TRANSIENT CONTACT INTERACTION OF SOLID BODIES AND THIN SHELLS

DMITRY V. TARLAKOVSKII[1], GREGORY V. FEDOTENKOV[2]

[1] Lomonosov Moscow State University, Moscow Aviation Institute
125993, Moscow, Volokolamskoe shosse, 4
tdvhome@mail.ru

[2] Lomonosov Moscow State University, Moscow Aviation Institute
125993, Moscow, Volokolamskoe shosse, 4
greghome@mail.ru

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Abstract. A spatial transient contact problem with moving boundaries for a thin elastic cylindrical shell and a rigid indenter bounded by a smooth convex surface is considered. A closed mathematical formulation is given and a system of resolving equations is constructed. The main integral equation follows from the principle of superposition and contact conditions. The core of this equation is the transient function for the cylindrical shell. To a closed system of resolving equations, it is supplemented by a kinematic relation for determining the moving boundary of the contact region and the equation of motion of the indenter as a rigid body. An algorithm for solving the spatial non-stationary contact problem for an infinitely long cylindrical shell and rigid indenter in the case of a normal impact on the side surface of the shell is constructed and implemented. Examples of calculations are given.

1 INTRODUCTION

Transient dynamics of continuous media is one of the most difficult areas of mechanics. The relevance of its study is because, as known, all natural phenomena are nonstationary. The concepts of static and stationary processes are often used as nothing more than an approximation (usually justified) of real phenomena. In many cases, dynamic properties of the medium are necessary, both qualitatively and quantitatively. The study of non-stationary processes is relevant for modern space and aviation technology, shipbuilding, construction of engineering structures, etc.

One has to deal with the issues of unsteady dynamics of elastic shells when solving a wide variety of problems. They find important practical applications in the aerospace, automotive and shipbuilding industries. Their successful solution is associated with the further harmonious development of various sciences: aerohydrodynamics, nonstationary contact problems, acoustics, applied and computational mathematics, and others.

One of the most difficult and least studied problems are transient contact problems with moving boundaries. The complexity of solving problems of this class is due to the mixed and

nonlinear nature of the boundary conditions, as well as the unknown position of the boundary of the contact area, which moves in time. To solve such problems, approaches based on the method of transient functions (Green's functions) are used [1−7]. Note that these approaches to the solution are based on a preliminary analytical solution of problems on the construction of influence functions and are a convenient tool for solving problems of nonstationary dynamics of elastic bodies, including nonstationary inverse problems. Approaches based on the finite element method and other numerical methods are also possible.

In this paper, we propose a method and an algorithm for solving transient contact problems for cylindrical shell and rigid indenter. It is based on the application of the method of transient functions, according to which, first, by analytical methods, a fundamental solution (transient function) for a shell is constructed [1, 7]. Then, based on the principle of superposition, the problem is reduced to a system of resolving equations. The resulting system is solved using a numerical-analytical algorithm based on the grid method. Note that, despite the large number of works devoted to the mechanics of shells and contact problems, there are practically no publications devoted to solving spatial transient contact problems. The constructed technique can serve as a basis for the development of mathematical formulations and methods for solving new transient contact problems as well as for contact problems taking into account anisotropic combined dry friction [8-12].

2 STATEMENT OF THE PROBLEM

To describe the motion of the shell, we use the equations of motion of the S. P. Timoshenko model in displacements, record associated with the directions of the main curvatures of the middle surface [13]:

$$\partial^2 \mathbf{W}/\partial \tau^2 = \mathbf{L}\mathbf{W} + \mathbf{P}, \quad \mathbf{L} = \left(L_{ij}\right)_{5\times5}, \quad \mathbf{W} = \left(u, v, w, \chi_\alpha, \chi_z\right)^{\mathrm{T}}, \quad \mathbf{P} = \left(0, 0, p, 0, 0\right)^{\mathrm{T}} \tag{1}$$

$$L_{11} = \frac{\partial^2}{\partial \alpha^2} + \eta^2 \left(\frac{\partial^2}{\partial z^2} - k^2 \right), \ L_{12} = \left(1 - \eta^2\right) \frac{\partial^2}{\partial \alpha \partial z}, \ L_{13} = \left(1 + k\eta^2\right) \frac{\partial}{\partial \alpha}$$

$$L_{14} = -\gamma^2 \frac{\partial^2}{\partial \alpha^2} + \eta^2 k^2, \ L_{15} = -\gamma^2 \left(1 - 2\eta^2\right) \frac{\partial^2}{\partial \alpha \partial z}, \ L_{21} = L_{12}, \ L_{22} = \frac{\partial^2}{\partial z^2} + \eta^2 \frac{\partial^2}{\partial \alpha^2}$$

$$L_{23} = \left(1 - 2\eta^2\right) \frac{\partial}{\partial z}, \ L_{24} = -\gamma^2 \eta^2 \frac{\partial^2}{\partial \alpha \partial z}, \ L_{25} = -\gamma^2 \eta^2 \frac{\partial^2}{\partial \alpha^2}, \ L_{31} = -L_{13}$$

$$L_{32} = -L_{23}, \ L_{33} = \eta^2 k^2 \left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial z^2} \right) - 1, \ L_{34} = \eta^2 k^2 \frac{\partial}{\partial \alpha}, \ L_{35} = \eta^2 k^2 \frac{\partial}{\partial z}$$

$$L_{41} = \gamma^{-2} L_{14}, \ L_{42} = \gamma^{-2} L_{24}, \ L_{43} = -\gamma^{-2} L_{34}, \ L_{44} = \frac{\partial^2}{\partial \alpha^2} + \eta^2 \left(\frac{\partial^2}{\partial z^2} - k^2 \gamma^{-2} \right)$$

$$L_{45} = L_{12}, \ L_{51} = \gamma^{-2} L_{15}, \ L_{52} = \gamma^{-2} L_{25}, \ L_{53} = -\gamma^{-2} L_{35}$$

$$L_{54} = L_{12}, \ L_{55} = \frac{\partial^2}{\partial z^2} + \eta^2 \left(\frac{\partial^2}{\partial \alpha^2} - k^2 \gamma^{-2} \right)$$

u, v − tangential displacements, w − normal displacement, χ_α, χ_z − normal fiber rotation angles, $\mathbf{P}$ − external forces, τ − dimensionless time, $\mathbf{L}$ is the matrix-operator. All variables and parameters are reduced hereinafter to a dimensionless form as follows (primed variables

correspond to dimensional ones):

$$u = R^{-1}u', v = R^{-1}v', w = R^{-1}w', u_c = R^{-1}u_c', z = R^{-1}z', \tau = R^{-1}c_1 t, \eta^2 = c_2^2 c_1^{-2}, c_1^2 = \rho^{-1}(\lambda + 2\mu)$$

ρ – mass density, λ, μ – Lame parameters, R and h – radius and thickness of the shell, u_c – displacement of the center of mass of the indenter, c_1 and c_2 – speed of the tension-compression waves and shear waves in the shell material, t – dimensional time.

At the initial moment of time, an absolutely solid impactor, moving with a given initial velocity, comes into contact with the lateral surface of the infinitely long thin elastic circular cylindrical shell. The impactor is a solid body with a smooth convex surface, the vector of its initial velocity is directed normal to the side surface of the shell (Fig. 1).
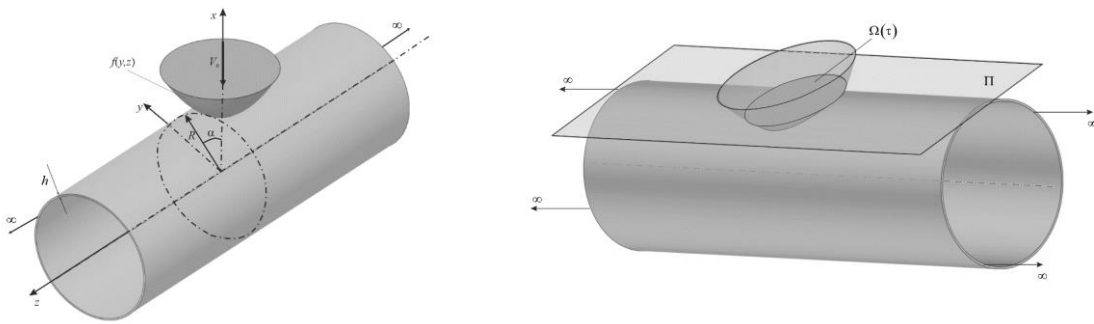
Figure 1: Statement of the problem, contact area

The equation of motion of the striker as an absolutely rigid body has the form (2):

$$m_u \ddot{u}_c(\tau) = P(\tau) = \int_{\Omega(\tau)} p(\xi, \zeta, \tau) d\xi d\zeta, \ u_c(0) = 0, \ \dot{u}_c(0) = V_0 \tag{2}$$

P – the resulting force of contact pressure p, $\Omega(\tau)$ – the contact area, the boundary of which depends on time.

We assume that the contact between the shell and the striker occurs under conditions of free sliding. Let Π_1 and Π_2 be the surfaces bounding the shell and the striker at the moment of time $\tau > 0$. With respect to an inertial Cartesian rectangular frame whose axis coincides with the axis of the shell, they are given by the equations

$$\Pi_k : x = f_k(y, z, \tau), (y, z) \in \Omega_k, k = 1, 2$$

Ω_k – projections of surfaces Π_1 and Π_2 on a plane Oyz. The true contact area is defined by the conditions (3):

$$\Pi_* : \Delta(y, z, \tau) = f_1(y, z, \tau) - f_2(y, z, \tau) = 0, \ p(y, z, \tau) < 0, (y, z) \in \Omega_k \tag{3}$$

where Δ – is the gap between the boundary surfaces Π_1 and Π_2. The equations that implicitly define the boundary $\partial \Pi_*$ of the contact area follow from (3):

$$\partial \Pi_* : \Delta(y, z, \tau) = 0, p(y, z, \tau) = 0$$

In the linearized formulation, the true contact area Π_* is replaced by a fictitious area $\Omega(\tau)$ belonging to the plane that is a common tangent to the surfaces Π_1, Π_2 at the initial time of interaction (Fig. 1). Suppose that the position of the surface bounding the striker at the time τ is given by the equation referred to the frame $Oxyz$:

$$x = f(y,z) + u_c(\tau) \tag{4}$$

The boundary $\partial\Omega(\tau)$ of the area $\Omega(\tau)$ is determined from the condition of intersection of the surface bounding the striker with the plane Π:

$$\partial\Omega(\tau): f(y,z) + u_c(\tau) = 1 \tag{5}$$

Taking into account (4), the contact conditions of the indenter and the shell have the form

$$f(y,z) + u_c(\tau) = w(\alpha, z, \tau), \; p(\alpha, z, \tau) < 0, \; y, z \in \Omega(\tau), \; y = \sin\alpha \tag{6}$$

At the initial time, the shell is in an unreformed state, which corresponds to zero initial conditions

$$u|_{\tau=0} = v|_{\tau=0} = w|_{\tau=0} = \chi_\alpha|_{\tau=0} = \chi_z|_{\tau=0} = \dot{u}|_{\tau=0} = \dot{v}|_{\tau=0} = \dot{w}|_{\tau=0} = \dot{\chi}_\alpha|_{\tau=0} = \dot{\chi}_z|_{\tau=0} = 0 \tag{7}$$

Equations and relations (1), (2), (4)-(7) are a closed mathematical formulation of the non-stationary contact problem with movable boundaries for a cylindrical shell and a rigid indenter.

3. SYSTEM OF RESOLVING EQUATIONS

To reduce the formulation of the contact problem we use the transient function for an infinitely long circular cylindrical shell. It represents the normal movement of the shell as a solution to the following problem:

$$\partial^2 \mathbf{G}/\partial\tau^2 = \mathbf{L}\mathbf{G} + \mathbf{Q}, \; \mathbf{L} = (L_{tj})_{5\times5}, \; \mathbf{G} = (G_u, G_v, G_w, G_{\chi1}, G_{\chi2})^{\mathrm{T}}, \mathbf{Q} = [0,0,\delta(\alpha)\delta(z)\delta(\tau),0,0]^{\mathrm{T}}$$

$\delta(\alpha)$, $\delta(z)$, $\delta(\tau)$ are the Dirac functions. The transient function was found in [7].

Based on the superposition principle [1-7], deflections are related to contact pressures by means of the convolution

$$w(\alpha, z, \tau) = \int_0^\tau dt \iint_{\Pi_*(t)} G_w(\alpha - \xi, \, z - \zeta, \tau - t) \, p(\xi, \zeta, t) \, dS \tag{8}$$

Replace the contact area $\Pi_*(\tau)$ with a dummy area $\Omega(\tau)$. Since $\mathrm{diam}\,\Omega(\tau) \ll 1$, we assume

$$\sin\alpha \approx \alpha \approx y, \; (y,z) \in \Omega(\tau) \tag{9}$$

From contact condition (6), (8) and (9), the main integral equation for the sought contact pressure is implies

$$\int_0^\tau dt \iint_{\Omega(t)} G_w(y - \xi, \, z - \zeta, \tau - t) \, p(\xi, \zeta, t) \, dS = f(y,z) + u_c(\tau) \tag{10}$$

In equation (10), in addition to the contact pressure, the contact area $\Omega(\tau)$ and the

displacement of the center of mass of the indenter $u_c(\tau)$ are also unknown. To obtain a closed system, we add (5) which determines the position of the boundary of the contact region, and the equation of motion of the striker (2), written in the integral form

$$u_c(\tau) = V_0\tau + \frac{1}{m_u}\int_0^\tau (\tau - t)\,dt \iint_{\Omega(t)} p(y, z, t)\,dS \tag{11}$$

3. SOLUTION METHOD

To solve the system of resolving equations, we use a numerical-analytical algorithm based on the method of mechanical quadrature. We divide the time interval $[0, \tau]$ into M uniform steps $\Delta_t = \tau/M$: $t_m = m\Delta_t$, $m = \overline{1, M}$, $t_M = M\Delta_t = \tau$. We replace the time integrals in (10) and (11) by the sum of the integrals over the intervals $[\tau_{m-1}, \tau_m]$ and assume that the contact pressure remains constant within each time interval:

$$\sum_{m=1}^{M}\int_{t_{m-1}}^{t_m} dt \iint_{\Omega(t)} G_w(y - \xi, z - \zeta, \tau - t)\,p_m(\xi, \zeta)\,dS = f(y, z) + u_{cM} \tag{12}$$

$$u_{cM} = V_0\tau + \frac{1}{m_u}\sum_{m=1}^{M}\int_{\tau_{m-1}}^{\tau_m}(\tau - t)\,dt \iint_{\Omega(t)} p_m(\xi, \zeta)\,dS, \; u_{cM} = u_c(\tau), \; p_m(\xi, \zeta) = p(\xi, \zeta, t_m), \; t_m = m\Delta_t$$

The contact area $\Omega(t) \in \Pi$ is approximately replaced by the union of rectangular subareas K_{mij}:

$$\Omega(t) \approx \Omega_m \in \Pi, \; \Omega_m = \bigcup_{i,j} K_{mij}, \; K_{mij} = \left\{(\xi, \zeta) : \xi_{i-1} \leq \xi \leq \xi_i, \zeta_{j-1} \leq \zeta \leq \zeta_j, t = t_m\right\} \tag{13}$$

$$\xi_i - \xi_{i-1} = \zeta_j - \zeta_{j-1} = \Delta, \; \xi_i = b_{m1} + i\Delta, \; \zeta_j = d_{i1} + j\Delta, \; i \in [1, I]$$

$$I = \left[(b_{m2} - b_{m1})/\Delta\right]; j \in [1, J_i], J_i = \left[(d_{i2} - d_{i1})/\Delta\right]$$

where $\left[(b_{m2} - b_{m1})/\Delta\right]$, $\left[(d_{i2} - d_{i1})/\Delta\right]$ are the integer parts of numbers $(b_{m2} - b_{m1})/\Delta$ и $(d_{i2} - d_{i1})/\Delta$.

To secure the stability of the numerical scheme, the step Δ in the spatial variables is selected from the condition $\Delta > \Delta_t$ [6]. In particular, we can put $\Delta = 2\Delta_t$. The boundaries b_{m1}, b_{m2}, d_{i1}, d_{i2} changes in discrete values of spatial variables ξ_i and ζ_j are solutions of the equations

$$f(b_m, 0) + u_{cm} = 1 \Rightarrow b_{m1}, b_{m2}, \; f(\xi_i, d_i) + u_{cm} = 1 \Rightarrow d_{i1}, b_{i2}, \; u_{cm} = u(t_m) \tag{14}$$

Using (13), we obtain numerical analogs of equations (12). To do this, we assume that the contact pressure remains constant within an elementary square K_{mij} : $p_m(\xi, \zeta)\big|_{(\xi, \zeta) \in K_{ij}} = p_{mij} = \text{const}$. Then from (12) - (14) follows

$$\sum_{m=1}^{M}\sum_{i=1}^{I}\sum_{j=1}^{J_i} G_{mijkl}\, p_{mij} = f_{kl} + u_{cM}, \quad u_{cM} = V_0\tau + \frac{\Delta_t^2\Delta^2}{m_u}\sum_{m=1}^{M} A_m \sum_{i=1}^{I}\sum_{j=1}^{J_i} p_{mij}, f(b_m,0) + u_{cm} = 1 \Rightarrow \tag{15}$$

$$b_{m1}, b_{m2}, f(\xi_i, d_i) + u_{cm} = 1 \Rightarrow d_{i1}, d_{i2}, \; k,i \in [1,I], I = \left[(b_{m2} - b_{m1})/\Delta\right]$$

$$l, j \in [1, J_i], J_i = \left[(d_{i2} - d_{i1})/\Delta\right]$$

$$G_{mijkl} = \int_{t_{m-1}}^{t_m} \int_{\xi_{i-i}}^{\xi_i} \int_{\zeta_{j-i}}^{\zeta_j} G_w(y_k - \xi, z_l - \zeta, M\Delta_t - t)\, d\xi\, d\zeta\, dt, \quad A_m = M - m + \frac{1}{2}$$

4. NUMERICAL EXAMPLES

As an example, consider the unsteady contact interaction of a circular cylindrical shell, of radius $R = 1$ and thickness $h = 0.025$, with an absolutely rigid indenter bounded by a surface in the form of a paraboloid of revolution: $f(y,z) = y^2 + z^2 + 1$. As a shell material, we take steel with the following characteristics: $\rho = 7850 \text{ kg/m}^3$, $\lambda = 1.15 \times 10^{11}$ Pa, $\mu = 7.69 \times 10^{10}$ Pa, which is advised by dimensionless parameters: $\eta = 0.53$, $\gamma = 1.3 \times 10^{-6}$.

Let us consider the influence of the initial velocity of the striker on the process of unsteady contact interaction. Let's take three of its possible values for analysis $V_0 = 0.01$, $V_0 = 0.05$ and $V_0 = 0.1$. The contact surface in this case is a circle whose boundary coincides with the section of the surface $x = y^2 + z^2 + 1 - u_c(\tau)$ by a plane $x = 1$ at the current time.

Fig. 2a-2c shows the time dependence of the displacement of the center of mass of the striker (Fig. 2a), the radius of the contact area on time (Fig. 2b) and the resulting contact force $P(\tau)$ versus time (Fig. 2c). Here the solid curves correspond to $V_0 = 0.01$, dashed - $V_0 = 0.05$, dash-dotted - $V_0 = 0.1$.

Fig. 3a-3c shows the distribution of the contact pressure along the angular coordinate α at the time $\tau = 0.3$. Fig. 3a corresponds to the coordinate $z = 0$, 3b - $z = 0.15$, 3c - $z = 0.3$. Solid curves correspond $V_0 = 0.01$, dashed - $V_0 = 0.05$, dash-dotted - $V_0 = 0.1$.

Fig. 4a-4c shows the distribution of the contact pressure along the longitudinal coordinate z at the time $\tau = 0.3$. Fig. 4a corresponds to the value of the coordinate $\alpha = 0$, 4b - $\alpha = 0.15$, 4c - $\alpha = 0.3$. Solid curves correspond $V_0 = 0.01$, dashed - $V_0 = 0.05$, dash-dotted - $V_0 = 0.1$.

Fig. 5a and 5b shows the distributions of normal displacements along the angular coordinate (Fig. 5a) and along the longitudinal coordinate (Fig. 5b). Here, the results correspond to the point in time $\tau = 0.6$ and the initial speed $V_0 = 0.1$. In fig. 5a, solid curves correspond $z = 0$, dashed - $z = 0.15$, dash-dotted - $z = 0.3$. Solid curves in Fig. 5b correspond $\alpha = 0$, dashed - $\alpha = 0.15$, dash-dotted - $\alpha = 0.3$.

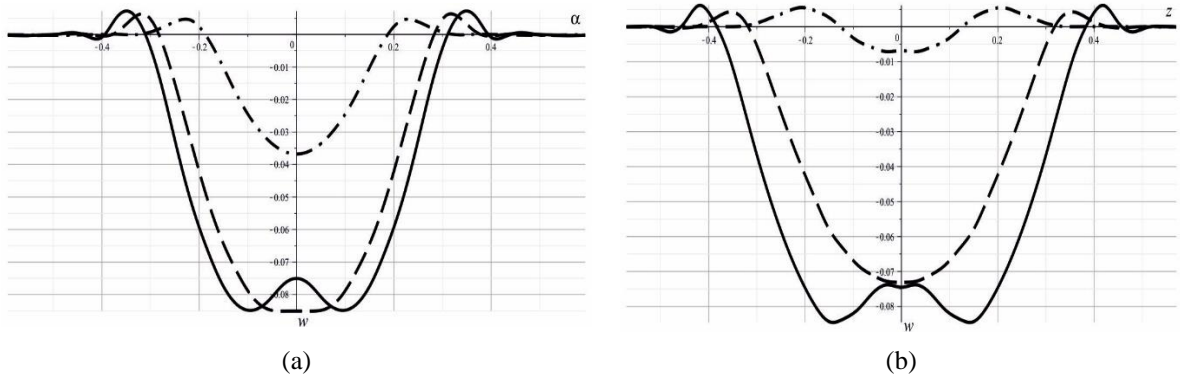| (a)	(b)	(c)
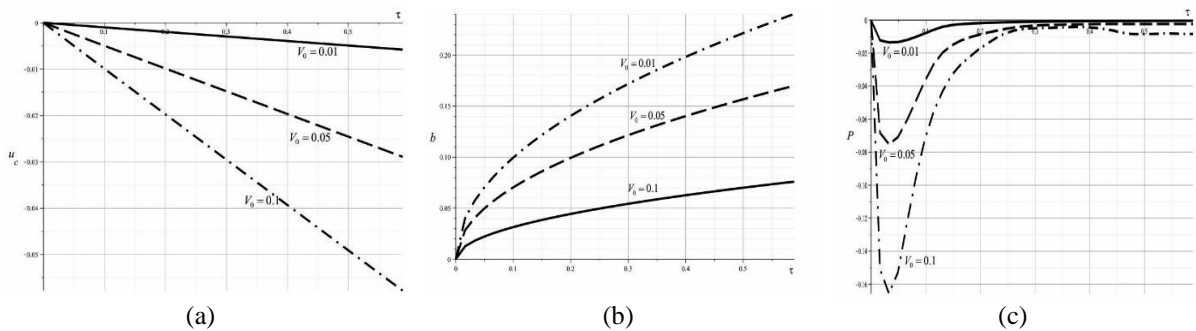
Figure 2: Dependences of the displacement of the center of mass of the striker, the radius of the contact area and the resulting contact force on time.

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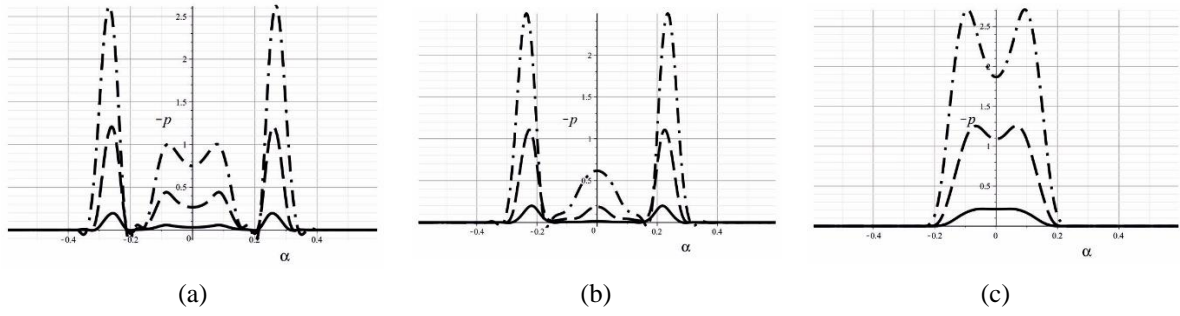
Figure 3: Distribution of contact pressure along the angular coordinate at $\tau = 0.3$.
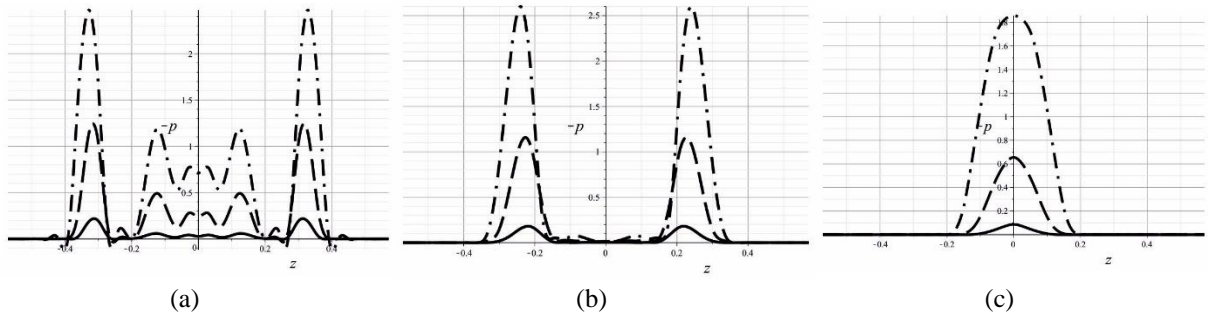
Figure 4: Distribution of contact pressure along the longitudinal coordinate at $\tau = 0.3$.

Figure 5: Distribution of normal displacements along the angular and longitudinal coordinates.

6. ACKNOWLEDGMENTS

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