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ARTICLE

Garhy Distribution with Different Estimation Methods and Applications to Engineering and Medical Data

Mohammed Elgarhy[1,2,3,*]

[1]Faculty of Computers and Information Systems, Egyptian Chinese University, Nasr City, Egypt
[2]Department of Basic Sciences, Higher Institute of Administrative Sciences, Belbeis, AlSharkia, Egypt
[3]Department of Computer Engineering, Biruni University, Istanbul, Turkey
[*]Corresponding Author: Mohammed Elgarhy.Email: m_elgarhy85@sva.edu.eg

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ABSTRACT: In this article, we introduce and investigate a new one-parameter mixture distribution called the "Garhy distribution". The probability density function is very adaptable, as it may take on right skewed, unimodal, and heavy tailed patterns. In addition, the hazard rate function indicates that data with increasing shaped failure rates may be adapted by the Garhy distribution. Several fundamental statistical and mathematical properties are calculated including mode, quantile function, moments, mean, variance, skewness, kurtosis, moment-generating function, incomplete moments, inequality measures, order statistics, and extropy measures. The scale parameter of the Garhy distribution is estimated employing twelve different estimation approaches, maximum likelihood, maximum product of spacings, least-squares, weighted least-squares, Anderson darling, right-tail Anderson darling, left-tail Anderson darling, Cramér von-Misses, and the least-squares method. The effectiveness of these strategies is evaluated using a detailed simulation study. Furthermore, we used the Garhy distribution to examine two real-world data sets, demonstrating its superior performance compared to specific competitors.

KEYWORDS: Garhy distribution; Mixture; Extropy; Estimation; Simulation.

1 Introduction

In many disciplines, such as medicine, engineering, finance, and insurance, modeling and analysis of lifetime data are fundamental. Well-known probability distributions that are used to analyze lifetime datasets are the exponential and gamma probability distributions, along with their extensions. Both distributions have interesting structural aspects, like the exponential distribution's memoryless and constant hazard rate function (HRF).

Mixture distributions are an essential and extremely adaptable type of statistical model prompted by the requirement to represent populations that are fundamentally diverse or made up of multiple separate subpopulations. The main motivation derives from the limitations of standard distributions, which frequently fail to reflect complicated data properties such as multimodality, skewness, and heavy tails that are ubiquitous in real-world datasets. A mixture model handles this by considering that the data is derived from a limited or infinite set of component distributions, each representing a distinct subpopulation or data domain; the overall model is a convex combination (weighted sum) of these components.

A novel continuous probability distribution can be proposed utilizing a mixture of two previously established probability distributions. A random variable X is said to have the next mixture distribution if its probability density function (PDF) takes the following formula

$$f(x) = wg_1(x) + (1 - w)g_2(x), \quad x \in R, \tag{1}$$

where $g_i(x)$ and $i = 1, 2$ is the PDF and w is the mixing proportion. Several authors used Equation (1) to introduce new distributions and some important literature reviews are mentioned in Table 1.

Table 1: A literature review of some mixture distributions

Distribution	$g_1(x)$	$g_2(x)$	w	Density function	Ref.
Lindley (L)	E(η)	G(2,η)	$\frac{\eta}{1+\eta}$	$f(x;\eta) = \frac{\eta^2}{\eta+1}(1+x)e^{-\eta x}, \quad x, \eta > 0.$	[1]
Shanker (SH)	E(η)	G(4,η)	$\frac{\eta^2}{1+\eta^2}$	$f(x;\eta) = \frac{\eta^2}{\eta^2+1}(\eta + x)e^{-\eta x}, \quad x,\ \eta > 0.$	[2]
Akash	E(η)	G(3,η)	$\frac{\eta^2}{1+\eta^2}$	$f(x;\eta) = \frac{\eta^3}{\eta^2+2}(1 + x^2)e^{-\eta x}, \quad x,\ \eta > 0.$	[3]
Ishita	E(η)	G(3,η)	$\frac{\eta^3}{1+\eta^3}$	$f(x;\eta) = \frac{\eta^3}{\eta^3+2}(\eta + x^2)e^{-\eta x}, \quad x,\ \eta > 0.$	[4]
Rani	E(η)	G(5,η)	$\frac{\eta^5}{24+\eta^5}$	$f(x;\eta) = \frac{\eta^5}{\eta^5+24}(\eta + x^4)e^{-\eta x}, \quad x,\ \eta > 0.$	[5]
Pranav	E(η)	G(4,η)	$\frac{\eta^4}{6+\eta^4}$	$f(x;\eta) = \frac{\eta^4}{\eta^4+6}(\eta + x^3)e^{-\eta x}, \quad x,\ \eta > 0.$	[6]
Shukla	E(η)	G($\alpha + 1$,η)	$\frac{\eta^{\alpha+1}}{\Gamma(\alpha+1)+\eta^{\alpha+1}}$	$f(x;\eta,\alpha) = \frac{\eta^{\alpha+1}}{\eta^{\alpha+1}+\Gamma(\alpha+1)}(\eta + x^{\alpha})e^{-\eta x}, \quad x,\ \eta,\ \alpha > 0.$	[7]
Ram Awadh	E(η)	G(6,η)	$\frac{\eta^6}{120+\eta^6}$	$f(x;\eta) = \frac{\eta^6}{\eta^6+120}(\eta + x^5)e^{-\eta x}, \quad x,\ \eta > 0.$	[8]
Rama	E(η)	G(4,η)	$\frac{\eta^3}{6+\eta^3}$	$f(x;\eta) = \frac{\eta^4}{\eta^3+6}(1 + x^3)e^{-\eta x}, \quad x,\ \eta > 0.$	[9]
Xgamma (XG)	E(η)	G(3,η)	$\frac{\eta}{1+\eta}$	$f(x;\eta) = \frac{\eta^2}{1+\eta}(1 + \frac{\eta}{2}x^2)e^{-\eta x}, \quad x,\ \eta > 0.$	[10]
Chris-Jerry	E(η)	G(3,η)	$\frac{\eta}{2+\eta}$	$f(x;\eta) = \frac{\eta^2}{\eta+2}(1 + \eta x^2)e^{-\eta x}, \quad x,\ \eta > 0.$	[11]
XLindley	E(η)	L(η)	$\frac{\eta}{1+\eta}$	$f(x;\eta) = \frac{\eta^2}{(1+\eta)^2}(2 + \eta + x)e^{-\eta x}, \quad x,\ \eta > 0.$	[12]
Zeghdoudi	G(2,η)	G(3,η)	$\frac{\eta}{2+\eta}$	$f(x;\eta) = \frac{\eta^3}{2+\eta}x(1 + x)e^{-\eta x}, \quad x,\ \eta > 0.$	[13]
Haq	E(η)	XG(η)	$\frac{\eta}{1+\eta}$	$f(x;\eta) = \left(\frac{\eta}{1+\eta}\right)^2(2 + \eta + \frac{\eta x^2}{2})e^{-\eta x}, \quad x,\ \eta > 0.$	[14]
Hamza	E(η)	G(7,η)	$\frac{\alpha\eta^5}{120+\alpha\eta^5}$	$f(x;\eta,\alpha) = \frac{\eta^6}{120+\alpha\eta^5}\left(\alpha + \frac{\eta x^6}{6}\right)e^{-\eta x}, \quad x,\ \eta, \alpha > 0.$	[15]
XRama (XR)	E(η)	SH(η)	$\frac{\eta^3}{6+\eta^3}$	$f(x;\eta) = \frac{\eta^4}{(6+\eta^3)^2}(\eta^3 + 6x^3 + 12)e^{-\eta x}, \quad x,\ \eta > 0.$	[16]
Double XRama	E(η)	XR(η)	$\frac{\eta^3}{6+\eta^3}$	$f(x;\eta) = \frac{\eta^4}{(6+\eta^3)^2}(\eta^6 + 18\eta^3 + 108 + 36x^3)e^{-\eta x}, x,\ \eta > 0.$	[17]
Komal	E(η)	G(2,η)	$\frac{\eta(\eta+1)}{\eta^2+\eta+1}$	$f(x;\eta) = \frac{\eta^2}{\eta^2+\eta+1}(1 + \eta + x)e^{-\eta x}, \quad x,\ \eta > 0.$	[18]

where E(.) and G(.,.) are the exponential and gamma distributions.

Each of these continuous lifetime statistical distributions in Table 1 has advantages and disadvantages based on their form of the HRF. However, in many cases, these distributions are not suitable for modeling lifetime data either theoretically or practically. As a result, the purpose of this study is to develop a novel distribution that is more flexible than these continuous lifetime statistical distributions to model lifetime data in terms of reliability and HRF shapes. To address this limitation, we propose a new one-parameter mixture distribution called the "Garhy distribution". This study emphasizes the following objectives:

1. Create a flexible distribution that can represent right-skewed, unimodal, and heavy-tailed data. In addition, the HRF indicates that data with up-side-down-shaped failure rates may be adapted by the Garhy distribution.
2. Several fundamental statistical and mathematical properties are calculated including mode, quantile function, moments, mean, variance, skewness, kurtosis, moment-generating function, incomplete moments, inequality measures, order statistics, and extropy measures.
3. Examine the unknown parameter of the Garhy distribution employing twelve estimation strategies including percentile estimation, least squares estimation, maximum likelihood estimation, Cramér-von-Mises estimation, weighted least squares estimation and maximum product of spacings estimation. The effectiveness of these strategies is evaluated using a detailed simulation study.
4. The flexibility and adaptability of the Garhy distribution allows it to fit two real-world datasets related to medical and engineering sciences better than other well-known continuous statistical distributions, such as the Lindley, Shanker, Akash, Ishita, Rani, Pranav, Shukla, Ram Awadh, Rama, xgamma, Chris-Jerry, XLindley, Zeghdoudi, Haq and Hamza distributions.

The remainder of this article is structured and designed as follows: The construction of the Garhy distribution is described in Section 2. Several statistical properties of the Garhy distribution are computed in Section 3. Section 4 presents the parameter estimates of the Garhy distribution employing twelve different classical estimation methods. Section 5 examines the behavior of estimated parameters utilizing a Monte Carlo simulation. In Section 6, two real datasets related to engineering and medical are fitted to the Garhy distribution to assess its flexibility. Finally, some concluding remarks are mentioned in Section 7.

2 A New Garhy Distribution

In this section, we introduce the new one-parameter lifetime distribution called Garhy distribution as a mixture of G(2,η), G(3,η) and $w = \frac{\eta^3}{6+\eta^3}$. Then, the PDF and cumulative distribution function (CDF) of the Garhy distribution are given by

$$f(x;\eta) = \frac{\eta^3 x}{6+\eta^3}(\eta^2 + 3x)e^{-\eta x}, \quad x, \eta > 0, \tag{2}$$

and

$$F(x;\eta) = 1 - \left[\frac{3\eta^2x^2}{6+\eta^3} + (\eta x + 1)\right]e^{-\eta x}. \tag{3}$$

The survival function (SF), hazard rate function (HRF), reversed HRF and cumulative HRF of Garhy distribution are provided via

$$S(x;\eta) = \left[\frac{3\eta^2x^2}{6+\eta^3} + (\eta x + 1)\right]e^{-\eta x},$$

$$h(x;\eta) = \frac{\eta^3 x(\eta^2+3x)}{[3\eta^2x^2 + (6+\eta^3)(\eta x+1)]},$$

$$\tau(x;\eta) = \frac{\eta^3 x(\eta^2+3x)}{(6+\eta^3)e^{\eta x} - [3\eta^2x^2 + (6+\eta^3)(\eta x+1)]},$$

and

$$H(x;\eta) = \log[6+\eta^3] - \log\left[3\eta^2x^2 + (\eta x+1)(6+\eta^3)\right] - \eta x.$$

Figures 1, 2 and 3 show some different curves for PDF, CDF, and HRF of the Garhy distribution using different values of the parameter η. From Figure 1, we can note that the behavior of the PDF curves can be unimodal, right-skewed, and heavy-tailed shaped. But from Figure 3, we can note that the behavior of the PDF curves can be up-side-down. Figures 4 and 5 show 3D Plots and contour plots of PDF and HRF for the Garhy distribution. We can notes from these figures the PDF is unimodal and the maximum value is reached at approximately $\eta = 2, x = 1$, but the HRF is increasing.

3 Statistical Properties of Garhy Distribution

In this section, we look at several significant properties of the Garhy distribution. These make its probabilistic features easier to understand.

3.1 Mode

The mode of the Garhy distribution corresponds to the maximum point of the PDF in the support $(0,\infty)$. It can be identified by equating $\frac{d\log[f(x;\eta)]}{dx} = 0$, as follows:

$$\frac{d\log[f(x;\eta)]}{dx} = -\frac{1}{x} + \frac{3}{\eta^2+3x} - \eta = 0. \tag{4}$$

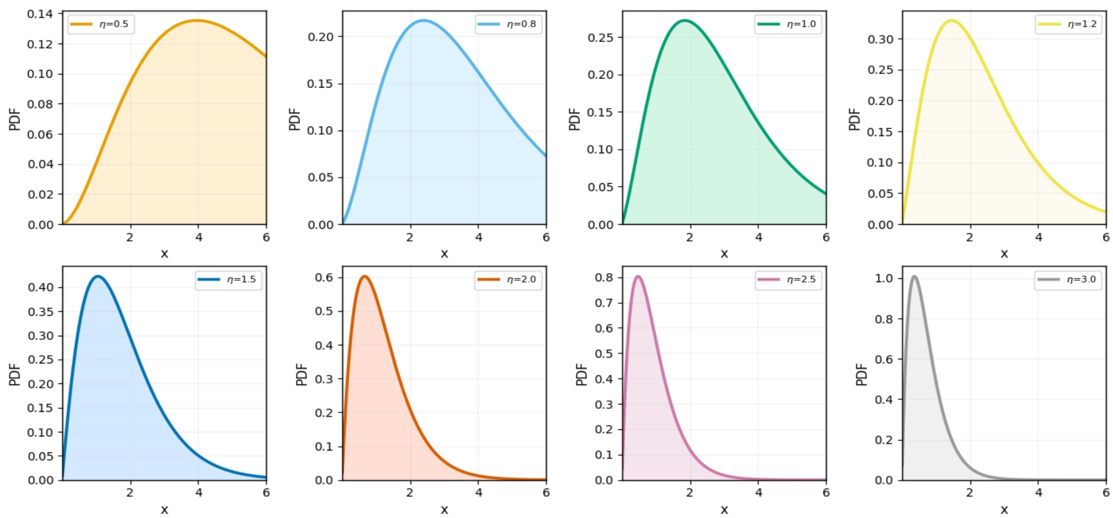

Figure 1: 2D Plots of PDF for the Garhy distribution

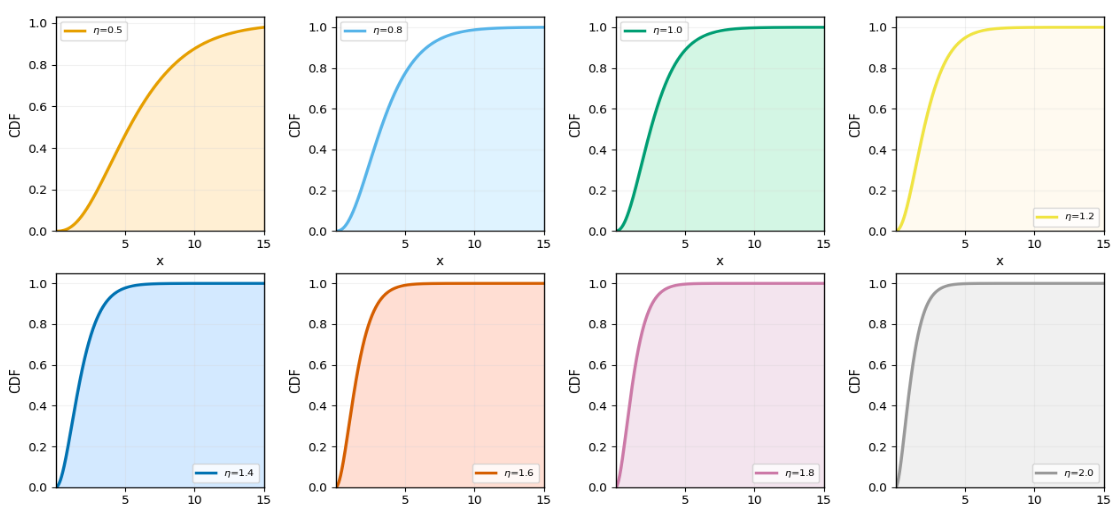

Figure 2: 2D Plots of CDF for the Garhy distribution

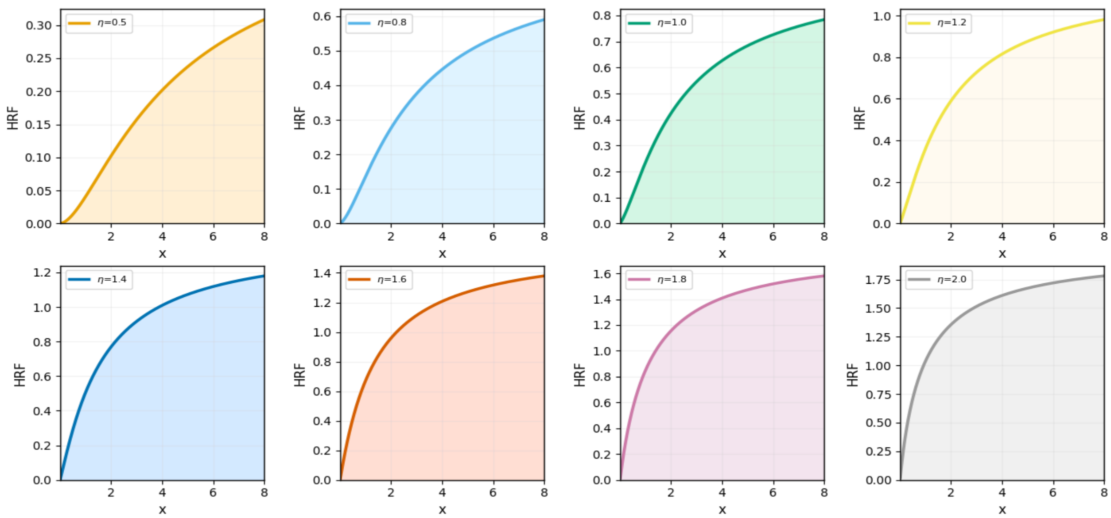

Figure 3: 2D Plots of HRF for the Garhy distribution

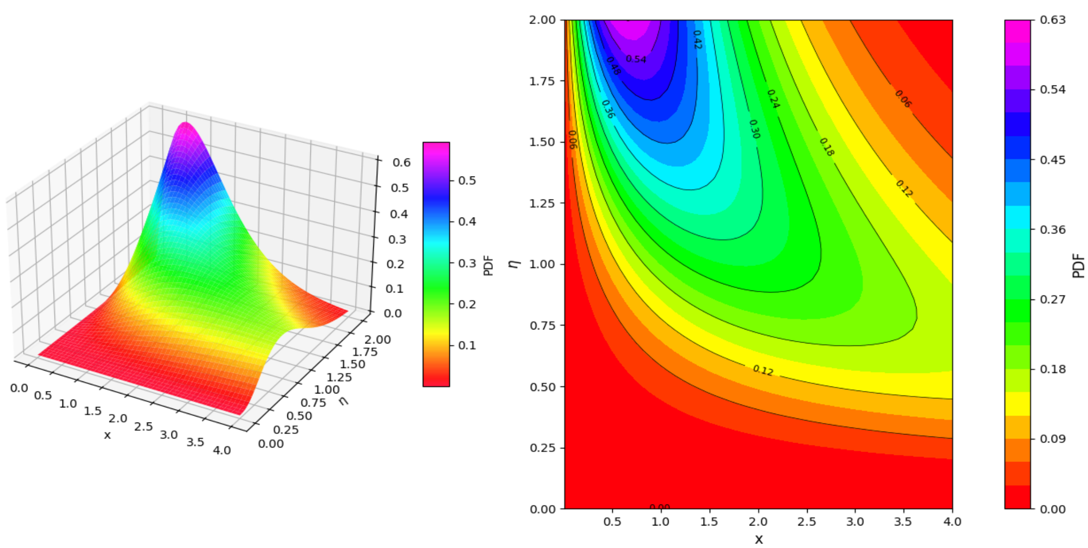

Figure 4: 3D Plots of PDF for the Garhy distribution

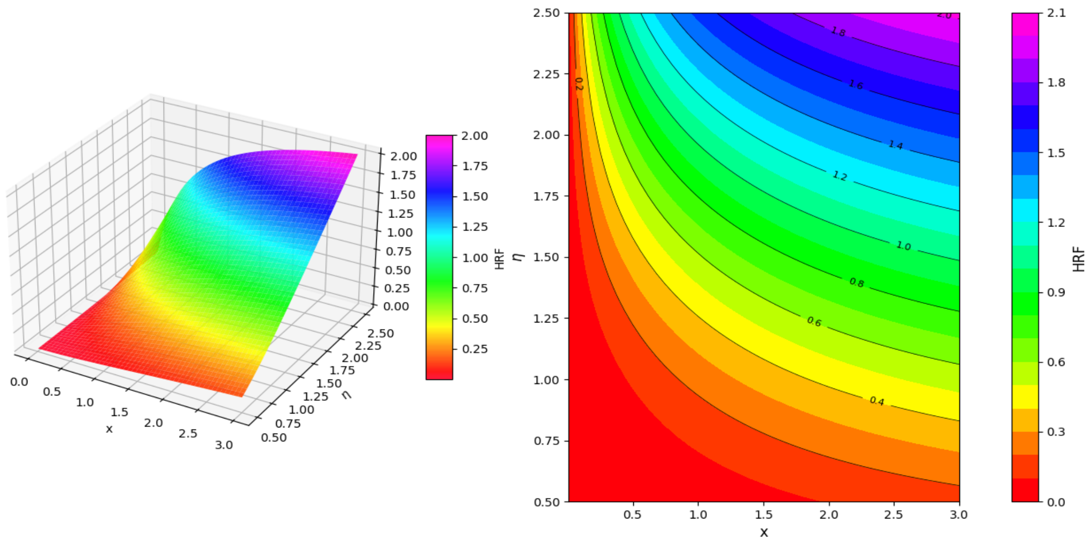

Figure 5: 3D Plots of HRF for the Garhy distribution

After simplifying, Equation (4) becomes $3\eta x^2 + (\eta^3 - 6)x - \eta^2 = 0$, from which we acquire a solution that is provided as

$$x_* = \frac{(6 - \eta^3) + \sqrt{\eta^6 + 36}}{6\eta}.$$

3.2 Quantile Function

The quantile function of the Garhy distribution, shown as $Q(u)$ for $u \in (0, 1)$, is the solution of equation $F(Q(u)) = u$.

$$1 - u = \left[\frac{3\eta^2 Q(u)^2}{6 + \eta^3} + (\eta Q(u) + 1)\right] e^{-\eta Q(u)}. \tag{5}$$

The above Equation (5) does not have a closed form. The problem will be solved numerically to obtain the median, skewness, and kurtosis.

3.3 Moments

In this subsection, we calculate the p_{th} moment of the Garhy distribution. The p_{th} ordinary moment can be computed by utilizing the next formula

$$\mu'_p = E(X^p) = \int_0^\infty x^p f(x; \eta) dx = \frac{\eta^3}{6 + \eta^3} \int_0^\infty x^{p+1} (\eta^2 + 3x) e^{-\eta x} dx.$$

After some calculations, then the p_{th} moment of the Garhy distribution is given by

$$\mu'_p = \frac{\left(\eta^3 + 3p + 6\right)\Gamma(p+2)}{\eta^p(6+\eta^3)}. \tag{6}$$

The mean and variance of the Garhy distribution can be investigated employing p = 1 and 2 in Equation (6), as below:

$$\mu = \mu'_1 = E(X) = \frac{2\left(\eta^3 + 9\right)}{\eta(6+\eta^3)},$$

and

$$var(x) = \mu'_2 - {\mu'_1}^2 = \frac{6\left(\eta^3 + 12\right)}{\eta^2(6+\eta^3)} - \frac{4\left(\eta^3 + 9\right)^2}{\eta^2(6+\eta^3)^2}.$$

Additionally, the corresponding skewness is indicated as $E\{[X - E(X)]^3/[var(X)]^{3/2}\}$. The kurtosis is described as $E\{[X - E(X)]^4/[var(X)]^2\}$, and the coefficient of variation (CV) is expressed as $[SD(X)]/E(X)$, where SD is the standard deviation.

The moment generating function of the Garhy distribution can be computed using the following formula

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x;\eta)dx = \frac{\eta^3}{6+\eta^3}\int_0^\infty x(\eta^2 + 3x)e^{-(\eta-t)x}dx.$$

After some simplifications, then the moment generating function of the Garhy distribution is given by

$$M_X(t) = \frac{\eta^3}{6+\eta^3}\left[\frac{\eta^2(\eta-t)+6}{(\eta-t)^3}\right], \quad t < \eta.$$

3.4 Incomplete Moments and Inequality Measures

Incomplete moments are key components required to measure inequality. Measurements like the Lorenz curve and Gini coefficients rely on them, demonstrating how vital they are for quantifying wealth disparities. The s_{th} lower incomplete moment of the Garhy distribution is provided via

$$\varkappa_s(t) = \int_0^t x^s f(x;\eta)dx = \frac{\eta^3}{6+\eta^3}\int_0^t x^{s+1}(\eta^2 + 3x)e^{-\eta x}dx.$$

After some simplifications, the s_{th} lower incomplete moment of the Garhy distribution is given by

$$\varkappa_s(t) = \frac{\eta^3\,\gamma(s+2,\eta t) + 3\,\gamma(s+3,\eta t)}{\eta^s\,(6+\eta^3)}.$$

Using the same steps as above, then the s_{th} upper incomplete moment of the Garhy distribution can computed as follows

$$\varphi_s(t) = \frac{\eta^3\,\Gamma(s+2,\eta t) + 3\,\Gamma(s+3,\eta t)}{\eta^s\,(6+\eta^3)}.$$

The Lorenz curve (LC) and Bonferroni curve (BC) curves play important roles in demography, dependability, economics, insurance, and medicine. They can also be used in a unit data analysis setting. As a result, we have formulated them inside the framework of the Garhy distribution. The LC and BC can be calculated simply as

$$LC = \frac{\varkappa_1(t)}{E(X)} = \frac{\eta^3\,\Gamma(3,\eta t) + 3\,\Gamma(4,\eta t)}{2\,(9+\eta^3)},$$

and

$$BC = \frac{LC}{F(t;\eta)} = \frac{\varkappa_1(t)}{E(X)} = \frac{\eta^3\,\Gamma(3,\eta t) + 3\,\Gamma(4,\eta t)}{2\,(9+\eta^3)\left[1-\left[\frac{3\eta^2x^2}{6+\eta^3}+(\eta x+1)\right]e^{-\eta x}\right]},$$

respectively.

3.5 Extropy

As an alternative measure of uncertainty, Lad et al. [19] created extropy (EXT), which is the twofold complement of the classical entropy. EXT is a useful tool in statistical applications for evaluating predicted accuracy of distributions, usually using the total log score approach. The EXT of X is stated as:

$$\Psi_1 = \frac{-1}{2}\int_0^\infty f^2(x;\eta)dx. \tag{7}$$

By combining PDF (2) to Equation (7), the EXT of the Garhy distribution may be determined as follows

$$\Psi_1 = \frac{\eta^6}{(6+\eta^3)^2}\int_0^\infty x^2(\eta^4+6\eta^2x+9x^2)e^{-2\eta x}dx. \tag{8}$$

After some reduction, then the EXT of the Garhy distribution is given by

$$\Psi_1 = \frac{-\eta(\eta^6+9\eta^3+27)}{8(6+\eta^3)^2}.$$

Ref. [20] introduced the concept of Weighted EXT (WEXT) as a substitute for traditional entropy. Unlike entropy, the WEXT provides a different perspective on uncertainty by giving higher values in a probability distribution more weight. It is depicted as

$$\Psi_2 = \frac{-1}{2}\int_0^1 x f^2(x;\eta)dx = \frac{\eta^6}{(6+\eta^3)^2}\int_0^\infty x^3(\eta^4+6\eta^2 x+9x^2)e^{-2\eta x}dx. \tag{9}$$

Employing a similar approach as described above, the WEXT of the Garhy distribution acquires the following form:

$$\Psi_2 = \frac{-(3\eta^6+36\eta^3+135)}{[16(6+\eta^3)]^2}.$$

3.6 Order Statistics

Assume that $X_1, X_2, \ldots, X_n$ be a random sample from the Garhy distribution, with order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. The PDF of $X_{i:n}$ of order statistics is given by

$$f_{X_{i:n}}(x) = \frac{n!}{(i-1)!(n-i)!} f(x;\eta)[F(x;\eta)]^{i-1}[1-F(x;\eta)]^{n-i}, \tag{10}$$

The PDF for $x_{i:n}$ can be determined by using the following formula:

$$f_{X_{i:n}}(x) = \frac{n!\eta^3 x(\eta^2+3x)e^{-(n-i+1)\eta x}}{(i-1)!(n-i)!(6+\eta^3)}\left[1-\left[\frac{3\eta^2x^2}{6+\eta^3}+(\eta x+1)\right]e^{-\eta x}\right]^{i-1}\left[\frac{3\eta^2x^2}{6+\eta^3}+(\eta x+1)\right]^{n-i}. \tag{11}$$

The lowest and highest order statistics of the Garhy distribution can be computed by putting $i = 1$ and n in Equation (11), respectively, as follows

$$f_{X_{1:n}}(x) = \frac{n\eta^3 x(\eta^2+3x)e^{-n\eta x}}{(6+\eta^3)}\left[\frac{3\eta^2x^2}{6+\eta^3}+(\eta x+1)\right]^{n-1},$$

and

$$f_{X_{n:n}}(x) = \frac{n\eta^3 x(\eta^2+3x)e^{-\eta x}}{(6+\eta^3)}\left[1-\left[\frac{3\eta^2x^2}{6+\eta^3}+(\eta x+1)\right]e^{-\eta x}\right]^{n-1}.$$

4 Estimation Methods

The parameter η of the Garhy distribution is estimated using traditional methods. These estimation techniques are based on the minimizing or maximizing of specific objective functions. The following estimates are provided: CM estimates (CMEs), percentiles estimates (PEs), ADLTS estimates (ADLTSEs), ML estimates (MLEs), AD estimates (ADEs), MSALD estimates (MSALDEs), LS estimates (LSEs), RAD estimates (RADEs), MPS estimates (MPSEs), MSSL estimates (MSSLEs), WLS estimates (WLSEs), and LAD estimates (LADEs). Many authors used some or all of all previous estimation methods in their studies to compare the behavior of the parameters of their models, such as [21–25].

4.1 Maximum Likelihood Method

The ML method is a powerful traditional method for estimating the parameters of any distribution [26,27]. Estimation of the Garhy parameter using complete samples is done by maximizing the log-likelihood function. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be a random sample of size n from the Garhy distribution with PDF (2). The log-likelihood function is given by:

$$l(\eta) = 3n\log(\eta) - n\log(6+\eta^3) + \sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\log(\eta^2+3x_i) - \eta\sum_{i=1}^{n}x_i.$$

The log-likelihood function must be maximized to obtain the MLE $\hat{\eta}_1$. The derivative of l with respect to η is:

$$\frac{\partial l}{\partial \eta} = \frac{3n}{\eta} - \frac{3n\eta^2}{6+\eta^3} + \sum_{i=1}^{n}\frac{2\eta}{\eta^2+3x_i} - \sum_{i=1}^{n}x_i.$$

This equation lacks a closed form and requires a numerical technique for solution.

4.2 Maximum Product of Spacings Method

The MPS approach was presented by Cheng and Amin [28]. The MPSE $\hat{\eta}_9$ is produced by maximizing:

$$MPS(\eta) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\tau_i, \quad \text{where} \quad \tau_i = F_{(i)} - F_{(i-1)},$$

with $F_{(0)} = 0$ and $F_{(n+1)} = 1$. The derivative is:

$$\begin{aligned}\frac{\partial MPS}{\partial \eta} &= \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{\tau_i}\cdot\frac{\partial \tau_i}{\partial \eta}\\ &= \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{F_{(\eta_i)} - F_{(\eta_{i-1})}}{\tau_i}.\end{aligned}$$

The MPSE is obtained by solving $\frac{\partial MPS}{\partial \eta} = 0$.

4.3 Least Squares Method

The LS method was originated by Swain et al. [29]. Let $F_{X_{(i)}}(x;\eta)$ be the distribution function for ordered random variables $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, where the CDF is:

$$F(x;\eta) = 1 - \left[\frac{3\eta^2 x^2}{6+\eta^3} + (\eta x + 1)\right] e^{-\eta x}, \quad x > 0, \eta > 0.$$

The LSE is derived by minimizing:

$$LSE(\eta) = \sum_{i=1}^{n} \left[F(X_{(i)};\eta) - \frac{i}{n+1}\right]^2.$$

Define $F_{(i)} = F(X_{(i)};\eta)$ for brevity. The derivative of $F_{(i)}$ with respect to η is:

$$F_{(\eta_i)} = \frac{\partial F_{(i)}}{\partial \eta} = -\left[\frac{6\eta x_{(i)}^2}{6+\eta^3} - \frac{3\eta^4 x_{(i)}^2}{(6+\eta^3)^2} - \frac{3\eta^2 x_{(i)}^3}{6+\eta^3} - \eta x_{(i)}^2\right] e^{-\eta x_{(i)}}.$$

The derivative of the LSE function is:

$$\frac{\partial LSE}{\partial \eta} = 2\sum_{i=1}^{n} F_{(\eta_i)} \left[F_{(i)} - \frac{i}{n+1}\right].$$

The LSE $\hat{\eta}_2$ is obtained by solving $\frac{\partial LSE}{\partial \eta} = 0$.

4.4 Weighted Least Squares Method

The WLSE [29] is derived by minimizing:

$$WLSE(\eta) = \sum_{i=1}^{n} w_i \left[F(X_{(i)};\eta) - \frac{i}{n+1}\right]^2, \quad \text{where} \quad w_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

The derivative is:

$$\frac{\partial WLSE}{\partial \eta} = 2\sum_{i=1}^{n} w_i F_{(\eta_i)} \left[F_{(i)} - \frac{i}{n+1}\right].$$

The WLSE $\hat{\eta}_3$ is obtained by solving $\frac{\partial WLSE}{\partial \eta} = 0$.

4.5 Anderson-Darling Method

The ADE [30] is generated by minimizing:

$$ADE(\eta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F_{(i)} + \log(1 - F_{(n+1-i)})\right].$$

The derivative is:

$$\frac{\partial ADE}{\partial \eta} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{1}{F_{(i)}}\cdot\frac{\partial F_{(i)}}{\partial \eta} + \frac{1}{1-F_{(n+1-i)}}\cdot\left(-\frac{\partial F_{(n+1-i)}}{\partial \eta}\right)\right]$$
$$= -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{F_{(\eta_i)}}{F_{(i)}} - \frac{F_{(\eta_{n+1-i})}}{1-F_{(n+1-i)}}\right].$$

The ADE $\hat{\eta}_4$ is obtained by solving $\frac{\partial ADE}{\partial \eta} = 0$.

4.6 Right-Tail Anderson-Darling

The RADE $\hat{\eta}_5$ is generated by minimizing:

$$RADE(\eta) = \frac{n}{2} - 2\sum_{i=1}^{n}F_{(i)} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S_{(n+1-i)},$$

where $S_{(i)} = 1 - F_{(i)}$. The derivative is:

$$\frac{\partial RADE}{\partial \eta} = -2\sum_{i=1}^{n}\frac{\partial F_{(i)}}{\partial \eta} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\cdot\frac{1}{S_{(n+1-i)}}\cdot\left(-\frac{\partial F_{(n+1-i)}}{\partial \eta}\right)$$
$$= -2\sum_{i=1}^{n}F_{(\eta_i)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{F_{(\eta_{n+1-i})}}{S_{(n+1-i)}}.$$

The RADE is obtained by solving $\frac{\partial RADE}{\partial \eta} = 0$.

4.7 Left-Tail Anderson-Darling (LAD)

The LADE $\hat{\eta}_6$ is generated by minimizing:

$$LADE(\eta) = -\frac{3}{2}n + 2\sum_{i=1}^{n}F_{(i)} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log F_{(i)}.$$

The derivative is:

$$\frac{\partial LADE}{\partial \eta} = 2\sum_{i=1}^{n}\frac{\partial F_{(i)}}{\partial \eta} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\cdot\frac{1}{F_{(i)}}\cdot\frac{\partial F_{(i)}}{\partial \eta}$$
$$= 2\sum_{i=1}^{n}F_{(\eta_i)} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{F_{(\eta_i)}}{F_{(i)}}.$$

The LADE is obtained by solving $\frac{\partial LADE}{\partial \eta} = 0$.

4.8 Cramér-von Mises Method

The CM method was proposed by [31]. The CME $\hat{\eta}_8$ is determined by minimizing:

$$CME(\eta) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F_{(i)} - \frac{2i-1}{2n}\right]^2.$$

The derivative is:

$$\begin{aligned}\frac{\partial CME}{\partial \eta} &= 2\sum_{i=1}^{n}\left[F_{(i)} - \frac{2i-1}{2n}\right]\cdot\frac{\partial F_{(i)}}{\partial \eta}\\ &= 2\sum_{i=1}^{n}F_{(\eta_i)}\left[F_{(i)} - \frac{2i-1}{2n}\right].\end{aligned}$$

The CME is obtained by solving $\frac{\partial CME}{\partial \eta} = 0$.

4.9 Minimum Spacing Absolute Distance Method

The MSAD method determines the optimal parameter values by minimizing the MSAD statistic. Following the approach developed by Ahmed [31], the Garhy distribution parameter η is estimated through minimization of the following expression:

$$MSAD(\eta) = \sum_{i=1}^{n+1}\left|\tau_i - \frac{1}{n+1}\right|,$$

where $\tau_i = F(X_{(i)};\eta) - F(X_{(i-1)};\eta)$, with boundary conditions $F(X_{(0)};\eta) = 0$ and $F(X_{(n+1)};\eta) = 1$. The resulting estimator $\hat{\eta}_{10}$ is obtained by minimizing $MSAD(\eta)$ over the parameter space.

4.10 Kolmogorov Method

The Kolmogorov estimation technique, widely recognized through the work of Aguilar et al. [32], offers an effective approach for parameter estimation by quantifying the maximum deviation between empirical and theoretical distributions. For the Garhy distribution, parameter estimation involves minimizing the Kolmogorov statistic defined as:

$$KM(\eta) = \max_{1\le i\le n}\left[\frac{i}{n} - F(x_{(i)};\eta),\ F(x_{(i)};\eta) - \frac{i-1}{n}\right].$$

The Kolmogorov estimator $\hat{\eta}_{11}$ is derived as the solution to the minimization problem $\min_{\eta} KM(\eta)$.

4.11 Minimum Spacing Square Distance Method

Employing the same fundamental principles as the ADE method, the Garhy distribution parameter can be estimated by minimizing the Minimum Spacing Square Distance statistic. The corresponding MSSD estimator is obtained by optimizing the objective function:

$$MSSD(\eta) = \sum_{i=1}^{n+1} \left(\tau_i - \frac{1}{n+1} \right)^2,$$

where $\tau_i = F(X_{(i)};\eta) - F(X_{(i-1)};\eta)$. The MSSD estimator $\hat{\eta}_{12}$ results from minimizing $MSSD(\eta)$ with respect to the parameter η.

4.12 Minimum Spacing Linex Distance Method

Building upon the methodology established for the ADE approach, the Garhy distribution parameter estimation can be performed by minimizing the Minimum Spacing Linex Distance statistic. The MSLD estimator is derived through optimization of the following objective function:

$$MSLND(\eta) = \sum_{i=1}^{n+1} \left[e^{\left(\tau_i - \frac{1}{n+1}\right)} - \left(\tau_i - \frac{1}{n+1} \right) - 1 \right],$$

where $\tau_i = F(X_{(i)};\eta) - F(X_{(i-1)};\eta)$, with $F(X_{(0)};\eta) = 0$ and $F(X_{(n+1)};\eta) = 1$. The MSLND estimator $\hat{\eta}_{13}$ is obtained by minimizing $MSLND(\eta)$ over the parameter domain.

5 Numerical Outcomes

Using the R statistical platform [33], a Monte Carlo simulation study was carried out to assess the performance of the twelve different approaches for estimating the parameter η of the Garhy distribution. The goal of the experimental framework was to evaluate each estimation method to find which of them has the best performance. This comprised seven different sample sizes, ranging from small to moderate ($n = 20, 50, 100, 150, 200, 250, 300$), and six different values of the true parameter ($\eta = 0.5, 0.8, 1.3, 1.7, 2.5, 3.5$). For every unique combination of these factors, the simulation was repeated $N = 1000$ times to ensure reliable and stable results. The process of generating random data adhered to the inverse transform method. The quantile function was mathematically calculated using R's uniroot function, and using the `optim()` function iteratively, combining several optimization algorithms-L-BFGS-B, Nelder–Mead, and BFGS. This multi-algorithm procedure was designed to improve numerical stability and ensure convergence for all estimation methods considered in the study. For each estimator $\hat{\eta}$ Mean and MSE were estimated, The MSE served as the primary criterion for ranking estimators, For each (n, η) combination, estimators were ranked from 1 (best) to 12 (worst) based on their MSE values. The Ranks were superscript for each estimated MSE in in simulation tables from Table 2 to Table 7, Table 8 contains the sum and overall ranks of all estimation methods, Figures 6 and 7 plot MSE against sample size, for all twelve methods for $\eta = 0.5$ and $\eta = 0.8$. Based on simulation tables from Table 2 to Table 7, and Figures 6 and 7, all estimators demonstrated the property of consistency, with MSE values decreasing monotonically as the sample size increased across all parameter values, as expected theoretically, also the estimated mean is closer to the true mean value of η as n increases.

Table 8, shows that the MPS estimator has the smallest sum of ranks 48 with overall rank 1, indicating that the MPS is the best estimator for the Garhy distribution, closely followed by the MLE as the second rank with total rank sum 78. The RADE, has overall rank 3, with sum of ranks 130.5, ADE, has overall rank 4, with sum of ranks 173.5, and WLSE, has overall rank 5 with sum of ranks 200. formed a distinct middle-performing group, showing reasonable but not exceptional performance. The MSAD, MSSD and MSLD consistently demonstrated the lowest performance, with MSAD ranking 10, MSSD ranking 11 and MSLD ranking 12 the last rank. Remarkably, the relative ranking of the estimators remained largely invariant with the true parameter value η. The consistent superiority of MPS and MLE across the wide range of η values (0.5 to 3.5), suggests that these methods are robust to distributional characteristics. Also, From Table 2 to Table 7, we can observe that the MSE decreases as the sample size n increases.

Table 2: Estimated values at $\eta = 0.5$

n	Estimate	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE	MSAD	KM	MSSD	MSLD
20	Mean	0.50259	0.49599	0.50158	0.50139	0.50207	0.50132	0.50044	0.50314	0.49767	0.50216	0.49614	0.49615
	MSE	0.00167$^{\{2\}}$	0.00165$^{\{1\}}$	0.00193$^{\{6\}}$	0.00181$^{\{5\}}$	0.00193$^{\{7\}}$	0.00179$^{\{4\}}$	0.00174$^{\{3\}}$	0.00220$^{\{9\}}$	0.00288$^{\{10\}}$	0.00198$^{\{8\}}$	0.00306$^{\{11\}}$	0.00311$^{\{12\}}$
50	Mean	0.50053	0.49664	0.49988	0.49980	0.50012	0.49991	0.49921	0.50112	0.49740	0.50017	0.49637	0.49637
	MSE	0.00087$^{\{1\}}$	0.00087$^{\{2\}}$	0.00100$^{\{7\}}$	0.00094$^{\{5\}}$	0.00099$^{\{6\}}$	0.00093$^{\{4\}}$	0.00091$^{\{3\}}$	0.00112$^{\{9\}}$	0.00141$^{\{10\}}$	0.00100$^{\{8\}}$	0.00170$^{\{11\}}$	0.00171$^{\{12\}}$
100	Mean	0.50064	0.49818	0.50019	0.50027	0.50036	0.50015	0.49970	0.50108	0.49815	0.50023	0.49779	0.49779
	MSE	0.00044$^{\{2\}}$	0.00044$^{\{1\}}$	0.00050$^{\{7\}}$	0.00046$^{\{4\}}$	0.00050$^{\{6\}}$	0.00047$^{\{5\}}$	0.00046$^{\{3\}}$	0.00058$^{\{9\}}$	0.00069$^{\{10\}}$	0.00053$^{\{8\}}$	0.00088$^{\{11\}}$	0.00089$^{\{12\}}$
150	Mean	0.50127	0.49937	0.50141	0.50140	0.50148	0.50137	0.50104	0.50194	0.50083	0.50173	0.49836	0.49835
	MSE	0.00032$^{\{2\}}$	0.00031$^{\{1\}}$	0.00038$^{\{6\}}$	0.00036$^{\{5\}}$	0.00038$^{\{7\}}$	0.00035$^{\{4\}}$	0.00033$^{\{3\}}$	0.00046$^{\{9\}}$	0.00054$^{\{10\}}$	0.00040$^{\{8\}}$	0.00063$^{\{11\}}$	0.00063$^{\{12\}}$
200	Mean	0.50075	0.49972	0.50058	0.50054	0.50064	0.50054	0.50052	0.50073	0.49963	0.50054	0.49969	0.49969
	MSE	0.00016$^{\{2\}}$	0.00016$^{\{1\}}$	0.00020$^{\{6\}}$	0.00018$^{\{4\}}$	0.00020$^{\{7\}}$	0.00018$^{\{5\}}$	0.00017$^{\{3\}}$	0.00023$^{\{9\}}$	0.00026$^{\{10\}}$	0.00020$^{\{8\}}$	0.00032$^{\{11\}}$	0.00032$^{\{12\}}$
250	Mean	0.50026	0.49952	0.49990	0.50010	0.49999	0.50003	0.50010	0.50004	0.49970	0.49997	0.49915	0.49915
	MSE	0.00010$^{\{2\}}$	0.00010$^{\{1\}}$	0.00012$^{\{7\}}$	0.00011$^{\{5\}}$	0.00012$^{\{6\}}$	0.00011$^{\{4\}}$	0.00011$^{\{3\}}$	0.00014$^{\{9\}}$	0.00017$^{\{10\}}$	0.00012$^{\{8\}}$	0.00019$^{\{11\}}$	0.00019$^{\{12\}}$
300	Mean	0.50026	0.49968	0.50004	0.50018	0.50007	0.50013	0.50021	0.50014	0.49970	0.50024	0.49945	0.49945
	MSE	0.00008$^{\{2\}}$	0.00008$^{\{1\}}$	0.00010$^{\{7\}}$	0.00009$^{\{5\}}$	0.00010$^{\{6\}}$	0.00009$^{\{4\}}$	0.00009$^{\{3\}}$	0.00011$^{\{9\}}$	0.00014$^{\{10\}}$	0.00010$^{\{8\}}$	0.00016$^{\{11\}}$	0.00016$^{\{12\}}$

where {.} indicates the rank of MSE for different methods of estimation.

Table 3: Estimated values at $\eta = 0.8$

n	Estimate	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE	MSAD	KM	MSSD	MSLD
20	Mean	0.80343	0.79444	0.80208	0.80180	0.80279	0.80171	0.80036	0.80444	0.79626	0.80286	0.79466	0.79467
	MSE	0.00395$^{\{2\}}$	0.00393$^{\{1\}}$	0.00456$^{\{7\}}$	0.00428$^{\{5\}}$	0.00455$^{\{6\}}$	0.00423$^{\{4\}}$	0.00415$^{\{3\}}$	0.00515$^{\{9\}}$	0.00682$^{\{10\}}$	0.00466$^{\{8\}}$	0.00730$^{\{11\}}$	0.00741$^{\{12\}}$
50	Mean	0.80062	0.79527	0.79960	0.79977	0.79997	0.79967	0.79861	0.80142	0.79616	0.79997	0.79485	0.79484
	MSE	0.00207$^{\{1\}}$	0.00207$^{\{2\}}$	0.00236$^{\{8\}}$	0.00223$^{\{5\}}$	0.00236$^{\{7\}}$	0.00220$^{\{4\}}$	0.00218$^{\{3\}}$	0.00261$^{\{9\}}$	0.00340$^{\{10\}}$	0.00235$^{\{6\}}$	0.00407$^{\{11\}}$	0.00411$^{\{12\}}$
100	Mean	0.80089	0.79752	0.80035	0.80048	0.80056	0.80025	0.79946	0.80152	0.79754	0.80028	0.79692	0.79692
	MSE	0.00105$^{\{1\}}$	0.00105$^{\{2\}}$	0.00120$^{\{7\}}$	0.00112$^{\{5\}}$	0.00120$^{\{6\}}$	0.00112$^{\{4\}}$	0.00110$^{\{3\}}$	0.00137$^{\{9\}}$	0.00164$^{\{10\}}$	0.00125$^{\{8\}}$	0.00211$^{\{11\}}$	0.00212$^{\{12\}}$
150	Mean	0.80186	0.79926	0.80216	0.80214	0.80226	0.80204	0.80156	0.80279	0.80128	0.80254	0.79777	0.79775
	MSE	0.00076$^{\{2\}}$	0.00075$^{\{1\}}$	0.00091$^{\{6\}}$	0.00085$^{\{5\}}$	0.00091$^{\{7\}}$	0.00084$^{\{4\}}$	0.00078$^{\{3\}}$	0.00106$^{\{9\}}$	0.00129$^{\{10\}}$	0.00094$^{\{8\}}$	0.00151$^{\{11\}}$	0.00151$^{\{12\}}$
200	Mean	0.80110	0.79967	0.80102	0.80089	0.80089	0.80086	0.80077	0.80106	0.79956	0.80079	0.79970	0.79970
	MSE	0.00039$^{\{2\}}$	0.00039$^{\{1\}}$	0.00047$^{\{7\}}$	0.00044$^{\{5\}}$	0.00047$^{\{6\}}$	0.00043$^{\{4\}}$	0.00041$^{\{3\}}$	0.00053$^{\{9\}}$	0.00063$^{\{10\}}$	0.00048$^{\{8\}}$	0.00076$^{\{11\}}$	0.00076$^{\{12\}}$
250	Mean	0.80035	0.79933	0.79995	0.80007	0.80002	0.80006	0.80014	0.80005	0.79950	0.79992	0.79881	0.79881
	MSE	0.00025$^{\{2\}}$	0.00025$^{\{1\}}$	0.00029$^{\{7\}}$	0.00027$^{\{5\}}$	0.00029$^{\{6\}}$	0.00027$^{\{4\}}$	0.00026$^{\{3\}}$	0.00032$^{\{9\}}$	0.00041$^{\{10\}}$	0.00029$^{\{8\}}$	0.00046$^{\{11\}}$	0.00047$^{\{12\}}$
300	Mean	0.80039	0.79956	0.80019	0.80017	0.80022	0.80023	0.80029	0.80021	0.79943	0.80034	0.79923	0.79923
	MSE	0.00020$^{\{1\}}$	0.00020$^{\{2\}}$	0.00023$^{\{6\}}$	0.00022$^{\{5\}}$	0.00023$^{\{7\}}$	0.00022$^{\{4\}}$	0.00022$^{\{3\}}$	0.00026$^{\{9\}}$	0.00034$^{\{10\}}$	0.00023$^{\{8\}}$	0.00038$^{\{11\}}$	0.00038$^{\{12\}}$

where {.} indicates the rank of MSE for different methods of estimation.

Table 4: Estimated values at $\eta = 1.3$

n	Estimate	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE	MSAD	KM	MSSD	MSLD
20	Mean	1.30454	1.29157	1.30276	1.30209	1.30369	1.30221	1.30013	1.30610	1.29422	1.30385	1.29311	1.29316
	MSE	0.00909$^{\{2\}}$	0.00897$^{\{1\}}$	0.01036$^{\{7\}}$	0.00973$^{\{5\}}$	0.01034$^{\{6\}}$	0.00963$^{\{4\}}$	0.00957$^{\{3\}}$	0.01155$^{\{9\}}$	0.01567$^{\{10\}}$	0.01055$^{\{8\}}$	0.01652$^{\{11\}}$	0.01677$^{\{12\}}$
50	Mean	1.30059	1.29297	1.29922	1.29930	1.29977	1.29933	1.29766	1.30208	1.29409	1.29981	1.29274	1.29272
	MSE	0.00475$^{\{1\}}$	0.00475$^{\{2\}}$	0.00535$^{\{8\}}$	0.00508$^{\{5\}}$	0.00535$^{\{7\}}$	0.00501$^{\{3\}}$	0.00503$^{\{4\}}$	0.00587$^{\{9\}}$	0.00763$^{\{10\}}$	0.00532$^{\{6\}}$	0.00923$^{\{11\}}$	0.00931$^{\{12\}}$
100	Mean	1.30117	1.29632	1.30045	1.30069	1.30072	1.30029	1.29908	1.30208	1.29630	1.30009	1.29582	1.29582
	MSE	0.00241$^{\{1.5\}}$	0.00241$^{\{1.5\}}$	0.00272$^{\{7\}}$	0.00255$^{\{5\}}$	0.00272$^{\{6\}}$	0.00255$^{\{4\}}$	0.00254$^{\{3\}}$	0.00306$^{\{9\}}$	0.00374$^{\{10\}}$	0.00283$^{\{8\}}$	0.00481$^{\{11\}}$	0.00483$^{\{12\}}$
150	Mean	1.30275	1.29896	1.30303	1.30313	1.30333	1.30301	1.30230	1.30414	1.30185	1.30372	1.29698	1.29696
	MSE	0.00176$^{\{2\}}$	0.00175$^{\{1\}}$	0.00205$^{\{6\}}$	0.00193$^{\{5\}}$	0.00206$^{\{7\}}$	0.00191$^{\{4\}}$	0.00181$^{\{3\}}$	0.00239$^{\{9\}}$	0.00292$^{\{10\}}$	0.00212$^{\{8\}}$	0.00345$^{\{11\}}$	0.00347$^{\{12\}}$
200	Mean	1.30156	1.29941	1.30129	1.30130	1.30147	1.30124	1.30113	1.30152	1.29933	1.30112	1.29974	1.29974
	MSE	0.00090$^{\{2\}}$	0.00089$^{\{1\}}$	0.00107$^{\{6\}}$	0.00098$^{\{4\}}$	0.00108$^{\{7\}}$	0.00098$^{\{5\}}$	0.00095$^{\{3\}}$	0.00118$^{\{9\}}$	0.00143$^{\{10\}}$	0.00109$^{\{8\}}$	0.00175$^{\{11\}}$	0.00175$^{\{12\}}$
250	Mean	1.30047	1.29903	1.29991	1.29993	1.30000	1.30003	1.30017	1.29999	1.29903	1.29986	1.29833	1.29833
	MSE	0.00057$^{\{2\}}$	0.00057$^{\{1\}}$	0.00066$^{\{7\}}$	0.00061$^{\{4\}}$	0.00065$^{\{6\}}$	0.00061$^{\{5\}}$	0.00060$^{\{3\}}$	0.00073$^{\{9\}}$	0.00095$^{\{10\}}$	0.00066$^{\{8\}}$	0.00107$^{\{11\}}$	0.00107$^{\{12\}}$
300	Mean	1.30050	1.29926	1.30028	1.30034	1.30027	1.30028	1.30042	1.30030	1.29914	1.30049	1.29891	1.29891
	MSE	0.00046$^{\{1.5\}}$	0.00046$^{\{1.5\}}$	0.00052$^{\{6\}}$	0.00049$^{\{3\}}$	0.00053$^{\{7\}}$	0.00049$^{\{4\}}$	0.00050$^{\{5\}}$	0.00057$^{\{9\}}$	0.00078$^{\{10\}}$	0.00053$^{\{8\}}$	0.00087$^{\{11\}}$	0.00087$^{\{12\}}$

where {.} indicates the rank of MSE for different methods of estimation.

Table 5: Estimated values at $\eta = 1.7$

n	Estimate	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE	MSAD	KM	MSSD	MSLD
20	Mean	1.70695	1.68876	1.70448	1.70371	1.70579	1.70367	1.70093	1.70917	1.69409	1.70610	1.69098	1.69108
	MSE	0.01534$^{\{2\}}$	0.01494$^{\{1\}}$	0.01732$^{\{6\}}$	0.01629$^{\{5\}}$	0.01734$^{\{7\}}$	0.01616$^{\{4\}}$	0.01593$^{\{3\}}$	0.01963$^{\{9\}}$	0.02608$^{\{10\}}$	0.01777$^{\{8\}}$	0.02734$^{\{11\}}$	0.02776$^{\{12\}}$
50	Mean	1.70116	1.69032	1.69944	1.69960	1.70012	1.69953	1.69754	1.70327	1.69224	1.70031	1.69034	1.69033
	MSE	0.00792$^{\{2\}}$	0.00786$^{\{1\}}$	0.00892$^{\{7.5\}}$	0.00845$^{\{5\}}$	0.00892$^{\{7.5\}}$	0.00836$^{\{4\}}$	0.00831$^{\{3\}}$	0.00990$^{\{9\}}$	0.01259$^{\{10\}}$	0.00890$^{\{6\}}$	0.01523$^{\{11\}}$	0.01537$^{\{12\}}$
100	Mean	1.70171	1.69479	1.70080	1.70083	1.70118	1.70056	1.69901	1.70308	1.69489	1.70055	1.69436	1.69436
	MSE	0.00402$^{\{2\}}$	0.00400$^{\{1\}}$	0.00452$^{\{6\}}$	0.00424$^{\{5\}}$	0.00452$^{\{7\}}$	0.00424$^{\{4\}}$	0.00420$^{\{3\}}$	0.00515$^{\{9\}}$	0.00620$^{\{10\}}$	0.00473$^{\{8\}}$	0.00799$^{\{11\}}$	0.00804$^{\{12\}}$
150	Mean	1.70375	1.69841	1.70433	1.70411	1.70448	1.70403	1.70308	1.70562	1.70236	1.70499	1.69582	1.69579
	MSE	0.00292$^{\{2\}}$	0.00288$^{\{1\}}$	0.00341$^{\{6\}}$	0.00321$^{\{5\}}$	0.00342$^{\{7\}}$	0.00318$^{\{4\}}$	0.00299$^{\{3\}}$	0.00402$^{\{9\}}$	0.00487$^{\{10\}}$	0.00356$^{\{8\}}$	0.00572$^{\{11\}}$	0.00574$^{\{12\}}$
200	Mean	1.70212	1.69916	1.70182	1.70156	1.70199	1.70169	1.70152	1.70210	1.69901	1.70157	1.69953	1.69953
	MSE	0.00149$^{\{2\}}$	0.00148$^{\{1\}}$	0.00179$^{\{7\}}$	0.00164$^{\{4\}}$	0.00178$^{\{6\}}$	0.00164$^{\{5\}}$	0.00157$^{\{3\}}$	0.00198$^{\{9\}}$	0.00238$^{\{10\}}$	0.00181$^{\{8\}}$	0.00290$^{\{11\}}$	0.00291$^{\{12\}}$
250	Mean	1.70067	1.69855	1.70001	1.70015	1.70003	1.70012	1.70027	1.70006	1.69881	1.69988	1.69771	1.69771
	MSE	0.00095$^{\{2\}}$	0.00094$^{\{1\}}$	0.00109$^{\{6.5\}}$	0.00100$^{\{4\}}$	0.00109$^{\{6.5\}}$	0.00101$^{\{5\}}$	0.00099$^{\{3\}}$	0.00122$^{\{9\}}$	0.00157$^{\{10\}}$	0.00110$^{\{8\}}$	0.00177$^{\{11\}}$	0.00178$^{\{12\}}$
300	Mean	1.70071	1.69904	1.70038	1.70042	1.70044	1.70046	1.70058	1.70042	1.69881	1.70069	1.69853	1.69853
	MSE	0.00077$^{\{2\}}$	0.00077$^{\{1\}}$	0.00087$^{\{6\}}$	0.00082$^{\{3\}}$	0.00087$^{\{7\}}$	0.00082$^{\{4.5\}}$	0.00082$^{\{4.5\}}$	0.00096$^{\{9\}}$	0.00129$^{\{10\}}$	0.00088$^{\{8\}}$	0.00144$^{\{11\}}$	0.00144$^{\{12\}}$

where {.} indicates the rank of MSE for different methods of estimation.

Table 6: Estimated values at $\eta = 2.5$

n	Estimate	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE	MSAD	KM	MSSD	MSLD
20	Mean	2.51562	2.48056	2.51074	2.50945	2.51352	2.50917	2.50475	2.51988	2.49218	2.51415	2.48297	2.48323
	MSE	$0.03952^{\{2\}}$	$0.03786^{\{1\}}$	$0.04526^{\{6\}}$	$0.04234^{\{5\}}$	$0.04542^{\{7\}}$	$0.04196^{\{4\}}$	$0.04040^{\{3\}}$	$0.05306^{\{9\}}$	$0.06603^{\{10\}}$	$0.04700^{\{8\}}$	$0.06998^{\{11\}}$	$0.07111^{\{12\}}$
50	Mean	2.50381	2.48313	2.50090	2.50091	2.50233	2.50085	2.49758	2.50763	2.48695	2.50272	2.48191	2.48184
	MSE	$0.02008^{\{2\}}$	$0.01977^{\{1\}}$	$0.02309^{\{6\}}$	$0.02176^{\{5\}}$	$0.02312^{\{7\}}$	$0.02151^{\{4\}}$	$0.02085^{\{3\}}$	$0.02653^{\{9\}}$	$0.03158^{\{10\}}$	$0.02335^{\{8\}}$	$0.03809^{\{11\}}$	$0.03840^{\{12\}}$
100	Mean	2.50373	2.49060	2.50212	2.50220	2.50294	2.50165	2.49926	2.50622	2.49041	2.50209	2.48950	2.48950
	MSE	$0.01015^{\{2\}}$	$0.01005^{\{1\}}$	$0.01164^{\{6\}}$	$0.01085^{\{5\}}$	$0.01165^{\{7\}}$	$0.01083^{\{4\}}$	$0.01048^{\{3\}}$	$0.01366^{\{9\}}$	$0.01569^{\{10\}}$	$0.01229^{\{8\}}$	$0.02025^{\{11\}}$	$0.02037^{\{12\}}$
150	Mean	2.50663	2.49654	2.50743	2.50726	2.50800	2.50704	2.50535	2.51015	2.50414	2.50903	2.49197	2.49192
	MSE	$0.00728^{\{2\}}$	$0.00716^{\{1\}}$	$0.00882^{\{6\}}$	$0.00822^{\{5\}}$	$0.00884^{\{7\}}$	$0.00814^{\{4\}}$	$0.00748^{\{3\}}$	$0.01067^{\{9\}}$	$0.01224^{\{10\}}$	$0.00932^{\{8\}}$	$0.01431^{\{11\}}$	$0.01437^{\{12\}}$
200	Mean	2.50387	2.49831	2.50336	2.50300	2.50365	2.50306	2.50272	2.50396	2.49818	2.50295	2.49838	2.49838
	MSE	$0.00373^{\{2\}}$	$0.00369^{\{1\}}$	$0.00459^{\{6\}}$	$0.00419^{\{5\}}$	$0.00459^{\{7\}}$	$0.00418^{\{4\}}$	$0.00392^{\{3\}}$	$0.00525^{\{9\}}$	$0.00602^{\{10\}}$	$0.00472^{\{8\}}$	$0.00725^{\{11\}}$	$0.00727^{\{12\}}$
250	Mean	2.50145	2.49749	2.50016	2.50070	2.50035	2.50043	2.50067	2.50038	2.49848	2.50009	2.49576	2.49575
	MSE	$0.00238^{\{2\}}$	$0.00237^{\{1\}}$	$0.00280^{\{6.5\}}$	$0.00259^{\{5\}}$	$0.00280^{\{6.5\}}$	$0.00258^{\{4\}}$	$0.00249^{\{3\}}$	$0.00322^{\{9\}}$	$0.00397^{\{10\}}$	$0.00285^{\{8\}}$	$0.00445^{\{11\}}$	$0.00445^{\{12\}}$
300	Mean	2.50142	2.49830	2.50080	2.50112	2.50094	2.50093	2.50111	2.50093	2.49847	2.50134	2.49733	2.49732
	MSE	$0.00194^{\{2\}}$	$0.00194^{\{1\}}$	$0.00225^{\{6\}}$	$0.00211^{\{5\}}$	$0.00225^{\{7\}}$	$0.00210^{\{4\}}$	$0.00205^{\{3\}}$	$0.00254^{\{9\}}$	$0.00327^{\{10\}}$	$0.00230^{\{8\}}$	$0.00362^{\{11\}}$	$0.00363^{\{12\}}$

where {.} indicates the rank of MSE for different methods of estimation.

Table 7: Estimated values at $\eta = 3.5$

n	Estimate	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE	MSAD	KM	MSSD	MSLD
20	Mean	3.52794	3.46716	3.51910	3.51699	3.52406	3.51645	3.50938	3.53461	3.48582	3.52500	3.46917	3.46933
	MSE	$0.09726^{\{2\}}$	$0.09298^{\{1\}}$	$0.11347^{\{6\}}$	$0.10556^{\{5\}}$	$0.11391^{\{7\}}$	$0.10445^{\{4\}}$	$0.09943^{\{3\}}$	$0.13492^{\{9\}}$	$0.16243^{\{10\}}$	$0.11843^{\{8\}}$	$0.17471^{\{11\}}$	$0.17764^{\{12\}}$
50	Mean	3.50757	3.47167	3.50249	3.50270	3.50504	3.50233	3.49734	3.51352	3.47991	3.50567	3.46944	3.46935
	MSE	$0.04923^{\{2\}}$	$0.04846^{\{1\}}$	$0.05785^{\{6\}}$	$0.05412^{\{5\}}$	$0.05793^{\{7\}}$	$0.05348^{\{4\}}$	$0.05113^{\{3\}}$	$0.06744^{\{9\}}$	$0.07697^{\{10\}}$	$0.05881^{\{8\}}$	$0.09388^{\{11\}}$	$0.09465^{\{12\}}$
100	Mean	3.50674	3.48394	3.50386	3.50407	3.50532	3.50306	3.49936	3.51064	3.48558	3.50406	3.48176	3.48174
	MSE	$0.02492^{\{2\}}$	$0.02467^{\{1\}}$	$0.02911^{\{6\}}$	$0.02697^{\{5\}}$	$0.02914^{\{7\}}$	$0.02687^{\{4\}}$	$0.02569^{\{3\}}$	$0.03463^{\{9\}}$	$0.03873^{\{10\}}$	$0.03088^{\{8\}}$	$0.04988^{\{11\}}$	$0.05017^{\{12\}}$
150	Mean	3.51079	3.49328	3.51217	3.51198	3.51320	3.51147	3.50888	3.51679	3.50660	3.51486	3.48584	3.48575
	MSE	$0.01773^{\{2\}}$	$0.01742^{\{1\}}$	$0.02209^{\{6\}}$	$0.02039^{\{5\}}$	$0.02213^{\{7\}}$	$0.02019^{\{4\}}$	$0.01835^{\{3\}}$	$0.02706^{\{9\}}$	$0.02975^{\{10\}}$	$0.02350^{\{8\}}$	$0.03500^{\{11\}}$	$0.03516^{\{12\}}$
200	Mean	3.50647	3.49681	3.50556	3.50522	3.50607	3.50505	3.50447	3.50665	3.49692	3.50493	3.49638	3.49637
	MSE	$0.00909^{\{2\}}$	$0.00899^{\{1\}}$	$0.01147^{\{6\}}$	$0.01040^{\{5\}}$	$0.01148^{\{7\}}$	$0.01037^{\{4\}}$	$0.00959^{\{3\}}$	$0.01332^{\{9\}}$	$0.01476^{\{10\}}$	$0.01187^{\{8\}}$	$0.01768^{\{11\}}$	$0.01772^{\{12\}}$
250	Mean	3.50262	3.49573	3.50042	3.50090	3.50076	3.50085	3.50122	3.50090	3.49783	3.50031	3.49266	3.49265
	MSE	$0.00583^{\{2\}}$	$0.00580^{\{1\}}$	$0.00700^{\{6\}}$	$0.00643^{\{5\}}$	$0.00700^{\{7\}}$	$0.00641^{\{4\}}$	$0.00610^{\{3\}}$	$0.00815^{\{9\}}$	$0.00974^{\{10\}}$	$0.00716^{\{8\}}$	$0.01087^{\{11\}}$	$0.01089^{\{12\}}$
300	Mean	3.50246	3.49704	3.50140	3.50169	3.50166	3.50161	3.50188	3.50165	3.49716	3.50231	3.49535	3.49534
	MSE	$0.00476^{\{2\}}$	$0.00475^{\{1\}}$	$0.00563^{\{6\}}$	$0.00522^{\{5\}}$	$0.00563^{\{7\}}$	$0.00522^{\{4\}}$	$0.00503^{\{3\}}$	$0.00645^{\{9\}}$	$0.00797^{\{10\}}$	$0.00578^{\{8\}}$	$0.00887^{\{11\}}$	$0.00888^{\{12\}}$

where {.} indicates the rank of MSE for different methods of estimation.

Table 8: Sum and overall ranks of tables from Table 2 to Table 7

Table	n	MLE	MPS	LSE	WLSE	CME	ADE	RADE	LADE	MSAD	KM	MSSD	MSLD
	20	2	1	6	5	7	4	3	9	10	8	11	12
	50	1	2	7	5	6	4	3	9	10	8	11	12
	100	2	1	7	4	6	5	3	9	10	8	11	12
2	150	2	1	6	5	7	4	3	9	10	8	11	12
	200	2	1	6	4	7	5	3	9	10	8	11	12
	250	2	1	7	5	6	4	3	9	10	8	11	12
	300	2	1	7	5	6	4	3	9	10	8	11	12
	20	2	1	7	5	6	4	3	9	10	8	11	12
	50	1	2	8	5	7	4	3	9	10	6	11	12
	100	1	2	7	5	6	4	3	9	10	8	11	12
3	150	2	1	6	5	7	4	3	9	10	8	11	12
	200	2	1	7	5	6	4	3	9	10	8	11	12
	250	2	1	7	5	6	4	3	9	10	8	11	12
	300	1	2	6	5	7	4	3	9	10	8	11	12
	20	2	1	7	5	6	4	3	9	10	8	11	12
	50	1	2	8	5	7	3	4	9	10	6	11	12
	100	1.5	1.5	7	5	6	4	3	9	10	8	11	12
4	150	2	1	6	5	7	4	3	9	10	8	11	12
	200	2	1	6	4	7	5	3	9	10	8	11	12
	250	2	1	7	4	6	5	3	9	10	8	11	12
	300	1.5	1.5	6	3	7	4	5	9	10	8	11	12
	20	2	1	6	5	7	4	3	9	10	8	11	12
	50	2	1	7.5	5	7.5	4	3	9	10	6	11	12
	100	2	1	6	5	7	4	3	9	10	8	11	12
5	150	2	1	6	5	7	4	3	9	10	8	11	12
	200	2	1	7	4	6	5	3	9	10	8	11	12
	250	2	1	6.5	4	6.5	5	3	9	10	8	11	12
	300	2	1	6	3	7	4.5	4.5	9	10	8	11	12
	20	2	1	6	5	7	4	3	9	10	8	11	12
	50	2	1	6	5	7	4	3	9	10	8	11	12
	100	2	1	6	5	7	4	3	9	10	8	11	12
6	150	2	1	6	5	7	4	3	9	10	8	11	12
	200	2	1	6	5	7	4	3	9	10	8	11	12
	250	2	1	6.5	5	6.5	4	3	9	10	8	11	12
	300	2	1	6	5	7	4	3	9	10	8	11	12
	20	2	1	6	5	7	4	3	9	10	8	11	12
	50	2	1	6	5	7	4	3	9	10	8	11	12
	100	2	1	6	5	7	4	3	9	10	8	11	12
7	150	2	1	6	5	7	4	3	9	10	8	11	12
	200	2	1	6	5	7	4	3	9	10	8	11	12
	250	2	1	6	5	7	4	3	9	10	8	11	12
	300	2	1	6	5	7	4	3	9	10	8	11	12
$\sum$ Ranks		78	48	270.5	200	281.5	173.5	130.5	378	420	330	462	504
Overall Rank		2	1	6	5	7	4	3	9	10	8	11	12

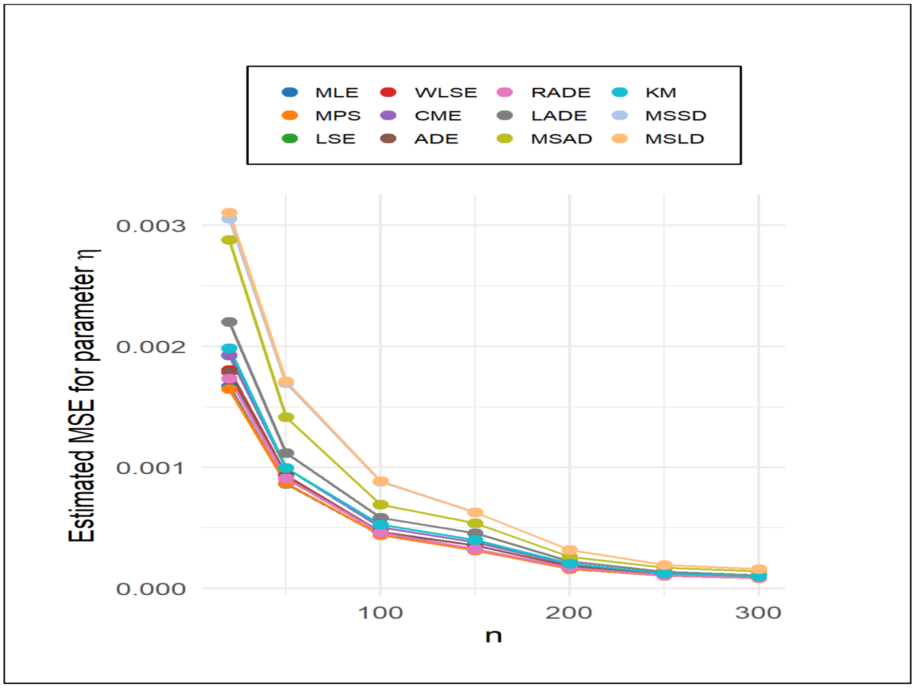

Figure 6: Estimated MSE for η in Table 2

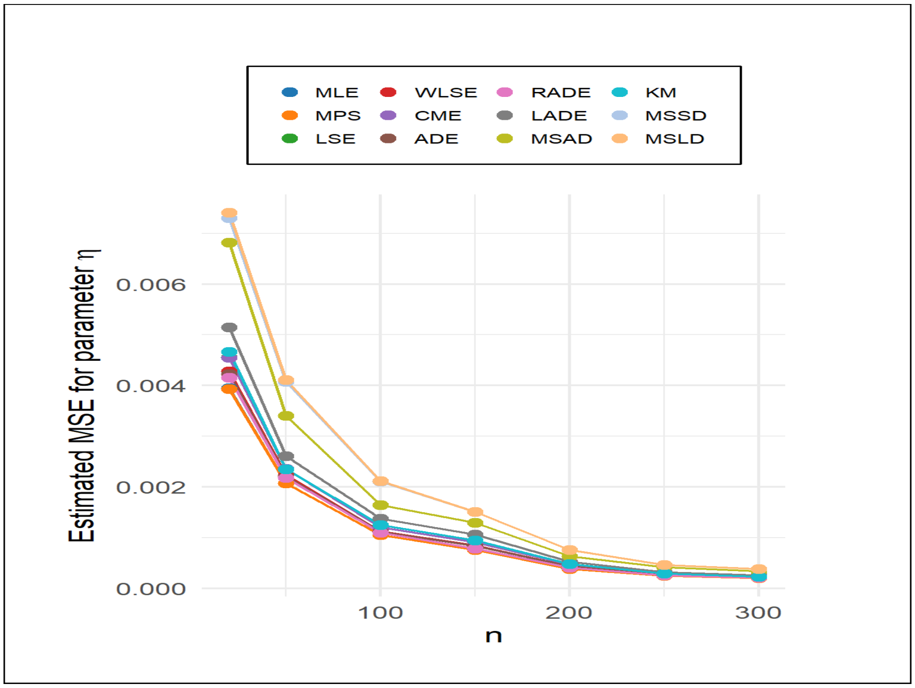

Figure 7: Estimated MSE for η in Table 3

6 Applications

In this section, we evaluate the performance of the proposed Garhy distribution by comparing it with fifteen well-known competing distributions using two real data applications. The competing models include: Lindley, Shanker, Akash, Ishita, Rani, Pranav, Rama, Xgamma, ChrisJerry, XLindley, Komal, RamAwadh, Zeghdoudi, Haq, and Hamza distributions.

For each data set, we compute the MLEs along with their corresponding standard errors (SEs). To assess the goodness-of-fit (GOF), we consider several statistical measures: the log-likelihood value (LogLik), Akaike information criterion (AIC), Bayesian information criterion (BIC), corrected AIC (CAIC), Hannan–Quinn information criterion (HQIC), and the absolute standard average error (ASAE). Additionally, we report the classical GOF statistics-Kolmogorov–Smirnov (KS), Cramér–von Mises (CvM), and Anderson–Darling (AD)-along with their associated p–values. Graphical comparisons based on the estimated probability density function (PDF), cumulative distribution function (CDF), and probability–probability (P–P) plots are also provided to visually assess the adequacy of the fitted models.

6.1 First Data Set: Minneapolis/St Paul Precipitation

The first data set, first reported by Hinkley [34], contains thirty consecutive values of March precipitation (inches) in Minneapolis/St Paul. The observations are as follows: 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Table 9 describes this data set.

Table 9: Descriptive statistics for (precipitation) data set.

Statistic	Value
Sample size (n)	30
Minimum	0.32
Maximum	4.75
Range	4.43
Mean	1.720
Median	1.515
Variance	1.063
Standard deviation	1.031
Skewness	1.133
Kurtosis	2.451
Coefficient of variation (CV)	0.599

Table 10 summarizes the MLEs, SEs, and GOF statistics for all fitted models. Of the sixteen models that we considered, our proposed Garhy distribution exhibits a very competitive performance. It also has the lowest AIC (79.1896) and ASAE (0.2020)—a better trade-off of model complexity and fit. Additionally, the Garhy distribution has the minimum values of KS (0.0802), CvM (0.0226) and AD (0.1802) statistics and the maximum p-values (0.9904, 0.9946, and 0.9950, respectively), strongly backing its suitability towards the precipitation data.

Graphic diagnostics further confirm the suitability of the Garhy distribution. Figure 8 displays the estimated PDFs, where the Garhy model closely follows the empirical histogram. The estimated CDF in Figure 9 and the P p plots in Figure 10 show that the Garhy distribution provides an excellent fit, and its CDF aligns closely with the empirical distribution.

Table 10: MLEs with SEs and GOF statistics for for (precipitation) data set.

Model	$\hat{\eta}$ SE($\hat{\eta}$)	$\hat{\alpha}$ SE($\hat{\alpha}$)	LogLik	AIC	BIC	CAIC	HQIC	ASAE	KS (KS-pval)	CvM-Stat (CvM-pval)	AD-Stat (AD-pval)
Garhy	1.5504	–	38.5948	79.1896	80.5908	79.3324	79.6378	0.2020	0.0802	0.0226	0.1802
	(0.1440)	–							(0.9904)	(0.9946)	(0.9950)
Lindley	0.9096	–	43.1437	88.2875	89.6886	88.4303	88.7357	0.2840	0.1882	0.2618	1.5907
	(0.1247)	–							(0.2384)	(0.1740)	(0.1564)
Shanker	0.8984	–	42.9874	87.9748	89.3760	88.1176	88.4230	0.2873	0.1743	0.2216	1.4199
	(0.1088)	–							(0.3216)	(0.2297)	(0.1968)
Akash	1.2617	–	43.4282	88.8563	90.2575	88.9992	89.3046	0.2835	0.1839	0.2435	1.5362
	(0.1308)	–							(0.2620)	(0.1971)	(0.1682)
Ishita	1.2002	–	43.8529	89.7057	91.1069	89.8486	90.1540	0.2905	0.1822	0.2486	1.5921
	(0.1087)	–							(0.2724)	(0.1903)	(0.1561)
Rani	1.8334	–	47.4985	96.9970	98.3982	97.1398	97.4452	0.3049	0.2353	0.4217	2.4662
	(0.1106)	–							(0.0722)	(0.0627)	(0.0520)
Pranav	1.5121	–	45.4904	92.9809	94.3821	93.1237	93.4291	0.2975	0.2068	0.3271	2.0035
	(0.1094)	–							(0.1538)	(0.1132)	(0.0918)
Rama	1.6341	–	44.4400	90.8800	92.2812	91.0229	91.3283	0.2856	0.1969	0.2695	1.6926
	(0.1344)	–							(0.1950)	(0.1652)	(0.1368)
Xgamma	1.1901	–	44.5735	91.1471	92.5483	91.2899	91.5953	0.2870	0.2227	0.3825	2.1710
	(0.1517)	–							(0.1021)	(0.0798)	(0.0746)
ChrisJerry	1.3608	–	42.7668	87.5335	88.9347	87.6764	87.9818	0.2733	0.1827	0.2338	1.4438
	(0.1619)	–							(0.2692)	(0.2109)	(0.1905)
XLindley	0.7799	–	44.5478	91.0955	92.4967	91.2384	91.5438	0.2948	0.2142	0.3601	2.0852
	(0.1108)	–							(0.1276)	(0.0918)	(0.0829)
Komal	0.8242	–	44.0513	90.1026	91.5038	90.2454	90.5508	0.2924	0.2021	0.3132	1.8653
	(0.1106)	–							(0.1722)	(0.1238)	(0.1094)
RamAwadh	2.1619	–	49.7720	101.5439	102.9451	101.6868	101.9922	0.3112	0.2603	0.5180	2.9152
	(0.1118)	–							(0.0343)	(0.0352)	(0.0306)
Zeghdoudi	1.5320	–	38.6705	79.3410	80.7422	79.4839	79.7893	0.2427	0.0877	0.0271	0.2089
	(0.1668)	–							(0.9751)	(0.9860)	(0.9878)
Haq	0.9279	–	46.3296	94.6593	96.0605	94.8021	95.1075	0.3036	0.2495	0.5163	2.8118
	(0.1255)	–							(0.0477)	(0.0355)	(0.0345)
Hamza	3.8183	0.0166	44.1476	92.2953	95.0977	92.7397	93.1918	0.2114	0.1387	0.1334	0.9347
	(0.3404)	(0.0186)							(0.6107)	(0.4465)	(0.3926)

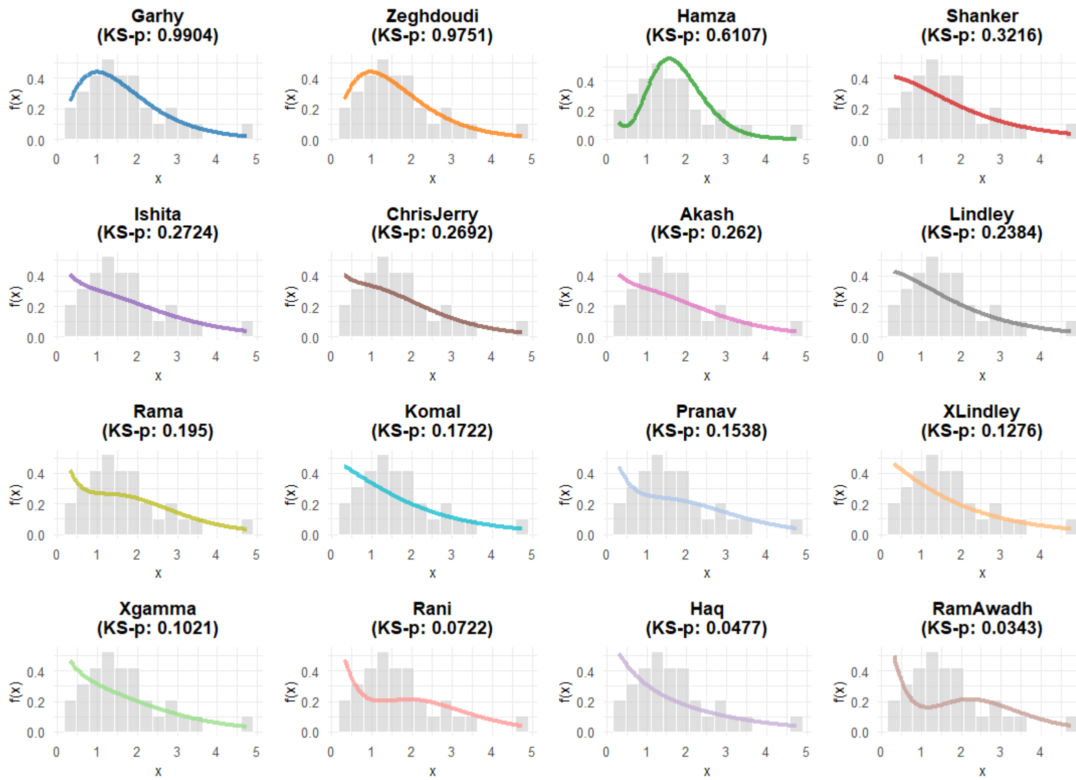

Figure 8: Estimated PDFs for the competing models—(precipitation) data set

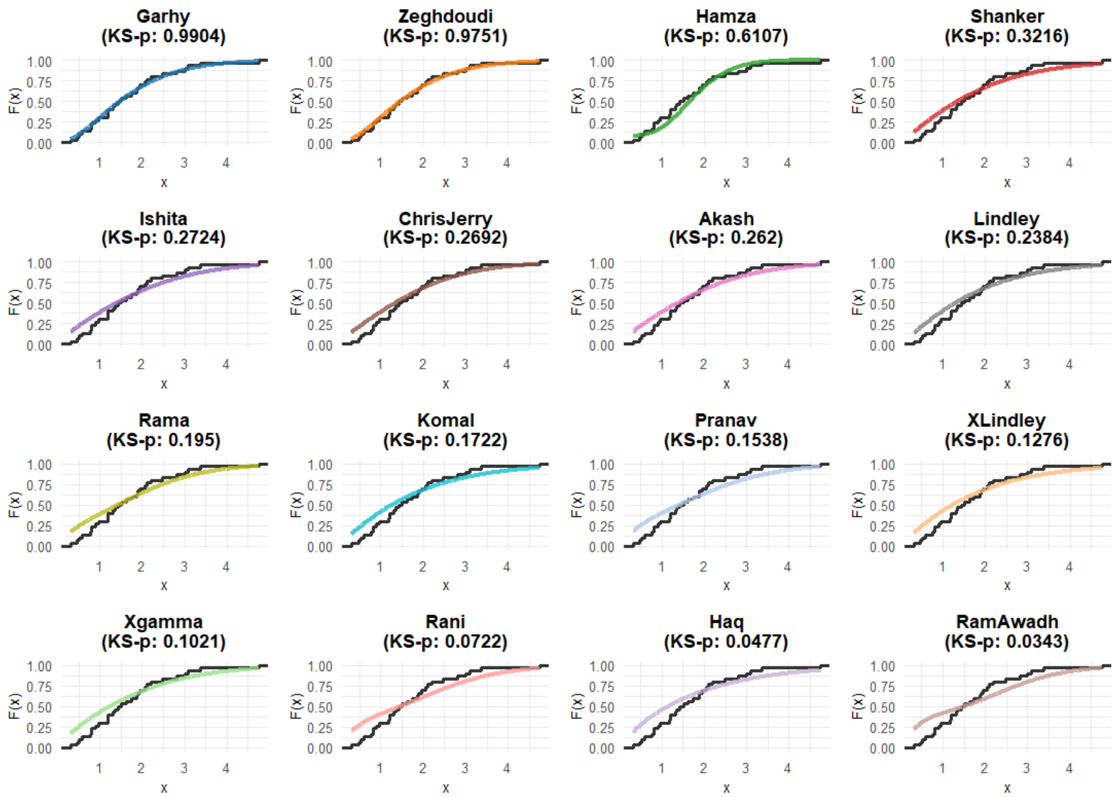

Figure 9: Estimated CDFs for the competing models—(precipitation) data set

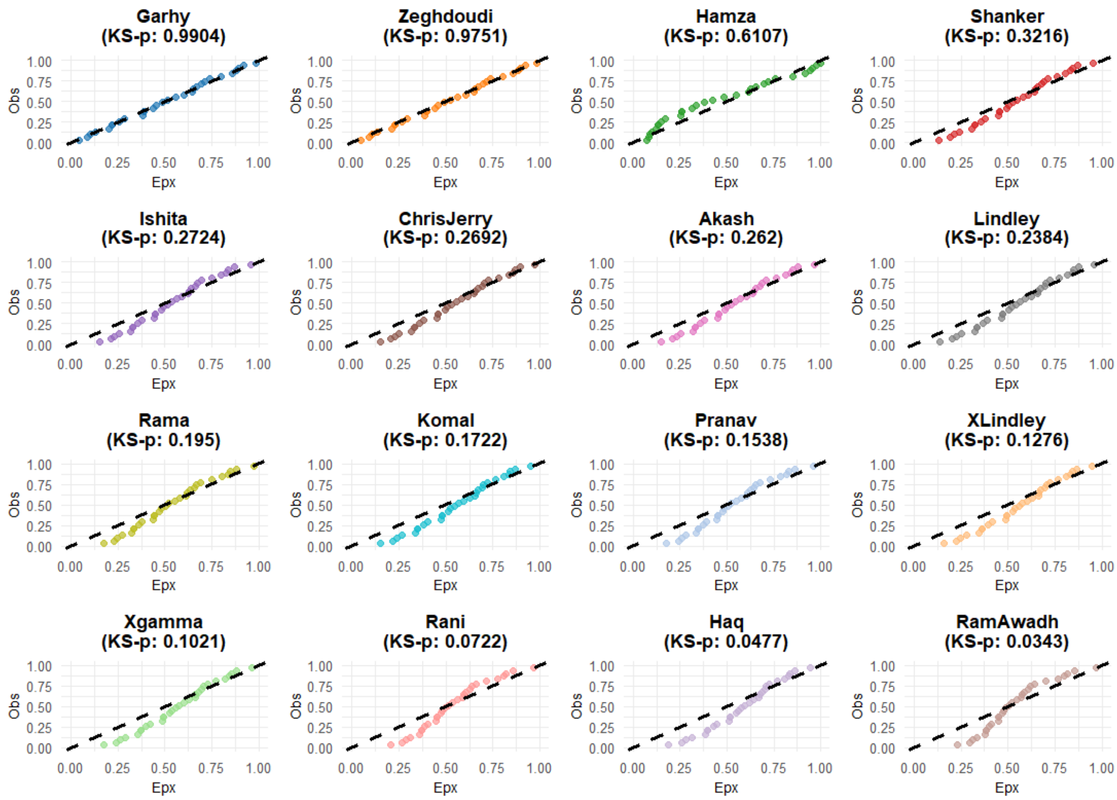

Figure 10: P-P plots for the competing models—(precipitation) data set

6.2 Second Data Set: Times to First Infection of Kidney Dialysis Patients

The second data set represents the times to the first infection (in days) for kidney dialysis patients, originally reported by Klein and Moeschberger [35]. This data set has been widely used in several recent studies to evaluate performance; see, for example, [36–39].

The original data consist of $n = 28$ observations:

{ 2.5, 2.5, 3.5, 3.5, 3.5, 4.5, 5.5, 6.5, 6.5, 7.5, 7.5, 7.5, 7.5, 8.5, 9.5, 10.5, 11.5, 12.5, 12.5, 13.5, 14.5, 14.5, 21.5, 21.5, 22.5, 22.5, 25.5, 27.5 }.

Following the standard preprocessing adopted by previous authors, and in order to make the observations compatible with probability models defined on the bounded interval $(0, 1)$, we normalize the data by dividing each value by 30. The transformed data set becomes:

{ 0.0833, 0.0833, 0.1167, 0.1167, 0.1167, 0.1500, 0.1833, 0.2167, 0.2167, 0.2500, 0.2500, 0.2500, 0.2500, 0.2833, 0.3167, 0.3500, 0.3833, 0.4167, 0.4167, 0.4500, 0.4833, 0.4833, 0.7167, 0.7167, 0.7500, 0.7500, 0.8500, 0.9167 }.

Table 11 describes this data set.

Their normalized data set has become a standard against which to assess bounded distribution goodness-of-fit. MLEs, SEs and GOF statistics for the fitted models are presented in Table 12. The suggested Garhy distribution triumphs over every other rival model with both the lowest AIC value (-5.8265) and ASAE (0.0336). and also generates the smallest statistics of KS (0.1067), CvM (0.0364) and AD (0.3215), with the highest associated p-values (0.9072, 0.9536, and 0.9206, respectively). These findings suggest that the Garhy distribution is the best fit to the normalized infection time data.

Visual comparisons also support these results. As with the previous data, the estimated PDFs in Figure 11 demonstrate that the Garhy distribution does a good job in capturing the shape of the data. The CDF plots in Figure 12 again show good agreement between the Garhy model and the empirical distribution, and the P–P plots in Figure 13 verify that the Garhy distribution fits the data well, with points close to the diagonal.

Table 11: Descriptive statistics for (infection times of kidney dialysis patients) data set.

Statistic	Value
Sample size (n)	28
Minimum	0.0833
Maximum	0.9167
Range	0.8334
Mean	0.3598
Median	0.3167
Variance	0.0632
Standard deviation	0.2513
Skewness	1.517
Kurtosis	3.110
Coefficient of variation (CV)	0.699

Table 12: MLEs with SEs and GOF statistics for (infection times of kidney dialysis patients) data set.

Model	$\hat{\eta}$ SE($\hat{\eta}$)	$\hat{\alpha}$ SE($\hat{\alpha}$)	LogLik	AIC	BIC	CAIC	HQIC	ASAE	KS (KS-pval)	CvM-Stat (CvM-pval)	AD-Stat (AD-pval)
Garhy	5.3963	–	-3.9132	-5.8265	-4.4943	-5.6726	-5.4192	0.0336	0.1067	0.0364	0.3215
	(0.6852)	–							(0.9072)	(0.9536)	(0.9206)
Lindley	3.2704	–	0.2942	2.5884	3.9206	2.7423	2.9957	0.0768	0.1900	0.2350	1.4952
	(0.5199)	–							(0.2645)	(0.2091)	(0.1777)
Shanker	2.9160	–	0.6257	3.2513	4.5835	3.4052	3.6586	0.0820	0.1957	0.2590	1.6154
	(0.4686)	–							(0.2340)	(0.1773)	(0.1514)
Akash	3.4228	–	1.4588	4.9176	6.2498	5.0714	5.3248	0.0948	0.2124	0.3349	1.9788
	(0.4835)	–							(0.1597)	(0.1077)	(0.0948)
Ishita	3.0300	–	1.1896	4.3793	5.7115	4.5331	4.7866	0.0927	0.2083	0.3234	1.9128
	(0.4450)	–							(0.1759)	(0.1159)	(0.1030)
Rani	3.3228	–	2.3412	6.6823	8.0145	6.8362	7.0896	0.1188	0.2346	0.4701	2.5640
	(0.3931)	–							(0.0918)	(0.0467)	(0.0463)
Pranav	3.1567	–	1.7207	5.4414	6.7737	5.5953	5.8487	0.1049	0.2191	0.3901	2.2106
	(0.4199)	–							(0.1359)	(0.0761)	(0.0710)
Rama	3.5736	–	2.4898	6.9795	8.3117	7.1334	7.3868	0.1150	0.2300	0.4377	2.4469
	(0.4480)	–							(0.1034)	(0.0568)	(0.0533)
Xgamma	3.7789	–	1.4513	4.9026	6.2349	5.0565	5.3099	0.0926	0.2145	0.3356	1.9843
	(0.5917)	–							(0.1521)	(0.1072)	(0.0941)
ChrisJerry	4.3935	–	1.0699	4.1398	5.4720	4.2936	4.5470	0.0830	0.2126	0.3084	1.8528
	(0.6356)	–							(0.1589)	(0.1277)	(0.1112)
XLindley	2.8271	–	0.6762	3.3524	4.6846	3.5062	3.7596	0.0831	0.1972	0.2665	1.6495
	(0.4866)	–							(0.2263)	(0.1685)	(0.1448)
Komal	2.8633	–	0.6585	3.3171	4.6493	3.4709	3.7244	0.0827	0.1967	0.2641	1.6382
	(0.4802)	–							(0.2288)	(0.1713)	(0.1469)
RamAwadh	3.5293	–	3.1137	8.2274	9.5596	8.3812	8.6346	0.1368	0.2533	0.5704	3.0071
	(0.3674)	–							(0.0551)	(0.0258)	(0.0275)
Zeghdoudi	5.9656	–	-3.8627	-5.7254	-4.3932	-5.5716	-5.3182	0.0339	0.1090	0.0371	0.3225
	(0.7219)	–							(0.8934)	(0.9505)	(0.9197)
Haq	2.9744	–	1.1611	4.3221	5.6543	4.4760	4.7294	0.0916	0.2058	0.3081	1.8488
	(0.5000)	–							(0.1866)	(0.1279)	(0.1118)
Hamza	2.6498	25.8673	0.7140	5.4280	8.0924	5.9080	6.2425	0.0838	0.1981	0.2713	1.6714
	(0.5008)	(27.9863)							(0.2216)	(0.1631)	(0.1407)

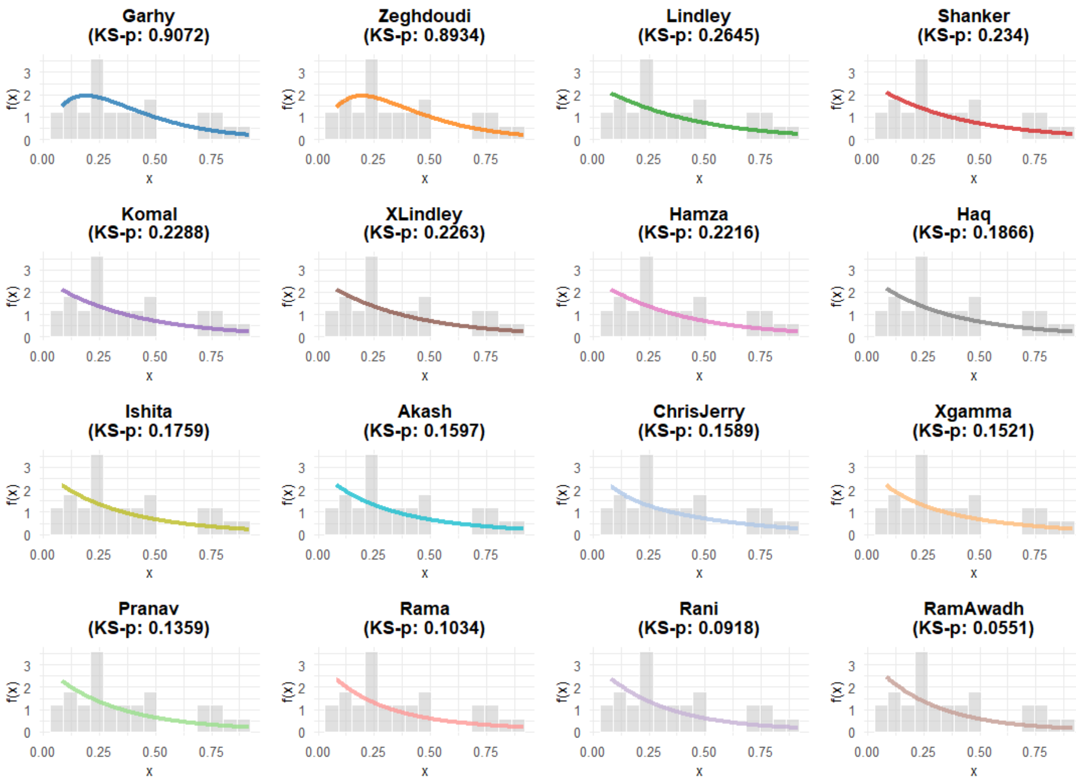

Figure 11: Estimated PDFs for the competing models—(infection times of kidney dialysis patients) data

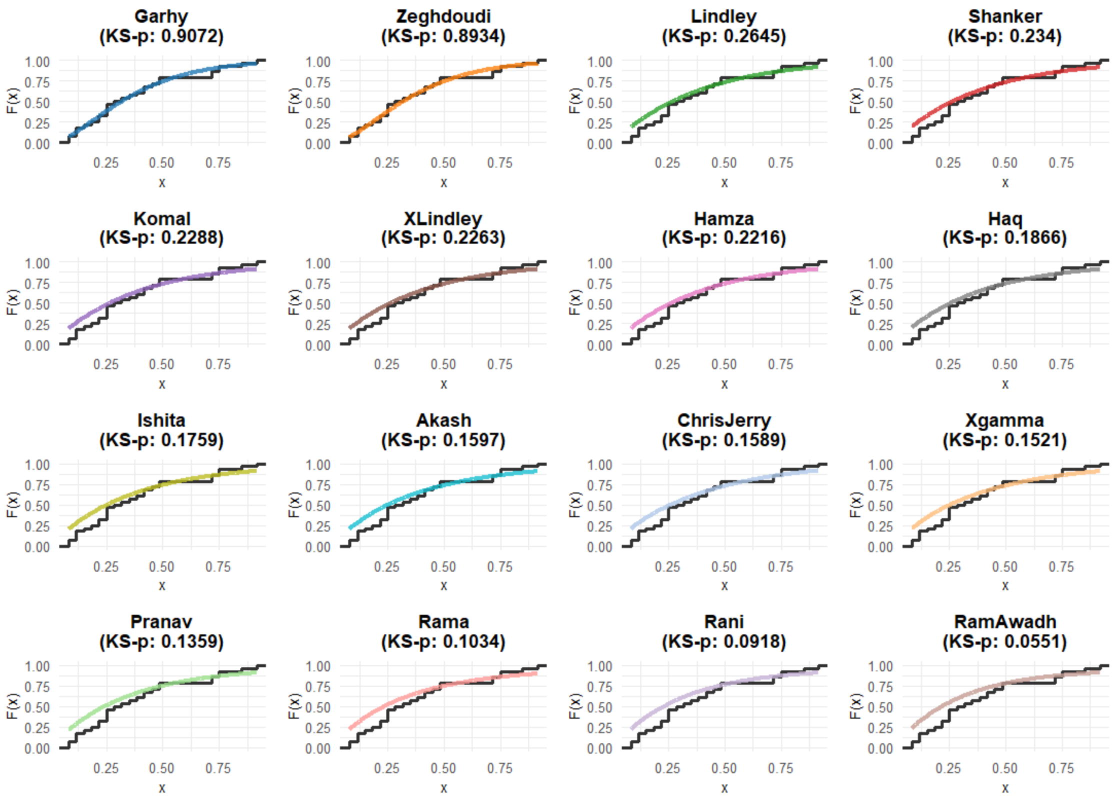

Figure 12: Estimated CDFs for the competing models—(infection times of kidney dialysis patients) data

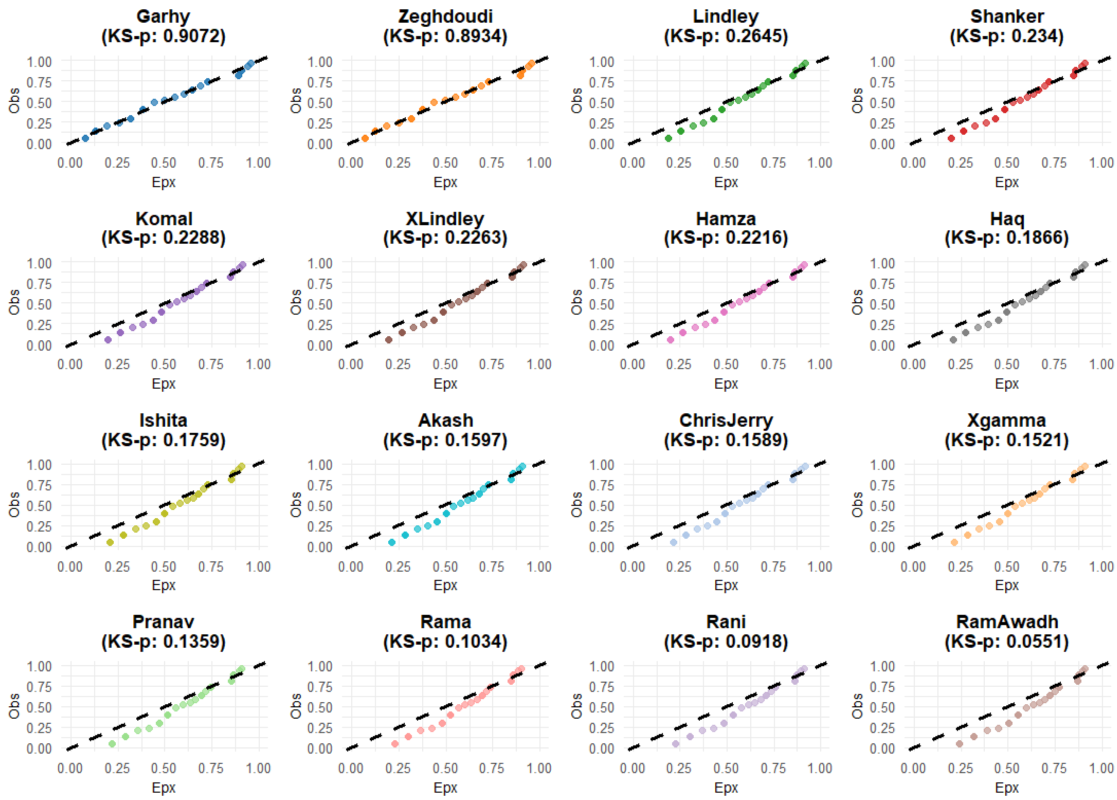

Figure 13: P-P plots for the competing models—(infection times of kidney dialysis patients) data

Across both data applications, the proposed Garhy distribution consistently exhibits excellent performance. Not only do it achieve competitive values in information criteria and goodness-of-fit statistics, but it also provides a visually accurate representation of the data, as confirmed by PDF, CDF and P-P plots. These results highlight the flexibility and applicability of the Garhy distribution to modeling real-world data that span different domains.

7 Concluding Remarks

In this paper, we introduced and examined a new one-parameter mixing distribution known as the "Garhy distribution". The PDF is quite versatile, as it can take on right-skewed, unimodal, and heavy-tailed patterns. Furthermore, the hazard rate function shows that data with rising shaped failure rates can be fitted using the Garhy distribution. Numerous basic statistical and mathematical features are calculated, including mode, quantile function, moments, mean, variance, skewness, kurtosis, moment-generating function, incomplete moments, inequality measures, order statistics, and extropy measures. The scale parameter of the Garhy distribution is estimated using twelve different methods, including maximum likelihood, maximum product of spacings, least-squares, weighted least-squares, Anderson darling, right-tail Anderson darling, left-tail Anderson darling, Cramér von Misses, and least-squares. A comprehensive simulation study is used to determine the efficiency of these approaches. In addition, we assessed the importance and flexibility of the Garhy distribution using two real-world datasets and demonstrated its exceptional ability to match the data perfectly. Our results show that, compared to other existing distributions, the Garhy distribution has a higher quality of fit.

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References

1. Lindley DV. Fiducial models and Bayes' theorem. J R Stat Soc S. 1958;20(1):102–107.
2. Shanker R. Shanker Distribution and Its Applications. Int J Stat Appl. 2015;5(6):338–348.
3. Shanker R. Akash model and its applications. Int J Probab Stat. 2015;4(3):65–75.
4. Shanker R, Shukla KK. Ishita distribution and its applications. Biom Biostat Int J. 2017;5(2):39–46.
5. Shanker R. Rani distribution and its application. Biom Biostat Int J. 2017;6(1):1–10.
6. Shukla KK. Pranav model with properties and its applications. Biom Biostat Int J. 2018;7(3):244–254.
7. Shukla KK, Shanker R. Shukla Distribution And Its Application. Reliab Theory Appl. 2019;14(3):46–55.
8. Shukla KK. Ram Awadh model with properties and applications. Biom Biostat Int J. 2018;7(6):515–523.
9. Shanker R. Rama Distribution and Its Application. Int J Stat Appl. 2017;7(1):26–35.
10. Sen S, Maiti S, Chandra N. The Xgamma model: statistical properties and application. J Mod Appl Stat Methods. 2016;15(1):774–788.
11. Onyekwere CK, Obulezi OJ. Chris-Jerry model and its applications. Asian J Probab Stat. 2022;20:16–30.
12. Chouia S, Zeghdoudi H. The XLindley model: properties and application. J Stat Theory Appl. 2021;20(2):318–327.
13. Messaadia H, Zeghdoudi H. Zeghdoudi model and its applications. Int J Comput Sci Math. 2018;9:58–65.
14. Ahsan-ul-Haq M. Statistical analysis of Haq distribution: estimation and applications. Pak J Statist. 2022;38(4):473–490.
15. Aijaz A, Jallal M, Ain SQU, Tripathi R. The Hamza distribution with statistical properties and applications. Asian J Probab Stat. 2020;8(1):28–42.
16. Etaga HO, Onyekwere CK, Lydia OI, Nwankwo MP, Oramulu DO, Obulezi OJ. Estimation of the Xrama distribution parameter under complete and progressive type-II censored schemes. Sch J Phys Math Stat. 2023;10:203–219.
17. Gideon DU, Ibeakuzie PO, Ekemezie DFN, Nwankwo MP, Oramulu DO, Etaga HO. The double XRAMA distribution: theory and applications. Earthline J Math Sci. 2024;14(3):477–500.
18. Shanker R. Komal distribution with properties and application in survival analysis. Biom Biostat Int J. 2023;12(2):40–44.
19. Lad F, Sanfilippo G, Agr G. Extropy: complementary dual of entropy. Statist Sci. 2015;30:40–58.
20. Balakrishnan N, Buono F, Longobardi M. On weighted extropies. Commun Stat Theory Methods. 2022;51:6250–6267.
21. Karakaya K, Tanış C. Different methods of estimation for the one parameter Akash distribution. Cumhuriyet Science Journal. 2020;41(4):944–950.
22. Karakaya K, Tanış C. Estimating the parameters of Xgamma Weibull distribution. Adıyaman University Journal of Science. 2020;10(2):557–571.
23. Sindhu TN, Abd Elgawad MA, Shafiq A, Abushal TA. Entropy-transformed teissier distribution: A modern statistical framework for engineering, pharmaceutical, and metrological applications. J Radiat Res Appl Sci. 2025;18(3):101773.
24. Onyekwere CK, Aguwa OC, Obulezi OJ. An updated Lindley distribution: Properties, estimation, acceptance sampling, actuarial risk assessment and applications. Innov Stat Probab. 2025;1(1):1–27.

25. Obulezi OJ. Obulezi distribution: A novel one-parameter distribution for lifetime data modeling. Mod J Stat. 2025;2(1):32–74.
26. Fisher RA. On the mathematical foundations of theoretical statistics. Philos Trans R Soc Lond A. 1922;222:309–368.
27. Fisher RA. Theory of statistical estimation. Math Proc Camb Philos Soc. 1925;22(5):700–725.
28. Cheng RCH, Amin NAK. Estimating parameters in continuous univariate distributions with a shifted origin. J R Stat Soc B. 1983;45(3):394–403.
29. Swain JJ, Venkatraman S, Wilson JR. Least-squares estimation of distribution functions in Johnson's translation system. J Stat Comput Simul. 1988;29(4):271–297.
30. Anderson TW, Darling DA. A test of goodness of fit. J Am Stat Assoc. 1954;49(268):765–769.
31. Ahmed M, Alsadat N, Chesneau C, Elgarhy M. Power unit inverse Lindley distribution with different measures of uncertainty, estimation and applications. AIMS Math. 2024;9(8):20976–21024.
32. Aguilar GAS, Moala FA, Cordeiro GM. Zero-truncated Poisson exponentiated gamma distribution: application and estimation methods. J Stat Theory Pract. 2019;13:1–20.
33. R Core Team. R: a language and environment for statistical computing. Vienna: R Foundation for Statistical Computing; 2025.
34. Hinkley D. On quick choice of power transformation. J R Stat Soc C. 1977;26(1):67–69.
35. Klein JP, Moeschberger ML. Survival analysis: techniques for censored and truncated data. New York: Springer; 2006.
36. Gemeay AM, Sapkota LP, Tashkandy YA, Bakr ME, Balogun OS, Hussam E. New bounded probability model: properties, estimation, and applications. Heliyon. 2024;10(23):e38965.
37. Chesneau C, Tomy L, Gillariose J. On a new distribution based on the arccosine function. Arab J Math. 2021;10(3):589–598.
38. Chesneau C, Jamal F. Some new facts about the unit-Rayleigh distribution with applications. Mathematics. 2020;8(11):1954.
39. Alotaibi N, Hashem AF, Elbatal I, Alyami SA, Al-Moisheer AS, Elgarhy M. Inference for a Kavya–Manoharan inverse length biased exponential distribution under progressive-stress model based on progressive type-II censoring. Entropy. 2022;24(8):1033.