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Finite Elements in Analysis and Design 21 (1995) 45 – 74

FINITE ELEMENTS IN ANALYSIS AND DESIGN

Parametric unification of matrix structural analysis: classical formulation and d-connected mixed elements

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Abstract

Concepts and techniques from the field of Parametrized Variational Principles (PVPs) are extended to Matrix Structural Analysis (MSA). Free parameters are used as weighting factors of governing discrete equations. Combining this idea with matrix manipulation techniques yields a continuous spectrum of supermatrix equations. Setting parameters to numerical values provides specific solution methods, some of which are well known whereas others are not. The approach is applied to the classical MSA of truss and framework structures as well as to displacement-connected FE models generated by a parametrized mixed functional. The main advantage of this "top down" derivation of solution schemes is the unification of seemingly disjoint methods for instructional and classification purposes. In addition, the question of duality between range-space and null-space representations is clarified.

1. Introduction

Work in Parametrized Variational Principles or PVPs, as surveyed in a recent article [1], has assisted in the development of *families* of mixed and hybrid variational principles. The number of free parameters in a particular family depends on the structure of the generating functional and the selection of independently varied fields. To date, the most studied application is compressible linear hyperelasticity. For this application it has been shown [2, 3] that *three* free parameters are sufficient to generate all mixed and hybrid principles in which displacements are independently varied. Canonical principles, such as Total Potential Energy or Hu–Washizu's, are obtained by setting those three parameters to specific values.

The original development of PVPs [4–6] was prompted by applications to the Finite Element Method (FEM). Development thrusts have so far focused on the *element level*, and include the formulation of high-performance elements, the derivation of element-level error indicators, and the construction of finite element templates [7]. The common ingredient among these applications is that PVPs allow the construction of element families. As in the case of functionals, element instances are obtained by setting parameters to specific values.

SSDI 0168-874X(95)00027-5

The present paper explores the application of parametrized formulations at the *assembly level.* That is, we consider the analytical representation of the assembled finite element equations in which *parameters are still left free.* Starting from this representation, and proceeding through a combination of matrix algebra and parameter selection we are able to generate all recognized methods of Matrix Structural Analysis (MSA), as well as others that are comparatively unknown. While for MSA the latter are often curiosities, they are shown for completeness and as potential source of instructional exercise material. For expediency, some nomenclature and techniques are borrowed without proof from the field of constrained quadratic optimization, a topic that can be formulated in one-to-one correspondence with MSA [8].

Several structural models can be treated by this methodology:

(1) The classical Matrix Structural Analysis (MSA) of skeletal (truss and framework) structures.
(2) Displacement-connected FEM models generated by mixed PVPs.
(3) Displacement-stress-connected FEM models generated by mixed PVPs.
(4) Individual elements and superelements constructed through hybrid PVP formulations.

For space reasons this paper concentrates on the formulation of items (1) and (2), leaving the more complex (3) and (4) to be considered in future articles.

The present study, being the first of its kind, should be regarded as exploratory. Thus duplications and possibly unproductive dead ends can be expected. Nevertheless it will be shown that the use of parameters illuminates common features of seemingly disparate methods.

2. Classical matrix structural analysis

The classical treatment of MSA deals with the numerical simulation, by matrix methods, of linear truss and framework structures modeled as assemblies of bar and beam elements. As further discussed in Section 4.1, the advent of this branch of Structural Mechanics can be traced back to the European aeronautical industry in the 1930s as a means to systematize repetitive, tabular computations of aircraft stress, vibration and flutter on desk calculators. It thus predates the development of the Finite Element Method (FEM), and of commercial digital computers, by over two decades.

In engineering disciplines that cover structures, such as Aerospace, Civil and Mechanical, classical MSA is normally taught to US college undergraduates at the junior (third year) level. This instruction provides a bridge to the use of FEM-based analysis in more advanced senior and graduate courses. The whole subject can be readily covered using basic energy methods such as the Principle of Virtual Work, and no proficiency in variational calculus is required or expected from the students. Proceeding in a similar vein, classical MSA is used here to gently introduce fundamental matrix-manipulation techniques as well as the use of free parameters in the role of equation-weighting factors, while avoiding the complications of the general FEM. Several features of the general case are studied in Section 3.

2.1. State and data vectors

Consider a linear-elastic truss or framework structure under conservative static loading. A discrete mathematical model is constructed as an assembly of bar and beam elements connected at

joints. Such elements are collectively called *members*. Discretization unknowns are collected in the following three state vectors that have direct physical significance:

- $\boldsymbol{f}$ Array of n_f mechanical internal forces (also called resultant forces or generalized stresses). These are obtained by integration of stresses over member cross sections. Internal forces are *not* continuous across elements, and indeed such continuity would make no sense at joints where several members meet.
- $\boldsymbol{g}$ Array of $n_g = n_f$ internal deformations (also called generalized strains) dual to $\boldsymbol{f}$.
- $\boldsymbol{v}$ Array containing n_v nodal displacement components. Nodes are placed at joints and sometimes at intermediate member locations.

Unless the contrary is stated, it is assumed that *prescribed zero displacements have been removed* from $\boldsymbol{v}$. In particular, all rigid body motions of the structure are thereby explicitly precluded. Formulations that incorporate this assumption are collectively identified in the sequel by the acronym EPD, which stands for *Excluded Prescribed Displacements*. Of course, should *nonzero* joint displacements be prescribed, EPD formulations cannot be used, and the generalization discussed in Section 2.4 is required.

Full linear independence among the entries in $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{v}$ is assumed. As a result, if $n_f = n_v$ the discrete structural model is called *statically determinate* or *isostatic*. If $n_f > n_v$ the model is called *statically indeterminate* or *hyperstatic*. The case $n_f < n_v$, which identifies internal mechanisms (meaning that the structure is kinematically unstable), is not considered here.

The problem data is collected in the following vectors:

- $\boldsymbol{p}$ Array of n_v mechanical nodal forces corresponding to the node displacements $\boldsymbol{v}$.
- $\boldsymbol{g}^0$ Array of $n_g = n_f$ initial deformations: value taken by $\boldsymbol{g}$ if all member forces $\boldsymbol{f}$ vanish. This is sometimes called the *prestrain vector*.
- $\boldsymbol{f}^0$ Array of $n_g = n_f$ initial forces: value taken by $\boldsymbol{f}$ if all member deformations $\boldsymbol{g}$ vanish. This is often called the *prestress vector*.

Initial deformations and forces account for effects due to temperature changes, prestresses, lack of fit, or nonlinear behavior treated by pseudo-force methods. They are connected by the relations $\boldsymbol{g}^0 = -\boldsymbol{C}\boldsymbol{f}^0$ and $\boldsymbol{f}^0 = -\boldsymbol{S}\boldsymbol{g}^0$, where $\boldsymbol{C}$ and $\boldsymbol{S}$ are the compliance and rigidity matrices, respectively, defined below.

2.2. Range-space form of EPD equations

The governing equations that link $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{v}$ to the problem data can be organized into three groups, which are labeled as $\{\mathrm{K}_r\}$, $\{\mathrm{C}\}$ and $\{\mathrm{E}_r\}$ for kinematic, constitutive and equilibrium equations, respectively. (Subscript r stands for the qualifier "range" explained below.) The reciprocal forms (deformation-to-force and force-to-deformation) of the constitutive equations are labeled $\{\mathrm{C}_f\}$ and $\{\mathrm{C}_g\}$, respectively. Borrowing a term from linear algebra, the three groups will be collectively called the *range-space* form of the governing equations:

$$\begin{aligned} \{K_r\}: &\quad \boldsymbol{g} = \boldsymbol{A}\boldsymbol{v}, \\ \{C\} \equiv \{C_f, C_g\}: &\quad \boldsymbol{f} = \boldsymbol{S}\boldsymbol{g} + \boldsymbol{f}^0, \quad \boldsymbol{g} = \boldsymbol{C}\boldsymbol{f} + \boldsymbol{g}^0, \\ \{E_r\}: &\quad \boldsymbol{A}^{\mathrm{T}}\boldsymbol{f} = \boldsymbol{p}. \end{aligned} \tag{1}$$

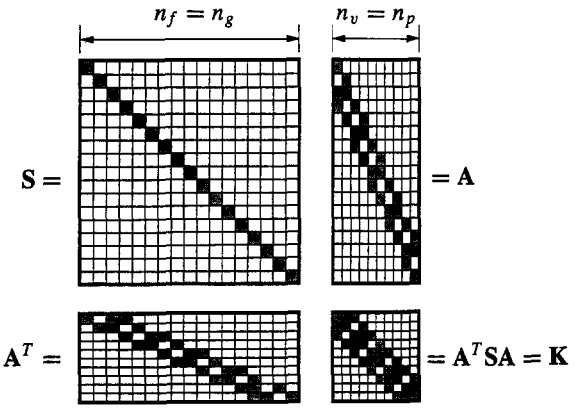

Fig. 1. Sparseness pattern of matrices S, A and K for a small regular 2D structure. Falk's representation of matrix products [9] is used in the above arrangement.

The following matrices make their appearance in (1):

A	The $n_g \times n_v$ *connection matrix*, which is also called geometric matrix, assembly matrix, topology matrix and deformation-displacement matrix in the literature. Since (in the EPD version) rigid body modes have been explicitly removed from v, A can be assumed to have *full rank* n_v. This matrix is square if the structure is statically determinate ($n_g = n_f = n_v$) and rectangular (with more rows than columns) otherwise.
A^T	The *equilibrium matrix* is the transpose of the connection matrix and has the same rank.
$S = C^{-1}$	The $n_f \times n_f \equiv n_g \times n_g$ *rigidity matrix*, which relates internal element forces to element deformations. This matrix is square, symmetric positive definite, diagonal or block-diagonal. Since it is positive definite it has full rank $n_f = n_g$.
$C = S^{-1}$	The $n_g \times n_g \equiv n_f \times n_f$ *compliance matrix*, which relates element deformations to internal element forces. Like its inverse, C is square, symmetric positive definite, diagonal or block diagonal.

S and C are sometimes identified by the confusing names of element-disassembled stiffness and flexibility matrices, respectively, in the old (pre-1970) MSA literature.

Matrix and vector combinations that appear frequently in the sequel and thus deserve special identifiers are:

$$K = A^T S A, \quad F = K^{-1}, \quad p^0 = A^T f^0, \quad p^* = p - p^0. \tag{2}$$

K is the *structural stiffness* matrix, which has order $n_v \times n_v$ and full rank. Since S is symmetric, so is K. Its inverse F, which exists because of the foregoing rank assumptions on S and A, is the *structural flexibility* matrix. Vector p^0 collects initial node forces. Finally, p^* is called the effective force vector.

Typically, matrices A, S and K are highly sparse. Their block sparsity patterns for a small regular structure with 10 unconstrained nodes and 17 elements are illustrated in Fig. 1.

In the sequel symbols $\boldsymbol{I}$ and $\boldsymbol{0}$ denote the identity and null matrix, respectively, of the appropriate order.

The invariance condition equating external and internal energies measured from the undeformed state ($\boldsymbol{g} = \boldsymbol{0}$, $\boldsymbol{v} = \boldsymbol{0}$) is

$$\tfrac{1}{2}(\boldsymbol{p} + \boldsymbol{p}^0)^{\mathrm{T}}\boldsymbol{v} = \tfrac{1}{2}(\boldsymbol{f} + \boldsymbol{f}^0)^{\mathrm{T}}\boldsymbol{g}, \tag{3}$$

in which factor $\frac{1}{2}$ is inconsequential. It may be verified that the set of governing equations (1) satisfies (3) identically.

The discrete forms of the principles of Virtual Work $\boldsymbol{p}^{\mathrm{T}}\delta\boldsymbol{v} = \boldsymbol{f}^{\mathrm{T}}\delta\boldsymbol{g}$ and Complementary Virtual Work $\boldsymbol{v}^{\mathrm{T}}\delta\boldsymbol{p} = \boldsymbol{g}^{T}\delta\boldsymbol{f}$ readily follow from (3)

2.3. Null space form of EPD equations

An alternative expression of the governing EPD equations (1) called the *null-space* representation, relies on the a priori satisfaction of the equilibrium equations $\{\mathrm{E}_r\}$. The general solution of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{f} = \boldsymbol{p}$ can be expressed as the sum of its particular and homogeneous components:

$$\boldsymbol{f} = \boldsymbol{B}_0\boldsymbol{p} + \boldsymbol{B}_1\boldsymbol{y}, \tag{4}$$

where $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ satisfy

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}_0 = \boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{I}, \qquad \boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}_1 = \boldsymbol{0}, \quad \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{0}. \tag{5}$$

The following matrices and vectors appear in these expressions:

- $\boldsymbol{y}$ An array of $n_y = n_f - n_v$ force redundants, implicitly defined through (4).
- $\boldsymbol{B}_0$ The $n_f \times n_v$ loads-influence matrix. Because of the first of (5), $\boldsymbol{B}_0$ may be interpreted as a generalized inverse of $\boldsymbol{A}^{\mathrm{T}}$.
- $\boldsymbol{B}_1$ The $n_f \times n_y$ self-strain matrix. $\boldsymbol{B}_1$ contains (as columns) a basis for the null space of $\boldsymbol{A}^{\mathrm{T}}$ if $n_f > n_v$. The name "self-stress" used by some authors is incorrect, since $\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{f} \neq \boldsymbol{0}$ if $\boldsymbol{g}^0 \neq \boldsymbol{0}$.

Conditions (5) are graphically illustrated in Fig. 2. If the structure is statically determinate, $n_v = n_f$, $n_y = 0$, $\boldsymbol{B}_1$ and $\boldsymbol{y}$ are void, and $\boldsymbol{B}_0^{\mathrm{T}} = \boldsymbol{A}^{-1}$. Fig. 2 illustrates the fact that although matrix $\boldsymbol{A}^{\mathrm{T}}$ is sparse, $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ are generally full. This can be seen from the physical interpretation of (4): each column of $\boldsymbol{B}_0$ contains the internal forces generated by applying an individual nodal force component while holding $\boldsymbol{y} = \boldsymbol{0}$. The column is typically full because the effect of a single force usually propagates throughout the structure unless "blocked" through clever selection of redundants in $\boldsymbol{y}$.

Applying the energy invariance condition (3) it is easy to show that $\boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{g} = \boldsymbol{v}$ and $\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g} = \boldsymbol{0}$. The latter relation justifies the label "self-strain" attached to $\boldsymbol{B}_1$ (although "self-deformation" would be more precise in the MSA context). We can therefore display the null-space form of the governing equations as

$$\begin{aligned}
\{K_n\}: &\quad \begin{bmatrix} \boldsymbol{B}_0^{\mathrm{T}} \\ \boldsymbol{B}_1^{\mathrm{T}} \end{bmatrix} \boldsymbol{g} = \begin{Bmatrix} \boldsymbol{v} \\ \boldsymbol{0} \end{Bmatrix}. \\
\{C\} \equiv \{C_g, C_f\}: &\quad \boldsymbol{g} = \boldsymbol{C}\boldsymbol{f} + \boldsymbol{g}^0, \qquad \boldsymbol{f} = \boldsymbol{S}\boldsymbol{g} + \boldsymbol{f}^0. \\
\{E_n\}: &\quad \boldsymbol{f} = \boldsymbol{B}_0\boldsymbol{p} + \boldsymbol{B}_1\boldsymbol{y}.
\end{aligned} \tag{6}$$

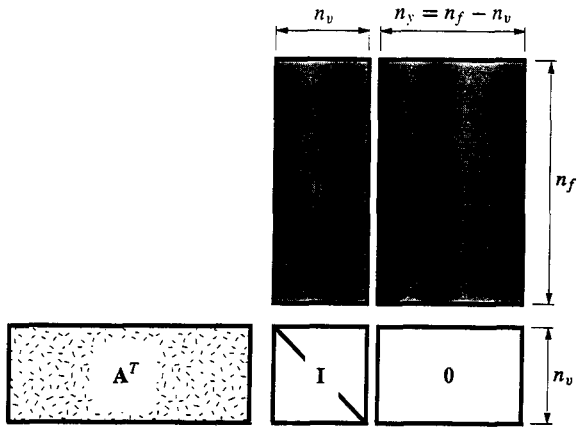

Fig. 2. Graphical representation of the constraint relations (5) using Falk's matrix product visualization. The fill patterns aim to convey the fact that while $\boldsymbol{A}$ is a highly sparse matrix (cf. Fig. 1), $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ are generally full. Relative matrix dimensions are typical of a two-dimensional structure. Matrix $\boldsymbol{K}$ would have the same dimensions as $\boldsymbol{I}$ above.

A general method to compute $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ given $\boldsymbol{A}$ is discussed by Fletcher [10] in the context of constrained optimization by Lagrangian methods. If $n_y = n_f - n_v > 0$, augment $\boldsymbol{A}$ with a $n_f \times n_y$ matrix $\boldsymbol{V}$ such that the augmented square matrix has full rank. Then

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{V} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{B}_0^{\mathrm{T}} \\ \boldsymbol{B}_1^{\mathrm{T}} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \boldsymbol{A}^{\mathrm{T}} \\ \boldsymbol{V}^{\mathrm{T}} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{B}_0 & \boldsymbol{B}_1 \end{bmatrix}. \tag{7}$$

As illustrated in Fig. 3, a complete inversion of the augmented matrix is not actually needed.

Equation (7) makes clear that $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ are far from unique. A wide spectrum of null-space methods, surveyed in Fletcher's book, can be obtained depending on how $\boldsymbol{V}$ is chosen and the inverse (7) represented. As noted, in structural analysis $\boldsymbol{A}$ is highly sparse while $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ are generally full. It is therefore important to chose $\boldsymbol{V}$ and an implicit (factored) representation of the inverse to conserve sparsity.

2.4. *Including prescribed nodal displacements*

In the foregoing subsections all prescribed nodal displacements and associate nodal (reaction) forces have been explicitly removed from $\boldsymbol{v}$, $\boldsymbol{p}$ and $\boldsymbol{A}$. This choice, labeled EPD, has exposition advantages because it leads to cleaner formulas. Furthermore, the order of systems to be submitted to the equation solvers is reduced, although this advantage becomes progressively insignificant as discrete models become large.

Alternatively, one may retain prescribed node displacements in the discrete governing equations. This choice is labeled as IPD, which stands for *Included Prescribed Displacements*. In this case, arrays $\boldsymbol{v}$ and $\boldsymbol{p}$ are formally partitioned as

$$\boldsymbol{v} = \begin{Bmatrix} \boldsymbol{v}_0 \\ \boldsymbol{v}_1 \end{Bmatrix}, \qquad \boldsymbol{p} = \begin{Bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \end{Bmatrix}. \tag{8}$$

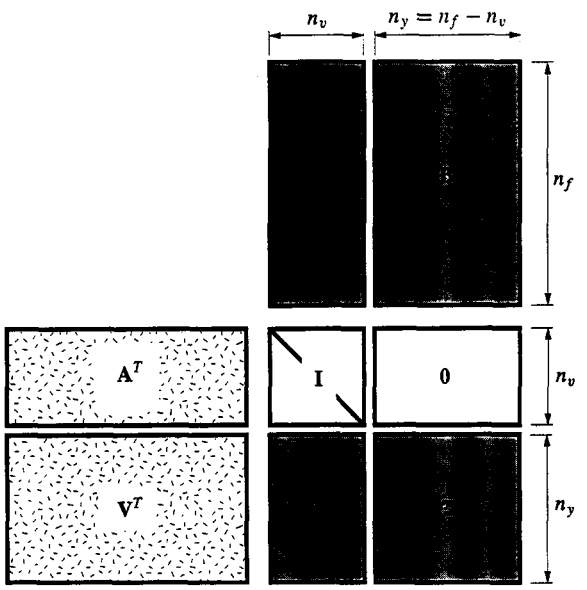

Fig. 3. Graphical representation of Fletcher's scheme (7) to construct $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ by inverting a $\boldsymbol{V}^{\mathrm{T}}$-augmented $\boldsymbol{A}^{\mathrm{T}}$. Note that completion of the inversion process below the n_vth row is not required. Consequently products $\boldsymbol{V}^{\mathrm{T}}\boldsymbol{B}_0$ and $\boldsymbol{V}^{\mathrm{T}}\boldsymbol{B}_1$ have been left undefined.

Here array $\boldsymbol{v}_1$, of dimension $n_{v1} < n_v$, collects prescribed displacements whereas array $\boldsymbol{p}_0$ collects known nodal forces. The unknown displacement and force arrays are $\boldsymbol{v}_0$ and $\boldsymbol{p}_1$, respectively. The latter are called *reaction forces*. It should be stressed that in practical computer implementations an explicit partition of $\boldsymbol{v}$ and $\boldsymbol{p}$ as per (8) is rarely done as it entails expensive equation rearrangements, but it is convenient for matrix notation.

The partition (8) induces the following decomposition of the kinematic and equilibrium equations:

$$\boldsymbol{g} = \boldsymbol{A}\boldsymbol{v} = \boldsymbol{A}_0\boldsymbol{v}_0 + \boldsymbol{A}_1\boldsymbol{v}_1, \qquad \begin{bmatrix} \boldsymbol{A}_0^{\mathrm{T}} \\ \boldsymbol{A}_1^{\mathrm{T}} \end{bmatrix} \boldsymbol{f} = \begin{Bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \end{Bmatrix}, \tag{9}$$

where $\boldsymbol{p}_1$ and $\boldsymbol{v}_0$ are unknown. Collecting these equations we can express the range-space IPD form as

$$\begin{aligned} \{K_r\}: &\quad \boldsymbol{g} = \boldsymbol{A}_0\boldsymbol{v}_0 + \boldsymbol{A}_1\boldsymbol{v}_1. \\ \{C\} \equiv \{C_f, C_g\}: &\quad \boldsymbol{f} = \boldsymbol{S}\boldsymbol{g} + \boldsymbol{f}^0, \qquad \boldsymbol{g} = \boldsymbol{C}\boldsymbol{f} + \boldsymbol{g}^0, \\ \{E_r\}: &\quad \begin{bmatrix} \boldsymbol{A}_0^{\mathrm{T}} \\ \boldsymbol{A}_1^{\mathrm{T}} \end{bmatrix} \boldsymbol{f} = \begin{Bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \end{Bmatrix}. \end{aligned} \tag{10}$$

As for the null-space representation, the internal force decomposition (4) changes slightly to

$$\boldsymbol{f} = \boldsymbol{B}_0\boldsymbol{p}_0 + \boldsymbol{B}_1\boldsymbol{y}, \tag{11}$$

where $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ now satisfy

$$\boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{B}_0 = \boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{A}_0 = \boldsymbol{I}, \quad \boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{B}_1 = \boldsymbol{0}, \quad \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{B}_0 = \boldsymbol{0}, \quad \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{B}_1 = \boldsymbol{R}. \tag{12}$$

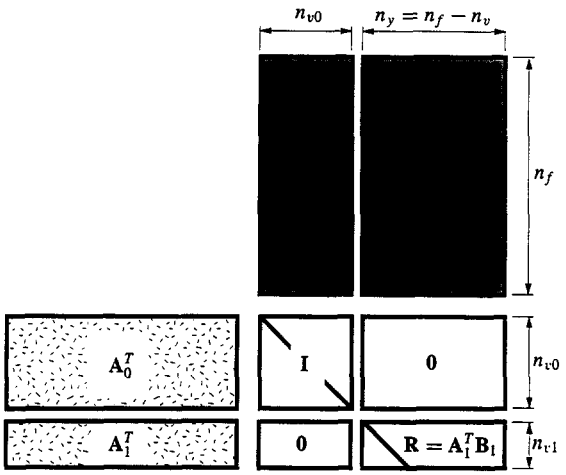

Fig. 4. The A and B matrices when prescribed nodal displacements are included, v is formally partitioned as per (8) and constraints (12) are imposed.

These constraints are illustrated in Fig. 4. Note that R could be chosen to be highly sparse (as in Fig. 4) but is otherwise arbitrary. Inserting (11) and (12) into the energy invariance equation (3) one finds the relations

$$B_0^{\mathrm{T}} g = v_0, \quad B_1^{\mathrm{T}} g = h = R^{\mathrm{T}} v_1, \quad p_1 = Ry, \tag{13}$$

in which the last one may be also obtained by premultiplying both sides of (11) by A_1^{T}, then using $A_1^{\mathrm{T}} B_0 = 0$ and $A_1^{\mathrm{T}} f = p_1$. Note that y and h are energy conjugate because $y^{\mathrm{T}} h = p_1^{\mathrm{T}} v_1$. Therefore, h may be visualized as "dislocations" or "gaps" that open up as a result of the individual action of the redundant forces.

Combining the preceding relations, the null-space IPD form can be presented as

$$\begin{aligned} \{K_n\}: &\quad \begin{bmatrix} B_0^{\mathrm{T}} \\ B_1^{\mathrm{T}} \end{bmatrix} g = \begin{Bmatrix} v_0 \\ h \end{Bmatrix}, \\ \{C\} \equiv \{C_g, C_f\}: &\quad g = Cf + g^0, \qquad f = Sg + f^0, \\ \{E_n\}: &\quad f = B_0 p_0 + B_1 y. \end{aligned} \tag{14}$$

When are IPD forms preferable over EPD ones? Several scenarios may be mentioned:

(1) There are *nonzero* prescribed nodal displacements, i.e. $v_1 \neq 0$. If so EPD forms are insufficient.
(2) Recovery of reaction node forces is of interest.
(3) Rearrangement of matrix equations to explicitly delete prescribed-displacement equations is inconvenient or costly. For example, if such matrices are in secondary (disk) storage, or piecewise distributed into local memories of a massively parallel computer.
(4) Duality between range and null space formulations is to be exhibited, as discussed next.

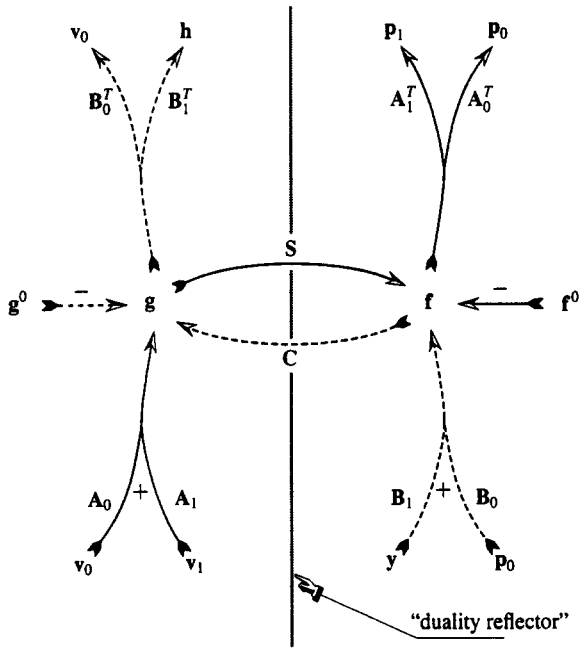

Fig. 5. Graphical representation of the duality scheme (15) for IPD forms. Kinematic relations appear on the left and equilibrium relations on the right. Full (dashed) paths depict relations primarily used in the range-space (null-space) forms. The $\leftrightarrow$ correspondences of (15) result on "flipping" the diagram about the duality-reflector vertical line.

2.5. Duality

The duality between range-space and null-space representations of governing equations has been the subject of a large volume of papers across many disciplines. In the present (MSA) context that property can be displayed by considering the IPD forms (10) and (14), and observing that the following formal substitutions convert one into the other:

$$\boldsymbol{g} \leftrightarrow \boldsymbol{f}, \quad \boldsymbol{g}^0 \leftrightarrow \boldsymbol{f}^0, \quad \boldsymbol{S} \leftrightarrow \boldsymbol{C}, \quad \boldsymbol{v}_0 \leftrightarrow \boldsymbol{p}_0, \quad \boldsymbol{A}_0 \leftrightarrow \boldsymbol{B}_0, \quad \boldsymbol{A}_1 \leftrightarrow \boldsymbol{B}_1, \quad \boldsymbol{v}_1 \leftrightarrow \boldsymbol{y}, \quad \boldsymbol{p}_1 \leftrightarrow \boldsymbol{h}. \tag{15}$$

This scheme is diagrammed in Fig. 5. The correspondences $\boldsymbol{v}_1 \leftrightarrow \boldsymbol{y}$ and $\boldsymbol{p}_1 \leftrightarrow \boldsymbol{h}$ deserve some comment. For structural models in which $\boldsymbol{y}$ and $\boldsymbol{p}_1$ have the same dimension, it is possible to take $\boldsymbol{y} \equiv \boldsymbol{p}_1$ (in physical terms, reaction forces are chosen as redundants), whereupon $\boldsymbol{R} = \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{B}_1 = \boldsymbol{I}$ and $\boldsymbol{h} \equiv \boldsymbol{v}_1$. If so those two correspondences coalesce into one: $\boldsymbol{v}_1 \leftrightarrow \boldsymbol{p}_1$, and duality is perfect. But this is generally impossible for arbitrary structures in which n_{v1} is different from $n_y = n_f - n_v$ (cf. Fig. 4).

Note that no two-way correspondence can be established between the EPD forms (1) and (6). Reason: there is no null-space counterpart of the process of removing columns of $\boldsymbol{A}$ and rows of $\boldsymbol{A}^{\mathrm{T}}$ to exclude prescribed nodal displacements. Thus questions such as "what is the dual of the EPD stiffness equations (22) of the Displacement Method?" make no sense.

As a corollary of the foregoing remark, there are no null-space forms of free-free structures or, in general, of insufficiently supported structures.

2.6. Parametrization of range-space EPD equations

The range-space EPD equations (1) can be grouped to form the following parametrized supermatrix equation:

$$\begin{bmatrix} (s_2+s_3)\boldsymbol{C} & -s_3\boldsymbol{I} & -s_2\boldsymbol{A} \\ -s_3\boldsymbol{I} & (s_1+s_3)\boldsymbol{S} & -s_1\boldsymbol{SA} \\ -s_2\boldsymbol{A}^{\mathrm{T}} & -s_1\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S} & (1+s_1+s_2)\boldsymbol{K} \end{bmatrix} \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \\ \boldsymbol{v} \end{Bmatrix} = \begin{Bmatrix} -(s_2+s_3)\boldsymbol{g}^0 \\ -s_3\boldsymbol{f}^0 \\ \boldsymbol{p}-(1+s_2)\boldsymbol{A}^{\mathrm{T}}\boldsymbol{f}^0 \end{Bmatrix}, \tag{16}$$

where s_1, s_2 and s_3 are free scalar parameters. In compact form:

$$\boldsymbol{Y}\,\boldsymbol{w} = \boldsymbol{r}, \tag{17}$$

in which $\boldsymbol{w}$ and $\boldsymbol{r}$ group left-hand and right-hand side vectors, respectively, of (16) and the symmetric coefficient matrix $\boldsymbol{Y}$ has order $n_f + n_g + n_v = 2n_f + n_v$. It is shown below that $\boldsymbol{Y}$ has full rank if $s_1s_2 + s_2s_3 + s_3s_1 \neq 0$.

The particular form (16) is suggested by the configuration of FEM equations generated by the Parametrized Variational Principles (PVPs) mentioned in the Introduction. Within the context of classical MSA, however, no recourse to variational formulations need to be made. Indeed the three matrix equations in (16) can be directly constructed by residual-weighting techniques as follows:

$$\begin{aligned} &s_2(\boldsymbol{Cf}-\boldsymbol{Av}+\boldsymbol{g}^0)+s_3(\boldsymbol{Cf}-\boldsymbol{g}+\boldsymbol{g}^0)=0, \qquad \text{or} \quad s_2\{C_g\star K_r\}+s_3\{C_g\}, \\ &s_1\boldsymbol{S}(\boldsymbol{g}-\boldsymbol{Av})+s_3(\boldsymbol{Sg}-\boldsymbol{f}+\boldsymbol{f}^0)=0, \qquad \text{or} \quad s_1\boldsymbol{S}\{K_r\}+s_3\{C_f\}, \\ &s_1\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}(\boldsymbol{Av}-\boldsymbol{g})+s_2\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{SAv}-\boldsymbol{f}+\boldsymbol{f}^0)+(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{SAv}-\boldsymbol{p}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{f}^0)=0, \\ &\quad \text{or} \quad s_1\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\{K_r\}+s_2\boldsymbol{A}^{\mathrm{T}}\{C_f\star K_r\}+\{E_r\star C_f\star K_r\}. \end{aligned} \tag{18}$$

Here by symbol $\{C_f\star K_r\}$ is meant a combination of the constitutive equation $\boldsymbol{f}=\boldsymbol{S}(\boldsymbol{g}-\boldsymbol{f})$ and the kinematic equation $\boldsymbol{g}=\boldsymbol{Av}$, and likewise for the other compound symbols. The particular weighting arrangement (18) is chosen so that the coefficient matrix of (16) is *symmetric* for any combination of parameters.

2.7. Parametrized solution and specializations

Under the assumptions stated below the inverse of the coefficient matrix in (17) can be written

$$\boldsymbol{Y}^{-1} = \begin{bmatrix} q_1\boldsymbol{SQS}-\boldsymbol{S} & q_2\boldsymbol{SQ}-\boldsymbol{I} & \boldsymbol{SAK}^{-1} \\ q_2\boldsymbol{QS}-\boldsymbol{I} & q_3\boldsymbol{Q} & \boldsymbol{AK}^{-1} \\ \boldsymbol{K}^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S} & \boldsymbol{K}^{-1}\boldsymbol{A}^{\mathrm{T}} & \boldsymbol{K}^{-1} \end{bmatrix}, \tag{19}$$

where $\boldsymbol{Q}=\boldsymbol{Q}^{\mathrm{T}}=\boldsymbol{AK}^{-1}\boldsymbol{A}^{\mathrm{T}}=\boldsymbol{AFA}^{\mathrm{T}}$, and

$$q_1 = 2+\frac{s_1+s_3}{s_1s_2+s_2s_3+s_3s_1}, \quad q_2 = 2+\frac{s_3}{s_1s_2+s_2s_3+s_3s_1}, \quad q_3 = 1+\frac{s_2+s_3}{s_1s_2+s_2s_3+s_3s_1}. \tag{20}$$

These coefficients exist if

$$s_{\det} = s_1 s_2 + s_2 s_3 + s_3 s_1 = \det \begin{bmatrix} s_2 + s_3 & -s_3 & -s_2 \\ -s_3 & s_1 + s_3 & -s_1 \\ -s_2 & -s_1 & 1 + s_1 + s_2 \end{bmatrix} \neq 0. \tag{21}$$

If $s_{\det} = 0$ the coefficient matrix $\boldsymbol{Y}$ of (16) is singular because it contains dependent equations. Linear independence can be recovered by dropping those equations, as done below. Special forms of practical or historical interest may be precipitated by setting s_1, s_2 and s_3 appropriately, as illustrated next.

Setting $s_1 = s_2 = s_3 = 0$ ($s_{\det} = 0$) yields the stiffness equation of the Displacement Method:

$$\boldsymbol{K}\boldsymbol{v} = \boldsymbol{p} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{f}^0 = \boldsymbol{p} - \boldsymbol{p}^0 = \boldsymbol{p}^*. \tag{22}$$

The Direct Stiffness Method (DSM) is a highly efficient implementation of the stiffness-matrix assembly on an element-by-element basis:

$$\boldsymbol{K} = \sum_e \left(\mathscr{L}^{(e)}\right)^{\mathrm{T}} \left(\boldsymbol{A}^{(e)}\right)^{\mathrm{T}} \boldsymbol{S}^{(e)} \boldsymbol{A}^{(e)} \mathscr{L}^{(e)} = \sum_e \left(\mathscr{L}^{(e)}\right)^{\mathrm{T}} \boldsymbol{K}^{(e)} \mathscr{L}^{(e)}, \tag{23}$$

in which e is an element index, $\boldsymbol{A}^{(e)}$, $\boldsymbol{S}^{(e)}$ and $\boldsymbol{K}^{(e)}$ are element-level counterparts of $\boldsymbol{A}$, $\boldsymbol{S}$ and $\boldsymbol{K}$, respectively, and $\mathscr{L}^{(e)}$ are element-to-global-freedom localization matrices with entries 0 or 1. Because of sparsity exploitation, (23) is more efficient than the naive use of the triple matrix product $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A}$. The DSM is the assembly technique used by the overwhelming majority of general-purpose finite element codes.

Setting $s_1 = s_3 = 0$ and $s_2 = -1$ ($s_{\det} = 0$) yields the Force-Displacement Method, sometimes called the Combined Method in the old (pre-1970) literature:

$$\begin{bmatrix} -\boldsymbol{C} & \boldsymbol{A} \\ \boldsymbol{A}^{\mathrm{T}} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{v} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{g}^0 \\ \boldsymbol{p} \end{Bmatrix}. \tag{24}$$

Setting $s_2 = s_3 = 0$ and $s_1 = -1$ ($s_{\det} = 0$) yields the Deformation-Displacement Method:

$$\begin{bmatrix} -\boldsymbol{S} & \boldsymbol{A}\boldsymbol{S} \\ \boldsymbol{S}\boldsymbol{A}^{\mathrm{T}} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{g} \\ \boldsymbol{v} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{p}^* \end{Bmatrix}. \tag{25}$$

Setting $s_1 = 0$, $s_3 = -s_2 = 1$ ($s_{\det} = 1$) yields the Force-Deformation-Displacement Method:

$$\begin{bmatrix} \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{A} \\ -\boldsymbol{I} & \boldsymbol{S} & \boldsymbol{0} \\ \boldsymbol{A}^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \\ \boldsymbol{v} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ -\boldsymbol{f}^0 \\ \boldsymbol{p} \end{Bmatrix}. \tag{26}$$

Equivalent forms of (24) through (26) are obtained if the signs of all s_i's are reversed.

Use of the explicit inverse (19) and right hand side of (16) yields the general solution

$$\boldsymbol{z} = \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \\ \boldsymbol{v} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{S}(\boldsymbol{g} - \boldsymbol{g}^0) \\ \boldsymbol{A}\boldsymbol{v} \\ \boldsymbol{K}^{-1}\boldsymbol{p}^* \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{S}(\boldsymbol{A}\boldsymbol{K}^{-1}\boldsymbol{p}^* - \boldsymbol{g}^0) \\ \boldsymbol{A}\boldsymbol{K}^{-1}\boldsymbol{p}^* \\ \boldsymbol{K}^{-1}\boldsymbol{p}^* \end{Bmatrix}. \tag{27}$$

As can be expected from the construction (18), this solution is independent of the free parameters. In fact, through cancellations, (27) is valid even if $s_{\det} = 0$, e.g. (22), (24) and (25). As discussed

in Section 3 such independence carries over to selected FEM formulations based on mixed PVPs, but not to the general case.

The following sensitivities are important in some applications such as computer-based member optimization:

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{p}} = \boldsymbol{K}^{-1} = \boldsymbol{F}, \quad \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{f}^0} = -\boldsymbol{A}^{\mathrm{T}}, \quad \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{g}^0} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}, \quad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{p}} = \boldsymbol{SAF}, \quad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{g}^0} = \boldsymbol{SQS} - \boldsymbol{S}. \tag{28}$$

2.8. Parametrization of range-space IPD equations

In accordance with (8) the stiffness matrix $\boldsymbol{K}$ partitions as

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{00} & \boldsymbol{K}_{01} \\ \boldsymbol{K}_{10} & \boldsymbol{K}_{11} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A}_0 & \boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A}_1 \\ \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A}_0 & \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A}_1 \end{bmatrix}. \tag{29}$$

The parametrized supermatrix equations (16) become

$$\begin{bmatrix} (s_2+s_3)\boldsymbol{C} & -s_3\boldsymbol{I} & -s_2\boldsymbol{A}_0 & -s_2\boldsymbol{A}_1 \\ -s_3\boldsymbol{I} & (s_1+s_3)\boldsymbol{S} & -s_1\boldsymbol{S}\boldsymbol{A}_0 & -s_1\boldsymbol{S}\boldsymbol{A}_1 \\ -s_2\boldsymbol{A}_0^{\mathrm{T}} & -s_1\boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{S} & s_K\boldsymbol{K}_{00} & s_K\boldsymbol{K}_{01} \\ -s_2\boldsymbol{A}_1^{\mathrm{T}} & -s_1\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{S} & s_K\boldsymbol{K}_{10} & s_K\boldsymbol{K}_{11} \end{bmatrix} \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \\ \boldsymbol{v}_0 \\ \boldsymbol{v}_1 \end{Bmatrix} = \begin{Bmatrix} -(s_2+s_3)\boldsymbol{g}^0 \\ -s_3\boldsymbol{f}^0 \\ \boldsymbol{p}_0 - (1+s_2)\boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{f}^0 \\ \boldsymbol{p}_1 - (1+s_2)\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{f}^0 \end{Bmatrix} \tag{30}$$

in which for compactness $s_K = 1 + s_1 + s_2$. Upon transferring all known terms to the right hand side we obtain

$$\begin{bmatrix} (s_2+s_3)\boldsymbol{C} & -s_3\boldsymbol{I} & -s_2\boldsymbol{A}_0 & \boldsymbol{0} \\ -s_3\boldsymbol{I} & (s_1+s_3)\boldsymbol{S} & -s_1\boldsymbol{S}\boldsymbol{A}_0 & \boldsymbol{0} \\ -s_2\boldsymbol{A}_0^{\mathrm{T}} & -s_1\boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{S} & s_K\boldsymbol{K}_{00} & \boldsymbol{0} \\ -s_2\boldsymbol{A}_1^{\mathrm{T}} & -s_1\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{S} & s_K\boldsymbol{K}_{10} & -\boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \\ \boldsymbol{v}_0 \\ \boldsymbol{p}_1 \end{Bmatrix} = \begin{Bmatrix} -(s_2+s_3)\boldsymbol{g}^0 + s_2\boldsymbol{A}_1\boldsymbol{v}_1 \\ -s_3\boldsymbol{f}^0 + s_1\boldsymbol{S}\boldsymbol{A}_1\boldsymbol{v}_1 \\ \boldsymbol{p}_0 - (1+s_2)\boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{f}^0 - s_K\boldsymbol{K}_{01}\boldsymbol{v}_1 \\ -(1+s_2)\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{f}^0 - s_K\boldsymbol{K}_{11}\boldsymbol{v}_1 \end{Bmatrix}. \tag{31}$$

Setting $s_1 = s_2 = s_3 = 0$ gives immediately the equation for $\boldsymbol{v}_0$:

$$\boldsymbol{K}_{00}\boldsymbol{v}_0 = \boldsymbol{p}_0 - \boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{f}^0 - \boldsymbol{K}_{01}\boldsymbol{v}_1 = \boldsymbol{p}_0 - \boldsymbol{A}_0^{\mathrm{T}}(\boldsymbol{f}^0 + \boldsymbol{S}\boldsymbol{A}_1\boldsymbol{v}_1) = \boldsymbol{p}_0^*, \tag{32}$$

where $\boldsymbol{p}_0^*$ is an adjusted effective-nodal-force vector that accounts for the effect of both prestress $\boldsymbol{f}^0$ and prescribed displacements $\boldsymbol{v}_1$. Solving and backsubstituting yields the solution

$$\begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \\ \boldsymbol{v} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \\ \boldsymbol{v}_0 \\ \boldsymbol{v}_1 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{S}\boldsymbol{g} + \boldsymbol{f}^0 \\ \boldsymbol{A}_0\boldsymbol{v}_0 + \boldsymbol{A}_1\boldsymbol{v}_1 \\ \boldsymbol{K}_{00}^{-1}\boldsymbol{p}_0^* \\ \boldsymbol{v}_1 \end{Bmatrix}. \tag{33}$$

The unknown reaction forces may be recovered from the last matrix equation in (31)

$$\boldsymbol{p}_1 = \boldsymbol{K}_{10}\boldsymbol{v}_0 + \boldsymbol{K}_{11}\boldsymbol{v}_1 + \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{f}^0. \tag{34}$$

The interesting sensitivity derivatives are

$$\frac{\partial \boldsymbol{v}_0}{\partial \boldsymbol{p}_0} = \boldsymbol{K}_{00}^{-1}, \quad \frac{\partial \boldsymbol{v}_0}{\partial \boldsymbol{v}_1} = -\boldsymbol{K}_{00}^{-1}\boldsymbol{K}_{01}, \quad \frac{\partial \boldsymbol{v}_0}{\partial \boldsymbol{g}^0} = \boldsymbol{K}_{00}^{-1}\boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{S},$$
$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{p}_0} = \boldsymbol{S}\boldsymbol{A}_0\boldsymbol{K}_{00}^{-1}, \quad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{v}_1} = \boldsymbol{S}\boldsymbol{A}_0\boldsymbol{K}_{00}^{-1}\boldsymbol{K}_{01}, \quad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{g}^0} = \boldsymbol{S}\boldsymbol{Q}_{00}\boldsymbol{S} - \boldsymbol{S}, \tag{35}$$

where $\boldsymbol{Q}_{00} = \boldsymbol{A}_0\boldsymbol{K}_{00}^{-1}\boldsymbol{A}_0^{\mathrm{T}}$.

Modification of other well known solution methods, such as (24)–(26), to explicitly account for prescribed displacements can be obtained by appropriate choice of parameters as previously illustrated.

2.9. Symmetric null-space EPD forms

Formal substitution of (4) into (16), followed by premultiplication of the first matrix equation by $\boldsymbol{B}_1^{\mathrm{T}}$ to restore symmetry, yields the parametrized null-space system of order $(n_f - n_v) + n_g + n_v = 2n_f$:

$$\begin{bmatrix} (s_2+s_3)\boldsymbol{F}_{11} & -s_3\boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{0} \\ -s_3\boldsymbol{B}_1 & (s_1+s_3)\boldsymbol{S} & -s_1\boldsymbol{S}\boldsymbol{A} \\ \boldsymbol{0} & -s_1\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S} & (1+s_1+s_2)\boldsymbol{K} \end{bmatrix} \begin{Bmatrix} \boldsymbol{y} \\ \boldsymbol{g} \\ \boldsymbol{v} \end{Bmatrix} = \begin{Bmatrix} -(s_2+s_3)(\boldsymbol{F}_{10}\boldsymbol{p} + \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g}^0) \\ s_3(\boldsymbol{B}_0\boldsymbol{p} - \boldsymbol{f}^0) \\ (1+s_2)(\boldsymbol{p} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{f}^0) \end{Bmatrix}, \tag{36}$$

where $\boldsymbol{F}_{11} = \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{C}\boldsymbol{B}_1$ and $\boldsymbol{F}_{10} = \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{C}\boldsymbol{B}_0$. System (36) is singular because it contains redundant equations inasmuch as the equilibrium equations are now superflous. To trim it by elimination of $\boldsymbol{v}$, set $s_1 = 0$ and $s_2 = -1$ while keeping $s = s_3$ as a free parameter:

$$\begin{bmatrix} (s-1)\boldsymbol{F}_{11} & -s\boldsymbol{B}_1^{\mathrm{T}} \\ -s\boldsymbol{B}_1 & s\boldsymbol{S} \end{bmatrix} \begin{Bmatrix} \boldsymbol{y} \\ \boldsymbol{g} \end{Bmatrix} = \begin{Bmatrix} (1-s)(\boldsymbol{F}_{10}\boldsymbol{p} + \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g}^0) \\ s(\boldsymbol{B}_0\boldsymbol{p} - \boldsymbol{f}^0) \end{Bmatrix}. \tag{37}$$

This is now a one-parameter form of order $n_y + n_v$ that represents a weighted combination of kinematic and constitutive equations. Two important specializations are obtained for $s = 0$ and $s = 1$:

$$-\boldsymbol{F}_{11}\boldsymbol{y} = \boldsymbol{F}_{10}\boldsymbol{p} + \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g}^0, \tag{38}$$

$$\begin{bmatrix} \boldsymbol{0} & -\boldsymbol{B}_1^{\mathrm{T}} \\ -\boldsymbol{B}_1 & \boldsymbol{S} \end{bmatrix} \begin{Bmatrix} \boldsymbol{y} \\ \boldsymbol{g} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{B}_0\boldsymbol{p} - \boldsymbol{f}^0 \end{Bmatrix}. \tag{39}$$

Eq. (38) is the venerable Redundant-Force Method, also referred to as the Flexibility Method or simply *the* Force Method in the MSA literature. (The last name may be open to question, as there are unsymmetric versions.) Eq. (39) is the Redundant-Force-Deformation Method. Elimination of $\boldsymbol{g}$ from the latter yields (38). Using this as basis for solution gives for the contribution $\boldsymbol{B}_1\boldsymbol{y}$ of redundants to internal forces:

$$\boldsymbol{B}_1\boldsymbol{y} = -\boldsymbol{G}(\boldsymbol{C}\boldsymbol{B}_0\boldsymbol{p} + \boldsymbol{g}^0), \tag{40}$$

where $\boldsymbol{G} = \boldsymbol{B}_1\boldsymbol{F}_{11}^{-1}\boldsymbol{B}_1^{\mathrm{T}}$ is a symmetric $n_f \times n_f$ matrix, which is null if the structure is statically determinate. Backsubtitution into the governing equations yields

$$\begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \\ \boldsymbol{v} \end{Bmatrix} = \begin{Bmatrix} (\boldsymbol{I} - \boldsymbol{GC})\boldsymbol{B}_0\boldsymbol{p} - \boldsymbol{G}\boldsymbol{g}^0 \\ \boldsymbol{C}(\boldsymbol{I} - \boldsymbol{GC})\boldsymbol{B}_0\boldsymbol{p} + (\boldsymbol{I} - \boldsymbol{G})\boldsymbol{g}^0 \\ \boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{C}\boldsymbol{B}_0\boldsymbol{p} + \boldsymbol{B}_0^{\mathrm{T}}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{g}^0) \end{Bmatrix}. \tag{41}$$

The interesting sensitivity derivatives are

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{p}} = \boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{C}\boldsymbol{B}_0, \quad \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{g}^0} = \boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{I} - \boldsymbol{G}), \quad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{p}} = (\boldsymbol{I} - \boldsymbol{GC})\boldsymbol{B}_0, \quad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{g}^0} = -\boldsymbol{G}. \tag{42}$$

Comparing this to (28), we see that

$$\boldsymbol{F} = \boldsymbol{K}^{-1} = \boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{C}\boldsymbol{B}_0, \quad \boldsymbol{SAF} = (\boldsymbol{I} - \boldsymbol{G})\boldsymbol{B}_0, \quad \boldsymbol{G} = \boldsymbol{S} - \boldsymbol{SQS}, \quad \text{etc.} \tag{43}$$

One speedup of null-space methods that should not be overlooked accrues when only few of the applied forces, say $\hat{n}_v$, in $\boldsymbol{p}$ are nonzero. If these are collected in $\hat{\boldsymbol{p}}$, $\boldsymbol{B}_0\boldsymbol{p}_0$ can be contracted to $\hat{\boldsymbol{B}}_0\hat{\boldsymbol{p}}$, in which $\hat{\boldsymbol{B}}_0$ has only $\hat{n}_v \ll n_v$ columns; cf. Section 4.2.

2.10. Symmetric null-space IPD forms

The most expedient way to get parametrized symmetric null-space IPD forms is by substituting (4) into (31) and symmetrizing. An alternative derivation, however, is followed below to illustrate the application of duality (which cannot be used for EPD forms). On making the formal substitutions (15) into (30) we obtain

$$\begin{bmatrix} (s_2 + s_3)\boldsymbol{S} & -s_3\boldsymbol{I} & -s_2\boldsymbol{B}_0 & -s_2\boldsymbol{B}_1 \\ -s_3\boldsymbol{I} & (s_1 + s_3)\boldsymbol{C} & -s_1\boldsymbol{C}\boldsymbol{B}_0 & -s_1\boldsymbol{C}\boldsymbol{B}_1 \\ -s_2\boldsymbol{B}_0^{\mathrm{T}} & -s_1\boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{C} & s_F\boldsymbol{F}_{00} & s_F\boldsymbol{F}_{01} \\ -s_2\boldsymbol{B}_1^{\mathrm{T}} & -s_1\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{C} & s_F\boldsymbol{F}_{10} & s_F\boldsymbol{F}_{11} \end{bmatrix} \begin{Bmatrix} \boldsymbol{g} \\ \boldsymbol{f} \\ \boldsymbol{p}_0 \\ \boldsymbol{y} \end{Bmatrix} = \begin{Bmatrix} -(s_2 + s_3)\boldsymbol{f}^0 \\ -s_3\boldsymbol{g}^0 \\ \boldsymbol{v}_0 - (1 + s_2)\boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{g}^0 \\ \boldsymbol{R}^{\mathrm{T}}\boldsymbol{v}_1 - (1 + s_2)\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g}^0 \end{Bmatrix} \tag{44}$$

in which $s_F = 1 + s_1 + s_2 = s_K$, and the dual of the stiffness partitions are the flexibilities

$$\boldsymbol{K}_{00} \to \boldsymbol{F}_{00} = \boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{C}\boldsymbol{B}_0, \quad \boldsymbol{K}_{01} \to \boldsymbol{F}_{01} = \boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{C}\boldsymbol{B}_1, \quad \boldsymbol{K}_{10} \to \boldsymbol{F}_{10} = \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{C}\boldsymbol{B}_0, \quad \boldsymbol{K}_{11} \to \boldsymbol{F}_{11} = \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{C}\boldsymbol{B}_1. \tag{45}$$

To get rid of $\boldsymbol{f}$, set $s_1 = 0$, $s_3 = 0$ and $s_2 = -s$:

$$\begin{bmatrix} -s\boldsymbol{S} & s\boldsymbol{B}_0 & s\boldsymbol{B}_1 \\ s\boldsymbol{B}_0^{\mathrm{T}} & (1 - s)\boldsymbol{F}_{00} & (1 - s)\boldsymbol{F}_{01} \\ s\boldsymbol{B}_1^{\mathrm{T}} & (1 - s)\boldsymbol{F}_{10} & (1 - s)\boldsymbol{F}_{11} \end{bmatrix} \begin{Bmatrix} \boldsymbol{g} \\ \boldsymbol{p}_0 \\ \boldsymbol{y} \end{Bmatrix} = \begin{Bmatrix} s\boldsymbol{f}^0 \\ \boldsymbol{v}_0 - (1 - s)\boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{g}^0 \\ \boldsymbol{R}\boldsymbol{v}_1 - (1 - s)\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g}^0 \end{Bmatrix}. \tag{46}$$

This may be transformed through the use of the identity $\boldsymbol{v}_0 = \boldsymbol{F}_{00}\boldsymbol{p}_0 + \boldsymbol{F}_{01}\boldsymbol{y} + \boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{g}^0$ and rearrangement of unknowns, to get

$$\begin{bmatrix} (s - 1)\boldsymbol{F}_{11} & -s\boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{R}^{\mathrm{T}} \\ -s\boldsymbol{B}_1 & s\boldsymbol{S} & \boldsymbol{0} \\ \boldsymbol{R} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{y} \\ \boldsymbol{g} \\ \boldsymbol{v}_1 \end{Bmatrix} = \begin{Bmatrix} (1 - s)(\boldsymbol{F}_{10}\boldsymbol{p}_0 + \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g}^0) \\ s(\boldsymbol{B}_0\boldsymbol{p}_0 - \boldsymbol{f}^0) \\ \boldsymbol{p}_1 \end{Bmatrix}, \tag{47}$$

which is the IPD version of the one-parameter form (37). Setting $s = 0$ gives immediately the Force-Redundant-Method equation

$$-\boldsymbol{F}_{11}\boldsymbol{y} = \boldsymbol{F}_{10}\boldsymbol{p}_0 + \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g}^0 - \boldsymbol{R}^{\mathrm{T}}\boldsymbol{v}_1 \tag{48}$$

which is (38) corrected for the effect of nonzero prescribed displacements through the "dislocation vector" term $\boldsymbol{h} = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{v}_1$. State vectors $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{v}_0$ can now be computed through the governing equations, while the reaction force vector may be recovered as $\boldsymbol{p}_1 = \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{f} = \boldsymbol{R}\boldsymbol{y}$.

Setting $s = 1$ in (47) gives the IPD version of the Redundant-Force-Deformation Method (37).

2.11. Unsymmetric forms

Consider again the parametrized range-space EPD form (16). On scaling the three matrix equations by $\boldsymbol{W}_f$, $\boldsymbol{W}_g$ and $\boldsymbol{W}_v$, respectively, we obtain

$$\begin{bmatrix} \boldsymbol{W}_f & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{W}_g & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{W}_v \end{bmatrix} \begin{bmatrix} (s_2+s_3)\boldsymbol{C} & -s_3\boldsymbol{I} & -s_2\boldsymbol{A} \\ -s_3\boldsymbol{I} & (s_1+s_3)\boldsymbol{S} & -s_1\boldsymbol{S}\boldsymbol{A} \\ -s_2\boldsymbol{A}^{\mathrm{T}} & -s_1\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S} & s_K\boldsymbol{K} \end{bmatrix} \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \\ \boldsymbol{v} \end{Bmatrix} = \begin{Bmatrix} -(s_2+s_3)\boldsymbol{W}_f\boldsymbol{g}^0 \\ -s_3\boldsymbol{W}_e\boldsymbol{f}^0 \\ \boldsymbol{W}_v(\boldsymbol{p} - (1+s_2)\boldsymbol{A}\boldsymbol{f}^0) \end{Bmatrix}. \tag{49}$$

This system is generally unsymmetric. One particularly useful choice for the scaling matrices is $\boldsymbol{W}_f = \boldsymbol{B}_1^{\mathrm{T}}$, $\boldsymbol{W}_g = \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{C}$ and $\boldsymbol{W}_v = \boldsymbol{I}$. Because of the orthogonality condition $\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{0}$ this produces two identically zero submatrices:

$$\begin{bmatrix} (s_2+s_3)\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{C} & -s_3\boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{0} \\ -s_3\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{C} & (s_1+s_3)\boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{0} \\ -s_2\boldsymbol{A}^{\mathrm{T}} & -s_1\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S} & s_K\boldsymbol{K} \end{bmatrix} \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \\ \boldsymbol{v} \end{Bmatrix} = \begin{Bmatrix} -(s_2+s_3)\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g}^0 \\ s_3\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g}^0 \\ \boldsymbol{p} - (1+s_2)\boldsymbol{p}^0 \end{Bmatrix}. \tag{50}$$

Taking $s_1 = s_3 = 0$ and $s_2 = -1$ all submatrices multiplying $\boldsymbol{g}$ and $\boldsymbol{v}$ vanish and (50) reduces to

$$\begin{bmatrix} -\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{C} \\ \boldsymbol{A}^{\mathrm{T}} \end{bmatrix} \boldsymbol{f} = \begin{Bmatrix} \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g}^0 \\ \boldsymbol{p} \end{Bmatrix}. \tag{51}$$

Taking $s_2 = s_3 = 0$ and $s_1 = -1$ all submatrices multiplying $\boldsymbol{f}$ and $\boldsymbol{v}$ vanish and (50) reduces to

$$\begin{bmatrix} -\boldsymbol{B}_1^{\mathrm{T}} \\ \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S} \end{bmatrix} \boldsymbol{g} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{p} - \boldsymbol{p}^0 \end{Bmatrix} \quad \text{or} \quad \begin{bmatrix} \boldsymbol{B}_1^{\mathrm{T}} \\ \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S} \end{bmatrix} \boldsymbol{g} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{p} - \boldsymbol{p}^0 \end{Bmatrix}. \tag{52}$$

Eq. (51) and (52) are unsymmetric versions of the Force Method and Deformation Method, respectively. These selectively mix equations from the range-space and null-space sets. For example (51) combines $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{f} = \boldsymbol{p}$, $\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g} = \boldsymbol{0}$ and $\boldsymbol{g} = \boldsymbol{C}\boldsymbol{f} + \boldsymbol{g}^0$.

Setting $s_1 = s_2 = s_3 = 0$ recovers the standard Displacement Method (22) and thus yields nothing new.

The IPD versions are found to be

$$\begin{bmatrix} -\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{C} \\ \boldsymbol{A}_0^{\mathrm{T}} \end{bmatrix} \boldsymbol{f} = \begin{Bmatrix} \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g}^0 - \boldsymbol{h} \\ \boldsymbol{p}_0 \end{Bmatrix}, \qquad \begin{bmatrix} \boldsymbol{B}_1^{\mathrm{T}} \\ \boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{S} \end{bmatrix} \boldsymbol{g} = \begin{Bmatrix} \boldsymbol{h} \\ \boldsymbol{p}_0 - \boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{f}^0 \end{Bmatrix}, \tag{53}$$

in which $\boldsymbol{h} = \boldsymbol{R}\boldsymbol{v}_1$. These can be easily obtained as instances of the parametrized form

$$\begin{bmatrix} (s_2+s_3)\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{C} & -s_3\boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{0} & -s_2\boldsymbol{R}^{\mathrm{T}} \\ -s_3\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{C} & (s_1+s_3)\boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{0} & -s_1\boldsymbol{R}^{\mathrm{T}} \\ -s_2\boldsymbol{A}_0^{\mathrm{T}} & -s_1\boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{S} & s_K\boldsymbol{K}_{00} & s_K\boldsymbol{K}_{01} \\ -s_2\boldsymbol{A}_1^{\mathrm{T}} & -s_1\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{S} & s_K\boldsymbol{K}_{10} & s_K\boldsymbol{K}_{11} \end{bmatrix} \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \\ \boldsymbol{v}_0 \\ \boldsymbol{v}_1 \end{Bmatrix} = \begin{Bmatrix} -(s_2+s_3)\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g}^0 \\ s_3\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{g}^0 \\ \boldsymbol{p}_0 - (1+s_2)\boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{f}^0 \\ \boldsymbol{p}_1 - (1+s_2)\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{f}^0 \end{Bmatrix}. \tag{54}$$

Taking the duals of (53) yields

$$\begin{bmatrix} -\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{S} \\ \boldsymbol{B}_0^{\mathrm{T}} \end{bmatrix} \boldsymbol{g} = \begin{Bmatrix} \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{f}^0 - \boldsymbol{p}_1 \\ \boldsymbol{v}_0 \end{Bmatrix}, \qquad \begin{bmatrix} \boldsymbol{A}_1^{\mathrm{T}} \\ \boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{C} \end{bmatrix} \boldsymbol{f} = \begin{Bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{v}_0 - \boldsymbol{B}_0^{\mathrm{T}}\boldsymbol{g}^0 \end{Bmatrix}, \tag{55}$$

which are not useful because of the presence of unknowns in the right-hand side.

Symmetrization of any of these forms via premultiplication by the transpose of the coefficient matrix is not recommended because it squares the condition number and hinders sparsity.

2.12. *The debut of analysis methods*

The first applications of the matrix forms of the standard Force Method (38) and Displacement Method (22) cannot be pinned down with certainty, but are likely to have occurred in the 1930s as discussed in Section 4.1. The driving application in the period 1945–55 was to sweptback-wing aircraft analysis. The first journal article on the Matrix Force Method for this application is by Levy [11], followed by publications of Rand [12], Langefors [13], Wehle and Lansing [14], and Argyris and Kelsey [15]. The automated selection of redundants in the Force Method (38) by Gauss–Jordan elimination was further developed by Denke [16] and Robinson [17]. This technique is well covered in Przemieniecki's book [18], which also contains an exhaustive pre-1966 bibliography. A subsequent flurry of activity occurred in the 1980s, focused on the construction of maximally sparse $\boldsymbol{B}_1$ matrices [19–24].

Original publications of the Matrix Displacement Method for aircraft analysis are by Levy [25], Argyris and Kelsey [15] and Turner et al. [26]. The last paper marks also the start of the present Finite Element Method. The Direct Stiffness Method version (23) was further developed by Turner and coworkers at Boeing [27, 28] and by 1970 it had eclipsed the Force Method in general-purpose FEM programs.

The combined Force-Displacement Method (24) was apparently first noted by Kosko [29,30]. It is unclear where the combined Deformation-Displacement Method (25) and Force-Deformation-Displacement Method (26) appeared originally in print. The Redundant-Force-Deformation Method (39) is mentioned in passing by Fraeijs de Veubeke [31].

Apparently, the first journal publication that contains the unsymmetric form (51) in the structural mechanics literature is by Patnaik and Dayaratnam [32]. A similar form, with minor matrix rearrangements, is attributed to Thierauf and Topcu [33] (a reference not seen by the author) by Kaneko and Plemmonds [21].

The parametrized forms presented here are new.

3. *d*-Connected mixed finite elements

Moving up to the next step in complexity, this Section considers the finite element model of a linear-elastic structure treated as a continuum. The elements are derived through a three-field parametrized mixed functional in which displacements, strains and stresses may be independently varied. Elements are connected through common node displacements whereas strains and stresses are still *interelement discontinuous*. To simplify the exposition attention is restricted to *compressible linear hyperelasticity without initial strains*. A comparable treatment of incompressible elasticity requires a six-parameter functional [34].

3.1. Governing equations

Consider a body of volume V referred to a rectangular Cartesian coordinate system $\{x_i\}$, $i = 1,2,3$. The body is bounded by the surface S of external unit normal $\boldsymbol{n} \equiv \{n_i\}$. The surface is decomposed into $S : S_u \cup S_t$. Displacements $\hat{\boldsymbol{u}} \equiv \{\hat{u}_i\}$ are prescribed on S_u whereas surface tractions $\hat{\boldsymbol{t}} = \{\hat{t}_i\}$ are prescribed on S_t. Body forces $\boldsymbol{b} \equiv \{b_i\}$ are given in the volume V.

The three unknown internal fields are: displacements $\boldsymbol{u} \equiv \{u_i\}$, strains $\boldsymbol{e} \equiv \{e_{ij}\}$ and stresses $\boldsymbol{\sigma} \equiv \{\sigma_{ij}\}$. The stress traction vector on S is $\boldsymbol{\sigma}_n = \boldsymbol{t} \equiv \{t_i\} = \{\sigma_{ji} n_j\}$ (summation convention implied). To facilitate the construction of elasticity functionals in matrix form, stresses and strains are arranged in the usual 6-component vector forms

$$\begin{aligned} \boldsymbol{\sigma}^{\mathrm{T}} &= [\,\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{23}\ \sigma_{31}\,], \\ \boldsymbol{e}^{\mathrm{T}} &= [\,e_{11}\ e_{22}\ e_{33}\ 2e_{12}\ 2e_{23}\ 2e_{31}\,], \end{aligned} \tag{56}$$

These fields are connected by the kinematic, constitutive and internal-equilibrium equations, which on ignoring initial strain effects are

$$\boldsymbol{e} = \boldsymbol{D}\boldsymbol{u}, \qquad \boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{e}, \qquad \boldsymbol{D}^{\mathrm{T}}\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}. \tag{57}$$

Here $\boldsymbol{E}$ is the 6×6 stress–strain matrix of elastic moduli arranged in the usual manner, $\boldsymbol{D}$ is the 6×3 symmetric-gradient operator and its transpose the 3×6 tensor-divergence operator:

$$\boldsymbol{D}^{\mathrm{T}} = \begin{bmatrix} \partial/\partial x_1 & 0 & 0 & \partial/\partial x_2 & 0 & \partial/\partial x_3 \\ 0 & \partial/\partial x_2 & 0 & \partial/\partial x_1 & \partial/\partial x_3 & 0 \\ 0 & 0 & \partial/\partial x_3 & 0 & \partial/\partial x_2 & \partial/\partial x_1 \end{bmatrix}. \tag{58}$$

The boundary conditions are

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on} \quad S_u, \qquad \boldsymbol{\sigma}_n = \boldsymbol{t} = \hat{\boldsymbol{t}} \quad \text{on} \quad S_t. \tag{59}$$

3.2. The parametrized mixed functional

The following material is extracted from Refs. [1–6] to which the reader is referred for a more detailed exposition. In variational methods of approximation we do not work of course with the exact fields that satisfy the governing equations (57) and (59) but with *independent* (primary, parent) fields, which are subject to variations, and *dependent* (derived, secondary, associated) fields,

which are not. The approximation is determined by taking variations with respect to the independent fields.

An independently varied field will be identified by a superposed tilde, for example $\tilde{\boldsymbol{u}}$. A derived field is identified by writing the symbol of its parent independent field as superscript. For example, the strain and stress fields derived from displacements are $\boldsymbol{e}^u = \boldsymbol{D}\tilde{\boldsymbol{u}}$ and $\boldsymbol{\sigma}^u = \boldsymbol{E}\boldsymbol{e}^u = \boldsymbol{E}\boldsymbol{D}\tilde{\boldsymbol{u}}$. Similarly, $\boldsymbol{\sigma}^e = \boldsymbol{E}\tilde{\boldsymbol{e}}$ are strains derived from varied strains $\tilde{\boldsymbol{e}}$ while $\boldsymbol{e}^\sigma = \boldsymbol{E}^{-1}\tilde{\boldsymbol{\sigma}}$ are strains derived from varied stresses $\tilde{\boldsymbol{\sigma}}$. An advantage of this convention is that $\boldsymbol{u}$, $\boldsymbol{e}$ and $\boldsymbol{\sigma}$ may be reserved for the exact (or generic) fields.

The Parametrized Variational Principle (PVP) that produces the elements considered here is $\delta\Pi = 0$, where Π is the functional

$$\Pi = U - P^{\mathrm{c}}. \tag{60}$$

Here U is the generalized strain energy [2,3] that contains three free parameters s_1, s_2 and s_3:

$$U(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\sigma}},\tilde{\boldsymbol{e}}) = \tfrac{1}{2}\int_V \begin{Bmatrix} \tilde{\boldsymbol{\sigma}} \\ \boldsymbol{\sigma}^e \\ \boldsymbol{\sigma}^u \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} (s_2+s_3)\boldsymbol{I} & -s_3\boldsymbol{I} & -s_2\boldsymbol{I} \\ -s_3\boldsymbol{I} & (s_1+s_3)\boldsymbol{I} & -s_1\boldsymbol{I} \\ -s_2\boldsymbol{I} & -s_1\boldsymbol{I} & (1+s_1+s_2)\boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{e}^\sigma \\ \tilde{\boldsymbol{e}} \\ \boldsymbol{e}^u \end{Bmatrix} \mathrm{d}V, \tag{61}$$

where $\boldsymbol{I}$ is the 6×6 identity matrix, and P^{c} is the conventional potential of external loads, which for zero initial strains is

$$P^{\mathrm{c}}(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\sigma}},\tilde{\boldsymbol{e}}) = \int_V \boldsymbol{b}^{\mathrm{T}}\tilde{\boldsymbol{u}}\,\mathrm{d}V + \int_{S_u} \overset{\sqcup}{\boldsymbol{\sigma}}{}_n^{\mathrm{T}}(\tilde{\boldsymbol{u}}-\hat{\boldsymbol{u}})\,\mathrm{d}S + \int_{S_t} \hat{\boldsymbol{t}}^{\mathrm{T}}\tilde{\boldsymbol{u}}\,\mathrm{d}S. \tag{62}$$

where $\boldsymbol{\sigma}_n^u = (1+s_1+s_2)\boldsymbol{\sigma}_n^u - s_2\tilde{\boldsymbol{\sigma}}_n - s_1\boldsymbol{\sigma}_n^e$. Specific functionals result on setting s_1, s_2 and s_3 to numerical values. For example, the Hu–Washizu functional is obtained by taking $s_1 = 0$ and $s_3 = -s_2 = 1$.

Functional (60) is of *mixed* type. Parametrized *hybrid* functionals of displacement-connected and traction-connected types can be obtained by changing P^{c} to potentials P^{d} and P^{t}, respectively. These hybrid versions, studied in the foregoing references, are not considered in the present work.

3.3. Finite element equations

In the next four subsections the "body of volume V and surface S" of Sections 3.1 and 3.2 becomes either an *individual finite element* or an *arbitrary assembly of such elements*. The trial assumptions on stresses, strains and displacements are

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{N}_\sigma \boldsymbol{f}, \qquad \tilde{\boldsymbol{e}} = \boldsymbol{N}_e \boldsymbol{g}, \qquad \tilde{\boldsymbol{u}} = \boldsymbol{N}_u \boldsymbol{v}. \tag{63}$$

Here $\boldsymbol{f}$ is an array of n_f stress-amplitude parameters, $\boldsymbol{g}$ an array of n_g strain-amplitude parameters, and $\boldsymbol{v}$ is an array of n_v node displacements, whereas $\boldsymbol{N}_\sigma$, $\boldsymbol{N}_e$ and $\boldsymbol{N}_u$ denote the corresponding arrays of shape functions. For simplicity it is assumed that $\boldsymbol{N}_u\boldsymbol{v}$ can match exactly the displacement B.C.s on S_u whereupon the second integral in (62) drops out.

The derived fields are $\boldsymbol{\sigma}^e = \boldsymbol{E}\boldsymbol{N}_e\boldsymbol{g}$, $\boldsymbol{\sigma}^u = \boldsymbol{E}\boldsymbol{e}^u = \boldsymbol{E}\,\boldsymbol{D}\boldsymbol{N}_u\boldsymbol{v} = \boldsymbol{E}\boldsymbol{G}_u\boldsymbol{v}$, $\boldsymbol{e}^\sigma = \boldsymbol{E}^{-1}\boldsymbol{\sigma} = \boldsymbol{E}^{-1}\boldsymbol{N}_\sigma\boldsymbol{f}$, and $\boldsymbol{e}^u = \boldsymbol{D}\boldsymbol{N}_u\boldsymbol{v} = \boldsymbol{G}_u\boldsymbol{v}$, with $\boldsymbol{G}_u = \boldsymbol{D}\boldsymbol{N}_u$. For future use it is convenient to express the displacement-derived strain interpolation as the multiplicative splitting [35] $\boldsymbol{G}_u = \boldsymbol{N}_{eu}\boldsymbol{A}$, or

$$\boldsymbol{e}^u = \boldsymbol{N}_{eu}\boldsymbol{g}^u, \qquad \boldsymbol{g}^u = \boldsymbol{A}\boldsymbol{v}, \tag{64}$$

where g^u is an array of length n_{gu} that contains displacement-derived-strain amplitudes, (which could be appropriate nodal values) and matrix A is *position independent*. Matrix A serves a function similar to that discussed in Section 2. Inserting these assumptions into (60) and making Π stationary with respect to f, g and v, the following FE equations are obtained:

$$\begin{bmatrix} (s_2+s_3)C & -s_3H & -s_2H_\sigma A \\ -s_3H^{\mathrm{T}} & (s_1+s_3)S & -s_1S_eA \\ -s_2A^{\mathrm{T}}H_\sigma^{\mathrm{T}} & -s_1A^{\mathrm{T}}S_e^{\mathrm{T}} & (1+s_1+s_2)K \end{bmatrix} \begin{Bmatrix} f \\ g \\ v \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ p \end{Bmatrix}, \tag{65}$$

where

$$C = \int_V N_\sigma^{\mathrm{T}} E^{-1} N_\sigma \,\mathrm{d}V, \quad S = \int_V N_e^{\mathrm{T}} E N_e \,\mathrm{d}V, \quad K = A^{\mathrm{T}} S_u A, \quad S_u = \int_V N_{eu}^{\mathrm{T}} E N_{eu} \,\mathrm{d}V,$$
$$H = \int_V N_\sigma^{\mathrm{T}} N_e \,\mathrm{d}V, \quad H_\sigma = \int_V N_\sigma^{\mathrm{T}} N_{eu} \,\mathrm{d}V, \quad S_e = \int_V N_e^{\mathrm{T}} E N_{eu} \,\mathrm{d}V, \quad p = \int_V N_u^{\mathrm{T}} b \,\mathrm{d}V + \int_{S_t} N_u^{\mathrm{T}} \hat{t} \,\mathrm{d}S. \tag{66}$$

For this model class the compliance C and rigidity S are still element-block-diagonal square matrices. But they are not generally the inverse of each other, and in fact may have different order. The range-space and null-space equations cannot be cleanly extracted from (65) using the weighted-residual expressions (18), a point further discussed in Section 4.3 in conjunction with the potential use of the Force Method.

We now study specializations that lead to definite conclusions as regards parameter dependence.

3.4. Isomixed models

Elements based on (64)–(66) will be called *isomixed* if the three strain–stress approximation subspaces coincide, that is, $\tilde{\sigma} \equiv \sigma^e \equiv \sigma^u$ and $\tilde{e} \equiv e^\sigma \equiv e^u$. They are obtained by choosing identical interpolations:

$$N_\sigma = N_e = N_{eu} \equiv N_{\mathrm{iso}} \text{ (say)}, \tag{67}$$

which of course implies $n_f = n_g = n_{gu}$. Consequently $H = H_\sigma$, and $S = S_e = S_u$, all matrices being square and symmetric, reducing (65) to

$$\begin{bmatrix} (s_2+s_3)C & -s_3H & -s_2HA \\ -s_3H & (s_1+s_3)S & -s_1SA \\ -s_2A^{\mathrm{T}}H & -s_1A^{\mathrm{T}}S & (1+s_1+s_2)K \end{bmatrix} \begin{Bmatrix} f \\ g \\ v \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ p \end{Bmatrix}, \tag{68}$$

with $K = A^{\mathrm{T}}SA$. Elimination of the g and f unknowns, a task that may be carried out at the element level, yields the stiffness equation $K^*v = p$. The effective stiffness K^* can be presented in a compact form through the introduction of a "residual rigidity" $\varepsilon\Delta S$ defined by

$$C(S + \varepsilon\Delta S) = H^2, \tag{69}$$

where ε is a series-expansion scaling factor. Then

$$K^* = A^{\mathrm{T}}S^*A, \qquad S^* = S - s_\sigma\varepsilon\,\Delta S\,S\left(S - \frac{s_3^2}{s_{\mathrm{det}}}\varepsilon\Delta S\right)^{-1} = S - s_\sigma\varepsilon\,\Delta S + \mathrm{O}(\varepsilon^2), \tag{70}$$

in which $s_{\det} = s_1 s_2 + s_2 s_3 + s_3 s_1$ and $s_\sigma = s_2 + s_3$. In particular, if $s_3 = 0$, $\boldsymbol{K}^* = \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{S} - s_2 \varepsilon \Delta \boldsymbol{S})\boldsymbol{A}$ exactly. It can be seen that in general *the solution provided by an isomixed model is parameter dependent* unless $s_\sigma = 0$, or $s_{det} = 0$, or $\Delta \boldsymbol{S} = \boldsymbol{0}$. The latter case is further studied below. If parameter dependence occurs s_σ can be used to adjust the stiffness to try to improve element performance.

3.5. Commutative isomixed models

An isomixed element will be called *commutative* if the additional conditions (A)–(C) stated in Appendix I are satisfied. It is shown there that this subclass verifies

$$\boldsymbol{C}\boldsymbol{S} = \boldsymbol{S}\boldsymbol{C} = \boldsymbol{H}^2, \qquad \text{i.e.} \quad \Delta \boldsymbol{S} = \boldsymbol{0}, \tag{71}$$

and $\boldsymbol{K}^* = \boldsymbol{K} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A}$ becomes independent of the parameters. Because $\boldsymbol{S}$ and $\boldsymbol{C}$ now commute, $\boldsymbol{S}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ share the same eigensystem and all these matrices (and their powers) may be commuted at will (see Appendix A). Under these assumptions the inverse of the coefficient supermatrix in (68) becomes

$$\begin{bmatrix} \boldsymbol{H}^{-1}(q_1 \boldsymbol{S}\boldsymbol{Q}\boldsymbol{S} - \boldsymbol{S})\boldsymbol{H}^{-1} & \boldsymbol{H}^{-1}(q_2 \boldsymbol{S}\boldsymbol{Q} - \boldsymbol{I}) & \boldsymbol{H}^{-1}\boldsymbol{S}\boldsymbol{A}\boldsymbol{K}^{-1} \\ (q_2 \boldsymbol{Q}\boldsymbol{S} - \boldsymbol{I})\boldsymbol{H}^{-1} & q_3 \boldsymbol{Q} & \boldsymbol{A}\boldsymbol{K}^{-1} \\ \boldsymbol{K}^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{H}^{-1} & \boldsymbol{K}^{-1}\boldsymbol{A}^{\mathrm{T}} & \boldsymbol{K}^{-1} \end{bmatrix}, \tag{72}$$

where q_1, q_2 and q_3 are defined in (20) Comparing (72) with (19) shows that the discrete equations may be presented as

$$\begin{bmatrix} (s_2 + s_3)\boldsymbol{C} & -s_3 \boldsymbol{I} & -s_2 \boldsymbol{A} \\ -s_3 \boldsymbol{I} & (s_1 + s_3)\boldsymbol{S} & -s_1 \boldsymbol{S}\boldsymbol{A} \\ -s_2 \boldsymbol{A}^{\mathrm{T}} & -s_1 \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S} & (1 + s_1 + s_2)\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A} \end{bmatrix} \begin{Bmatrix} \boldsymbol{f}^* \\ \boldsymbol{g} \\ \boldsymbol{v} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{p} \end{Bmatrix}. \tag{73}$$

where $\boldsymbol{f}^* = \boldsymbol{H}\boldsymbol{f}$ may be called a vector of effective internal forces. But this has exactly the same configuration, for $\boldsymbol{f}^0 = \boldsymbol{g}^0 = \boldsymbol{0}$, as the parametrized supermatrix system (16). Consequently all techniques described in Section 2 for classical MSA are applicable to commutative isomixed models at both element and assembly levels, if $\boldsymbol{f}$ is redefined as $\boldsymbol{f}^*$. The equilibrium, constitutive and kinematic equations may be separated as

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{f}^* = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{f} = \boldsymbol{p}, \quad \boldsymbol{f}^* = \boldsymbol{H}\boldsymbol{f} = \boldsymbol{S}\boldsymbol{g}, \quad \boldsymbol{H}\boldsymbol{g} = \boldsymbol{C}\boldsymbol{f}, \quad \boldsymbol{g} = \boldsymbol{A}\boldsymbol{v}. \tag{74}$$

Initial strain effects may be readily introduced by changing the constitutive equations to $\boldsymbol{f}^* = \boldsymbol{H}\boldsymbol{f} = \boldsymbol{S}(\boldsymbol{g} - \boldsymbol{g}^0)$ and $\boldsymbol{H}\boldsymbol{g} = \boldsymbol{C}(\boldsymbol{f} - \boldsymbol{f}^0)$.

3.6. Hypermixed models and the limitation principle

Let $\boldsymbol{\sigma}_{\mathrm{iso}}$ and $\boldsymbol{e}_{\mathrm{iso}}$ denote the stress and strain fields, respectively, of a commutative isomixed model. Suppose that we want to further enrich these two fields by injecting additional interpolation modes:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\mathrm{iso}} + \boldsymbol{N}_x \boldsymbol{f}_x = \boldsymbol{N}_{\mathrm{iso}} \boldsymbol{f}_{\mathrm{iso}} + \boldsymbol{N}_x \boldsymbol{f}_x, \qquad \boldsymbol{e} = \boldsymbol{e}_{\mathrm{iso}} + \boldsymbol{N}_x \boldsymbol{g}_x = \boldsymbol{N}_{\mathrm{iso}} \boldsymbol{g}_{\mathrm{iso}} + \boldsymbol{N}_x \boldsymbol{g}_x. \tag{75}$$

Vectors f_x and g_x contain additional generalized stresses and strains, respectively, while the displacement field interpolation stays the same so that $e^u = N_{\text{iso}}Av$. This will be called a *hypermixed* model. Its finite element equations are

$$\begin{bmatrix} (s_2+s_3)C & (s_2+s_3)C_x & -s_3H & -s_3H_x & -s_2HA \\ (s_2+s_3)C_x^{\mathrm{T}} & (s_2+s_3)C_{xx} & -s_3H_x^{\mathrm{T}} & -s_3H_{xx} & -s_2H_x^{\mathrm{T}}A \\ -s_3H & -s_3H_x & (s_1+s_3)S & (s_1+s_3)S_x & -s_1SA \\ -s_3H_x^{\mathrm{T}} & -s_3H_{xx} & (s_1+s_3)S_x^{\mathrm{T}} & (s_1+s_3)S_{xx} & -s_1S_x^{\mathrm{T}}A \\ -s_2A^{\mathrm{T}}H & -s_2A^{\mathrm{T}}H_x & -s_1A^{\mathrm{T}}S & -s_1A^{\mathrm{T}}S_x & (1+s_1+s_2)K \end{bmatrix} \begin{Bmatrix} f \\ f_x \\ g \\ g_x \\ v \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ p \end{Bmatrix}, \quad (76)$$

where $C_x = \int_V N_x E^{-1} N_{\text{iso}}\,\mathrm{d}V$, $S_x = \int_V N_x E N_{\text{iso}}\,\mathrm{d}V$, etc. By hypothesis the first, third and fifth matrix equations are satisfied if $f = f_{\text{iso}}$, $g = g_{\text{iso}}$, $f_x = 0$ and $g_x = 0$. Using that property and the following identities derived in Appendix A: $S_x = TS$, $C_x = TC$ and $H_x = TH$, in which $T = H_x H^{-1}$, it is easily shown that the second and fourth matrix equations are satisfied only by the isomixed solution. Consequently nothing is gained by injecting extra stress/strain freedoms as per (75)

This is a statement of Fraeijs de Veubeke's limitation principle [36], here shown to hold for arbitrary functionals. The proof can be extended, with more algebra, to different interpolations for the additional stress and strain modes. A point generally overlooked is that principle is not valid for arbitrary right-hand sides of (76) as might be produced, for instance, by general initial-strain or initial-stress patterns. The question of whether the principle holds for noncommutative isomixed models is open. In addition, the principle is not generally applicable to hybrid models, or to displacement-stress-connected mixed models [37].

3.7. Hypomixed models

Mixed FE models that are not isomixed or hypermixed will be called *hypomixed.* For these the solution of the element-level *and* assembled equations depend on the choice of free parameters. Condensation of f and g (a process that can be carried out at the element level) gives the stiffness equations

$$K^* v = A^{\mathrm{T}} S^* A v = p, \quad (77)$$

where K^* and S^* are effective stiffness and rigidity matrices, respectively. This form shows that discrete equilibrium, constitutive and kinematic relations may be extracted by redefining f and g as their effective counterparts:

$$A^{\mathrm{T}} f^* = p, \qquad f^* = S^* g^*, \qquad g^* = Av. \quad (78)$$

But now, unlike the case (74) of commutative isomixed models, both S^* and f^* are *parameter dependent.*

In the general case this dependence is complicated. One situation that lends itself to simple analysis consists of assuming *constant* strains and stresses over each element. For brevity we consider here a one-parameter formulation in which $s_2 = s_3 = 0$ and $s_1 = -s$ to eliminate stresses as independent field, and assume that constitutive properties are uniform over elements. The element strain field is

taken as

$$\boldsymbol{e} = \bar{\boldsymbol{e}}, \tag{79}$$

where a bar over a quantity denotes its mean value over an element. The resulting discrete model is hypomixed unless the displacement-derived strains $\boldsymbol{e}^u$ are also constant, which happens only for simplex elements. Denoting by $v = \int_V \mathrm{d}V$ the element volume measure, the individual-element equations become

$$\begin{bmatrix} -sv\boldsymbol{E}^{-1} & s\boldsymbol{S}_e\boldsymbol{A} \\ s\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}_e^{\mathrm{T}} & (1-s)\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}_u\boldsymbol{A} \end{bmatrix} \begin{Bmatrix} \boldsymbol{g} \\ \boldsymbol{v} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{p} \end{Bmatrix}. \tag{80}$$

Condensation of the strain degrees of freedom $\boldsymbol{g} \equiv \bar{\boldsymbol{e}}$ yields (77), where

$$\boldsymbol{K}^* = \boldsymbol{A}^{\mathrm{T}}\left[(1-s)\boldsymbol{S}_u + s\boldsymbol{S}_e^*\right]\boldsymbol{A}, \qquad \boldsymbol{S}_e^* = v^{-1}\boldsymbol{S}_e^{\mathrm{T}}\boldsymbol{E}\boldsymbol{S}_e. \tag{81}$$

This shows that the effective stiffness matrix is simply a weighted linear combination of the displacement-assumed stiffness $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}_u\boldsymbol{A}$ and the constant-strain stiffness $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}_e^*\boldsymbol{A}$. The former is frequently overstiff while the latter is generally rank deficient and oversoft. Thus parameter s may be adjusted to try to achieve a balanced stiffness [4] and hence improve the overall performance of the element. The practical realization of this idea while requiring low geometric-distortion sensitivity has been far more successful, however, in *hybrid* elements because additional "performance control levers" are then available [38–41].

4. Historical vignettes

This final section correlates the foregoing unification results with selected turning points in the evolution of Matrix Structural Analysis methods.

4.1. Who first wrote down a stiffness or flexibility matrix?

The non-matrix formulation of Discrete Structural Mechanics, based on the ancestors of the present Force and Displacement Methods, can be traced back to the 1860s. By the early 1900s the essential developments were complete. A readable historical account is given by Timoshenko [42]. (Interestingly enough, the term "matrix" never appears in this book.)

The matrix formulation that preceded the modern FEM is generally ascribed to a group of aircraft engineers, cited in Section 2.12, whose work appeared over the period 1947 through 1956 following the World War II "blackout." These were the pioneers in the use of programmable (analog and digital) computers to aircraft analysis and design. But what happened between World Wars?

The author has found a comprehensive description of matrix methods in the monograph by Frazer et al. [43], the first edition of which appeared in 1938. It presents research material developed in support of the British aeronautical industry, and is clear from references to internal reports that those investigations were carried out since the late 1920s. The methods described are geared towards manually operated desk calculators, for which matrices are natural vehicles to organize repetitive tabular computations. Because the emphasis is on dynamic analysis (vibration and flutter) the Displacement

Method dominates. Flexibility matrices are only used to generate dynamical matrices in combination with mass matrices.

Going back along the mathematical path it should be recalled that matrices – or "determinants" as they were initially called – were invented in 1858 by Cayley at Cambridge, although Gibbs (the co-inventor of vector calculus) claimed priority for the German mathematician Grassmann. In any case, matrix algebra was initially cultivated in Great Britain and Germany. The application of matrices to structural problems may have been prompted by the 1925–1932 appearance of several comprehensive treatises in matrix algebra cited in [43, p. 401].

Thus the answer to the subtitle question remains so far unanswered. But circumstantial evidence suggests: somebody working for the aircraft industries in Britain or Germany, which were the leading aeronautical powers of the time, circa 1930.

4.2. *Transitional thinking: from humans to computers*

Matrix Structural Analysis evolved rapidly over the period 1950–1970. Readers trained in present FEM technology may find publications of that period difficult to follow on account of personalized notation and overreliance on physical arguments. Such difficulties may be placed in perpective by noting that those two decades were a *transitional period* between the human-driven calculations of the pre-computer era and the fully automatic procedures of present finite element programs.

For efficient digital computation, data organization (in terms of fast access as well as exploitation of sparseness, vectorization and parallelism) is of primary concern whereas raw problem size, up to certain computer-dependent bounds, is secondary. But for hand calculations minimal problem size is a key aspect. Most humans cannot comfortably solve linear systems of more than 5 or 6 equations by direct methods, and perhaps 5–10 times that through problem-oriented "relaxation" methods. Thus problem reduction techniques were clearly paramount before 1950, and exerted noticeable influence until the early 1970s.

One can recognize "transitional MSA thinking" by the elaborate *functional groupings* of static and kinematic variables. Most such schemes can be understood in terms of the following classification:

$$
\begin{aligned}
&\text{generalized forces} \left\{ \begin{array}{ll} \text{primary} & \left\{ \begin{array}{l} \text{applied forces} \\ \text{redundant forces} \end{array} \right. \\ \text{secondary} & \left\{ \begin{array}{l} \text{condensable forces} \\ \text{support reactions} \end{array} \right. \end{array} \right. \\
&\text{generalized displacements} \left\{ \begin{array}{ll} \text{primary} & \left\{ \begin{array}{l} \text{applied displacements} \\ \text{redundant displacements} \end{array} \right. \\ \text{secondary} & \left\{ \begin{array}{l} \text{condensable displacements} \\ \text{support conditions} \end{array} \right. \end{array} \right.
\end{aligned}
\tag{82}
$$

Here, *applied forces* are those acting with nonzero values, that is, the ones visibly drawn as arrows by an engineer or instructor. In transitional thinking (zero) forces on unloaded degrees of freedom

are classified as *condensable* because they can be reduced out through static condensation techniques. Similarly, nonzero *applied displacements* were clearly differentiated from zero-displacements arising from support conditions because the latter can be easily thrown out while the former must be retained. Redundant displacements, which are the counterpart of redundant forces, have been given many names, among them "kinematically indeterminate displacements" and "kinematic deficiencies."

The hierarchy of (82) can be accommodated within the notation of this article by introducing the symbols

$$\boldsymbol{p}^{\text{gen}} = \begin{Bmatrix} \text{applied:} & \boldsymbol{p}_a \\ \text{redundant:} & \boldsymbol{y} \\ \text{condensable:} & \boldsymbol{p}_c = \boldsymbol{0} \\ \text{reactions:} & \boldsymbol{p}_s \end{Bmatrix}, \qquad \boldsymbol{v}^{\text{gen}} = \begin{Bmatrix} \text{applied:} & \boldsymbol{v}_a \\ \text{redundant:} & \boldsymbol{z} \\ \text{condensable:} & \boldsymbol{v}_c \\ \text{support:} & \boldsymbol{v}_s = \boldsymbol{0} \end{Bmatrix}. \tag{83}$$

It should be noted that these two vectors are not energy conjugate. To further exacerbate the confusion, many transitional expositions embed $\boldsymbol{y}$ in $\boldsymbol{p}_s$ while merging $\boldsymbol{z}$ and $\boldsymbol{v}_c$.

In the influential unification of the Force and Displacement methods by Argyris and Kelsey [15] the decomposition of internal states may be expressed as

$$\boldsymbol{f} = \hat{\boldsymbol{B}}_0 \boldsymbol{p}_a + \hat{\boldsymbol{B}}_1 \boldsymbol{y}, \qquad \boldsymbol{g} = \hat{\boldsymbol{A}}_0 \boldsymbol{v}_a + \hat{\boldsymbol{A}}_1 \boldsymbol{v}_c, \tag{84}$$

Here $\boldsymbol{z} \equiv \boldsymbol{v}_c$, that is, condensable displacements were taken as redundant displacements (a differentiation was made by later authors). The relation of the $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{B}}$ matrices to those used in Section 2 is as follows. Matrix $\hat{\boldsymbol{B}}_0$ omits columns corresponding to condensable forces collected in $\boldsymbol{p}_c = \boldsymbol{0}$. Matrix $\hat{\boldsymbol{A}}_0$ is a subset of $\boldsymbol{A}_1$. Finally, $\hat{\boldsymbol{A}}_1 \equiv \boldsymbol{A}_0$, and $\hat{\boldsymbol{B}}_1 \equiv \boldsymbol{B}_1$. The resultant duality, presented in side-by-side fashion in Table II of [15], states that $\boldsymbol{p}_a \leftrightarrow \boldsymbol{v}_a$ and $\boldsymbol{y} \leftrightarrow \boldsymbol{v}_c$, as illustrated in Figure 6. This correspondence effectively "crosswires" subsets of (15), whereas support conditions were explicitly removed from the discrete governing equations. As such, (84) represents a mixture of what are here called EPD and IPD forms.

While important for hand computations (as well as for the memory-limited computers of that period), this reduction-oriented organization is unsuitable for FEM programming. In present FEM practice the distinction between applied and condensable forces, or between applied displacements and support conditions, is irrelevant. This computer oriented viewpoint governs the present unification, for example in the force/displacement partition (8).

Why then bother about these historical matters? Two arguments may be offered.

First, transitional MSA thinking, while irrelevant to FEM programmers, still pervades many undergraduate textbooks. This is often done with good pedagogical intentions, for hand computations do help in understanding key steps of structural modeling and analysis. Some students face difficulties, however, in moving to "computer thinking" in more advanced courses.

Second, problem size reduction techniques are still important in computer-algebra symbolic computations because symbolic expressions tend to "combinatorially explode" as the number of degrees of freedom increases. This kind of computation is becoming more popular in prototyping and design of structural systems, as powerful CAS programs become available on inexpensive personal computers.

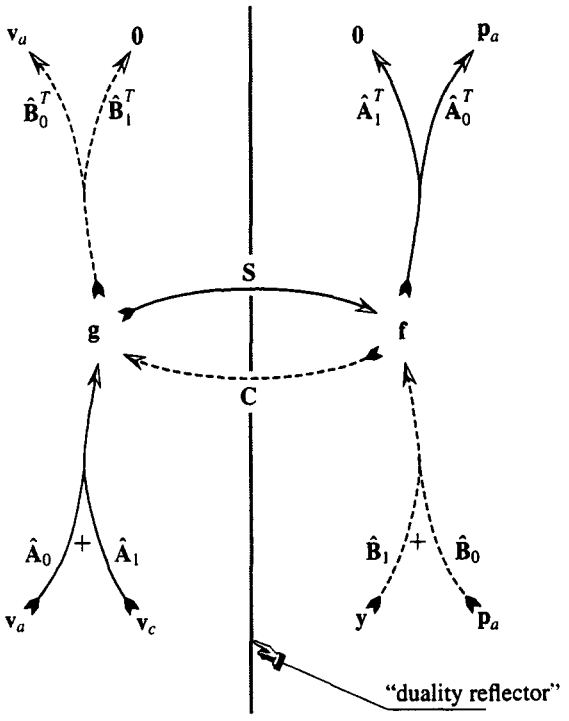

Fig. 6. Graphical representation of the Argyris–Kelsey duality scheme, which should be compared to Fig. 5. The relation of the hatted $\boldsymbol{A}$ and $\boldsymbol{B}$ matrices to those used in Section 2 is explained in the text.

4.3. *The decline and fall of the force method*

Before the advent of the digital computer, the non-matrix version of the standard Force Method had enjoyed a distinguished reputation since the source contributions by Maxwell, Mohr and Castigliano. The method provides directly the internal forces, which are of paramount interest in stress-driven design. It offers considerable scope of ingenuity to experienced structural engineers through clever selection of redundant force systems. By the late 1940s the matrix version had taken the lead in the aircraft applications of the time. As discussed in the previous subsection, the energy-based matrix transformation theory of structures developed by Argyris culminated (by 1954) in showing the formal duality of the Force and Displacement methods.

The duality statement, however, had a confusing aspect. If taken literally, it appears as if both methods go through exactly the same sequence of steps, which can be (and were) displayed side by side. The explanation, as illustrated in Fig. 6, is that this duality applies to an artificial situation: the same structure is analyzed by the Force Method under a system of applied loads $\boldsymbol{p}_a$, and then by the Displacement Method under applied displacements $\boldsymbol{v}_a$ corresponding to those loads. The second case is contrived. What practically counts is being able to solve the same problem by dual methods rather than dual problems by the same method.

Aerospace engineers trained in the 1950s Matrix Force Method – who were often in managerial positions in the 1960s – did not give up easily. By 1965 only Boeing and Bell, among the major aerospace companies, had made major investments in the Displacement Method. That tenacity was

eventually put to a severe test. The 1965 NASA request-for-proposal to build the NASTRAN finite element system called for the simultaneous development of Displacement and Force versions [44]. Each version was supposed to have identical modeling and solution capabilities, including dynamics and buckling, and contracts were awarded accordingly. Eventually the development of the Force version was cancelled in 1969. The following year may be taken as closing the transitional period discussed in Section 4.2, and as marking the end of the Force Method as a serious contender for general-purpose FEM programs.

Can the Force Method be revived? This was the subject of a 1987 Technical Note [8], where its potential for special applications was speculated upon. Breakthroughs along those directions may have to rely on a multifaceted attack, as outlined next.

One obstacle in extending the standard Force Method to mixed or hybrid FE models of continua is that the governing discrete equations become "diffuse" on account of element integration. Interweaving makes the separation of equilibrium, constitutive and kinematic relations less clear. For arbitrary d-connected mixed models this difficulty can be overcome by using effective internal forces $\boldsymbol{f}^*$ and deformations $\boldsymbol{g}^*$ as illustrated by (74) and (78). But stress-connected mixed elements and hybrid elements cause additional complications. This fuzziness also clouds duality.

The correct Force-Displacement duality for arbitrary *continuum* elastic systems was systematically pursued by Fraeijs de Veubeke and co-workers at Liège from 1965 through 1975, following suggestions from Zienkiewicz on the exploitation of Southwell's slab analogy [45]. The equivalence requires the introduction of stress functions as continuum duals of the displacement field [46]. This path, although elegant, is impractical because it requires the development of new potential-based elements with non-physical degrees of freedom. Those elements cannot be mixed with standard ones.

The Calgary Lecture Notes [31] hint at a more practical approach: the direct construction of sparse $\boldsymbol{B}$ matrices by consideration of weak compatibility forms on patches of displacement-connected elements. Systematic exploitation of this idea might have resulted in a serious revival effort. Unfortunately this research was closed by Fraeijs de Veubeke's untimely death in 1976. Several of the aforementioned topics have been systematically investigated by Patnaik since the early 1970s [47–49] within the framework of the so-called Integrated Force Method, independently from the work at Liège.

Acknowledgement

The research work in Parametrized Variational Principles used in Sections 2 and 3 has been supported by the Air Force Office of Scientific Research under Grant F49620-87-C-0074, NASA Langley Research Center under Grant NAG1-756, NASA Lewis Research Center under Grant NAG3-1273, and the National Science Foundation under Grant ASC-9217394. The author is grateful to Dr. Patnaik and Professors Clough, Plemmons and Zienkiewicz for valuable suggestions on historical points and associated references.

Appendix A. Some matrix relations for mixed finite element models

In this appendix matrix relations useful in the derivations of Sections 3.5 and 3.6 are obtained for isomixed and hypermixed FE models. Consider first the case of isomixed models defined by the

interpolations (67). Such models are called *commutative* if the following assumptions hold:

(A) All constitutive coefficients are constant over each element.

(B) Each component of stress and strain is interpolated by the same shape functions, N_s.

(C) If the elements are numerically integrated, the same quadrature rule is used for all stress and strain components.

If these assumptions hold it will be shown that the rigidity and compliance matrices $\boldsymbol{S}$ and $\boldsymbol{C}$ commute and that their product is $\boldsymbol{H}^2$. The proof is worked out for a plate element in plane stress rather than for a 3D element to save space. This 2D element has thickness h and area A. For such an element, assumption (B) reads

$$\sigma_{11} = N_s f_{11}, \quad \sigma_{22} = N_s f_{22}, \quad \sigma_{12} = N_s f_{12}, \quad e_{11} = N_s g_{11}, \quad e_{22} = N_s g_{22}, \quad e_{12} = N_s g_{12}, \tag{A.1}$$

where f_{11} through g_{12} are stress and strain generalized coordinates. Now consider an *individual element*. Let E_{ij} and E'_{ij} denote the entries of the constitutive modulus matrices $\boldsymbol{E}$ and $\boldsymbol{E}^{-1}$, respectively, which according to assumption (A) are constant. Then the element-level matrices $\boldsymbol{S}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ defined by (66) and (67) take the following block configurations:

$$\boldsymbol{S} = \begin{bmatrix} E_{11}\boldsymbol{M} & E_{12}\boldsymbol{M} & E_{13}\boldsymbol{M} \\ E_{12}\boldsymbol{M} & E_{22}\boldsymbol{M} & E_{23}\boldsymbol{M} \\ E_{13}\boldsymbol{M} & E_{23}\boldsymbol{M} & E_{33}\boldsymbol{M} \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} E'_{11}\boldsymbol{M} & E'_{12}\boldsymbol{M} & E'_{13}\boldsymbol{M} \\ E'_{12}\boldsymbol{M} & E'_{22}\boldsymbol{M} & E'_{23}\boldsymbol{M} \\ E'_{13}\boldsymbol{M} & E'_{23}\boldsymbol{M} & E'_{33}\boldsymbol{M} \end{bmatrix}, \quad \boldsymbol{H} = \begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M} \end{bmatrix}, \tag{A.2}$$

where

$$\boldsymbol{M} = \int_A h \boldsymbol{N}_s^{\mathrm{T}} \boldsymbol{N}_s \, \mathrm{d}A, \tag{A.3}$$

is a purely numeric, positive-definite symmetric matrix denoted by $\boldsymbol{M}$ because its configuration resembles that of a consistent mass matrix. The powers $\boldsymbol{M}^2$ and $\boldsymbol{H}^2$ are also symmetric. Matrices (A.2) can be compactly represented as the Kronecker products

$$\boldsymbol{S} = \boldsymbol{E} \otimes \boldsymbol{M}, \qquad \boldsymbol{C} = \boldsymbol{E}^{-1} \otimes \boldsymbol{M}, \qquad \boldsymbol{H} = \boldsymbol{I} \otimes \boldsymbol{M}. \tag{A.4}$$

The *mixed product rule* of Kronecker products, proven for example in [50, p. 24], states that

$$(\boldsymbol{T} \otimes \boldsymbol{U})(\boldsymbol{V} \otimes \boldsymbol{W}) = \boldsymbol{T}\boldsymbol{V} \otimes \boldsymbol{U}\boldsymbol{W}, \tag{A.5}$$

where $\boldsymbol{T}$ through $\boldsymbol{W}$ are arbitrary, but product conforming, matrices. Application of this rule with $\boldsymbol{T} \to \boldsymbol{E}$, $\boldsymbol{V} \to \boldsymbol{E}^{-1}$ and $\boldsymbol{U} = \boldsymbol{W} \to \boldsymbol{M}$ yields $(\boldsymbol{E} \otimes \boldsymbol{M})(\boldsymbol{E}^{-1} \otimes \boldsymbol{M}) = \boldsymbol{I} \otimes \boldsymbol{M}^2$, or

$$\boldsymbol{S}\boldsymbol{C} = \boldsymbol{C}\boldsymbol{S} = \boldsymbol{H}^2. \tag{A.6}$$

Thus not only $\boldsymbol{S}$ and $\boldsymbol{C}$ commute, but their product is independent of constitutive coefficients and depends only on the element geometry (and on the integration rule if the element is evaluated by numerical quadrature). The property (A.6) generalizes immediately to element assemblies for which $\boldsymbol{S}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ are block-diagonal because stresses and strains are interelement-discontinuous in the models considered in Section 3.

Because $\boldsymbol{C}$ and $\boldsymbol{S}$ commute, they must have the same eigensystem. Let $\boldsymbol{\Gamma}_C^2$ and $\boldsymbol{\Gamma}_S^2$ be the diagonal matrices of eigenvalues of $\boldsymbol{C}$ and $\boldsymbol{S}$, respectively (which must be positive because $\boldsymbol{E}$ and $\boldsymbol{E}^{-1}$ are positive definite), and $\boldsymbol{V} = (\boldsymbol{V}^{\mathrm{T}})^{-1}$ the orthogonal matrix of normalized eigenvectors. Then

$$\boldsymbol{C} = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{\Gamma}_C^2\boldsymbol{V}, \quad \boldsymbol{S} = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{\Gamma}_S^2\boldsymbol{V}, \quad \boldsymbol{H} = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{\Gamma}_C\boldsymbol{\Gamma}_S\boldsymbol{V}, \quad \boldsymbol{H}^{-1} = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{\Gamma}_C^{-1}\boldsymbol{\Gamma}_S^{-1}\boldsymbol{V}. \tag{A.7}$$

If assumption (B) fails while the others hold, commutativity of $\boldsymbol{S}$ and $\boldsymbol{C}$ (which still conform) continues to be verified if the interpolation for corresponding components of strain and stress is the same. But their product is no longer $\boldsymbol{H}^2$ and now depends on the constitutive coefficients. For example, suppose that in a four-noded quadrilateral plane stress element σ_{11}, σ_{22}, e_{11} and e_{22} are interpolated bilinearly whereas σ_{12} and e_{12} are kept constant. Then $\boldsymbol{SC} = \boldsymbol{CS} \neq \boldsymbol{H}^2$ because $\boldsymbol{S}$ and $\boldsymbol{C}$ are no longer Kronecker matrices and the mixed product rule does not apply. If corresponding strains and stresses are interpolated by different functions (in the above example, take σ_{12} as bilinear while e_{12} is constant) both commutativity and material-independence are lost; furthermore $\boldsymbol{S}$ and $\boldsymbol{C}$ would not generally conform.

If numerical integration is used, sometimes selective integration, which is the use of different integration rules for different stress and/or strain components, is used to improve element performance. Such schemes are equivalent to selective component interpolation [51], and the foregoing properties apply. Thus assumptions (B) and (C) are interrelated.

Another useful set of identities can be obtained for hypermixed elements defined by the enriched interpolations (75). Let $\boldsymbol{N}_{sx}$ denote the shape function submatrix for each component of stress and strain. The counterpart of $\boldsymbol{M}$ for the coupling of isomixed and enriched modes is

$$\boldsymbol{M}_x = \int_A h \boldsymbol{N}_{sx}^{\mathrm{T}} \boldsymbol{N}_s \, \mathrm{d}A, \tag{A.8}$$

which is generally a rectangular matrix. Then

$$\boldsymbol{S}_x = \begin{bmatrix} E_{11}\boldsymbol{M}_x & E_{12}\boldsymbol{M}_x & E_{13}\boldsymbol{M}_x \\ E_{12}\boldsymbol{M}_x & E_{22}\boldsymbol{M}_x & E_{23}\boldsymbol{M}_x \\ E_{13}\boldsymbol{M}_x & E_{23}\boldsymbol{M}_x & E_{33}\boldsymbol{M}_x \end{bmatrix}, \quad \boldsymbol{C}_x = \begin{bmatrix} E'_{11}\boldsymbol{M}_x & E'_{12}\boldsymbol{M}_x & E'_{13}\boldsymbol{M}_x \\ E'_{12}\boldsymbol{M}_x & E'_{22}\boldsymbol{M}_x & E'_{23}\boldsymbol{M}_x \\ E'_{13}\boldsymbol{M}_x & E'_{23}\boldsymbol{M}_x & E'_{33}\boldsymbol{M}_x \end{bmatrix}, \quad \boldsymbol{H}_x = \begin{bmatrix} \boldsymbol{M}_x & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_x & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_x \end{bmatrix}. \tag{A.9}$$

In Kronecker-product form: $\boldsymbol{S}_x = \boldsymbol{E} \otimes \boldsymbol{M}_x$, $\boldsymbol{C}_x = \boldsymbol{E}^{-1} \otimes \boldsymbol{M}_x$ and $\boldsymbol{H}_x = \boldsymbol{I} \otimes \boldsymbol{M}_x$. Applying the mixed product rule and using (A.4), one finds that $\boldsymbol{S}_x$ and $\boldsymbol{C}_x$ can be expressed as linear transformations of $\boldsymbol{S}$ and $\boldsymbol{C}$, respectively:

$$\begin{aligned} &(\boldsymbol{E} \otimes \boldsymbol{M}_x)(\boldsymbol{E}^{-1} \otimes \boldsymbol{M}) = \boldsymbol{I} \otimes \boldsymbol{M}_x\boldsymbol{M} \rightarrow \boldsymbol{S}_x\boldsymbol{C} = \boldsymbol{H}_x\boldsymbol{H}, \\ &(\boldsymbol{E}^{-1} \otimes \boldsymbol{M}_x)(\boldsymbol{E} \otimes \boldsymbol{M}) = \boldsymbol{I} \otimes \boldsymbol{M}_x\boldsymbol{M} \rightarrow \boldsymbol{C}_x\boldsymbol{S} = \boldsymbol{H}_x\boldsymbol{H}. \end{aligned} \tag{A.10}$$

Adding to these the identity $\boldsymbol{H}_x = \boldsymbol{H}_x\boldsymbol{H}\boldsymbol{H}^{-1}$ we obtain

$$\boldsymbol{S}_x = \boldsymbol{T}\boldsymbol{S}, \quad \boldsymbol{C}_x = \boldsymbol{T}\boldsymbol{C}, \quad \boldsymbol{H}_x = \boldsymbol{T}\boldsymbol{H} \quad \text{with} \quad \boldsymbol{T} = \boldsymbol{H}_x\boldsymbol{H}^{-1}. \tag{A.11}$$

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