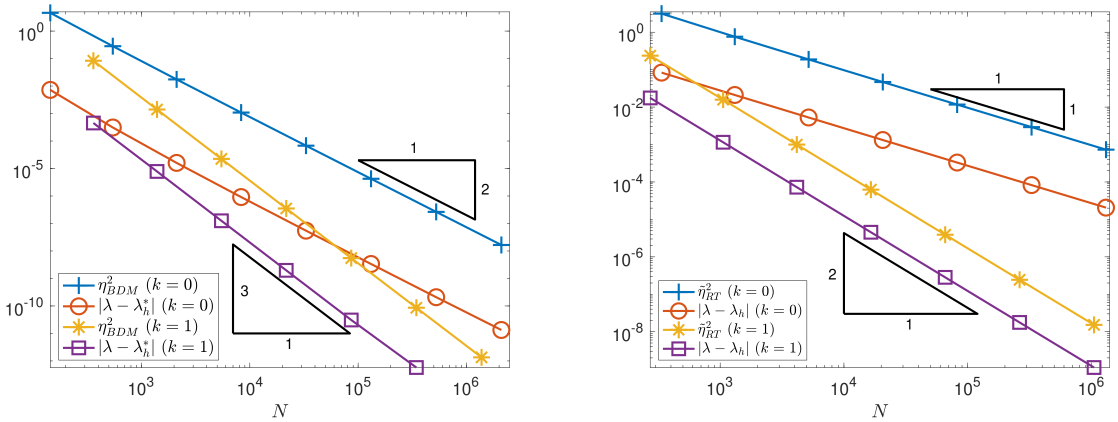

Figure 2: Convergence of eigenvalues and estimators for BDM (left) and RT (right) on the unit square

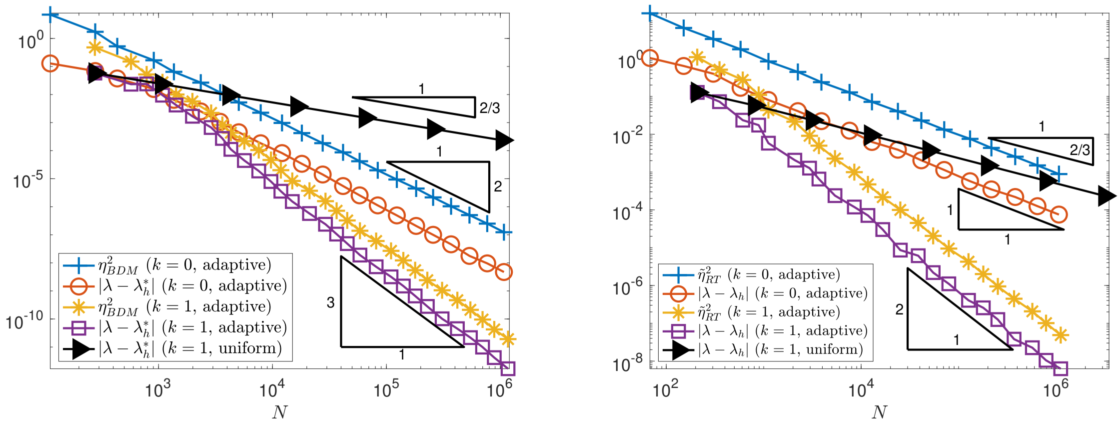

Figure 3: Convergence of eigenvalues and estimators for BDM (left) and RT (right) on the L-shaped domain

u_h^* for the BDM scheme. Note that for uniform refinement $O(h) = O(N^{-1/2})$, where $N := \dim(\Sigma_h) + \dim(U_h)$ denotes the number of degrees of freedom. Hence, we observe $O(h^{2(k+2)})$ convergence of the postprocessed eigenvalues λ_h^* as predicted by the theory. For the postprocessed eigenfunctions u_h^* we observe $O(h^{k+2})$ convergence in the H^1-norm, and for $k = 1$, $O(h^{k+3})$ convergence in the L^2-norm. Note that in the case $k = 0$, the convergence of u_h^* in the L^2-norm is only of order $O(h^2)$ as predicted by the theory.

In Figure 2 we demonstrate the reliability and efficiency of η_{BDM}^2 and $\tilde{\eta}_{RT}^2$ for $k = 0, 1$ and the eigenvalue error. Due to the improved eigenvalue convergence, the BDM scheme yields faster convergence for the same index k as the RT scheme. The estimator η_{BDM}^2 shows to be reliable and efficient also in the case $k = 0$.

4.2 L-shaped domain

The first eigenvalue for the L-shaped domain $\Omega = (-1,1)^2 \setminus ([0,1] \times [-1,0])$ reads $\lambda \approx 9.6397238440219$, where all but the last two digits have been proven to be correct [4]. Since the corresponding eigenfunction is singular, we observe suboptimal convergence rates of $O(N^{-2/3})$ for uniform meshes in Figure 3. We can improve the order of convergence using an adaptive finite element scheme. We consider the standard algorithm with the steps *solve*, *estimate*, *mark* and *refine*. We employ a direct solver to solve the saddle point systems, the bulk marking strategy with bulk parameter $\theta = 1/2$, and the red-green-blue refinement strategy. The estimators η^2_{BDM} and $\tilde{\eta}^2_{RT}$ prove to be reliable and efficient for the eigenvalue error on adaptively refined meshes. Moreover, we observe optimal orders of convergence $O(N^{-(k+2)})$ for the BDM adaptive scheme with postprocessed eigenvalues and $O(N^{-(k+1)})$ for the RT adaptive scheme.

ACKNOWLEDGEMENTS

Fleurianne Bertrand gratefully acknowledges support by the German Research Foundation (DFG) in the Priority Programme SPP 1748 *Reliable simulation techniques in solid mechanics* under grant number BE6511/1-1. Daniele Boffi is member of the INdAM Research group GNCS and his research is partially supported by IMATI/CNR and by PRIN/MIUR. Arbaz Khan has been supported by the faculty initiation grant MTD/FIG/100878 and Serb Matrics grant MTR/2020/000303.

REFERENCES

[1] Ana Alonso. Error estimators for a mixed method. *Numer. Math.*, 74(4):385–395, 1996.

[2] Fleurianne Bertrand, Daniele Boffi, and Rolf Stenberg. A posteriori error analysis for the mixed Laplace eigenvalue problem: investigations for the BDM-element. *PAMM - Proc. Appl. Math. Mech.*, 19(1), 2019.

[3] Fleurianne Bertrand, Daniele Boffi, and Rolf Stenberg. Asymptotically exact a posteriori error analysis for the mixed Laplace eigenvalue problem. *Comput. Methods Appl. Math.*, 20(2):215–225, 2020.

[4] Timo Betcke and Lloyd N. Trefethen. Reviving the method of particular solutions. *SIAM Rev.*, 47(3):469–491, 2005.

[5] Daniele Boffi. Finite element approximation of eigenvalue problems. *Acta Numer.*, 19:1–120, 2010.

[6] Daniele Boffi, Dietmar Gallistl, Francesca Gardini, and Lucia Gastaldi. Optimal convergence of adaptive FEM for eigenvalue clusters in mixed form. *Math. Comp.*, 86(307):2213–2237, 2017.

[7] Carsten Carstensen. A posteriori error estimate for the mixed finite element method. *Math. Comp.*, 66(218):465–476, 1997.

[8] Carsten Carstensen and Sören Bartels. Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids. I. Low order conforming, nonconforming, and mixed FEM. *Math. Comp.*, 71(239):945–969, 2002.

[9] Carstensen Carstensen. A unifying theory of a posteriori finite element error control. *Numer. Math.*, 100(4):617–637, 2005.

[10] Ricardo G. Durán, Lucia Gastaldi, and Claudio Padra. A posteriori error estimators for mixed approximations of eigenvalue problems. *Math. Models Methods Appl. Sci.*, 9(8):1165–1178, 1999.

[11] Alexandre Ern and Martin Vohralík. Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed discretizations. *SIAM J. Numer. Anal.*, 53(2):1058–1081, 2015.

[12] Joscha Gedicke and Arbaz Khan. Arnold–Winther mixed finite elements for Stokes eigenvalue problems. *SIAM J. Sci. Comput.*, 40(5):A3449–A3469, 2018.

[13] Carlo Lovadina and Rolf Stenberg. Energy norm a posteriori error estimates for mixed finite element methods. *Math. Comp.*, 75(256):1659–1674, 2006.

[14] Rolf Stenberg. Postprocessing schemes for some mixed finite elements. *RAIRO Modél. Math. Anal. Numér.*, 25(1):151–167, 1991.