# Preserving Steady-State in One-Dimensional Finite-Volume Computations of River Flow

E. Bladé[1]; M. Gómez-Valentín[2]; M. Sánchez-Juny[3]; and J. Dolz[4]

**Abstract:** When using finite-volume methods and the conservative form of the Saint Venant equations in one-dimensional flow computations, it is important to establish the correct balance between the discretized flux vector and the geometric source terms. Over the last few years various improvements to numerical schemes have been presented to achieve this correct balance, focusing on the capability to simulate water at rest on irregular geometries (C-property). In this paper it is shown that common schemes can lead to energy-violating solutions in the case of steady flow. We present developments based on the Roe TVD finite-volume scheme for one-dimensional Saint Venant equations, which results in a method that not only satisfies the C-property, but also preserves the correct steady flow when stationary boundary conditions are used. We also present a totally irregular channel test case for the verification of the method.

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## Introduction

Finite-volume methods are used to predict free surface water flow because they respond well in the presence of shocks. Several recent studies have presented solutions for treating source terms so as to achieve the correct balance in the discretization of the flux gradient. Important examples are Vázquez-Cendón (1999) and Hubbard and García-Navarro (2000). The first of these works includes validation tests for both rapidly varied flow and quiescent flows to ensure that the correct balance is achieved between the source term and the flux gradient (exact C-property). The tests presented in the second work also concern steady flow in a nonprismatic rectangular channel. More recently, Burguete and García-Navarro (2004) and Tseng (2004) examined the capability of numerical schemes to converge to a nonquiescent steady state, but their validation tests were performed on either rectangular channels or channels with a constant cross section. Sanders et al. (2003) used a MUSCL-type scheme for non-rectangular and nonprismatic channels and Vukovic and Sopta (2003) developed Roe TVD schemes with the exact conservation property for the same channel types.

Although it has been more than 25 years since Cunge et al. (1980) demonstrated that some discretizations of the unsteady flow equations are unable to converge to a steady state that satisfies the energy equation, the above-mentioned works did not consider the convergence of the aforementioned finite volume schemes to the correct energy-compatible steady state. In the present work we describe and verify a new numerical method based on the Roe TVD scheme with source term upwinding that not only achieves the correct balance (C-property), but also converges to steady states that satisfy the energy equation for irregular geometry.

## Numerical Scheme

The one-dimensional Saint Venant equations in conservative form for irregular channels are

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{H}$$

$$\mathbf{U} = \begin{pmatrix} A \\ Q \end{pmatrix}; \quad \mathbf{F} = \begin{pmatrix} Q \\ \dfrac{Q^2}{A} + gI_1 \end{pmatrix}; \quad \mathbf{H} = \begin{pmatrix} 0 \\ gI_2 + gA(S_0 - S_f) \end{pmatrix}$$

$$I_1 = \int_0^h (h-\eta) b(x,\eta) d\eta; \quad I_2 = \int_0^h (h-\eta) \frac{\partial b(x,\eta)}{\partial x} d\eta \quad (1)$$

where $\mathbf{U}$=vector of conserved variables; $\mathbf{F}$=flux vector; $\mathbf{H}$=source term, $A$ represents the wetted cross-sectional area; $Q$ the discharge; $g$ gravity, $S_0$ the channel slope, $S_f$ the friction slope, $h$ the water depth and $b$ the channel width. A finite-volume numerical scheme for these equations can be written as

[1]Assistant Professor, FLUMEN Research Group, Escola Tècnica Superior d'Enginyers de Camins, Canals i Ports de Barcelona, Universitat Politècnica de Catalunya, Jordi Girona 1-3, D-1, 08034, Barcelona, Spain (corresponding author). E-mail: ernest.blade@upc.edu

[2]Professor, FLUMEN Research Group, Escola Tècnica Superior d'Enginyers de Camins, Canals i Ports de Barcelona, Universitat Politècnica de Catalunya, Jordi Girona 1-3, D-1, 08034, Barcelona, Spain. E-mail: manuel.gomez@upc.edu

[3]Associate Professor, FLUMEN Research Group, Escola Tècnica Superior d'Enginyers de Camins, Canals i Ports de Barcelona, Universitat Politècnica de Catalunya, Jordi Girona 1-3, D-1, 08034, Barcelona, Spain. E-mail: marti.sanchez@upc.edu

[4]Professor, FLUMEN Research Group, Escola Tècnica Superior d'Enginyers de Camins, Universitat Politècnica de Catalunya, Jordi Girona 1-3, D-1, 08034, Barcelona, Spain. E-mail: j.dolz@upc.edu

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J. Hydraul. Eng., 2008, 134(9): 1343-1347

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}(\mathbf{F}^*_{i+1/2} - \mathbf{F}^*_{i-1/2}) + \frac{\Delta t}{\Delta x}\mathbf{H}_i^* \quad (2)$$

where $\mathbf{F}^*$ is the numerical flux and $\mathbf{H}^*$ is the numerical expression of the integral of the source term on the finite volume.

### Numerical Flux

The Saint Venant equations have a spatial dependence on the flux vector that derives from the pressure term $I_1$ as $dI_1/dx = (\partial I_1/\partial A)(\partial A/\partial x) + \partial I_1/\partial x$. This yields the discrete form

$$g\Delta I_{1,i+1/2} = \tilde{c}^2(\Delta A_{i+1/2}) + g(\Delta I_1|_{\tilde{A}})_{i+1/2} \quad (3)$$

where the celerity is $\tilde{c} = \sqrt{g(\partial I_1/\partial A)|_{x=\text{const}}}$ and $(\Delta I_1|_{\tilde{A}})_{i+1/2}$ =variation of pressure forces between finite volumes for a constant area $\tilde{A}_{i+1/2}$. The flux difference can be written as (Hubbard and García-Navarro 2000)

$$\Delta\mathbf{F}_{i+1/2} = \tilde{\mathbf{J}}_{i+1/2}\Delta\mathbf{U}_{i+1/2} + \mathbf{V}_{i+1/2} = \sum_{j=1}^{2}\tilde{\alpha}_j\tilde{\lambda}_j\tilde{\mathbf{e}}_j + \sum_{j=1}^{2}\tilde{\gamma}_j\tilde{\mathbf{e}}_j \quad (4)$$

where $\tilde{\mathbf{J}}$=Roe approximation to the Jacobian matrix of $\mathbf{F}$; $\tilde{\alpha}_j$ =Roe averages of the wave strengths; $\tilde{\lambda}_j$=eigenvalues of $\tilde{\mathbf{J}}_{i+1/2}$; and $\tilde{\mathbf{e}}_j$=its eigenvectors, as can be seen in Toro (1997). $\mathbf{V}$ =contribution of the irregular geometry while $\tilde{\gamma}_j$ is its decomposition

$$\mathbf{V} = \frac{\partial\mathbf{F}}{\partial x} = \begin{pmatrix} 0 \\ \left.\dfrac{\partial I_1}{\partial x}\right|_{A=\text{const}} \end{pmatrix}; \tilde{\gamma}_1 = \frac{1}{2\tilde{c}}g(\Delta I_1|_{\tilde{A}})_{i+1/2}; \tilde{\gamma}_2 = -\tilde{\gamma}_1 \quad (5)$$

Therefore, the numerical flux of the Roe TVD scheme is

$$\mathbf{F}^*_{i+1/2} = \frac{1}{2}(\mathbf{F}_i + \mathbf{F}_{i+1}) - \frac{1}{2}\left(\sum_{j=1}^{2}\tilde{\alpha}_j\varphi_j\tilde{\mathbf{e}}_j + \sum_{j=1}^{2}\tilde{\gamma}_j sign(\tilde{\lambda}_j)\tilde{\mathbf{e}}_j\right) + \frac{1}{2}\left(\sum_{j=1}^{2}\psi_j\tilde{\alpha}_j\varphi_j\left(1 - \left|\tilde{\lambda}_j\frac{\Delta t}{\Delta x}\right|\right)\tilde{\mathbf{e}}_j + \sum_{j=1}^{2}\psi_j\tilde{\gamma}_j sign(\tilde{\lambda}_j) \times\left(1 - \left|\tilde{\lambda}_j\frac{\Delta t}{\Delta x}\right|\right)\tilde{\mathbf{e}}_j\right) \quad (6)$$

where $\varphi_j$=Harten and Hyman entropy fix as presented by Toro (1997); and $\psi_j$=Minmod flux limiter as used by Alcrudo (1992).

The works referred to above and other studies in the literature use different formulations for $\tilde{c}$: $\tilde{c} = \sqrt{g\tilde{A}/\tilde{B}}$, where $\tilde{A} = (A_i + A_{i+1})/2$ and $\tilde{B} = (B_i + B_{i+1})/2$; $\tilde{c} = (c_i + c_{i+1})/2$, $\tilde{c}^2 = (c_i^2 + c_{i+1}^2)/2$. and $\tilde{c} = \sqrt{g(\Delta I/\Delta A)_{i+1/2}}$. By substituting Eq. (6) into Eq. (4) it can be seen that, for irregular channels, none of these expressions of $\tilde{c}$ reproduce the pressure force jump decomposition of Eq. (3). Instead, this can be achieved by using the following expressions for $\tilde{c}^2$ and $(\Delta I_1|_{\tilde{A}})_{i+1/2}$:

$$\tilde{c}^2 = g\frac{(I_{1,i+1} - I_{1,i}) - (\Delta I_1|_{\tilde{A}})_{i+1/2}}{A_{i+1} - A_i}; \quad (\Delta I_1|_{\tilde{A}})_{i+1/2} = (I_1|_{\tilde{A}})_{i+1} - (I_1|_{\tilde{A}})_i \quad (7)$$

These expressions maintain the physical meaning of celerity, which is the variation of pressure forces in a cross section with regard to the variation of the flow area in the same section. If $A_{i+1} = A_i$, Eq. (7) cannot be numerically used, but Eq. (3) reduces to $g\Delta I_{1,i+1/2} = g(\Delta I_1|_{\tilde{A}})_{i+1/2}$ and any of the above definitions of celerity is possible. In such cases $\tilde{c}^2 = (c_i^2 + c_{i+1}^2)/2$ has been adopted.

### Source Term

The slope source term has an obvious balance with the flux vector, but this does not apply to the part of the source term due to friction (Hubbard and García-Navarro 2000). For the purposes of the present work we consider $S_f = 0$. As proposed by Vázquez-Cendón (1999), for the scheme to satisfy the exact conservation property, $\mathbf{H}_i^*$ can be divided into two contributions at finite-volume boundaries (as was done for $\mathbf{F}^*$) and decomposed on the eigenvectors of $\tilde{\mathbf{J}}$

$$\mathbf{H}_i^* = \mathbf{H}^*_{i,i-1/2} + \mathbf{H}^*_{i,i+1/2} \quad (8)$$

$$\mathbf{H}^*_{i,i-1/2} = \frac{1}{2}\left(\sum_{j=1}^{2}\tilde{\beta}_j\left(1 + \text{sign}(\tilde{\lambda}_j)\left[1 - \psi_j\left(1 - \left|\tilde{\lambda}_j\frac{\Delta t}{\Delta x}\right|\right)\right]\right)\tilde{\mathbf{e}}_j\right)_{i-1/2}$$

$$\mathbf{H}^*_{i,i+1/2} = \frac{1}{2}\left(\sum_{j=1}^{2}\tilde{\beta}_j\left(1 - \text{sign}(\tilde{\lambda}_j)\left(1 - \psi_j\left(1 - \left|\tilde{\lambda}_j\frac{\Delta t}{\Delta x}\right|\right)\right)\right)\tilde{\mathbf{e}}_j\right)_{i+1/2} \quad (9)$$

$$\tilde{\beta}_1 = -\frac{1}{2\tilde{c}}g\tilde{A}(\Delta z + \Delta h) + \frac{1}{2\tilde{c}}g\Delta I_1; \quad \tilde{\beta}_2 = -\tilde{\beta}_1 \quad (10)$$

where $z = z(x)$=channel bottom elevation.

## Steady-State Preservation

In all steady flows $\mathbf{U}_i^{n+1} = \mathbf{U}_i^n$. Therefore, combining the general Eq. (2) with Eq. (8) it follows

$$\mathbf{F}^*_{i-1/2} - \mathbf{F}^*_{i+1/2} + \mathbf{H}^*_{i,i-1/2} + \mathbf{H}^*_{i,i+1/2} = 0 \quad (11)$$

In the case of zero bed friction, for a quiescent flow $Q = 0$, $\Delta h = -\Delta z$, and $S_f = 0$. By incorporating these values into the expressions for $\mathbf{F}^*$ and $\mathbf{H}^*$ and Eq. (11), a direct operation yields $(\Delta z + \Delta h)_{i+1/2} = 0$. Therefore, the numerical scheme verifies the exact conservation property. The same flux vector and source term balance (11) must hold for steady, gradually varied flow. In this more general case, by incorporating the expressions for $\mathbf{F}^*$ and $\mathbf{H}^*$ into Eq. (11) we obtain

$$\frac{Q^2}{\tilde{A}_{i+1/2}}\left(\frac{A_{i+1} - A_i}{A_{i+1}A_i}\right) = g(\Delta z + \Delta h)_{i+1/2} \quad (12)$$

Convergence to the correct steady state means that the energy conservation equation must be satisfied between two cross sections, $i$ and $i+1$, whenever there is no hydraulic jump

$$(\Delta z + \Delta h)_{i+1/2} = \frac{1}{g}\left(\frac{Q^2}{2A_i^2} - \frac{Q^2}{2A_{i+1}^2}\right) \quad (13)$$

Finally, if Eqs. (12) and (13) are combined, it follows that $\tilde{A}$, which is the cross-sectional area at the intercells, must be the harmonic mean of the area of the neighboring cross sections

$$\tilde{A} = \frac{2A_iA_{i+1}}{A_i + A_{i+1}} \quad (14)$$



J. Hydraul. Eng., 2008, 134(9): 1343-1347

**Table 1.** Definition of the Irregular Trapezoidal Channel

| Distance to upstream (m) | Point coordinates (m) | | | |
|---|---|---|---|---|
| | Point 1 | Point 2 | Point 3 | Point 4 |
| 0 | (−3,10) | (0,0.4) | (2,0.4) | (5,10) |
| 1 | (−3,10) | (0,0.3) | (3,0.3) | (5,10) |
| 2 | (−3,10) | (0,1) | (2,0) | (5,10) |
| 3 | (−3,10) | (0,0) | (2,0) | (5,10) |
| 4 | (−5,10) | (0,1) | (1,1) | (4,10) |
| 5 | (−1,10) | (0,1) | (1,0.5) | (4,10) |
| 6 | (−3,10) | (0,1) | (0.5,1.1) | (4,10) |
| 7 | (−3,10) | (0,1) | (1,1) | (4,10) |
| 8 | (−3,10) | (0,0.3) | (3,0.3) | (5,10) |
| 9 | (−3,10) | (0,1) | (2,0) | (5,10) |
| 10 | (−3,10) | (0,0) | (2,0) | (5,10) |
| 11 | (−5,10) | (0,1) | (1,1) | (4,10) |
| 12 | (−1,10) | (0,1) | (1,0.5) | (4,10) |
| 13 | (−3,10) | (0,1) | (1,1) | (4,10) |
| 14 | (−3,10) | (0,1) | (1,1) | (4,10) |

## Validation: 1D Steady-State Preservation Test Cases

Bladé and Gómez Valentín (2006) performed validation tests of the scheme presented here for unsteady flows. Results of the proposed method for steady flow in an irregular trapezoidal channel are presented here and compared with other common schemes. The trapezoidal channel proposed for the test case consists of a 13 m-long frictionless channel defined with 14 cross sections shown in Table 1 (coordinates are distance to channel axis and elevation). The boundary conditions are a constant discharge of 2 $m^3/s$ upstream and a water elevation of 2 m downstream. The water surface profile of the exact solution can be calculated from energy conservation [Eq. (13)]. It is compared with the following numerical schemes or variations:

1. A classical Roe TVD scheme with no spatial variation of the flux vector [that is, without the second right-hand term of expression (4)] and the arithmetic mean for $\widetilde{A}$.
2. A Roe TVD scheme with spatial variation of $\mathbf{F}$, as in Eq. (4), and the arithmetic mean for $\widetilde{A}$.
3. The optimized Lax-Friedrichs scheme, based on a quasi-conservative form of the equation, as proposed by Burguete and García-Navarro (2004).
4. A Roe TVD scheme with spatial variation of the flux vector, as in Eq. (4), and the harmonic mean for $\widetilde{A}$, as in Eq. (14).

Figure 1 shows the results for Case (a): there are considerable errors if no spatial dependence of the flux vector is taken into account. Fig. 2 shows the results for Cases (b) and (c), with schemes that satisfy the exact C-property and produce very similar results, although in both cases the results differ from the exact

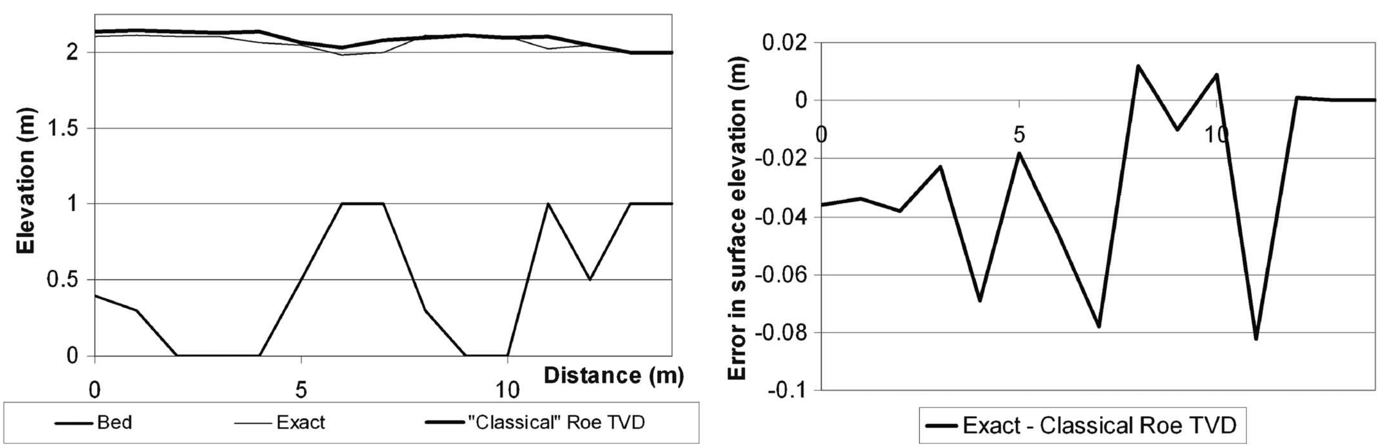


**Fig. 1.** Comparison of the exact surface with the Roe TVD scheme without spatial flux dependence

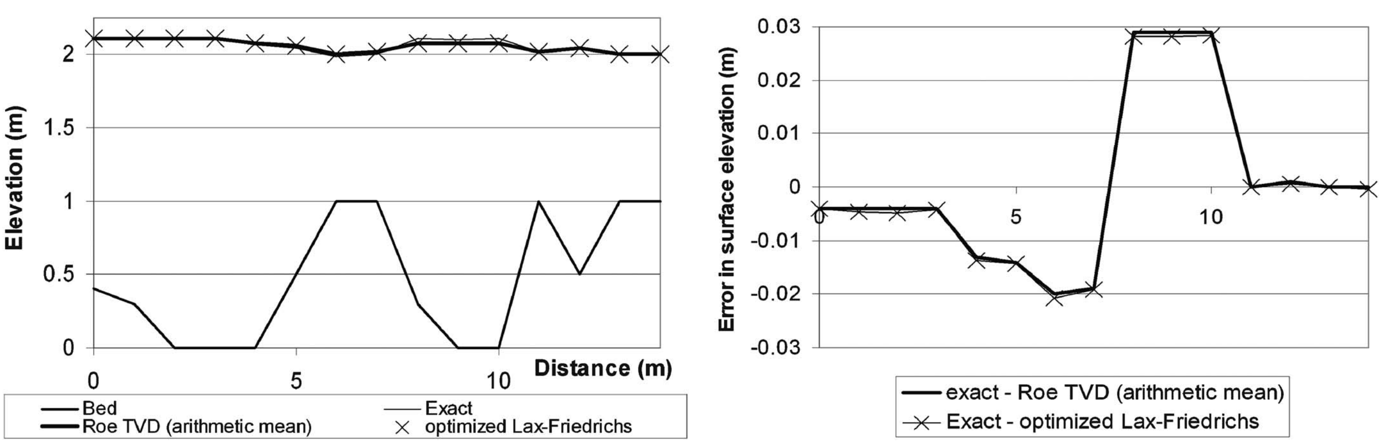


**Fig. 2.** Comparison of the exact surface with the Roe TVD scheme with the arithmetic mean for $\widetilde{A}$ and the optimized Lax-Friedrichs scheme



J. Hydraul. Eng., 2008, 134(9): 1343-1347

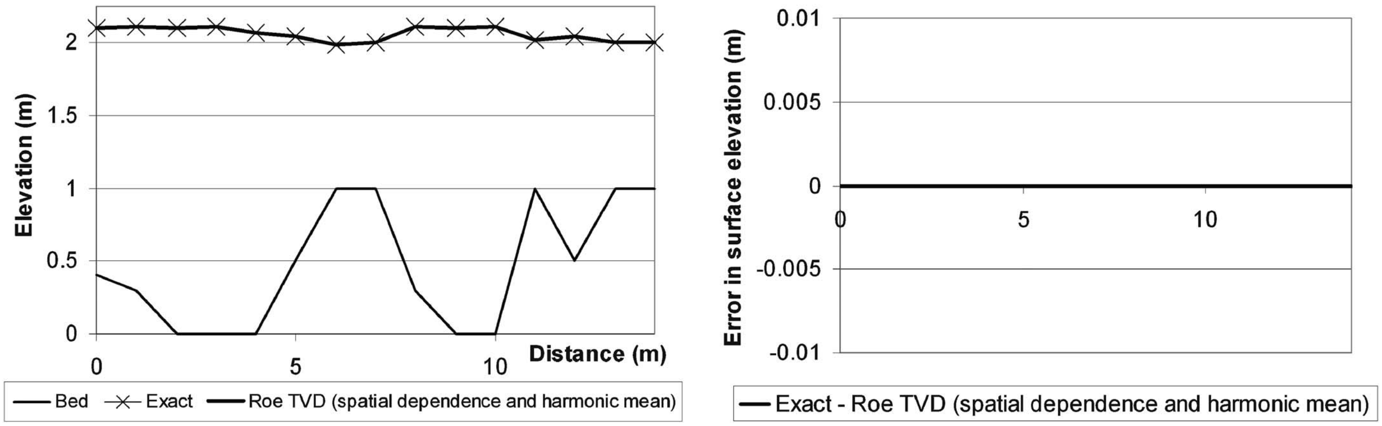


**Fig. 3.** Comparison of the exact surface with the Roe TVD scheme with the harmonic mean for $\widetilde{A}$

solution. Finally, Fig. 3 shows the ability of the Roe TVD scheme as proposed in this work to converge to the exact solution.

## Conclusions

The present work shows that some common and recent numerical methods fail to converge to the correct steady state, even in the absence of shocks. A correction of the discretized celerity in the Roe TVD scheme is presented to take into account the spatial dependence of the flux vector on geometry. This, together with the harmonic mean to approximate the cross-sectional area at intercells results in a method that produces solutions which are compatible with the energy equation when applied to steady flows.

## Notation

*The following symbols are used in this technical note:*

$A$ = wetted area of the cross section;
$\widetilde{A}_{i+1/2}$ = wetted area approximation at intercell $i+1/2$;
$B_i$ = channel surface width at cross section $i$;
$\widetilde{B}_{i+1/2}$ = approximation of $B$ at intercell $i+1/2$;
$b$ = channel width;
$\widetilde{c}$ = celerity approximation at an intercell;
$\widetilde{e}_j$ = $j$ component of the Roe averages of the eigenvectors of $\mathbf{A}$;
$\mathbf{F}$ = flux vector of the Saint Venant equations;
$\mathbf{F}^*_{i+1/2}$ = numerical flux at intercell $i+1/2$;
$g$ = gravity;
$\mathbf{H}$ = source term of the Saint Venant equations;
$\mathbf{H}^*$ = numerical expression of the integral of the source term on a finite volume;
$h$ = depth;
$I_1$ = pressure forces on the cross section;
$I_1|_{\widetilde{A}}$ = value of $I_1$ for a wetted area value of $\widetilde{A}$;
$I_2$ = variation with distance of the resultant pressure forces exerted by the river bed;
$\widetilde{\mathbf{J}}_{i+1/2}$ = Roe's approximation to the Jacobian of $\mathbf{F}$ at intercell $i+1/2$;
$Q$ = discharge;
$S_f$ = friction slope;
$S_0$ = river slope;
$\mathbf{U}$ = vector of conserved variables;
$\mathbf{U}_i^n$ = mean value of $\mathbf{U}$ over finite volume $i$ at time step $n$;
$\mathbf{V}$ = contribution of irregular geometry to the numerical flux difference;
$x$ = channel axis coordinate;
$z$ = channel bottom elevation;
$\widetilde{\alpha}_j$ = Roe average of the $j$th wave strength;
$\widetilde{\beta}_j$ = $j$th coefficients of the decomposition of the geometric source term in $\widetilde{\mathbf{e}}$;
$\Delta\mathbf{F}_{i+1/2}$ = jump of $\mathbf{F}$ across intercell $i+1/2$;
$\Delta t$ = time step;
$\Delta x$ = space increment;
$\Delta\mathbf{U}_{i+1/2}$ = jump of $\mathbf{U}$ across intercell $i+1/2$;
$\varphi$ = Harten and Hyman's entropy fix;
$\widetilde{\gamma}_j$ = $j$th coefficients of the decomposition of $\Delta(gI_1|_{\widetilde{A}})$ in the $\widetilde{\mathbf{e}}$ vector base;
$\widetilde{\lambda}_j$ = Roe averages of the $j$ eigenvalue of $\mathbf{A}$; and
$\Psi$ = flux limiter function.

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J. Hydraul. Eng., 2008, 134(9): 1343-1347

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