# Modeling Engineering Reliability Data Using the Modified Generalized Exponential Distribution

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**ABSTRACT**

This study introduces the modified generalized exponential (MGEx) distribution, a flexible probabilistic model designed to capture complex failure behaviors commonly encountered in engineering systems. The MGEx distribution accommodates various hazard rate shapes, including unimodal, increasing, and decreasing forms, making it suitable for modeling reliability and lifetime data. We derive the mathematical properties of the model and focus on computational techniques for parameter estimation using multiple frequentist approaches. To evaluate the numerical performance of these methods, we conduct extensive Monte Carlo simulations, comparing their accuracy and robustness through partial and overall ranking metrics. The practical utility of the MGEx model is demonstrated by applying it to two real-world engineering datasets. The results highlight the model's potential in enhancing simulation accuracy and supporting data-driven decision-making in reliability analysis, system design, and failure prediction tasks.

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## 1 Introduction

Lifetime data analysis and modeling are crucial in many disciplines, such as economics, biomedicine, engineering, insurance, and behavioral sciences. Recent studies, however, suggest that classical distributions often require improvement in engineering, biology, finance, and environmental research. Due to this restriction, there is a need for more flexible versions of these distributions, which has resulted in notable progress in the generalisation of well-established models. In applications spanning these domains, these extended distributions have proven successful.

The exponential (Ex) distribution has long been one of the most fundamental lifetime models due to its mathematical simplicity and constant hazard rate. However, this assumption often proves restrictive in practical applications where data exhibit non-constant failure patterns. To address this limitation, Numerous researchers have suggested creative generalisations and modifications of the Ex distribution to improve its usefulness and adaptability.

Among the notable examples are the Marshall-Olkin exponential [1], the modified Ex [2], the Marshall-Olkin logistic Ex [3], the Marshall-Olkin alpha-power Ex [4], the type-I heavy tailed Ex [5],

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the generalised odd log-logistic Ex [6], the extended odd Weibull Ex [7], the power-modified Kies Ex [8], Poisson moment Ex [9], extended Ex [10], unit extended Ex [11], exponentiated inverse Ex [12], and Ramos-Louzada Ex [13].

Building upon this evolutionary framework, the present study introduces the modified generalized exponential (MGEx) distribution as a natural and powerful extension of these earlier models. The MGEx distribution preserves the desirable analytical tractability of the exponential-based family while offering improved flexibility to capture a wide range of density and hazard rate shapes observed in real-world lifetime data.

The MGEx distribution builds upon the Ex model and incorporates features from the modified generalized (MG) family introduced by Shama et al. [14]. It discusses the essential desirable qualities and reasoning behind the MGEx distribution while going into detail into a number of mathematical properties and particular examples of the model.

The hazard rate function (HRF) of the MGEx model accommodates a variety of important shapes, including increasing, decreasing, unimodal, reversed-$J$-shaped, and $J$-rshaped HRF patterns. In addition, the MGEx model provides a right-skewed, reversed-$J$-shaped, or concave-down density function. It consistently delivers better fits compared to other competing distributions, including the modified Ex (MEx) by Rasekhi et al. [2], the Marshall–Olkin alpha power exponential (MOAPEx) by Nassar et al. [4], the odd log-logistic Lindley exponential (OLLLiEx) by Alizadeh et al. [15], the transmuted generalized exponential (TGEx) by Khan et al. [16], the alpha power exponential (APEx) by Mahdavi and Kundu [17], the logistic exponential (LEx) by Lan and Leemis [18], the exponentiated exponential (EEx) by Gupta and Kundu [19], and the standard exponential distribution.

Furthermore, six frequent estimation techniques estimate the MGEx model's parameters. This feature might be especially pertinent for engineers, applied statisticians, and practitioners looking for the best estimating technique. Comprehensive simulation studies examine the effectiveness of different methods, assessing accuracy using metrics including absolute bias, mean square errors, and mean relative mistakes. The best estimation technique for the MGEx parameters is then determined by analysing and ranking the outcomes of these measurements using partial and total ranks. The literature contains numerous discussions on estimating parameters for generalized distributions using classical approaches. For further insights, readers may refer to works such as [20,21].

This paper is organized as follows: Section 2 introduces the MGEx distribution and its properties. Section 3 presents the estimators for the unknown parameters of the MGEx model, utilizing six classical estimation methods. Simulation studies are discussed in Section 4. Section 5 demonstrates the applicability of the MGEx model to real-life datasets. Finally, Section 6 provides a summary of the article.

## 2 The MGEx Distribution

Consider the baseline exponential (Ex) distribution with parameter $\lambda > 0$, characterized by its probability density function (PDF), which is given by

$$g(x;\lambda) = \lambda e^{-\lambda x}, \quad x > 0, \lambda > 0. \tag{1}$$

The cumulative distribution function (CDF) of the Ex reduces to

$$G(x;\lambda) = 1 - e^{-\lambda x}. \tag{2}$$

The $r$th ordinary and incomplete moments, as well as the moment generating function (MGF) for the exponential distribution, are defined as follows

$\mu_r = \lambda^{-r}\Gamma(r+1)$, $\psi_r(x) = \lambda^{-r}\gamma(r+1,\lambda x)$ and $M_x(t) = \frac{\lambda}{\lambda-t}; t \neq 0$, respectively, where $\Gamma(a+1) = \int_0^1 w^a e^{-\omega} dw$ is the gamma function (GF) and $\gamma(r+1,\lambda x) = \int_0^{\lambda x} w^a e^{-\omega} dw$ is the lower incomplete GF and the generating function is $f^{-1}(G/\lambda) = \dfrac{-\ln(1-U)}{\lambda}, \quad 0 < U < 1$ –see Afify and Mohamed [7], for more information. The MG family's CDF and PDf are now presented, following the description provided by [14]. Consider the two baseline CDFs delivered by $G_1(x;\boldsymbol{\psi}_1)$ and $G_2(x;\boldsymbol{\psi}_2)$ with parameter vectors $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$, respectively. The CDF of the MG family is then expressed as

$$F(x;\rho,\eta,\boldsymbol{\psi}_1,\boldsymbol{\psi}_2) = 1 - \overline{G}_1(x;\boldsymbol{\psi}_1)^{\rho} e^{-G_2(x;\boldsymbol{\psi}_2)^{\eta}}, \tag{3}$$

where $\overline{G}_1(x;\boldsymbol{\psi}_1) = 1 - G_1(x;\boldsymbol{\psi}_1)$ refers to the baseline survivl function (SF), and $\rho$ and $\eta$ are positive shape parameters. The MG family density can be written as

$$f(x;\rho,\eta,\boldsymbol{\psi}_1,\boldsymbol{\psi}_2) = \left[\rho g_1(x;\boldsymbol{\psi}_1) + \eta g_2(x;\boldsymbol{\psi}_2)\overline{G}_1(x;\boldsymbol{\psi}_1) G_2(x;\boldsymbol{\psi}_2)^{\eta-1}\right] \times \overline{G}_1(x;\boldsymbol{\psi}_1)^{\rho-1} e^{-G_2(x;\boldsymbol{\psi}_2)^{\eta}}. \tag{4}$$

We can define the MGEx distribution's functions to achieve this. With a scale parameter $\lambda > 0$, let $G_1(x;\lambda) = G_2(x;\lambda) = 1 - e^{-\lambda x}$ represent a typical baseline exponential distribution. The CDF of the MGEx distribution can be obtained by inserting the CDF of the exponential distribution (2) into Eq. (3),

$$F(x;\rho,\eta,\lambda) = 1 - e^{-\rho\lambda x - \left(1-e^{-\lambda x}\right)^{\eta}}, \quad x > 0, (\rho,\eta,\lambda) > 0. \tag{5}$$

The following forms provide the MGEx distribution's PDF and HRF,

$$f(x;\rho,\eta,\lambda) = \left(\rho\lambda + \eta\lambda\left(1-e^{-\lambda x}\right)^{\eta-1} e^{-\lambda x}\right) e^{-\rho\lambda x - \left(1-e^{-\lambda x}\right)^{\eta}}. \tag{6}$$

Let, we defined a new function as

$$\kappa(x;\rho,\eta,\lambda) = \rho\lambda + \eta\lambda\left(1-e^{-\lambda x}\right)^{\eta-1} e^{-\lambda x}. \tag{7}$$

*Remark (1):*

1. It can be observed that when $\rho = \lambda = \eta = 1$, the resulting distribution is a simplified exponential distribution with a single parameter, which exhibits a decreasing HRF.

2. Setting $\rho = \eta = 1$ in (7), the result is given

$$\kappa(x;\rho,\eta,\lambda) = \lambda + \lambda e^{-\lambda x} = \lambda + g(x;\lambda).$$

Fig. 1 shows various shapes of the MGEx PDF for different parameter values. The plots indicate that the MGEx density can exhibit right skewness, a reversed-$J$ shape, or a concave-down form. Fig. 2 presents several shapes of the MGEx HRF, revealing that the hazard rate can take on increasing, decreasing, unimodal, reversed-$J$-shaped, and $J$-shaped patterns.

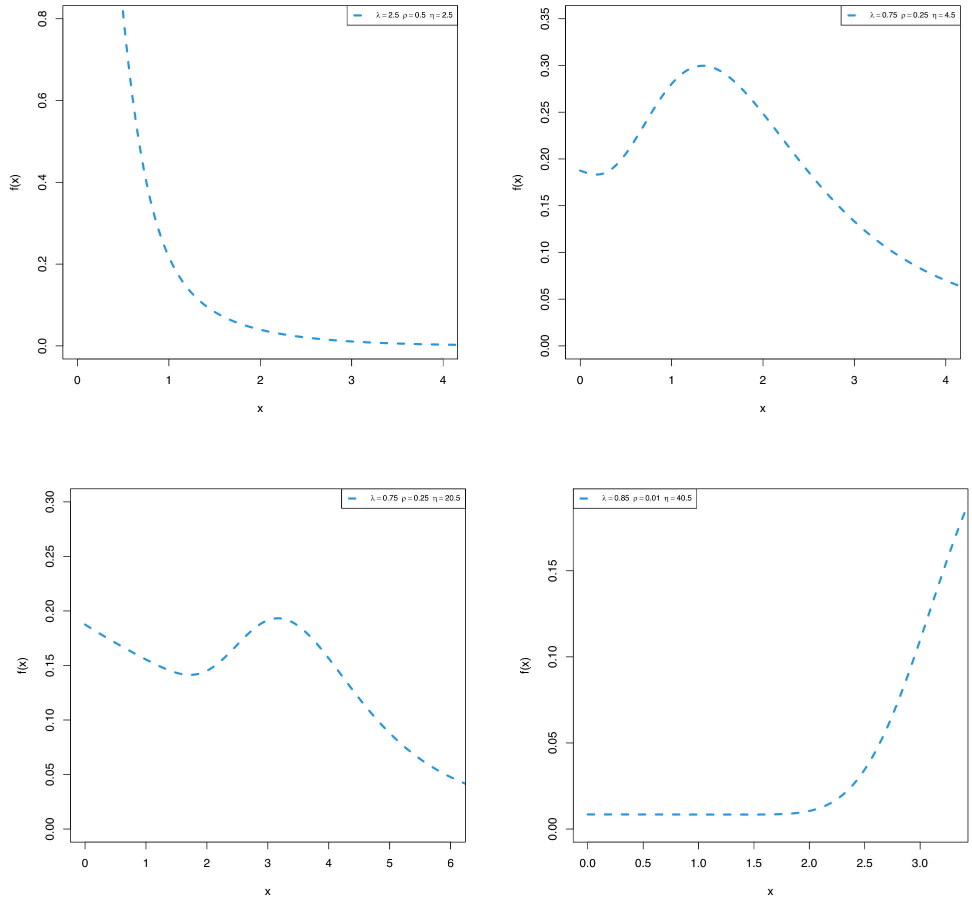


**Figure 1:** The plots of the MGEx PDF for different values of the parameters $\rho$, $\lambda$, and $\eta$

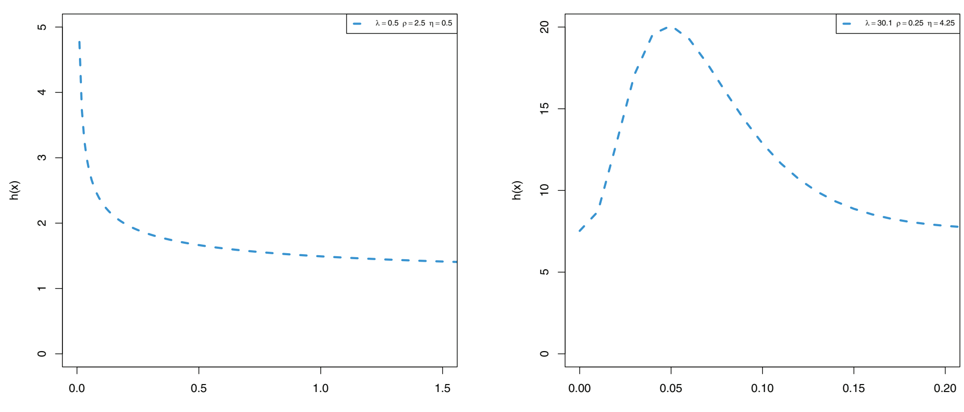


**Figure 2:** (Continued)

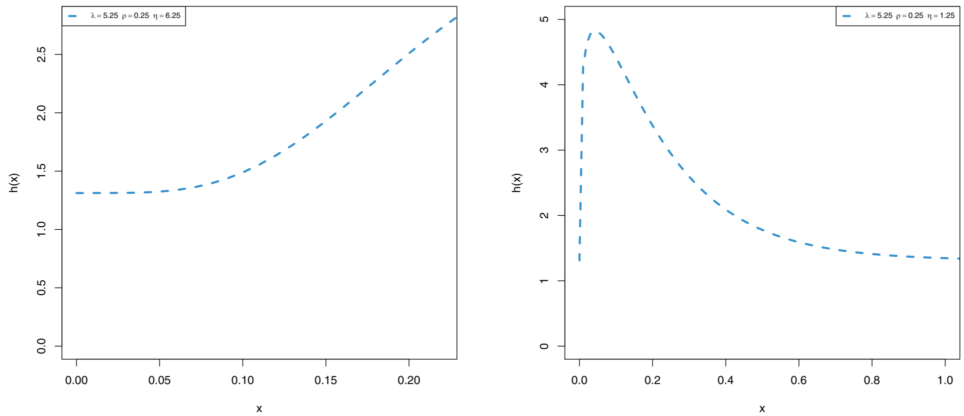


**Figure 2:** The plots of the MGEx HRF for different values of the parameters $\rho$, $\lambda$, and $\eta$

### *2.1 Moments*

The following theorem presents a simplified expansion of the $r$ th raw moment of $X$, which is based on the values of the standard GF.

**Theorem 1.** *For any integer $r \geq 0$, the rth raw moment of $X$ has the form*

$$\mu_r' = E\left(X^r\right) = \frac{1}{\lambda^r}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{i!\,j!}\left[\nu_k\Gamma(r+j+1)+\rho_t\,\Gamma(r+j+1)\right], \tag{8}$$

where

$$\nu_k = (\rho\lambda)^{j+1}\sum_{k=0}^{\infty}\frac{\binom{i\eta}{k}(-1)^k}{k^{r+j+1}}, \quad \text{and} \quad \rho_t = \eta\rho^j\sum_{t=0}^{\infty}\frac{\binom{(i+1)\eta-1}{t}(-1)^k}{(1+t)^{r+j+1}}.$$

**Proof:** By Maclaurin rule, the expression $e^x$ can be rewritten as

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}. \tag{9}$$

By the generalized binomial series, for $|z| < 1$ and $m$ is a positive real non-integer, the expression $(1-z)^m$ can be expressed as

$$(1-z)^m = \sum_{i=0}^{\infty}\binom{m}{i}(-1)^i z^i. \tag{10}$$

Using the expansion in (9), we obtain

$$e^{-\left(1-e^{-\lambda x}\right)^{\eta}} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(1-e^{-\lambda x}\right)^{i\eta}, \tag{11}$$

and

$$e^{-\rho\lambda x} = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(\rho\lambda x)^j, \tag{12}$$

for any $x \in \mathbb{R}$. Substituting in (6), we can write

$$f(x;\rho,\eta,\lambda)=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{i!\,j!}\left\{\rho\lambda+\eta\lambda\left(1-e^{-\lambda x}\right)^{\eta-1}e^{-\lambda x}\right\}(\rho\lambda x)^{j}\left(1-e^{-\lambda x}\right)^{i\eta}. \tag{13}$$

The $r$ th moment of $X$ reduces to

$$\begin{aligned}\mu_r' &= \int_0^{\infty} x^r f(x;\rho,\eta,\lambda)\,dx\\ &=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{i!\,j!}\left\{(\rho\lambda)^{j+1}\int_0^{\infty}x^{r+j}A^{i\eta}dx+\eta\rho^{j}\lambda^{j+1}\int_0^{\infty}x^{r+j}e^{-\lambda x}A^{(i+1)\eta-1}dx\right\},\end{aligned} \tag{14}$$

where $A=(1-e^{-\lambda x})$. Since $\exp\{-\lambda x\}<1$, using (10), the following quantities can be written as

$$\left(1-e^{-\lambda x}\right)^{(i+1)\eta-1}=\sum_{t=0}^{\infty}\binom{(i+1)\eta-1}{t}(-1)^{t}\exp\{-\lambda t x\}, \tag{15}$$

and

$$\left(1-e^{-\lambda x}\right)^{i\eta}=\sum_{k=0}^{\infty}\binom{i\eta}{k}(-1)^{k}\exp\{-\lambda k x\}. \tag{16}$$

Hence, we have

$$\mu_r'=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{i!\,j!}\left\{H_K\int_0^{\infty}x^{r+j}\exp\{-\lambda k x\}dx+M_T\int_0^{\infty}x^{r+j}\exp\{-\lambda(1+t)x\}dx\right\}. \tag{17}$$

where $H_K=(\rho\lambda)^{j+1}\sum_{k=0}^{\infty}\binom{i\eta}{k}(-1)^k$ and $M_T=\eta\rho^j\lambda^{j+1}\sum_{t=0}^{\infty}\binom{(i+1)\eta-1}{t}(-1)^t$. Using direct integration, the $r$ th moment of $X$ reduces to

$$\mu_r'=E(X^r)=\frac{1}{\lambda^r}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{i!\,j!}\left[\nu_k\Gamma(r+j+1)+\varrho_t\Gamma(r+j+1)\right]. \tag{18}$$

This completes the proof. □

The moment generating function (MGF) of $X$ is defined by

$$M(t)=E(e^{tX})=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r', \tag{19}$$

where $t\in\mathbb{R}$. Making use of Theorem 1, the MGF of $X$ takes the form

$$M(t)=\sum_{r,i,j=0}^{\infty}\frac{(-1)^{i+j}t^r}{i!\,j!\,r!\,\lambda^r}\left[\nu_k\Gamma(r+j+1)+\varrho_t\Gamma(r+j+1)\right]. \tag{20}$$

Table 1 displays the mean ($\mu_X$), variance ($\sigma_X^2$), skewness ($\zeta_1(X)$), and kurtosis ($\zeta_2(X)$) of the MGEx distribution, calculated numerically for various parameter values using R software. The results confirm that the skewness values $\zeta_1(X)$ are positive for all examined parameter combinations, indicating that the MGEx distribution consistently exhibits right-skewed behavior. This property demonstrates the model's capability to effectively describe positively skewed data, making it a useful and flexible option for modeling asymmetric phenomena.

**Table 1:** Numerical measures of the MGEx distribution for $\lambda = 1$ various values of the parameters $\rho$ and $\eta$

| $\boldsymbol{\varphi}^{\intercal} = (\rho,\ \eta,\ \lambda)^{\intercal}$ | $\mu_X$ | $\sigma_X^2$ | $\zeta_1(X)$ | $\zeta_2(X)$ |
|---|---|---|---|---|
| $(\rho = 1.75,\ \eta = 0.25,\ \lambda = 1.0)$ | 0.2796935 | 0.1993039 | 2.8960810 | 15.1019500 |
| $(\rho = 3.25,\ \eta = 2.5,\ \lambda = 1.0)$ | 0.2909586 | 0.0785708 | 1.8968110 | 8.5250280 |
| $(\rho = 5.0,\ \eta = 1.5,\ \lambda = 1.0)$ | 0.1845679 | 0.0329115 | 1.9889340 | 9.0670430 |
| $(\rho = 1.5,\ \eta = 0.75,\ \lambda = 1.0)$ | 0.4262857 | 0.2673024 | 2.6518190 | 13.8769500 |
| $(\rho = 2.75,\ \eta = 10.25,\ \lambda = 1.0)$ | 0.3620839 | 0.1279659 | 1.8879800 | 8.0758810 |
| $(\rho = 2.75,\ \eta = 0.25,\ \lambda = 1.0)$ | 0.1889893 | 0.0834902 | 2.7787760 | 14.2049700 |
| $(\rho = 4.25,\ \eta = 1.25,\ \lambda = 1.0)$ | 0.2066802 | 0.0425755 | 2.068242 | 9.6394110 |
| $(\rho = 3.75,\ \eta = 1.25,\ \lambda = 1.0)$ | 0.2304921 | 0.0532602 | 2.0928000 | 9.8393790 |
| $(\rho = 3.95,\ \eta = 2.95,\ \lambda = 1.0)$ | 0.2460576 | 0.0572062 | 1.8801670 | 8.2747050 |

### *2.2 Mean Deviations*

The following theorem explores an expansion of the mean deviation (MD) of $X$ around the mean, denoted as $MD\,(\mu)$, as well as the mean deviation around the median, denoted as $MD\,(M)$.

**Theorem 2.** *The mean deviation (MD) of X relative to the mean and the median can be formulated as follows:*

$$MD\,(\mu) = E(|X-\mu|) = 2\mu F\,(\mu;\rho,\eta,\lambda)$$
$$-2\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{i!\,j!}\left[\aleph_k\gamma\,(j+1,\mu\lambda k)+\wp_t\gamma\,(j+1,\mu\lambda(1+t))\right], \tag{21}$$

*and*

$$MD\,(M) = E(|X-M|) = 2M\,F\,(M;\rho,\eta,\lambda)+\mu-M$$
$$-2\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{i!\,j!}\left[\aleph_k\gamma_M(j+1,M\lambda k)+\wp_t\gamma_M(j+1,M\lambda(1+t))\right], \tag{22}$$

*where* $\gamma_x\,(a,b)=\int_0^b t^{a-1}e^{-t}dt$ *is the incomplete GF and*

$$\aleph_k=\rho^{j+1}\sum_{k=0}^{\infty}\frac{\binom{i\eta}{k}(-1)^k}{k^{j+1}},\quad \text{and}\quad \wp_t=\eta\rho^j\sum_{t=0}^{\infty}\frac{\binom{(i+1)\eta-1}{t}(-1)^k}{(1+t)^{j+1}}. \tag{23}$$

**Proof:** By the definition of the MD about the mean is defined by

$$MD\,(\mu)=\int_0^{\infty}|x-\mu|f\,(x;\rho,\eta,\lambda)\,dx \tag{24}$$

$$=\int_0^{\mu}(\mu-x)f\,(x;\rho,\eta,\lambda)\,dx+\int_{\mu}^{\infty}(x-\mu)f\,(x;\rho,\eta,\lambda)\,dx. \tag{25}$$

After simplification, we get

$$MD(\mu) = 2\mu F(\mu;\rho,\eta,\lambda) - 2\int_0^{\mu} x f(x;\rho,\eta,\lambda)\,dx. \tag{26}$$

The result in Eq. (21) is derived by substituting Eq. (18) into Eq. (26). Additionally, based on the definition of the MD around the median, we can express it as follows

$$\begin{aligned} MD(M) &= \int_0^{\infty} |x-M| f(x;\rho,\eta,\lambda)\,dx \\ &= \int_0^{M} (M-x) f(x;\rho,\eta,\lambda)\,dx + \int_M^{\infty} (x-M) f(x;\rho,\eta,\lambda)\,dx \\ &= 2M\,F(M;\rho,\eta,\lambda) + \mu - M - 2\int_0^{M} x f(x;\rho,\eta,\lambda)\,dx. \end{aligned} \tag{27}$$

By substituting Eq. (18) into Eq. (27), we derive the result presented in Eq. (22). □

### *2.3 Quantile Function*

The quantile function (QF) of the MGEx distribution, as defined in (5), denoted as $Q(u)$ for $0 < u < 1$, can be determined by solving the equation $F(Q(u)) = 0$ in (5) for $Q(u)$ in terms of $u$. This implies that

$$F^{-1}(u) = \log(1-u) + \lambda\rho x + (1-\exp\{-\lambda x\})^{\eta} = 0. \tag{28}$$

The median of the MGEx distribution is determined by setting $u = 0.5$ in Eq. (28). Similarly, other quantiles can be calculated by substituting the corresponding values of $u$ into Eq. (28).

### *2.4 Order Statistics*

Let $X_1, X_2, \ldots, X_n$ be a random sample from (6), with the order statistics $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$. It is well-established that the PDF and CDF of the $i$ th order statistic, denoted as $X_{(i)}$, are given (for $k = 1, 2, \ldots, n$) by:

$$\begin{aligned} f_{X_{(i)}}(x) &= \frac{n!}{(i-1)!\,(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i} f(x) \\ &= \frac{n!}{(i-1)!\,(n-i)!}\sum_{u=0}^{n-i}(-1)^u \binom{n-i}{u}[F(x)]^{i-1+u} f(x), \end{aligned} \tag{29}$$

and

$$F_{X_{(i)}}(x) = \sum_{l=k}^{n}\binom{n}{l}[F(x)]^{l}[1-F(x)]^{n-l} = \sum_{l=k}^{n}\sum_{u=0}^{n-i}(-1)^u\binom{n}{l}\binom{n-i}{u}[F(x)]^{l+u}. \tag{30}$$

From Eqs. (29) and (30), it follows that the PDF and CDF of $X_{(i)}$ for the MGEx distribution simplify to:

$$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!\,(n-i)!} \sum_{u=0}^{n-i} (-1)^u \binom{n-i}{u} \left[1 - e^{-\rho\lambda x - \left(1-e^{-\lambda x}\right)^{\eta}}\right]^{i+u-1}$$
$$\times \left(\rho\lambda + \eta\lambda \left(1 - e^{-\lambda x}\right)^{\eta-1} e^{-\lambda x}\right) e^{-\rho\lambda x - \left(1-e^{-\lambda x}\right)^{\eta}}, \tag{31}$$

and

$$F_{X_{(i)}}(x) = \sum_{l=k}^{n} \sum_{u=0}^{n-i} (-1)^u \binom{n}{l} \binom{n-i}{u} \left[1 - e^{-\rho\lambda x - \left(1-e^{-\lambda x}\right)^{\eta}}\right]^{l+u}.$$

### 2.5 *Rényi Entropy*

The Rényi entropy of a random variable $X$ quantifies the variation of uncertainty associated with $X$. It is defined as follows

$$I_{\zeta}(X) = \frac{1}{1-\zeta} \log\left(\left(J_{\zeta}(X)\right), \zeta > 0 \text{ and } \zeta \neq 1, \tag{32}$$

where

$$J_{\zeta}(X) = \int_{-\infty}^{\infty} f^{\zeta}(x)dx. \tag{33}$$

Using (6), we can write

$$f^{\zeta}(x;\underline{\boldsymbol{\varphi}}) = \left(\rho\lambda + \eta\lambda \left(1 - e^{-\lambda x}\right)^{\eta-1} e^{-\lambda x}\right)^{\zeta} e^{-\zeta\rho\lambda x - \zeta\left(1-e^{-\lambda x}\right)^{\eta}}. \tag{34}$$

After simplification, the Rényi entropy of $X \sim \text{MGEx}(\underline{\boldsymbol{\varphi}})$ has the form

$$I_{\zeta}(X) = \frac{1}{1-\zeta} \log\left(\sum_{h=0}^{\zeta} \sum_{y,i=0}^{\infty} \zeta^{**}_{h,i,y}(\underline{\boldsymbol{\varphi}})\right), \tag{35}$$

where

$$\zeta^{**}_{h,i,y}(\underline{\boldsymbol{\varphi}}) = \binom{(\zeta-h)(\eta-1)+i\eta}{y} \binom{\zeta}{h} \frac{(-1)^{i+y+\zeta-h}}{\lambda\left(\zeta+\zeta\rho+y-h\right)} \eta^{\zeta-h} \zeta^{i} \lambda^{\zeta} \rho^{h}. \tag{36}$$

The $\eta$-entropy, say $H_{\zeta}(X)$, is defined by

$$H_{\zeta}(X) = \frac{1}{\zeta-1} \log\left[1 - J_{\zeta}(X)\right], \eta > 0$$
$$= \frac{1}{\zeta-1} \log\left[1 - \sum_{h=0}^{\zeta} \sum_{y,i=0}^{\infty} \zeta^{**}_{h,i,y}(\underline{\boldsymbol{\varphi}})\right]. \tag{37}$$

## 3 Estimation in the MGEx Model

The parameters of the MGEx distribution are estimated using six classical methods, including weighted least-squares estimators (WLSEs), maximum likelihood estimators (MLEs), Cramér–von

Mises estimators (CRVMEs), maximum product of spacing estimators (MPSEs), Anderson–Darling estimators (ADEs), and right-tail ADEs (RADEs). The methods are described below.

The MLEs are obtained by maximizing the log-likelihood function given below for the parameters, as derived from (38), using classical iterative methods.

$$\ell(\boldsymbol{\varphi}) = \sum_{i=1}^{n} \log\left\{\rho\lambda + \eta\lambda\delta_i^{\eta-1}e^{-\lambda x_i}\right\} + \sum_{i=1}^{n}\left\{-\rho\lambda x_i - \delta_i^{\eta}\right\}, \tag{38}$$

where $\delta_i = (1 - e^{-\lambda x_i})$.

The likelihood equations are given by

$$\frac{\partial}{\partial\rho}\ell(\boldsymbol{\varphi}) = \sum_{i=1}^{n} \frac{\lambda}{\left(\rho\lambda + \eta\lambda\delta_i^{\eta-1}e^{-\lambda x_i}\right)} - \sum_{i=1}^{n}\lambda x_i = 0, \tag{39}$$

$$\frac{\partial}{\partial\eta}\ell(\boldsymbol{\varphi}) = \sum_{i=1}^{n} \frac{\lambda e^{-\lambda x_i}\delta_i^{\eta-1}\left[\eta\ln\delta_i^{\eta-1}+1\right]}{\left(\rho\lambda + \eta\lambda\delta_i^{\eta-1}e^{-\lambda x_i}\right)} - \sum_{i=1}^{n}\delta_i^{\eta}\ln\delta_i^{\eta} = 0, \tag{40}$$

and

$$\frac{\partial}{\partial\lambda}\ell(\boldsymbol{\varphi}) = \sum_{i=1}^{n} \frac{\rho + \eta\delta_i^{\eta-1}e^{-\lambda x_i}\left\{1+\lambda(\eta-1)x_ie^{-\lambda x_i}\delta_i^{-1} - \lambda x_i\right\}}{\left(\rho\lambda + \eta\lambda\delta_i^{\eta-1}e^{-\lambda x_i}\right)} - \sum_{i=1}^{n}\left[\rho x_i + \eta x_i e^{-\lambda x_i}\delta_i^{\eta-1}\right] = 0. \tag{41}$$

The weighted least-squares estimates (WLSEs) minimize

$$W(\boldsymbol{\varphi}) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{(i)}) - \frac{i}{n+1}\right]^2, \tag{42}$$

which follow by solving

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{(i)}) - \frac{i}{n+1}\right]\tau_s(x_{(i)}) = 0, \quad s = 1,2,3. \tag{43}$$

where

$$\tau_1(x_{(i)}) = \frac{\partial F(x_{(i)})}{\partial\rho} = \lambda x_{(i)}e^{-\rho\lambda x_{(i)} - \delta_{(i)}^{\eta}}, \tag{44}$$

$$\tau_2(x_{(i)}) = \frac{\partial F(x_{(i)})}{\partial\eta} = \delta_{(i)}^{\eta}\ln\delta_{(i)}e^{-\rho\lambda x_{(i)} - \delta_{(i)}^{\eta}}, \tag{45}$$

and

$$\tau_3(x_{(i)}) = \frac{\partial F(x_{(i)})}{\partial\lambda} = \left[\rho x_{(i)} + \eta x_{(i)}e^{-\lambda x_{(i)}}\delta_{(i)}^{\eta-1}\right]e^{-\rho\lambda x_{(i)} - \delta_{(i)}^{\eta}}, \tag{46}$$

where $\delta_{(i)} = 1 - e^{-\lambda x_{(i)}}$.

The MPSEs provide strong alternatives to the MLEs. Let $D_i = D_i(\boldsymbol{\varphi}) = F(x_{(i)}) - F(x_{(i-1)})$ represent the uniform spacing for $i = 1, \ldots, n+1$, where $F(x_{(0)}) = 0$, $F(x_{(n+1)}) = 1$ and $\sum_{i=1}^{n+1} D_i = 1$.

The MPSEs are obtained by maximizing the following quantity:

$$H(\boldsymbol{\varphi}) = \sum_{i=1}^{n+1}\log\left[D_i\right], \tag{47}$$

which can be determined by solving the following non-linear equations

$$\sum_{i=1}^{n+1} \frac{1}{D_i} \left[\tau_s(x_{(i)}) - \tau_s(x_{(i-1)})\right] = 0, \quad s = 1, 2, 3. \tag{48}$$

The CRVMEs minimize the quantity

$$C(\boldsymbol{\varphi}) = -\frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{(i)}) - \frac{2i-1}{2n}\right]^2, \tag{49}$$

which also follow by solving

$$\sum_{i=1}^{n} \left[F(x_{(i)}) - \frac{2i-1}{2n}\right] \tau_s(x_{(i)}) = 0, \quad s = 1, 2, 3. \tag{50}$$

The ADEs minimize

$$A(\boldsymbol{\varphi}) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left[\log F(x_{(i)}) + \log S(x_{(i)})\right], \tag{51}$$

which can be found as solutions of the system (for $s = 1, 2, 3$)

$$\sum_{i=1}^{n} (2i-1) \left[\frac{\tau_s(x_{(i)})}{F(x_{(i)})} - \frac{\tau_i(x_{(n+1-i)})}{S(x_{(n+1-i)})}\right] = 0. \tag{52}$$

The RADEs minimize

$$R(\boldsymbol{\varphi}) = \frac{n}{2} - 2\sum_{i=1}^{n} F(x_{(i)}) - \frac{1}{n}\sum_{i=1}^{n} (2i-1) \log S(x_{(n+1-i)}). \tag{53}$$

They can also be obtained from the non-linear equations

$$-2\sum_{i=1}^{n} \tau_s(x_{(i)}) + \frac{1}{n}\sum_{i=1}^{n} (2i-1) \frac{\tau_s(x_{(n+1-i)})}{S(x_{(n+1-i)})} = 0, \quad s = 1, 2, 3. \tag{54}$$

## 4 Simulation Study

The simulation study compares the estimates obtained from the six methods outlined in Section 4 by analyzing the averages of three key metrics are the absolute bias (|Bias|), mean square error (MSE), and mean relative error (MRE), which are defined as:

$$|\text{Bias}(\widehat{\boldsymbol{\varphi}})| = \frac{1}{N}\sum_{i=1}^{N} |\widehat{\boldsymbol{\varphi}} - \boldsymbol{\varphi}|, \tag{55}$$

$$\text{MSE} = \frac{1}{N}\sum_{i=1}^{N} (\widehat{\boldsymbol{\varphi}} - \boldsymbol{\varphi})^2, \tag{56}$$

and

$$\text{MRE} = \frac{1}{N}\sum_{i=1}^{N} |\widehat{\boldsymbol{\varphi}} - \boldsymbol{\varphi}|/\boldsymbol{\varphi}. \tag{57}$$

Simulated observations from the MGEx model are generated using Eq. (28), where $U$ represents a uniform random variable in the interval $(0, 1)$. A total of $N = 5000$ random samples $x_1, \ldots, x_n$ are created for $n = 20, 50, 150, 300, 500$ using the MGEx model with parameter sets $\lambda = \{0.15, 0.25, 0.5, 0.75, 1.25\}$, $\rho = \{0.15, 0.5, 0.75, 1.25\}$, and $\eta = \{0.15, 0.25, 0.5\}$. Simulations are carried out in R (R Core Team, 2020, version 4.0.3) using the `nlminb` function from the `stats` package.

We estimate the model parameters for various combinations of parameter values and sample sizes, calculating the MSEs and MREs for the resulting estimates. Four out of six simulation outcomes are summarized in Tables 2–9, where each row presents the ranks of the estimates across all methods, with superscripts indicating their rank positions, and $\sum$ *Ranks* showing the partial sum of these ranks. Finally, Table 10 provides both partial and overall rankings, highlighting that the MPS method consistently outperforms the other estimation techniques for the MGEx distribution, achieving an overall score of 77.

**Table 2:** Numerical results for the parameter set $\boldsymbol{\varphi} = (\lambda = 0.15, \rho = 0.15, \eta = 0.25)^\intercal$

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| 20 | \|BIAS\| | $\hat{\lambda}$ | $0.3785^{\{2\}}$ | $0.3911^{\{3\}}$ | $0.5000^{\{5\}}$ | $0.3394^{\{1\}}$ | $0.4499^{\{4\}}$ | $0.5075^{\{6\}}$ |
| | | $\hat{\rho}$ | $0.7151^{\{1\}}$ | $0.8202^{\{2\}}$ | $1.5549^{\{5\}}$ | $1.6433^{\{6\}}$ | $1.1789^{\{4\}}$ | $1.1205^{\{3\}}$ |
| | | $\hat{\eta}$ | $0.1939^{\{6\}}$ | $0.0712^{\{2\}}$ | $0.0908^{\{4\}}$ | $0.0661^{\{1\}}$ | $0.0724^{\{3\}}$ | $0.1079^{\{5\}}$ |
| | MSE | $\hat{\lambda}$ | $0.2679^{\{1\}}$ | $2.8995^{\{3\}}$ | $3.4510^{\{6\}}$ | $2.9109^{\{5\}}$ | $2.7617^{\{2\}}$ | $2.9065^{\{4\}}$ |
| | | $\hat{\rho}$ | $9.8140^{\{3\}}$ | $7.6294^{\{1\}}$ | $25.6388^{\{5\}}$ | $85.8021^{\{6\}}$ | $19.7209^{\{4\}}$ | $9.2654^{\{2\}}$ |
| | | $\hat{\eta}$ | $0.0809^{\{5\}}$ | $0.0115^{\{3\}}$ | $0.0388^{\{4\}}$ | $0.0085^{\{1\}}$ | $0.0109^{\{2\}}$ | $0.1466^{\{6\}}$ |
| | MRE | $\hat{\lambda}$ | $2.5232^{\{2\}}$ | $2.6073^{\{3\}}$ | $3.3334^{\{5\}}$ | $2.2629^{\{1\}}$ | $2.9997^{\{4\}}$ | $3.3832^{\{6\}}$ |
| | | $\hat{\rho}$ | $4.7674^{\{1\}}$ | $5.4681^{\{2\}}$ | $10.3662^{\{5\}}$ | $10.9553^{\{6\}}$ | $7.8598^{\{4\}}$ | $7.4701^{\{3\}}$ |
| | | $\hat{\eta}$ | $0.7755^{\{6\}}$ | $0.2848^{\{2\}}$ | $0.3631^{\{4\}}$ | $0.2645^{\{1\}}$ | $0.2897^{\{3\}}$ | $0.4315^{\{5\}}$ |
| | | $\sum$ *Ranks* | $27^{\{2\}}$ | $21^{\{1\}}$ | $43^{\{6\}}$ | $28^{\{3\}}$ | $30^{\{4\}}$ | $40^{\{5\}}$ |
| 50 | | $\hat{\lambda}$ | $0.3505^{\{6\}}$ | $0.1543^{\{2\}}$ | $0.1731^{\{4\}}$ | $0.1325^{\{1\}}$ | $0.1680^{\{3\}}$ | $0.1839^{\{5\}}$ |
| | \|BIAS\| | $\hat{\rho}$ | $0.3163^{\{5\}}$ | $0.2030^{\{1\}}$ | $0.2845^{\{4\}}$ | $0.2268^{\{2\}}$ | $0.2402^{\{3\}}$ | $0.3331^{\{6\}}$ |
| | | $\hat{\eta}$ | $0.1913^{\{6\}}$ | $0.0420^{\{3\}}$ | $0.0479^{\{4\}}$ | $0.0389^{\{1\}}$ | $0.0419^{\{2\}}$ | $0.0527^{\{5\}}$ |
| | | $\hat{\lambda}$ | $0.2392^{\{6\}}$ | $0.0741^{\{2\}}$ | $0.1107^{\{4\}}$ | $0.0666^{\{1\}}$ | $0.1101^{\{3\}}$ | $0.1132^{\{5\}}$ |
| | MSE | $\hat{\rho}$ | $0.3127^{\{2\}}$ | $0.2570^{\{1\}}$ | $0.8161^{\{5\}}$ | $0.4251^{\{3\}}$ | $0.4866^{\{4\}}$ | $0.9020^{\{6\}}$ |
| | | $\hat{\eta}$ | $0.0852^{\{6\}}$ | $0.0031^{\{2\}}$ | $0.0042^{\{4\}}$ | $0.0025^{\{1\}}$ | $0.0031^{\{3\}}$ | $0.0051^{\{5\}}$ |
| | | $\hat{\lambda}$ | $2.3369^{\{6\}}$ | $1.0284^{\{2\}}$ | $1.1539^{\{4\}}$ | $0.8834^{\{1\}}$ | $1.1202^{\{3\}}$ | $1.2264^{\{5\}}$ |
| | MRE | $\hat{\rho}$ | $2.1089^{\{5\}}$ | $1.3534^{\{1\}}$ | $1.8968^{\{4\}}$ | $1.5121^{\{2\}}$ | $1.6013^{\{3\}}$ | $2.2208^{\{6\}}$ |
| | | $\hat{\eta}$ | $0.7651^{\{6\}}$ | $0.1681^{\{3\}}$ | $0.1915^{\{4\}}$ | $0.1557^{\{1\}}$ | $0.1676^{\{2\}}$ | $0.2109^{\{5\}}$ |
| | | $\sum$ *Ranks* | $48^{\{5.5\}}$ | $17^{\{2\}}$ | $37^{\{4\}}$ | $13^{\{1\}}$ | $26^{\{3\}}$ | $48^{\{5.5\}}$ |
| 150 | | $\hat{\lambda}$ | $0.2164^{\{6\}}$ | $0.0784^{\{3\}}$ | $0.0815^{\{4\}}$ | $0.0632^{\{1\}}$ | $0.0776^{\{2\}}$ | $0.0881^{\{5\}}$ |
| | \|BIAS\| | $\hat{\rho}$ | $0.2235^{\{6\}}$ | $0.0765^{\{2\}}$ | $0.0830^{\{4\}}$ | $0.0724^{\{1\}}$ | $0.0785^{\{3\}}$ | $0.0994^{\{5\}}$ |
| | | $\hat{\eta}$ | $0.1319^{\{6\}}$ | $0.0230^{\{3\}}$ | $0.0259^{\{4\}}$ | $0.0209^{\{1\}}$ | $0.0225^{\{2\}}$ | $0.0285^{\{5\}}$ |
| | | $\hat{\lambda}$ | $0.1475^{\{6\}}$ | $0.0124^{\{2\}}$ | $0.0138^{\{4\}}$ | $0.0089^{\{1\}}$ | $0.0130^{\{3\}}$ | $0.0158^{\{5\}}$ |
| | MSE | $\hat{\rho}$ | $0.1451^{\{6\}}$ | $0.0145^{\{1\}}$ | $0.0204^{\{4\}}$ | $0.0145^{\{2\}}$ | $0.0169^{\{3\}}$ | $0.0362^{\{5\}}$ |
| | | $\hat{\eta}$ | $0.0508^{\{6\}}$ | $0.0009^{\{3\}}$ | $0.0011^{\{4\}}$ | $0.0007^{\{1\}}$ | $0.0009^{\{2\}}$ | $0.0014^{\{5\}}$ |

(Continued)

**Table 2 (continued)**

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| | MRE | $\hat{\lambda}$ | 1.4423{6} | 0.5225{3} | 0.5432{4} | 0.4216{1} | 0.5175{2} | 0.5874{5} |
| | | $\hat{\rho}$ | 1.4902{6} | 0.5101{2} | 0.5536{4} | 0.4824{1} | 0.5234{3} | 0.6626{5} |
| | | $\hat{\eta}$ | 0.5277{6} | 0.0921{3} | 0.1036{4} | 0.0835{1} | 0.0898{2} | 0.11406 {5} |
| | | $\sum$ *Ranks* | 54 {6} | 22{2.5} | 36{4} | 10{1} | 22{2.5} | 45{5} |
| 300 | \|BIAS\| | $\hat{\lambda}$ | 0.1433{6} | 0.0544{3} | 0.0558{4} | 0.0419{1} | 0.0518{2} | 0.0599{5} |
| | | $\hat{\rho}$ | 0.2043{6} | 0.0524{3} | 0.0545{4} | 0.0449{1} | 0.0509{2} | 0.0622{5} |
| | | $\hat{\eta}$ | 0.1111{6} | 0.0165{3} | 0.0181{4} | 0.0143{1} | 0.0155{2} | 0.0197{5} |
| | MSE | $\hat{\lambda}$ | 0.0917{6} | 0.0052{2.5} | 0.0056{4} | 0.0035{1} | 0.0052{2.5} | 0.0064{5} |
| | | $\hat{\rho}$ | 0.1457{6} | 0.0052{2} | 0.0061{4} | 0.0045{1} | 0.0057{3} | 0.0091{5} |
| | | $\hat{\eta}$ | 0.0409{6} | 0.0004{3} | 0.0005{4} | 0.0004{1} | 0.0004{2} | 0.0006{5} |
| | MRE | $\hat{\lambda}$ | 0.9550{6} | 0.3628{3} | 0.3719{4} | 0.2796{1} | 0.3451{2} | 0.3990{5} |
| | | $\hat{\rho}$ | 1.3622{6} | 0.3490{3} | 0.3636{4} | 0.2999{1} | 0.3399{2} | 0.4144{5} |
| | | $\hat{\eta}$ | 0.4443{6} | 0.0660{3} | 0.0723{4} | 0.0572{1} | 0.0619{2} | 0.0787{5} |
| | | $\sum$ *Ranks* | 54{6} | 25.5{3} | 36{4} | 9{1} | 19.5{2} | 45{5} |
| 500 | \|BIAS\| | $\hat{\lambda}$ | 0.1055{6} | 0.0410{3} | 0.0417{4} | 0.0275{1} | 0.0361{2} | 0.0455{5} |
| | | $\hat{\rho}$ | 0.1783{6} | 0.0379{3} | 0.0383{4} | 0.0273 {1} | 0.0338{2} | 0.0436{5} |
| | | $\hat{\eta}$ | 0.0904{6} | 0.0128{3} | 0.0140{4} | 0.0103{1} | 0.0112{2} | 0.0153{5} |
| | MSE | $\hat{\lambda}$ | 0.0659{6} | 0.0028{3} | 0.0029{4} | 0.0016{1} | 0.0026{2} | 0.0035{5} |
| | | $\hat{\rho}$ | 0.1322{6} | 0.0026{3} | 0.0027{4} | 0.0017{1} | 0.0024{2} | 0.0038{5} |
| | | $\hat{\eta}$ | 0.0332{6} | 0.0003{3} | 0.0003{4} | 0.0002{1} | 0.0002{2} | 0.0004{5} |
| | MRE | $\hat{\lambda}$ | 0.7035{6} | 0.2735{3} | 0.2783{4} | 0.1836{1} | 0.2409{2} | 0.3032{5} |
| | | $\hat{\rho}$ | 1.1889{6} | 0.2528{3} | 0.2549{4} | 0.1819{1} | 0.2253{2} | 0.2907{5} |
| | | $\hat{\eta}$ | 0.3614{6} | 0.0509{3} | 0.0561{4} | 0.0410{1} | 0.0447{2} | 0.0612{5} |
| | | $\sum$ *Ranks* | 54 {6} | 27{3} | 36{4} | 9{1} | 18{2} | 45{5} |

**Table 3:** Numerical results for the parameter set $\boldsymbol{\varphi} = (\lambda = 0.15, \rho = 0.15, \eta = 0.15)^{\intercal}$

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| 20 | \|BIAS\| | $\hat{\lambda}$ | 0.3744{1} | 1.6989{3} | 3.4398{6} | 0.7027{2} | 2.3084{5} | 1.8805{4} |
| | | $\hat{\rho}$ | 0.4241{1} | 1.0042{2} | 2.2428{6} | 1.2402{3} | 1.4858{5} | 1.4710{4} |
| | | $\hat{\eta}$ | 0.1877{6} | 0.0389{3} | 0.0493{4} | 0.0312{1} | 0.0357{2} | 0.0783{5} |
| | MSE | $\hat{\lambda}$ | 0.2657{1} | 177.1676{3} | 608.9593{6} | 17.6347{2} | 227.1962{5} | 218.7924{4} |
| | | $\hat{\rho}$ | 5.9745{1} | 10.8559{2} | 48.7141{6} | 35.0289{4} | 36.9578{5} | 14.4259{3} |
| | | $\hat{\eta}$ | 0.0967{5} | 0.0036{3} | 0.0091{4} | 0.0023{1} | 0.0032{2} | 2.5644{6} |
| | MRE | $\hat{\lambda}$ | 2.4959{1} | 11.3257{3} | 22.9322{6} | 4.6849{2} | 15.3892{5} | 12.5366{4} |
| | | $\hat{\rho}$ | 2.8271{1} | 6.6946{2} | 14.9517{6} | 8.2680{3} | 9.9051{5} | 9.8067{4} |
| | | $\hat{\eta}$ | 1.2510{6} | 0.2599{3} | 0.3285{4} | 0.2080{1} | 0.2381{2} | 0.5219{5} |

(Continued)

**Table 3 (continued)**

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| | | $\sum$ *Ranks* | 23$^{\{2\}}$ | 24$^{\{3\}}$ | 48$^{\{6\}}$ | 19$^{\{1\}}$ | 36$^{\{4\}}$ | 39$^{\{5\}}$ |
| 50 | | $\hat{\lambda}$ | 0.2937$^{\{2\}}$ | 0.3117$^{\{3\}}$ | 0.4845$^{\{6\}}$ | 0.1583$^{\{1\}}$ | 0.3233$^{\{4\}}$ | 0.3764$^{\{5\}}$ |
| | \|BIAS\| | $\hat{\rho}$ | 0.2048$^{\{2\}}$ | 0.3145$^{\{4\}}$ | 0.5634$^{\{5\}}$ | 0.1799$^{\{1\}}$ | 0.3138$^{\{3\}}$ | 0.5719$^{\{6\}}$ |
| | | $\hat{\eta}$ | 0.2000$^{\{6\}}$ | 0.0217$^{\{3\}}$ | 0.0257$^{\{4\}}$ | 0.0152$^{\{1\}}$ | 0.0176$^{\{2\}}$ | 0.0269$^{\{5\}}$ |
| | | $\hat{\lambda}$ | 0.2069$^{\{1\}}$ | 0.9975$^{\{3\}}$ | 3.0809$^{\{6\}}$ | 0.2589$^{\{2\}}$ | 1.4459$^{\{5\}}$ | 1.2411$^{\{4\}}$ |
| | MSE | $\hat{\rho}$ | 0.1118$^{\{1\}}$ | 0.9204$^{\{3\}}$ | 4.0994$^{\{6\}}$ | 0.5866$^{\{2\}}$ | 1.5546$^{\{4\}}$ | 2.7212$^{\{5\}}$ |
| | | $\hat{\eta}$ | 0.1199$^{\{6\}}$ | 0.0008$^{\{3\}}$ | 0.0012$^{\{4\}}$ | 0.0005$^{\{1\}}$ | 0.0007$^{\{2\}}$ | 0.0013$^{\{5\}}$ |
| | | $\hat{\lambda}$ | 1.9583$^{\{2\}}$ | 2.0778$^{\{3\}}$ | 3.2297$^{\{6\}}$ | 1.0556$^{\{1\}}$ | 2.1553$^{\{4\}}$ | 2.5095$^{\{5\}}$ |
| | MRE | $\hat{\rho}$ | 1.3656$^{\{2\}}$ | 2.0969$^{\{4\}}$ | 3.7557$^{\{5\}}$ | 1.1990$^{\{1\}}$ | 2.0918$^{\{3\}}$ | 3.8127$^{\{6\}}$ |
| | | $\hat{\eta}$ | 1.3334$^{\{6\}}$ | 0.1448$^{\{3\}}$ | 0.1716$^{\{4\}}$ | 0.1011$^{\{1\}}$ | 0.1173$^{\{2\}}$ | 0.1798$^{\{5\}}$ |
| | | $\sum$ *Ranks* | 28$^{\{2\}}$ | 29$^{\{3.5\}}$ | 46$^{\{5.5\}}$ | 11$^{\{1\}}$ | 29$^{\{3.5\}}$ | 46$^{\{5.5\}}$ |
| 150 | | $\hat{\lambda}$ | 0.1123$^{\{3\}}$ | 0.1203$^{\{4\}}$ | 0.1478$^{\{6\}}$ | 0.0235$^{\{1\}}$ | 0.0640$^{\{2\}}$ | 0.1348$^{\{5\}}$ |
| | \|BIAS\| | $\hat{\rho}$ | 0.1044$^{\{3\}}$ | 0.1125$^{\{4\}}$ | 0.1509$^{\{5\}}$ | 0.0202$^{\{1\}}$ | 0.0582$^{\{2\}}$ | 0.1653$^{\{6\}}$ |
| | | $\hat{\eta}$ | 0.0984$^{\{6\}}$ | 0.0123$^{\{3\}}$ | 0.0142$^{\{4\}}$ | 0.0050$^{\{1\}}$ | 0.0057$^{\{2\}}$ | 0.0148$^{\{5\}}$ |
| | | $\hat{\lambda}$ | 0.0767$^{\{6\}}$ | 0.0493$^{\{3\}}$ | 0.0739$^{\{5\}}$ | 0.0066$^{\{1\}}$ | 0.0372$^{\{2\}}$ | 0.0501$^{\{4\}}$ |
| | MSE | $\hat{\rho}$ | 0.0619$^{\{4\}}$ | 0.0432$^{\{3\}}$ | 0.1134$^{\{5\}}$ | 0.0047$^{\{1\}}$ | 0.0361$^{\{2\}}$ | 0.1303$^{\{6\}}$ |
| | | $\hat{\eta}$ | 0.0591$^{\{6\}}$ | 0.0002$^{\{3\}}$ | 0.0003$^{\{4\}}$ | 0.0001$^{\{1\}}$ | 0.0001$^{\{2\}}$ | 0.0004$^{\{5\}}$ |
| | | $\hat{\lambda}$ | 0.7489$^{\{3\}}$ | 0.8022$^{\{4\}}$ | 0.9852$^{\{6\}}$ | 0.1564$^{\{1\}}$ | 0.4266$^{\{2\}}$ | 0.8987$^{\{5\}}$ |
| | MRE | $\hat{\rho}$ | 0.6962$^{\{3\}}$ | 0.7499$^{\{4\}}$ | 1.0065$^{\{5\}}$ | 0.1346$^{\{1\}}$ | 0.3878$^{\{2\}}$ | 1.1018$^{\{6\}}$ |
| | | $\hat{\eta}$ | 0.6559$^{\{6\}}$ | 0.0818$^{\{3\}}$ | 0.0946$^{\{4\}}$ | 0.0335$^{\{1\}}$ | 0.0383$^{\{2\}}$ | 0.0988$^{\{5\}}$ |
| | | $\sum$ *Ranks* | 40$^{\{4\}}$ | 31$^{\{3\}}$ | 44$^{\{5\}}$ | 9$^{\{1\}}$ | 18$^{\{2\}}$ | 47$^{\{6\}}$ |
| 300 | | $\hat{\lambda}$ | 0.0298$^{\{3\}}$ | 0.0732$^{\{4\}}$ | 0.0855$^{\{6\}}$ | 0.0037$^{\{1\}}$ | 0.0164$^{\{2\}}$ | 0.0829$^{\{5\}}$ |
| | \|BIAS\| | $\hat{\rho}$ | 0.0352$^{\{3\}}$ | 0.0683$^{\{4\}}$ | 0.0826$^{\{5\}}$ | 0.0031$^{\{1\}}$ | 0.0135$^{\{2\}}$ | 0.0853$^{\{6\}}$ |
| | | $\hat{\eta}$ | 0.0302$^{\{6\}}$ | 0.0087$^{\{3\}}$ | 0.0099$^{\{4\}}$ | 0.0015$^{\{1\}}$ | 0.0017$^{\{2\}}$ | 0.0102$^{\{5\}}$ |
| | | $\hat{\lambda}$ | 0.0192$^{\{6\}}$ | 0.0124$^{\{3\}}$ | 0.0172$^{\{5\}}$ | 0.0004$^{\{1\}}$ | 0.0040$^{\{2\}}$ | 0.0153$^{\{4\}}$ |
| | MSE | $\hat{\rho}$ | 0.0226$^{\{6\}}$ | 0.0113$^{\{3\}}$ | 0.0193$^{\{4\}}$ | 0.0002$^{\{1\}}$ | 0.0024$^{\{2\}}$ | 0.0213$^{\{5\}}$ |
| | | $\hat{\eta}$ | 0.0173$^{\{6\}}$ | 0.0001$^{\{3\}}$ | 0.0002$^{\{4\}}$ | 0.0002$^{\{1\}}$ | 0.0003$^{\{2\}}$ | 0.0001$^{\{5\}}$ |
| | | $\hat{\lambda}$ | 0.1986$^{\{3\}}$ | 0.4879$^{\{4\}}$ | 0.5702$^{\{6\}}$ | 0.0243$^{\{1\}}$ | 0.1091$^{\{2\}}$ | 0.5531$^{\{5\}}$ |
| | MRE | $\hat{\rho}$ | 0.2347$^{\{3\}}$ | 0.4555$^{\{4\}}$ | 0.5503$^{\{5\}}$ | 0.0209$^{\{1\}}$ | 0.0900$^{\{2\}}$ | 0.5684$^{\{6\}}$ |
| | | $\hat{\eta}$ | 0.2013$^{\{6\}}$ | 0.0580$^{\{3\}}$ | 0.0658$^{\{4\}}$ | 0.0102$^{\{1\}}$ | 0.0116$^{\{2\}}$ | 0.0678$^{\{5\}}$ |
| | | $\sum$ *Ranks* | 42$^{\{4\}}$ | 31$^{\{3\}}$ | 43$^{\{5\}}$ | 9$^{\{1\}}$ | 18$^{\{2\}}$ | 46$^{\{6\}}$ |
| 500 | | $\hat{\lambda}$ | 0.0071$^{\{3\}}$ | 0.0549$^{\{4\}}$ | 0.0634$^{\{6\}}$ | 0.0006$^{\{1\}}$ | 0.0039$^{\{2\}}$ | 0.0614$^{\{5\}}$ |
| | \|BIAS\| | $\hat{\rho}$ | 0.0078$^{\{3\}}$ | 0.0502$^{\{4\}}$ | 0.0592$^{\{5\}}$ | 0.0006$^{\{1\}}$ | 0.0036$^{\{2\}}$ | 0.0603$^{\{6\}}$ |
| | | $\hat{\eta}$ | 0.0098$^{\{6\}}$ | 0.0068$^{\{3\}}$ | 0.0076$^{\{4\}}$ | 0.0005$^{\{1\}}$ | 0.0005$^{\{2\}}$ | 0.0077$^{\{5\}}$ |
| | | $\hat{\lambda}$ | 0.0045$^{\{3\}}$ | 0.0059$^{\{4\}}$ | 0.0081$^{\{6\}}$ | 0.0001$^{\{1\}}$ | 0.0005$^{\{2\}}$ | 0.0077$^{\{5\}}$ |
| | MSE | $\hat{\rho}$ | 0.0044$^{\{3\}}$ | 0.0050$^{\{4\}}$ | 0.0074$^{\{5\}}$ | 0.0001$^{\{1\}}$ | 0.0004$^{\{2\}}$ | 0.0091$^{\{6\}}$ |
| | | $\hat{\eta}$ | 0.0066$^{\{6\}}$ | 0.0007$^{\{3\}}$ | 0.0009$^{\{4\}}$ | 0.0001$^{\{1.5\}}$ | 0.0001$^{\{1.5\}}$ | 0.0001$^{\{5\}}$ |
| | | $\hat{\lambda}$ | 0.0472$^{\{3\}}$ | 0.3666$^{\{4\}}$ | 0.4227$^{\{6\}}$ | 0.0037$^{\{1\}}$ | 0.0259$^{\{2\}}$ | 0.4090$^{\{5\}}$ |

(Continued)

**Table 3 (continued)**

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| | MRE | $\hat{\rho}$ | $0.0517^{\{3\}}$ | $0.3346^{\{4\}}$ | $0.3946^{\{5\}}$ | $0.0037^{\{1\}}$ | $0.0239^{\{2\}}$ | $0.4019^{\{6\}}$ |
| | | $\hat{\eta}$ | $0.0653^{\{6\}}$ | $0.0451^{\{3\}}$ | $0.0508^{\{4\}}$ | $0.0031^{\{1\}}$ | $0.0033^{\{2\}}$ | $0.0516^{\{5\}}$ |

**Table 4:** Numerical results for the parameter set $\boldsymbol{\varphi} = (\lambda = 0.25, \rho = 0.15, \eta = 0.5)^{\intercal}$

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| 20 | \|BIAS\| | $\hat{\lambda}$ | $0.2915^{\{3\}}$ | $0.2762^{\{2\}}$ | $0.3184^{\{5\}}$ | $0.2661^{\{1\}}$ | $0.2945^{\{4\}}$ | $0.3575^{\{6\}}$ |
| | | $\hat{\rho}$ | $0.6465^{\{4\}}$ | $0.5323^{\{1\}}$ | $0.7391^{\{5\}}$ | $0.9623^{\{6\}}$ | $0.5446^{\{2\}}$ | $0.6236^{\{3\}}$ |
| | | $\hat{\eta}$ | $0.2019^{\{4\}}$ | $0.1908^{\{3\}}$ | $0.2665^{\{5\}}$ | $0.1687^{\{1\}}$ | $0.1855^{\{2\}}$ | $0.3099^{\{6\}}$ |
| | MSE | $\hat{\lambda}$ | $0.1748^{\{1\}}$ | $0.2513^{\{2\}}$ | $0.3744^{\{4\}}$ | $0.4084^{\{5\}}$ | $0.2993^{\{3\}}$ | $0.4375^{\{6\}}$ |
| | | $\hat{\rho}$ | $12.5118^{\{5\}}$ | $3.2705^{\{2\}}$ | $5.3376^{\{4\}}$ | $45.4028^{\{6\}}$ | $4.6545^{\{3\}}$ | $3.1888^{\{1\}}$ |
| | | $\hat{\eta}$ | $0.0841^{\{1\}}$ | $0.1312^{\{4\}}$ | $0.8324^{\{5\}}$ | $0.1161^{\{3\}}$ | $0.0979^{\{2\}}$ | $0.8369^{\{6\}}$ |
| | MRE | $\hat{\lambda}$ | $1.1661^{\{3\}}$ | $1.1047^{\{2\}}$ | $1.2735^{\{5\}}$ | $1.0645^{\{1\}}$ | $1.1781^{\{4\}}$ | $1.4301^{\{6\}}$ |
| | | $\hat{\rho}$ | $4.3099^{\{4\}}$ | $3.5485^{\{1\}}$ | $4.9272^{\{5\}}$ | $6.4155^{\{6\}}$ | $3.6304^{\{2\}}$ | $4.1569^{\{3\}}$ |
| | | $\hat{\eta}$ | $0.4039^{\{4\}}$ | $0.3815^{\{3\}}$ | $0.5329^{\{5\}}$ | $0.3374^{\{1\}}$ | $0.3709^{\{2\}}$ | $0.6198^{\{6\}}$ |
| | | $\sum$ *Ranks* | $29^{\{3\}}$ | $20^{\{1\}}$ | $43^{\{5.5\}}$ | $30^{\{4\}}$ | $24^{\{2\}}$ | $43^{\{5.5\}}$ |
| 50 | | $\hat{\lambda}$ | $0.2625^{\{6\}}$ | $0.1492^{\{2\}}$ | $0.1574^{\{4\}}$ | $0.1322^{\{1\}}$ | $0.1523^{\{3\}}$ | $0.1756^{\{5\}}$ |
| | \|BIAS\| | $\hat{\rho}$ | $0.2237^{\{6\}}$ | $0.1222^{\{1\}}$ | $0.1538^{\{4\}}$ | $0.1268^{\{3\}}$ | $0.1245^{\{2\}}$ | $0.1812^{\{5\}}$ |
| | | $\hat{\eta}$ | $0.1487^{\{6\}}$ | $0.0980^{\{3\}}$ | $0.1129^{\{4\}}$ | $0.0899^{\{1\}}$ | $0.0975^{\{2\}}$ | $0.1292^{\{5\}}$ |
| | | $\hat{\lambda}$ | $0.1666^{\{6\}}$ | $0.0485^{\{2\}}$ | $0.0553^{\{4\}}$ | $0.0382^{\{1\}}$ | $0.0515^{\{3\}}$ | $0.0673^{\{5\}}$ |
| | MSE | $\hat{\rho}$ | $0.1509^{\{3\}}$ | $0.0757^{\{1\}}$ | $0.1944^{\{4\}}$ | $0.2739^{\{5\}}$ | $0.0917^{\{2\}}$ | $0.2904^{\{6\}}$ |
| | | $\hat{\eta}$ | $0.0528^{\{6\}}$ | $0.0172^{\{3\}}$ | $0.0247^{\{4\}}$ | $0.0134^{\{1\}}$ | $0.0169^{\{2\}}$ | $0.0320^{\{5\}}$ |
| | | $\hat{\lambda}$ | $1.0502^{\{6\}}$ | $0.5966^{\{2\}}$ | $0.6296^{\{4\}}$ | $0.5288^{\{1\}}$ | $0.6092^{\{3\}}$ | $0.7025^{\{5\}}$ |
| | MRE | $\hat{\rho}$ | $1.4910^{\{6\}}$ | $0.8143^{\{1\}}$ | $1.0250^{\{4\}}$ | $0.8452^{\{3\}}$ | $0.8302^{\{2\}}$ | $1.2079^{\{5\}}$ |
| | | $\hat{\eta}$ | $0.2975^{\{6\}}$ | $0.1960^{\{3\}}$ | $0.2259^{\{4\}}$ | $0.1798^{\{1\}}$ | $0.1951^{\{2\}}$ | $0.2584^{\{5\}}$ |
| | | $\sum$ *Ranks* | $51^{\{6\}}$ | $18^{\{2\}}$ | $36^{\{4\}}$ | $17^{\{1\}}$ | $21^{\{3\}}$ | $46^{\{5\}}$ |
| 150 | | $\hat{\lambda}$ | $0.1592^{\{6\}}$ | $0.0811^{\{2\}}$ | $0.0831^{\{4\}}$ | $0.0719^{\{1\}}$ | $0.0818^{\{3\}}$ | $0.0944^{\{5\}}$ |
| | \|BIAS\| | $\hat{\rho}$ | $0.0989^{\{6\}}$ | $0.0501^{\{2\}}$ | $0.0532^{\{4\}}$ | $0.0492^{\{1\}}$ | $0.0504^{\{3\}}$ | $0.0603^{\{5\}}$ |
| | | $\hat{\eta}$ | $0.0673^{\{5\}}$ | $0.0543^{\{3\}}$ | $0.0605^{\{4\}}$ | $0.0512^{\{1\}}$ | $0.0540^{\{2\}}$ | $0.0687^{\{6\}}$ |
| | | $\hat{\lambda}$ | $0.1143^{\{6\}}$ | $0.0112^{\{2\}}$ | $0.0119^{\{4\}}$ | $0.0090^{\{1\}}$ | $0.0116^{\{3\}}$ | $0.0152^{\{5\}}$ |
| | MSE | $\hat{\rho}$ | $0.0516^{\{6\}}$ | $0.0049^{\{2\}}$ | $0.0062^{\{4\}}$ | $0.0049^{\{1\}}$ | $0.0051^{\{3\}}$ | $0.0086^{\{5\}}$ |
| | | $\hat{\eta}$ | $0.0202^{\{6\}}$ | $0.0049^{\{3\}}$ | $0.0062^{\{4\}}$ | $0.0041^{\{1\}}$ | $0.0049^{\{2\}}$ | $0.0080^{\{5\}}$ |
| | | $\hat{\lambda}$ | $0.6369^{\{6\}}$ | $0.3244^{\{2\}}$ | $0.3325^{\{4\}}$ | $0.2877^{\{1\}}$ | $0.3272^{\{3\}}$ | $0.3774^{\{5\}}$ |
| | MRE | $\hat{\rho}$ | $0.6595^{\{6\}}$ | $0.3337^{\{2\}}$ | $0.3549^{\{4\}}$ | $0.3277^{\{1\}}$ | $0.3362^{\{3\}}$ | $0.4018^{\{5\}}$ |
| | | $\hat{\eta}$ | $0.1347^{\{5\}}$ | $0.1085^{\{3\}}$ | $0.1209^{\{4\}}$ | $0.1023^{\{1\}}$ | $0.1080^{\{2\}}$ | $0.1375^{\{6\}}$ |
| | | $\sum$ *Ranks* | $52^{\{6\}}$ | $21^{\{2\}}$ | $36^{\{4\}}$ | $9^{\{1\}}$ | $24^{\{3\}}$ | $47^{\{5\}}$ |
| 300 | | $\hat{\lambda}$ | $0.0977^{\{6\}}$ | $0.0559^{\{2\}}$ | $0.0570^{\{4\}}$ | $0.0498^{\{1\}}$ | $0.0561^{\{3\}}$ | $0.0643^{\{5\}}$ |

(Continued)

**Table 4 (continued)**

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| | \|BIAS\| | $\hat{\rho}$ | $0.0473^{\{6\}}$ | $0.0335^{\{2\}}$ | $0.0346^{\{4\}}$ | $0.0326^{\{1\}}$ | $0.0337^{\{3\}}$ | $0.0387^{\{5\}}$ |
| | | $\hat{\eta}$ | $0.0339^{\{1\}}$ | $0.0377^{\{4\}}$ | $0.0418^{\{5\}}$ | $0.0355^{\{2\}}$ | $0.0376^{\{3\}}$ | $0.0475^{\{6\}}$ |
| | | $\hat{\lambda}$ | $0.0752^{\{6\}}$ | $0.0052^{\{2\}}$ | $0.0054^{\{4\}}$ | $0.0040^{\{1\}}$ | $0.0052^{\{3\}}$ | $0.0068^{\{5\}}$ |
| | MSE | $\hat{\rho}$ | $0.0224^{\{6\}}$ | $0.0020^{\{2\}}$ | $0.0022^{\{4\}}$ | $0.0019^{\{1\}}$ | $0.0020^{\{3\}}$ | $0.0029^{\{5\}}$ |
| | | $\hat{\eta}$ | $0.0092^{\{6\}}$ | $0.0023^{\{2\}}$ | $0.0028^{\{4\}}$ | $0.0019^{\{1\}}$ | $0.0023^{\{3\}}$ | $0.0037^{\{5\}}$ |
| | | $\hat{\lambda}$ | $0.3906^{\{6\}}$ | $0.2236^{\{2\}}$ | $0.2281^{\{4\}}$ | $0.1992^{\{1\}}$ | $0.2243^{\{3\}}$ | $0.2572^{\{5\}}$ |
| | MRE | $\hat{\rho}$ | $0.3151^{\{6\}}$ | $0.2234^{\{2\}}$ | $0.2305^{\{4\}}$ | $0.2176^{\{1\}}$ | $0.2247^{\{3\}}$ | $0.2578^{\{5\}}$ |
| | | $\hat{\eta}$ | $0.0679^{\{1\}}$ | $0.0753^{\{4\}}$ | $0.0836^{\{5\}}$ | $0.0710^{\{2\}}$ | $0.0753^{\{3\}}$ | $0.0951^{\{6\}}$ |
| | | $\sum$ *Ranks* | $44^{\{5\}}$ | $22^{\{2\}}$ | $38^{\{4\}}$ | $11^{\{1\}}$ | $27^{\{3\}}$ | $47^{\{6\}}$ |
| 500 | | $\hat{\lambda}$ | $0.0677^{\{6\}}$ | $0.0431^{\{2\}}$ | $0.0439^{\{4\}}$ | $0.0387^{\{1\}}$ | $0.0432^{\{3\}}$ | $0.0497^{\{5\}}$ |
| | \|BIAS\| | $\hat{\rho}$ | $0.0247^{\{2\}}$ | $0.0256^{\{3\}}$ | $0.0263^{\{5\}}$ | $0.0246^{\{1\}}$ | $0.0257^{\{4\}}$ | $0.0294^{\{6\}}$ |
| | | $\hat{\eta}$ | $0.0191^{\{1\}}$ | $0.0284^{\{3\}}$ | $0.0315^{\{5\}}$ | $0.0269^{\{2\}}$ | $0.0284^{\{4\}}$ | $0.0363^{\{6\}}$ |
| | | $\hat{\lambda}$ | $0.0564^{\{6\}}$ | $0.0030^{\{2\}}$ | $0.0031^{\{4\}}$ | $0.0024^{\{1\}}$ | $0.0031^{\{3\}}$ | $0.0040^{\{5\}}$ |
| | MSE | $\hat{\rho}$ | $0.0098^{\{6\}}$ | $0.0011^{\{2\}}$ | $0.0012^{\{4\}}$ | $0.0010^{\{1\}}$ | $0.0011^{\{3\}}$ | $0.0015^{\{5\}}$ |
| | | $\hat{\eta}$ | $0.0048^{\{6\}}$ | $0.0013^{\{2\}}$ | $0.0016^{\{4\}}$ | $0.0011^{\{1\}}$ | $0.0013^{\{3\}}$ | $0.0021^{\{5\}}$ |
| | | $\hat{\lambda}$ | $0.2708^{\{6\}}$ | $0.1723^{\{2\}}$ | $0.1754^{\{4\}}$ | $0.1546^{\{1\}}$ | $0.1728^{\{3\}}$ | $0.1988^{\{5\}}$ |
| | MRE | $\hat{\rho}$ | $0.1649^{\{2\}}$ | $0.1705^{\{3\}}$ | $0.1753^{\{5\}}$ | $0.1642^{\{1\}}$ | $0.1710^{\{4\}}$ | $0.1961^{\{6\}}$ |
| | | $\hat{\eta}$ | $0.0381^{\{1\}}$ | $0.0567^{\{3\}}$ | $0.0629^{\{5\}}$ | $0.0539^{\{2\}}$ | $0.0569^{\{4\}}$ | $0.0726^{\{6\}}$ |
| | | $\sum$ *Ranks* | $36^{\{4\}}$ | $22^{\{2\}}$ | $40^{\{5\}}$ | $11^{\{1\}}$ | $31^{\{3\}}$ | $49^{\{6\}}$ |

**Table 5:** Numerical results for the parameter set $\boldsymbol{\varphi} = (\lambda = 0.15, \rho = 0.75, \eta = 0.15)^{\intercal}$

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| 20 | \|BIAS\| | $\hat{\lambda}$ | $0.2605^{\{1\}}$ | $1.7494^{\{3\}}$ | $3.7499^{\{6\}}$ | $1.1219^{\{2\}}$ | $2.4729^{\{5\}}$ | $1.7653^{\{4\}}$ |
| | | $\hat{\rho}$ | $1.4314^{\{1\}}$ | $4.9449^{\{2\}}$ | $7.7997^{\{5\}}$ | $8.1121^{\{6\}}$ | $5.8404^{\{4\}}$ | $4.9635^{\{3\}}$ |
| | | $\hat{\eta}$ | $0.0647^{\{5\}}$ | $0.0406^{\{3\}}$ | $0.0511^{\{4\}}$ | $0.0323^{\{1\}}$ | $0.0364^{\{2\}}$ | $0.0709^{\{6\}}$ |
| | MSE | $\hat{\lambda}$ | $0.3889^{\{1\}}$ | $128.2774^{\{4\}}$ | $3878.1681^{\{6\}}$ | $38.4789^{\{2\}}$ | $865.7409^{\{5\}}$ | $62.0618^{\{3\}}$ |
| | | $\hat{\rho}$ | $50.5157^{\{1\}}$ | $157.7738^{\{3\}}$ | $365.5506^{\{4\}}$ | $1514.6342^{\{6\}}$ | $403.9405^{\{5\}}$ | $114.7354^{\{2\}}$ |
| | | $\hat{\eta}$ | $0.0167^{\{5\}}$ | $0.0043^{\{3\}}$ | $0.0079^{\{4\}}$ | $0.0023^{\{1\}}$ | $0.0032^{\{2\}}$ | $0.2256^{\{6\}}$ |
| | MRE | $\hat{\lambda}$ | $1.7363^{\{1\}}$ | $11.6624^{\{3\}}$ | $24.9994^{\{6\}}$ | $7.4790^{\{2\}}$ | $16.4863^{\{5\}}$ | $11.7687^{\{4\}}$ |
| | | $\hat{\rho}$ | $1.9086^{\{1\}}$ | $6.5932^{\{2\}}$ | $10.3997^{\{5\}}$ | $10.8161^{\{6\}}$ | $7.7872^{\{4\}}$ | $6.6181^{\{3\}}$ |
| | | $\hat{\eta}$ | $0.4316^{\{5\}}$ | $0.2704^{\{3\}}$ | $0.3404^{\{4\}}$ | $0.2153^{\{1\}}$ | $0.2429^{\{2\}}$ | $0.4732^{\{6\}}$ |
| | | $\sum$ *Ranks* | $21^{\{1\}}$ | $26^{\{2\}}$ | $44^{\{6\}}$ | $27^{\{3\}}$ | $34^{\{4\}}$ | $37^{\{5\}}$ |
| 50 | | $\hat{\lambda}$ | $0.1169^{\{1\}}$ | $0.3248^{\{4\}}$ | $0.4105^{\{6\}}$ | $0.2009^{\{2\}}$ | $0.2991^{\{3\}}$ | $0.3545^{\{5\}}$ |
| | \|BIAS\| | $\hat{\rho}$ | $0.2780^{\{1\}}$ | $1.7759^{\{4\}}$ | $3.2046^{\{6\}}$ | $0.9206^{\{2\}}$ | $1.4319^{\{3\}}$ | $2.4485^{\{5\}}$ |
| | | $\hat{\eta}$ | $0.0577^{\{6\}}$ | $0.0231^{\{3\}}$ | $0.0269^{\{4\}}$ | $0.0159^{\{1\}}$ | $0.0183^{\{2\}}$ | $0.0278^{\{5\}}$ |
| | | $\hat{\lambda}$ | $0.0763^{\{1\}}$ | $1.0021^{\{3\}}$ | $1.7801^{\{6\}}$ | $0.7186^{\{2\}}$ | $1.2655^{\{4\}}$ | $1.2873^{\{5\}}$ |
| | MSE | $\hat{\rho}$ | $1.4141^{\{1\}}$ | $23.0754^{\{2\}}$ | $77.4983^{\{6\}}$ | $24.6393^{\{3\}}$ | $33.4711^{\{4\}}$ | $35.0362^{\{5\}}$ |

(Continued)

**Table 5 (continued)**

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| | | $\hat{\eta}$ | 0.0197$^{\{6\}}$ | 0.0009$^{\{3\}}$ | 0.0014$^{\{4\}}$ | 0.0006$^{\{1\}}$ | 0.0008$^{\{2\}}$ | 0.0015$^{\{5\}}$ |
| | | $\hat{\lambda}$ | 0.7799$^{\{1\}}$ | 2.1654$^{\{4\}}$ | 2.7366$^{\{6\}}$ | 1.3397$^{\{2\}}$ | 1.9939$^{\{3\}}$ | 2.3633$^{\{5\}}$ |
| | MRE | $\hat{\rho}$ | 0.3707$^{\{1\}}$ | 2.3678$^{\{4\}}$ | 4.2728$^{\{6\}}$ | 1.2274$^{\{2\}}$ | 1.9092$^{\{3\}}$ | 3.2647$^{\{5\}}$ |
| | | $\hat{\eta}$ | 0.3843$^{\{6\}}$ | 0.1541$^{\{3\}}$ | 0.1797$^{\{4\}}$ | 0.1057$^{\{1\}}$ | 0.1218$^{\{2\}}$ | 0.1852$^{\{5\}}$ |
| | | $\sum$ *Ranks* | 24$^{\{2\}}$ | 30$^{\{4\}}$ | 48$^{\{6\}}$ | 16$^{\{1\}}$ | 26$^{\{3\}}$ | 45$^{\{5\}}$ |
| 150 | | $\hat{\lambda}$ | 0.0976$^{\{3\}}$ | 0.1139$^{\{5\}}$ | 0.1338$^{\{6\}}$ | 0.0239$^{\{1\}}$ | 0.0535$^{\{2\}}$ | 0.1061$^{\{4\}}$ |
| | \|BIAS\| | $\hat{\rho}$ | 0.0669$^{\{2\}}$ | 0.6146$^{\{4\}}$ | 0.8868$^{\{6\}}$ | 0.0636$^{\{1\}}$ | 0.2382$^{\{3\}}$ | 0.8024$^{\{5\}}$ |
| | | $\hat{\eta}$ | 0.0478$^{\{6\}}$ | 0.0132$^{\{3\}}$ | 0.0152$^{\{4\}}$ | 0.0051$^{\{1\}}$ | 0.0059$^{\{2\}}$ | 0.0155$^{\{5\}}$ |
| | | $\hat{\lambda}$ | 0.0753$^{\{6\}}$ | 0.0425$^{\{3\}}$ | 0.0587$^{\{5\}}$ | 0.0110$^{\{1\}}$ | 0.0196$^{\{2\}}$ | 0.0446$^{\{4\}}$ |
| | MSE | $\hat{\rho}$ | 0.0385$^{\{1\}}$ | 1.8818$^{\{4\}}$ | 4.8763$^{\{6\}}$ | 0.0578$^{\{2\}}$ | 0.4587$^{\{3\}}$ | 4.0904$^{\{5\}}$ |
| | | $\hat{\eta}$ | 0.0203$^{\{6\}}$ | 0.0003$^{\{3\}}$ | 0.0004$^{\{4\}}$ | 0.0001$^{\{1\}}$ | 0.0001$^{\{2\}}$ | 0.0004$^{\{5\}}$ |
| | | $\hat{\lambda}$ | 0.6509$^{\{3\}}$ | 0.7599$^{\{5\}}$ | 0.8918$^{\{6\}}$ | 0.1596$^{\{1\}}$ | 0.3566$^{\{2\}}$ | 0.7072$^{\{4\}}$ |
| | MRE | $\hat{\rho}$ | 0.0893$^{\{2\}}$ | 0.8194$^{\{4\}}$ | 1.1824$^{\{6\}}$ | 0.0848$^{\{1\}}$ | 0.3175$^{\{3\}}$ | 1.0698$^{\{5\}}$ |
| | | $\hat{\eta}$ | 0.3189$^{\{6\}}$ | 0.0883$^{\{3\}}$ | 0.1010$^{\{4\}}$ | 0.0339$^{\{1\}}$ | 0.0398$^{\{2\}}$ | 0.1034$^{\{5\}}$ |
| | | $\sum$ *Ranks* | 35$^{\{4\}}$ | 34$^{\{3\}}$ | 47$^{\{6\}}$ | 10$^{\{1\}}$ | 21$^{\{2\}}$ | 42$^{\{5\}}$ |
| 300 | | $\hat{\lambda}$ | 0.0499$^{\{3\}}$ | 0.0742$^{\{5\}}$ | 0.0829$^{\{6\}}$ | 0.0028$^{\{1\}}$ | 0.0143$^{\{2\}}$ | 0.0626$^{\{4\}}$ |
| | \|BIAS\| | $\hat{\rho}$ | 0.0227$^{\{2\}}$ | 0.3762$^{\{4\}}$ | 0.4686$^{\{6\}}$ | 0.0073$^{\{1\}}$ | 0.0687$^{\{3\}}$ | 0.4073$^{\{5\}}$ |
| | | $\hat{\eta}$ | 0.0238$^{\{6\}}$ | 0.0095$^{\{3\}}$ | 0.0107$^{\{4\}}$ | 0.0015$^{\{1\}}$ | 0.0017$^{\{2\}}$ | 0.0109$^{\{5\}}$ |
| | | $\hat{\lambda}$ | 0.0415$^{\{6\}}$ | 0.0143$^{\{4\}}$ | 0.0170$^{\{5\}}$ | 0.0003$^{\{1\}}$ | 0.0028$^{\{2\}}$ | 0.0115$^{\{3\}}$ |
| | MSE | $\hat{\rho}$ | 0.0124$^{\{2\}}$ | 0.4909$^{\{4\}}$ | 0.9134$^{\{5\}}$ | 0.0028$^{\{1\}}$ | 0.0843$^{\{3\}}$ | 0.9649$^{\{6\}}$ |
| | | $\hat{\eta}$ | 0.0114$^{\{6\}}$ | 0.0002$^{\{3\}}$ | 0.0002$^{\{4\}}$ | 0.0002$^{\{1\}}$ | 0.0003$^{\{2\}}$ | 0.0001$^{\{5\}}$ |
| | | $\hat{\lambda}$ | 0.3323$^{\{3\}}$ | 0.4948$^{\{5\}}$ | 0.5528$^{\{6\}}$ | 0.0186$^{\{1\}}$ | 0.0952$^{\{2\}}$ | 0.4171$^{\{4\}}$ |
| | MRE | $\hat{\rho}$ | 0.0302$^{\{2\}}$ | 0.5016$^{\{4\}}$ | 0.6248$^{\{6\}}$ | 0.0098$^{\{1\}}$ | 0.0916$^{\{3\}}$ | 0.5431$^{\{5\}}$ |
| | | $\hat{\eta}$ | 0.1585$^{\{6\}}$ | 0.0635$^{\{3\}}$ | 0.0715$^{\{4\}}$ | 0.0101$^{\{1\}}$ | 0.0116$^{\{2\}}$ | 0.0730$^{\{5\}}$ |
| | | $\sum$ *Ranks* | 36$^{\{4\}}$ | 35$^{\{3\}}$ | 46$^{\{6\}}$ | 9$^{\{1\}}$ | 21$^{\{2\}}$ | 42$^{\{5\}}$ |
| 500 | | $\hat{\lambda}$ | 0.0157$^{\{3\}}$ | 0.0552$^{\{5\}}$ | 0.0607$^{\{6\}}$ | 0.0006$^{\{1\}}$ | 0.0039$^{\{2\}}$ | 0.0442$^{\{4\}}$ |
| | \|BIAS\| | $\hat{\rho}$ | 0.0051$^{\{2\}}$ | 0.2635$^{\{5\}}$ | 0.3223$^{\{6\}}$ | 0.0011$^{\{1\}}$ | 0.0176$^{\{3\}}$ | 0.2594$^{\{4\}}$ |
| | | $\hat{\eta}$ | 0.0078$^{\{4\}}$ | 0.0073$^{\{3\}}$ | 0.0083$^{\{5\}}$ | 0.0004$^{\{1\}}$ | 0.0005$^{\{2\}}$ | 0.0084$^{\{6\}}$ |
| | | $\hat{\lambda}$ | 0.0131$^{\{6\}}$ | 0.0066$^{\{4\}}$ | 0.0074$^{\{5\}}$ | 0.0004$^{\{1\}}$ | 0.0005$^{\{2\}}$ | 0.0051$^{\{3\}}$ |
| | MSE | $\hat{\rho}$ | 0.0024$^{\{2\}}$ | 0.1768$^{\{4\}}$ | 0.2887$^{\{6\}}$ | 0.0002$^{\{1\}}$ | 0.0108$^{\{3\}}$ | 0.2626$^{\{5\}}$ |
| | | $\hat{\eta}$ | 0.0039$^{\{6\}}$ | 0.0008$^{\{3\}}$ | 0.0001$^{\{4.5\}}$ | 0.0001$^{\{1.5\}}$ | 0.0001$^{\{1.5\}}$ | 0.0001$^{\{4.5\}}$ |
| | | $\hat{\lambda}$ | 0.1044$^{\{3\}}$ | 0.3680$^{\{5\}}$ | 0.4011$^{\{6\}}$ | 0.0039$^{\{1\}}$ | 0.0257$^{\{2\}}$ | 0.2947$^{\{4\}}$ |
| | MRE | $\hat{\rho}$ | 0.0069$^{\{2\}}$ | 0.3514$^{\{5\}}$ | 0.4297$^{\{6\}}$ | 0.0014$^{\{1\}}$ | 0.0235$^{\{3\}}$ | 0.3459$^{\{4\}}$ |
| | | $\hat{\eta}$ | 0.0519$^{\{4\}}$ | 0.0489$^{\{3\}}$ | 0.0552$^{\{5\}}$ | 0.0029$^{\{1\}}$ | 0.0033$^{\{2\}}$ | 0.0557$^{\{6\}}$ |
| | | $\sum$ *Ranks* | 32$^{\{3\}}$ | 37$^{\{4\}}$ | 49.5$^{\{6\}}$ | 9.5$^{\{1\}}$ | 20.5$^{\{2\}}$ | 40.5$^{\{5\}}$ |

**Table 6:** Numerical results for the parameter set $\underline{\psi} = (\lambda = 0.5, \rho = 0.5, \eta = 0.5)^{\intercal}$

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| 20 | \|BIAS\| | $\hat{\lambda}$ | 0.6604{4} | 0.6066{1} | 0.7624{5} | 0.6569{2} | 0.6599{3} | 0.8116{6} |
| | | $\hat{\rho}$ | 3.9484{5} | 2.1965{2} | 2.5495{3} | 7.7843{6} | 3.0365{4} | 1.9943{1} |
| | | $\hat{\eta}$ | 10.1011{6} | 0.3064{2} | 0.5345{3} | 0.5849{4} | 0.2400{1} | 0.9447{5} |
| | MSE | $\hat{\lambda}$ | 1.1261{1} | 1.6684{2} | 2.4256{5} | 2.5147{6} | 2.0313{3} | 2.3662{4} |
| | | $\hat{\rho}$ | 216.9037{5} | 27.7137{2} | 31.5976{3} | 803.7337{6} | 68.5187{4} | 18.6237{1} |
| | | $\hat{\eta}$ | 321399.7257{6} | 12.8408{2} | 20.6400{3} | 395.8378{5} | 0.6535{1} | 103.7969{4} |
| | MRE | $\hat{\lambda}$ | 1.3208{4} | 1.2131{1} | 1.5247{5} | 1.3139{2} | 1.3197{3} | 1.6232{6} |
| | | $\hat{\rho}$ | 7.8968{5} | 4.3931{2} | 5.0989{3} | 15.5685{6} | 6.0729{4} | 3.9887{1} |
| | | $\hat{\eta}$ | 20.2022{6} | 0.6128{2} | 1.0689{3} | 1.1699{4} | 0.4803{1} | 1.8895{5} |
| | | $\sum$ *Ranks* | 42{6} | 16{1} | 33{3.5} | 41{5} | 24{2} | 33{3.5} |
| 50 | \|BIAS\| | $\hat{\lambda}$ | 0.3601{4} | 0.3237{1} | 0.3801{5} | 0.3255{2} | 0.3413{3} | 0.3974{6} |
| | | $\hat{\rho}$ | 0.9695{4} | 0.7032{1} | 1.1114{5} | 1.2352{6} | 0.8639{2} | 0.8869{3} |
| | | $\hat{\eta}$ | 0.1112{4} | 0.1066{2} | 0.1325{5} | 0.1002{1} | 0.1077{3} | 0.1526{6} |
| | MSE | $\hat{\lambda}$ | 0.2637{4} | 0.2203{1} | 0.3109{5} | 0.2515{3} | 0.2449{2} | 0.3435{6} |
| | | $\hat{\rho}$ | 13.6952{5} | 3.2501{1} | 7.5925{4} | 39.0288{6} | 6.7259{3} | 4.5537{2} |
| | | $\hat{\eta}$ | 0.0227{4} | 0.0207{2} | 0.0359{5} | 0.0170{1} | 0.0211{3} | 0.0745{6} |
| | MRE | $\hat{\lambda}$ | 0.7201{4} | 0.6474{1} | 0.7602{5} | 0.6509{2} | 0.6827{3} | 0.7948{6} |
| | | $\hat{\rho}$ | 1.9389{4} | 1.4065{1} | 2.2229{5} | 2.4703{6} | 1.7278{2} | 1.7739{3} |
| | | $\hat{\eta}$ | 0.2223{4} | 0.2132{2} | 0.2649{5} | 0.2004{1} | 0.2154{3} | 0.3051{6} |
| | | $\sum$ *Ranks* | 37{4} | 12{1} | 44{5.5} | 28{3} | 24{2} | 44{5.5} |
| 150 | \|BIAS\| | $\hat{\lambda}$ | 0.1815{4} | 0.1721{1} | 0.2012{5} | 0.1726{2} | 0.1814{3} | 0.2104{6} |
| | | $\hat{\rho}$ | 0.2633{4} | 0.2315{1} | 0.3669{6} | 0.2459{2} | 0.2605{3} | 0.3229{5} |
| | | $\hat{\eta}$ | 0.0582{3} | 0.0569{2} | 0.0697{5} | 0.0556{1} | 0.0587{4} | 0.0776{6} |
| | MSE | $\hat{\lambda}$ | 0.0563{3} | 0.0523{1} | 0.0698{5} | 0.0530{2} | 0.0572{4} | 0.0771{6} |
| | | $\hat{\rho}$ | 0.2838{4} | 0.1696{1} | 0.6898{6} | 0.2092{2} | 0.2252{3} | 0.4583{5} |
| | | $\hat{\eta}$ | 0.0059{4} | 0.0054{2} | 0.0081{5} | 0.0049{1} | 0.0057{3} | 0.0101{6} |
| | MRE | $\hat{\lambda}$ | 0.3630{4} | 0.3442{1} | 0.4024{5} | 0.3451{2} | 0.3629{3} | 0.4208{6} |
| | | $\hat{\rho}$ | 0.5266{4} | 0.4629{1} | 0.7339{6} | 0.4918{2} | 0.5209{3} | 0.6459{5} |
| | | $\hat{\eta}$ | 0.1163{3} | 0.1137{2} | 0.1395{5} | 0.1112{1} | 0.1174{4} | 0.1552{6} |
| | | $\sum$ *Ranks* | 33{4} | 12{1} | 48{5} | 15{2} | 30{3} | 51{6} |
| 300 | \|BIAS\| | $\hat{\lambda}$ | 0.1322{4} | 0.1147{1} | 0.1341{5} | 0.1187{2} | 0.1220{3} | 0.1398{6} |
| | | $\hat{\rho}$ | 0.1559{4} | 0.1373{1} | 0.1872{6} | 0.1491{2} | 0.1543{3} | 0.1790{5} |
| | | $\hat{\eta}$ | 0.0447{4} | 0.0382{1} | 0.0466{5} | 0.0383{2} | 0.0397{3} | 0.0520{6} |
| | MSE | $\hat{\lambda}$ | 0.0307{5} | 0.0222{1} | 0.0299{4} | 0.0236{2} | 0.0248{3} | 0.0327{6} |
| | | $\hat{\rho}$ | 0.0510{3} | 0.0387{1} | 0.1014{6} | 0.0497{2} | 0.0545{4} | 0.0909{5} |
| | | $\hat{\eta}$ | 0.0053{6} | 0.0024{2} | 0.0035{4} | 0.0023{1} | 0.0026{3} | 0.0044{5} |
| | MRE | $\hat{\lambda}$ | 0.2643{4} | 0.2294{1} | 0.2681{5} | 0.2373{2} | 0.2440{3} | 0.2796{6} |
| | | $\hat{\rho}$ | 0.3118{4} | 0.2746{1} | 0.3744{6} | 0.2983{2} | 0.3086{3} | 0.3581{5} |
| | | $\hat{\eta}$ | 0.0895{4} | 0.0764{1} | 0.0932{5} | 0.0766{2} | 0.0794{3} | 0.1040{6} |
| | | $\sum$ *Ranks* | 38{4} | 10{1} | 46{5} | 17{2} | 28{3} | 50{6} |

(Continued)

**Table 6 (continued)**

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| 500 | | $\hat{\lambda}$ | 0.1028{4} | 0.0899{1} | 0.1046{5} | 0.0922{2} | 0.0952{3} | 0.1089{6} |
| | \|BIAS\| | $\hat{\rho}$ | 0.1154{4} | 0.1029{1} | 0.1331{6} | 0.1094{2} | 0.1129{3} | 0.1289{5} |
| | | $\hat{\eta}$ | 0.0353{4} | 0.0304{2} | 0.0363{5} | 0.0298{1} | 0.0312{3} | 0.0401{6} |
| | | $\hat{\lambda}$ | 0.0205{6} | 0.0130{1} | 0.0174{4} | 0.0139{2} | 0.0145{3} | 0.0187{5} |
| | MSE | $\hat{\rho}$ | 0.0258{4} | 0.0195{1} | 0.0384{6} | 0.0228{2} | 0.0241{3} | 0.0343{5} |
| | | $\hat{\eta}$ | 0.0044{6} | 0.0015{2} | 0.0023{4} | 0.0014{1} | 0.0016{3} | 0.0027{5} |
| | | $\hat{\lambda}$ | 0.2055{4} | 0.1799{1} | 0.2091{5} | 0.1844{2} | 0.1904{3} | 0.2179{6} |
| | MRE | $\hat{\rho}$ | 0.2307{4} | 0.2059{1} | 0.2661{6} | 0.2188{2} | 0.2258{3} | 0.2578{5} |
| | | $\hat{\eta}$ | 0.0706{4} | 0.0601{2} | 0.0726{5} | 0.0597{1} | 0.0623{3} | 0.0802{6} |
| | | $\sum$ *Ranks* | 40{4} | 12{1} | 46{5} | 15{2} | 27{3} | 49{6} |

**Table 7:** Numerical results for the parameter set $\boldsymbol{\varphi} = (\lambda = 0.75, \rho = 0.75, \eta = 0.25)^{\intercal}$

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| 20 | \|BIAS\| | $\hat{\lambda}$ | 1.7448{1} | 2.4447{2} | 2.7904{4} | 2.8381{5} | 2.7386{3} | 2.9548{6} |
| | | $\hat{\rho}$ | 4.8963{3} | 4.5255{2} | 5.3351{4} | 10.5409{6} | 6.0602{5} | 3.9176{1} |
| | | $\hat{\eta}$ | 0.4163{6} | 0.0846{3} | 0.1088{4} | 0.0738{1} | 0.0760{2} | 0.1736{5} |
| | MSE | $\hat{\lambda}$ | 10.6689{1} | 120.7372{4} | 104.8674{3} | 172.3724{6} | 140.2894{5} | 101.9805{2} |
| | | $\hat{\rho}$ | 346.7641{5} | 113.0453{2} | 146.1383{3} | 1371.0272{6} | 318.5276{4} | 67.5664{1} |
| | | $\hat{\eta}$ | 569.3543{6} | 0.1181{3} | 0.1737{4} | 0.0145{2} | 0.0142{1} | 1.8531{5} |
| | MRE | $\hat{\lambda}$ | 2.3263{1} | 3.2596{2} | 3.7205{4} | 3.7841{5} | 3.6514{3} | 3.9397{6} |
| | | $\hat{\rho}$ | 6.5284{3} | 6.0340{2} | 7.1135{4} | 14.0546{6} | 8.0803{5} | 5.2234{1} |
| | | $\hat{\eta}$ | 1.6651{6} | 0.3385{3} | 0.4351{4} | 0.2952{1} | 0.3041{2} | 0.6944{5} |
| | | $\sum$ *Ranks* | 32{3.5} | 23{1} | 34{5} | 38{6} | 30{2} | 32{3.5} |
| 50 | | $\hat{\lambda}$ | 0.8339 {3} | 0.8123{1} | 0.9371{5} | 0.8311{2} | 0.8729{4} | 1.0104{6} |
| | \|BIAS\| | $\hat{\rho}$ | 1.0427{1} | 1.7197{2} | 2.6911{6} | 2.1885{5} | 2.0379{3} | 2.0794{4} |
| | | $\hat{\eta}$ | 0.0444{3} | 0.0449{4} | 0.0521{5} | 0.0418{1} | 0.0442{2} | 0.0581{6} |
| | | $\hat{\lambda}$ | 2.0905{1} | 2.5477{2} | 3.1729{4} | 4.8562{6} | 2.8643{3} | 3.8165{5} |
| | MSE | $\hat{\rho}$ | 11.6301{1} | 19.5903{2} | 46.5649{5} | 96.2298{6} | 34.1141{4} | 22.2665{3} |
| | | $\hat{\eta}$ | 0.0039{4} | 0.0035{3} | 0.0049{5} | 0.0029{1} | 0.0034{2} | 0.0065{6} |
| | | $\hat{\lambda}$ | 1.1119{3} | 1.0831{1} | 1.2495{5} | 1.1082{2} | 1.1639{4} | 1.3472{6} |
| | MRE | $\hat{\rho}$ | 1.3903{1} | 2.2929{2} | 3.5881{6} | 2.9179{5} | 2.7173{3} | 2.7725{4} |
| | | $\hat{\eta}$ | 0.1774{3} | 0.1795{4} | 0.2083{5} | 0.1673{1} | 0.1767{2} | 0.2324{6} |
| | | $\sum$ *Ranks* | 20{1} | 21{2} | 46{5.5} | 29{4} | 27{3} | 46{5.5} |
| 150 | | $\hat{\lambda}$ | 0.3537{2} | 0.3805{3} | 0.4446{5} | 0.3507{1} | 0.3945{4} | 0.4611{6} |
| | \|BIAS\| | $\hat{\rho}$ | 0.3513{1} | 0.5059{3} | 0.7024{6} | 0.4554{2} | 0.5223{4} | 0.6654{5} |
| | | $\hat{\eta}$ | 0.0256{4} | 0.0241{3} | 0.0279{5} | 0.0214{1} | 0.0231{2} | 0.0305{6} |
| | | $\hat{\lambda}$ | 0.3217{3} | 0.3100{2} | 0.4124{5} | 0.3026{1} | 0.3557{4} | 0.4611{6} |

(Continued)

**Table 7 (continued)**

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| | MSE | $\hat{\rho}$ | 0.36862$^{(1)}$ | 1.1913$^{(3)}$ | 2.6017$^{(6)}$ | 0.8958$^{(2)}$ | 1.2972$^{(4)}$ | 1.9856$^{(5)}$ |
| | | $\hat{\eta}$ | 0.0019$^{(6)}$ | 0.0009$^{(3)}$ | 0.0013$^{(4)}$ | 0.0007$^{(1)}$ | 0.0009$^{(2)}$ | 0.0015$^{(5)}$ |
| | | $\hat{\lambda}$ | 0.4715$^{(2)}$ | 0.5073$^{(3)}$ | 0.5928$^{(5)}$ | 0.4676$^{(1)}$ | 0.5259$^{(4)}$ | 0.6148$^{(6)}$ |
| | MRE | $\hat{\rho}$ | 0.4684$^{(1)}$ | 0.6745$^{(3)}$ | 0.9365$^{(6)}$ | 0.6072$^{(2)}$ | 0.6964$^{(4)}$ | 0.8872$^{(5)}$ |
| | | $\hat{\eta}$ | 0.1023$^{(4)}$ | 0.0962$^{(3)}$ | 0.1118$^{(5)}$ | 0.0857$^{(1)}$ | 0.0923$^{(2)}$ | 0.1220$^{(6)}$ |
| | | $\sum$ *Ranks* | 24$^{(2)}$ | 26$^{(3)}$ | 47$^{(5)}$ | 12$^{(1)}$ | 30$^{(4)}$ | 50$^{(6)}$ |
| 300 | | $\hat{\lambda}$ | 0.2169$^{(2)}$ | 0.2587$^{(4)}$ | 0.2989$^{(6)}$ | 0.2059$^{(1)}$ | 0.2424$^{(3)}$ | 0.2976$^{(5)}$ |
| | \|BIAS\| | $\hat{\rho}$ | 0.1999$^{(1)}$ | 0.3053$^{(4)}$ | 0.3994$^{(6)}$ | 0.2396$^{(2)}$ | 0.2967$^{(3)}$ | 0.3819$^{(5)}$ |
| | | $\hat{\eta}$ | 0.0192$^{(4)}$ | 0.0168$^{(3)}$ | 0.0197$^{(5)}$ | 0.0138$^{(1)}$ | 0.0151$^{(2)}$ | 0.0216$^{(6)}$ |
| | | $\hat{\lambda}$ | 0.1199$^{(2)}$ | 0.1251$^{(4)}$ | 0.1598$^{(5)}$ | 0.1031$^{(1)}$ | 0.1246$^{(3)}$ | 0.1718$^{(6)}$ |
| | MSE | $\hat{\rho}$ | 0.1058$^{(1)}$ | 0.2702$^{(3)}$ | 0.5499$^{(6)}$ | 0.1938$^{(2)}$ | 0.3645$^{(4)}$ | 0.5348$^{(5)}$ |
| | | $\hat{\eta}$ | 0.0019$^{(6)}$ | 0.0005$^{(3)}$ | 0.0006$^{(4)}$ | 0.0004$^{(1)}$ | 0.0004$^{(2)}$ | 0.0008$^{(5)}$ |
| | | $\hat{\lambda}$ | 0.2893$^{(2)}$ | 0.3449$^{(4)}$ | 0.3985$^{(6)}$ | 0.2745$^{(1)}$ | 0.3232$^{(3)}$ | 0.3968$^{(5)}$ |
| | MRE | $\hat{\rho}$ | 0.2665$^{(1)}$ | 0.4071$^{(4)}$ | 0.5325$^{(6)}$ | 0.3195$^{(2)}$ | 0.3956$^{(3)}$ | 0.5092$^{(5)}$ |
| | | $\hat{\eta}$ | 0.0766$^{(4)}$ | 0.0670$^{(3)}$ | 0.0788$^{(5)}$ | 0.0552$^{(1)}$ | 0.0605$^{(2)}$ | 0.0863$^{(6)}$ |
| | | $\sum$ *Ranks* | 23$^{(2)}$ | 32$^{(4)}$ | 49$^{(6)}$ | 12$^{(1)}$ | 25$^{(3)}$ | 48$^{(5)}$ |
| 500 | | $\hat{\lambda}$ | 0.1487$^{(2)}$ | 0.2023$^{(4)}$ | 0.2327$^{(6)}$ | 0.1328$^{(1)}$ | 0.1731$^{(3)}$ | 0.2256$^{(5)}$ |
| | \|BIAS\| | $\hat{\rho}$ | 0.1326$^{(1)}$ | 0.2145$^{(4)}$ | 0.2647$^{(6)}$ | 0.1401$^{(2)}$ | 0.1872$^{(3)}$ | 0.2551$^{(5)}$ |
| | | $\hat{\eta}$ | 0.0161$^{(5)}$ | 0.0129$^{(3)}$ | 0.0150$^{(4)}$ | 0.0099$^{(1)}$ | 0.0109$^{(2)}$ | 0.0162$^{(6)}$ |
| | | $\hat{\lambda}$ | 0.0642$^{(2)}$ | 0.0728$^{(4)}$ | 0.0930$^{(5)}$ | 0.0509$^{(1)}$ | 0.0649$^{(3)}$ | 0.0941$^{(6)}$ |
| | MSE | $\hat{\rho}$ | 0.0449$^{(1)}$ | 0.0944$^{(4)}$ | 0.1513$^{(5)}$ | 0.0603$^{(2)}$ | 0.0895$^{(3)}$ | 0.1637$^{(6)}$ |
| | | $\hat{\eta}$ | 0.0021$^{(6)}$ | 0.0003$^{(3)}$ | 0.0004$^{(4)}$ | 0.0002$^{(1)}$ | 0.0002$^{(2)}$ | 0.0004$^{(5)}$ |
| | | $\hat{\lambda}$ | 0.1983$^{(2)}$ | 0.2697$^{(4)}$ | 0.3102$^{(6)}$ | 0.1770$^{(1)}$ | 0.2308$^{(3)}$ | 0.3007$^{(5)}$ |
| | MRE | $\hat{\rho}$ | 0.1768$^{(1)}$ | 0.2860$^{(4)}$ | 0.3529$^{(6)}$ | 0.1867$^{(2)}$ | 0.2496$^{(3)}$ | 0.3401$^{(5)}$ |
| | | $\hat{\eta}$ | 0.0642$^{(5)}$ | 0.0519$^{(3)}$ | 0.0601$^{(4)}$ | 0.0397$^{(1)}$ | 0.0438$^{(2)}$ | 0.0649$^{(6)}$ |
| | | $\sum$ *Ranks* | 25$^{(3)}$ | 33$^{(4)}$ | 46$^{(5)}$ | 12$^{(1)}$ | 24$^{(2)}$ | 49$^{(6)}$ |

**Table 8:** Numerical results for the parameter set $\boldsymbol{\varphi} = (\lambda = 0.25, \rho = 1.25, \eta = 0.5)^{\intercal}$

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| 20 | \|BIAS\| | $\hat{\lambda}$ | 0.5154$^{(4)}$ | 0.4285$^{(1)}$ | 0.5564$^{(5)}$ | 0.4613$^{(3)}$ | 0.4583$^{(2)}$ | 0.5788$^{(6)}$ |
| | | $\hat{\rho}$ | 10.0551$^{(5)}$ | 5.0780$^{(3)}$ | 4.3570$^{(2)}$ | 22.6468$^{(6)}$ | 6.8176$^{(4)}$ | 3.4370$^{(1)}$ |
| | | $\hat{\eta}$ | 32.2623$^{(6)}$ | 1.0247$^{(1)}$ | 2.7692$^{(3)}$ | 5.9937$^{(5)}$ | 1.5283$^{(2)}$ | 5.9829$^{(4)}$ |
| | MSE | $\hat{\lambda}$ | 0.9371$^{(3)}$ | 0.6824$^{(1)}$ | 1.0776$^{(4)}$ | 1.2259$^{(6)}$ | 0.8069$^{(2)}$ | 1.1205$^{(5)}$ |
| | | $\hat{\rho}$ | 991.6864$^{(5)}$ | 92.3748$^{(3)}$ | 63.4480$^{(2)}$ | 3634.3874$^{(6)}$ | 202.8037$^{(4)}$ | 37.2467$^{(1)}$ |
| | | $\hat{\eta}$ | 389553.1128$^{(6)}$ | 240.3095$^{(1)}$ | 797.3789$^{(3)}$ | 27761.4340$^{(5)}$ | 540.7508$^{(2)}$ | 4819.9550$^{(4)}$ |
| | MRE | $\hat{\lambda}$ | 2.0617$^{(4)}$ | 1.7139$^{(1)}$ | 2.2258$^{(5)}$ | 1.8451$^{(3)}$ | 1.8331$^{(2)}$ | 2.3154$^{(6)}$ |
| | | $\hat{\rho}$ | 8.0441$^{(5)}$ | 4.0624$^{(3)}$ | 3.4856$^{(2)}$ | 18.1174$^{(6)}$ | 5.4541$^{(4)}$ | 2.7496$^{(1)}$ |

(Continued)

**Table 8 (continued)**

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| | | $\hat{\eta}$ | 64.5245$^{(6)}$ | 2.0493$^{(1)}$ | 5.5384$^{(3)}$ | 11.9874$^{(5)}$ | 3.0566$^{(2)}$ | 11.9657$^{(4)}$ |
| | | $\sum$ *Ranks* | 44$^{(5)}$ | 15$^{(1)}$ | 29$^{(3)}$ | 45$^{(6)}$ | 24$^{(2)}$ | 32$^{(4)}$ |
| 50 | \|BIAS\| | $\hat{\lambda}$ | 0.2067$^{(1)}$ | 0.2082$^{(2)}$ | 0.2576$^{(5)}$ | 0.2119$^{(3)}$ | 0.2207$^{(4)}$ | 0.2678$^{(6)}$ |
| | | $\hat{\rho}$ | 4.2692$^{(5)}$ | 2.7558$^{(2)}$ | 3.1491$^{(3)}$ | 7.7441$^{(6)}$ | 3.5226$^{(4)}$ | 2.6916$^{(1)}$ |
| | | $\hat{\eta}$ | 0.1345$^{(3)}$ | 0.1369$^{(4)}$ | 0.1844$^{(5)}$ | 0.1252$^{(1)}$ | 0.1344$^{(2)}$ | 0.2485$^{(6)}$ |
| | MSE | $\hat{\lambda}$ | 0.1057$^{(4)}$ | 0.0916$^{(1)}$ | 0.1438$^{(5)}$ | 0.0959$^{(2)}$ | 0.1002$^{(3)}$ | 0.1572$^{(6)}$ |
| | | $\hat{\rho}$ | 148.1554$^{(5)}$ | 28.5379$^{(2)}$ | 32.5326$^{(3)}$ | 589.1789$^{(6)}$ | 53.9749$^{(4)}$ | 22.8511$^{(1)}$ |
| | | $\hat{\eta}$ | 0.2438$^{(4)}$ | 0.0804$^{(2)}$ | 0.3229$^{(5)}$ | 0.0419$^{(1)}$ | 0.0999$^{(3)}$ | 1.5515$^{(6)}$ |
| | MRE | $\hat{\lambda}$ | 0.8267$^{(1)}$ | 0.8326$^{(2)}$ | 1.0302$^{(5)}$ | 0.8475$^{(3)}$ | 0.8828$^{(4)}$ | 1.0713$^{(6)}$ |
| | | $\hat{\rho}$ | 3.4153$^{(5)}$ | 2.2047$^{(2)}$ | 2.5193$^{(3)}$ | 6.1953$^{(6)}$ | 2.8180$^{(4)}$ | 2.1533$^{(1)}$ |
| | | $\hat{\eta}$ | 0.2689$^{(3)}$ | 0.2739$^{(4)}$ | 0.3687$^{(5)}$ | 0.2504$^{(1)}$ | 0.2689$^{(2)}$ | 0.4971$^{(6)}$ |
| | | $\sum$ *Ranks* | 31$^{(4)}$ | 21$^{(1)}$ | 39$^{(5.5)}$ | 29$^{(2)}$ | 30$^{(3)}$ | 39$^{(5.5)}$ |
| 150 | \|BIAS\| | $\hat{\lambda}$ | 0.1057$^{(1)}$ | 0.1101$^{(2)}$ | 0.1486$^{(5)}$ | 0.1151$^{(3)}$ | 0.1224$^{(4)}$ | 0.1490$^{(6)}$ |
| | | $\hat{\rho}$ | 1.0714$^{(2)}$ | 1.0682$^{(1)}$ | 1.6759$^{(6)}$ | 1.5487$^{(5)}$ | 1.3298$^{(3)}$ | 1.4775$^{(4)}$ |
| | | $\hat{\eta}$ | 0.0641$^{(1)}$ | 0.0675$^{(3)}$ | 0.0887$^{(5)}$ | 0.0672$^{(2)}$ | 0.0707$^{(4)}$ | 0.0970$^{(6)}$ |
| | MSE | $\hat{\lambda}$ | 0.0207$^{(1)}$ | 0.0217$^{(2)}$ | 0.0369$^{(5)}$ | 0.0234$^{(3)}$ | 0.02581 $^{(4)}$ | 0.0370$^{(6)}$ |
| | | $\hat{\rho}$ | 8.2570$^{(4)}$ | 4.4006$^{(1)}$ | 9.7468$^{(5)}$ | 25.9439$^{(6)}$ | 7.9591$^{(3)}$ | 7.1546$^{(2)}$ |
| | | $\hat{\eta}$ | 0.0074$^{(2)}$ | 0.0078$^{(3)}$ | 0.0129$^{(5)}$ | 0.0072$^{(1)}$ | 0.0084$^{(4)}$ | 0.0158$^{(6)}$ |
| | MRE | $\hat{\lambda}$ | 0.4226$^{(1)}$ | 0.4404$^{(2)}$ | 0.5945$^{(5)}$ | 0.4602$^{(3)}$ | 0.4898$^{(4)}$ | 0.5961$^{(6)}$ |
| | | $\hat{\rho}$ | 0.8571$^{(2)}$ | 0.8546$^{(1)}$ | 1.3407$^{(6)}$ | 1.2389$^{(5)}$ | 1.0639$^{(3)}$ | 1.1819$^{(4)}$ |
| | | $\hat{\eta}$ | 0.1283$^{(1)}$ | 0.1350$^{(3)}$ | 0.1773$^{(5)}$ | 0.1344$^{(2)}$ | 0.1413$^{(4)}$ | 0.1941$^{(6)}$ |
| | | $\sum$ *Ranks* | 15$^{(1)}$ | 18$^{(2)}$ | 47$^{(6)}$ | 30$^{(3)}$ | 33$^{(4)}$ | 46$^{(5)}$ |
| 300 | \|BIAS\| | $\hat{\lambda}$ | 0.0726$^{(1)}$ | 0.0774$^{(2)}$ | 0.1077$^{(6)}$ | 0.0781$^{(3)}$ | 0.0864$^{(4)}$ | 0.1049$^{(5)}$ |
| | | $\hat{\rho}$ | 0.5561$^{(1)}$ | 0.6079$^{(3)}$ | 0.9884$^{(6)}$ | 0.5787$^{(2)}$ | 0.6892$^{(4)}$ | 0.8493$^{(5)}$ |
| | | $\hat{\eta}$ | 0.0447$^{(1)}$ | 0.0458$^{(3)}$ | 0.0615$^{(5)}$ | 0.0453$^{(2)}$ | 0.0479$^{(4)}$ | 0.0671$^{(6)}$ |
| | MSE | $\hat{\lambda}$ | 0.0093$^{(1)}$ | 0.0103$^{(2)}$ | 0.0185$^{(6)}$ | 0.0112$^{(3)}$ | 0.0123$^{(4)}$ | 0.0179$^{(5)}$ |
| | | $\hat{\rho}$ | 1.3403$^{(2)}$ | 1.2331$^{(1)}$ | 3.2368$^{(6)}$ | 1.5352$^{(3)}$ | 1.6632$^{(4)}$ | 2.3349$^{(5)}$ |
| | | $\hat{\eta}$ | 0.0038$^{(3)}$ | 0.0035$^{(2)}$ | 0.0061$^{(5)}$ | 0.0033$^{(1)}$ | 0.0038$^{(4)}$ | 0.0073$^{(6)}$ |
| | MRE | $\hat{\lambda}$ | 0.2902$^{(1)}$ | 0.3095$^{(2)}$ | 0.4309$^{(6)}$ | 0.3125$^{(3)}$ | 0.3454$^{(4)}$ | 0.4195$^{(5)}$ |
| | | $\hat{\rho}$ | 0.4449$^{(1)}$ | 0.4864$^{(3)}$ | 0.7907$^{(6)}$ | 0.4629$^{(2)}$ | 0.5514$^{(4)}$ | 0.6794$^{(5)}$ |
| | | $\hat{\eta}$ | 0.0893$^{(1)}$ | 0.0917$^{(3)}$ | 0.1229$^{(5)}$ | 0.0906$^{(2)}$ | 0.0959$^{(4)}$ | 0.1342$^{(6)}$ |
| | | $\sum$ *Ranks* | 12$^{(1)}$ | 21$^{(2.5)}$ | 51$^{(6)}$ | 21$^{(2.5)}$ | 36$^{(4)}$ | 48$^{(5)}$ |
| 500 | \|BIAS\| | $\hat{\lambda}$ | 0.0575$^{(2)}$ | 0.0583$^{(3)}$ | 0.0818$^{(6)}$ | 0.0546$^{(1)}$ | 0.0654$^{(4)}$ | 0.0807$^{(5)}$ |
| | | $\hat{\rho}$ | 0.3767$^{(2)}$ | 0.4035$^{(3)}$ | 0.6499$^{(6)}$ | 0.3385$^{(1)}$ | 0.4570$^{(4)}$ | 0.5661$^{(5)}$ |
| | | $\hat{\eta}$ | 0.0348$^{(2)}$ | 0.0349$^{(3)}$ | 0.0470$^{(5)}$ | 0.0341$^{(1)}$ | 0.0367$^{(4)}$ | 0.0515$^{(6)}$ |
| | MSE | $\hat{\lambda}$ | 0.0058$^{(1)}$ | 0.0059$^{(2)}$ | 0.0108$^{(6)}$ | 0.0061$^{(3)}$ | 0.0071$^{(4)}$ | 0.0106$^{(5)}$ |
| | | $\hat{\rho}$ | 0.4062$^{(1)}$ | 0.4780$^{(3)}$ | 1.3193$^{(6)}$ | 0.4363$^{(2)}$ | 0.6049$^{(4)}$ | 0.8789$^{(5)}$ |
| | | $\hat{\eta}$ | 0.0027$^{(4)}$ | 0.0020$^{(2)}$ | 0.0036$^{(5)}$ | 0.0019$^{(1)}$ | 0.0022$^{(3)}$ | 0.0043$^{(6)}$ |
| | | $\hat{\lambda}$ | 0.2299$^{(2)}$ | 0.2330$^{(3)}$ | 0.3273$^{(6)}$ | 0.2184$^{(1)}$ | 0.2615$^{(4)}$ | 0.3229$^{(5)}$ |

(Continued)

**Table 8** (continued)

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| | MRE | $\hat{\rho}$ | 0.3014[2] | 0.3228[3] | 0.5199[6] | 0.2708[1] | 0.3656[4] | 0.4529[5] |
| | | $\hat{\eta}$ | 0.0696[2] | 0.0697[3] | 0.0941[5] | 0.0681[1] | 0.0734[4] | 0.1029[6] |
| | | $\sum$ *Ranks* | 18[2] | 25[3] | 51[6] | 12[1] | 35[4] | 48[5] |

**Table 9:** Numerical results for the parameter set $\boldsymbol{\varphi} = (\lambda = 1.25, \rho = 0.75, \eta = 0.15)^{\intercal}$

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| 20 | \|BIAS\| | $\hat{\lambda}$ | 2.3670[1] | 17.6638[4] | 20.6726[6] | 9.2136[2] | 17.7608[5] | 14.4901[3] |
| | | $\hat{\rho}$ | 1.7953[1] | 5.3986[3] | 6.8664[6] | 6.3146[5] | 5.9539[4] | 4.7985[2] |
| | | $\hat{\eta}$ | 0.0336[2] | 0.0439[4] | 0.0670[5] | 0.0306[1] | 0.0358[3] | 0.1329[6] |
| | MSE | $\hat{\lambda}$ | 30.5147[1] | 16702.2446[5] | 14601.1372[4] | 5434.8654[3] | 17919.7044[6] | 3605.7587[2] |
| | | $\hat{\rho}$ | 71.8577[1] | 184.0309[3] | 253.4864[4] | 708.2379[6] | 358.6152[5] | 103.9011[2] |
| | | $\hat{\eta}$ | 0.0030[2] | 0.0273[4] | 0.6041[5] | 0.0024[1] | 0.0035[3] | 7.3547[6] |
| | MRE | $\hat{\lambda}$ | 1.8936[1] | 14.1310[4] | 16.5381[6] | 7.3709[2] | 14.2086[5] | 11.5921[3] |
| | | $\hat{\rho}$ | 2.3937[1] | 7.1981[3] | 9.1552[6] | 8.4194[5] | 7.9387[4] | 6.3980[2] |
| | | $\hat{\eta}$ | 0.2243[2] | 0.2924[4] | 0.4468[5] | 0.2038[1] | 0.2388[3] | 0.8861[6] |
| | | $\sum$ *Ranks* | 12[1] | 34[4] | 47[6] | 26[2] | 38[5] | 32[3] |
| 50 | | $\hat{\lambda}$ | 0.8018[1] | 2.9301[4] | 3.8054[6] | 1.6784[2] | 2.5490[3] | 3.7819[5] |
| | \|BIAS\| | $\hat{\rho}$ | 0.2981[1] | 2.2006[4] | 3.3667[6] | 0.8663[2] | 1.6911[3] | 2.7716[5] |
| | | $\hat{\eta}$ | 0.0166[3] | 0.0236[4] | 0.0270[5] | 0.0140[1] | 0.0165[2] | 0.0294[6] |
| | | $\hat{\lambda}$ | 4.7779[1] | 113.6285[5] | 144.4272[6] | 72.4888[2] | 108.2826[4] | 105.6623[3] |
| | MSE | $\hat{\rho}$ | 3.5468[1] | 31.8663[3] | 77.5979[6] | 20.4871[2] | 36.0042[4] | 37.4428[5] |
| | | $\hat{\eta}$ | 0.0008[3] | 0.0009[4] | 0.0013[5] | 0.0005[1] | 0.0007[2] | 0.0017[6] |
| | | $\hat{\lambda}$ | 0.6415[1] | 2.3441[4] | 3.0443[6] | 1.3427[2] | 2.0392[3] | 3.0255[5] |
| | MRE | $\hat{\rho}$ | 0.3974[1] | 2.9341[4] | 4.4889[6] | 1.1551[2] | 2.2548[3] | 3.6955[5] |
| | | $\hat{\eta}$ | 0.1106[3] | 0.1571[4] | 0.1800[5] | 0.0934[1] | 0.1103[2] | 0.1957[6] |
| | | $\sum$ *Ranks* | 15[1.5] | 36[4] | 51[6] | 15[1.5] | 26[3] | 46[5] |
| 150 | | $\hat{\lambda}$ | 0.0989[1] | 0.9333[4] | 1.1936[5] | 0.1206[2] | 0.3617[3] | 1.2445[6] |
| | \|BIAS\| | $\hat{\rho}$ | 0.0404[1] | 0.6839[4] | 1.0018[6] | 0.0562[2] | 0.2096[3] | 0.9571[5] |
| | | $\hat{\eta}$ | 0.0044[3] | 0.0132[4] | 0.0152[5] | 0.0038[1] | 0.0044[2] | 0.0164[6] |
| | | $\hat{\lambda}$ | 0.2584[1] | 2.9905[4] | 4.2131[5] | 0.7434[2] | 1.3911[3] | 4.5594[6] |
| | MSE | $\hat{\rho}$ | 0.0151[1] | 2.5018[4] | 6.5495[6] | 0.0702[2] | 0.6978[3] | 4.4292[5] |
| | | $\hat{\eta}$ | 0.0001[2.5] | 0.0003[4] | 0.0004[5] | 0.0009[1] | 0.0001[2.5] | 0.0004[6] |
| | | $\hat{\lambda}$ | 0.0791[1] | 0.7467[4] | 0.9549[5] | 0.0965[2] | 0.2893[3] | 0.9956[6] |
| | MRE | $\hat{\rho}$ | 0.0538[1] | 0.9119[4] | 1.3358[6] | 0.0749[2] | 0.2795[3] | 1.2761[5] |
| | | $\hat{\eta}$ | 0.0296[3] | 0.0879[4] | 0.1016[5] | 0.0255[1] | 0.0295[2] | 0.1096[6] |
| | | $\sum$ *Ranks* | 14.5[1] | 36[4] | 48[5] | 15[2] | 24.5[3] | 51[6] |
| 300 | | $\hat{\lambda}$ | 0.0072[2] | 0.5653[4] | 0.7309[5] | 0.0039[1] | 0.0655[3] | 0.7627[6] |

(Continued)

**Table 9 (continued)**

| $n$ | Est. | Est. Par. | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|
| | \|BIAS\| | $\hat{\rho}$ | $0.0064^{\{2\}}$ | $0.3571^{\{4\}}$ | $0.5172^{\{5\}}$ | $0.0054^{\{1\}}$ | $0.0327^{\{3\}}$ | $0.5313^{\{6\}}$ |
| | | $\hat{\eta}$ | $0.0011^{\{3\}}$ | $0.0089^{\{4\}}$ | $0.0106^{\{5\}}$ | $0.0009^{\{1\}}$ | $0.0009^{\{2\}}$ | $0.0117^{\{6\}}$ |
| | | $\hat{\lambda}$ | $0.0040^{\{2\}}$ | $0.8395^{\{4\}}$ | $1.1787^{\{5\}}$ | $0.0017^{\{1\}}$ | $0.1110^{\{3\}}$ | $1.3044^{\{6\}}$ |
| | MSE | $\hat{\rho}$ | $0.0014^{\{2\}}$ | $0.3852^{\{4\}}$ | $0.8717^{\{5\}}$ | $0.0009^{\{1\}}$ | $0.0289^{\{3\}}$ | $0.9885^{\{6\}}$ |
| | | $\hat{\eta}$ | $0.0002^{\{2.5\}}$ | $0.0001^{\{4\}}$ | $0.0002^{\{5\}}$ | $0.0001^{\{1\}}$ | $0.0002^{\{2.5\}}$ | $0.0001^{\{6\}}$ |
| | | $\hat{\lambda}$ | $0.0058^{\{2\}}$ | $0.4522^{\{4\}}$ | $0.5847^{\{5\}}$ | $0.0032^{\{1\}}$ | $0.0524^{\{3\}}$ | $0.6102^{\{6\}}$ |
| | MRE | $\hat{\rho}$ | $0.0086^{\{2\}}$ | $0.4761^{\{4\}}$ | $0.6896^{\{5\}}$ | $0.0072^{\{1\}}$ | $0.0436^{\{3\}}$ | $0.7084^{\{6\}}$ |
| | | $\hat{\eta}$ | $0.0071^{\{3\}}$ | $0.0598^{\{4\}}$ | $0.0706^{\{5\}}$ | $0.0060^{\{1\}}$ | $0.0064^{\{2\}}$ | $0.0779^{\{6\}}$ |
| | | $\sum$ *Ranks* | $20.5^{\{2\}}$ | $36^{\{4\}}$ | $45^{\{5\}}$ | $9^{\{1\}}$ | $24.5^{\{3\}}$ | $54^{\{6\}}$ |
| 500 | | $\hat{\lambda}$ | $0.0006^{\{1\}}$ | $0.4155^{\{4\}}$ | $0.5537^{\{5\}}$ | $0.0007^{\{2\}}$ | $0.0093^{\{3\}}$ | $0.5702^{\{6\}}$ |
| | \|BIAS\| | $\hat{\rho}$ | $0.0007^{\{1\}}$ | $0.2589^{\{4\}}$ | $0.3663^{\{5\}}$ | $0.0009^{\{2\}}$ | $0.0051^{\{3\}}$ | $0.3703^{\{6\}}$ |
| | | $\hat{\eta}$ | $0.0006^{\{2\}}$ | $0.0069^{\{4\}}$ | $0.0083^{\{5\}}$ | $0.0001^{\{2\}}$ | $0.0002^{\{2\}}$ | $0.0091^{\{6\}}$ |
| | | $\hat{\lambda}$ | $0.0006^{\{1\}}$ | $0.3689^{\{4\}}$ | $0.5786^{\{5\}}$ | $0.0003^{\{2\}}$ | $0.0081^{\{3\}}$ | $0.6354^{\{6\}}$ |
| | MSE | $\hat{\rho}$ | $0.0005^{\{1\}}$ | $0.1792^{\{4\}}$ | $0.3289^{\{5\}}$ | $0.0009^{\{2\}}$ | $0.0027^{\{3\}}$ | $0.3536^{\{6\}}$ |
| | | $\hat{\eta}$ | $0.00002^{\{2\}}$ | $0.0002^{\{4\}}$ | $0.0001^{\{5\}}$ | $0.0003^{\{2\}}$ | $0.0007^{\{2\}}$ | $0.0001^{\{6\}}$ |
| | | $\hat{\lambda}$ | $0.0005^{\{1\}}$ | $0.3324^{\{4\}}$ | $0.4429^{\{5\}}$ | $0.0005^{\{2\}}$ | $0.0073^{\{3\}}$ | $0.4562^{\{6\}}$ |
| | MRE | $\hat{\rho}$ | $0.0009^{\{1\}}$ | $0.3452^{\{4\}}$ | $0.4884^{\{5\}}$ | $0.0012^{\{2\}}$ | $0.0068^{\{3\}}$ | $0.4937^{\{6\}}$ |
| | | $\hat{\eta}$ | $0.0011^{\{2\}}$ | $0.0465^{\{4\}}$ | $0.0542^{\{5\}}$ | $0.0011^{\{1\}}$ | $0.0011^{\{3\}}$ | $0.0604^{\{6\}}$ |
| | | $\sum$ *Ranks* | $12^{\{1\}}$ | $36^{\{4\}}$ | $45^{\{5\}}$ | $17^{\{2\}}$ | $25^{\{3\}}$ | $54^{\{6\}}$ |

**Table 10:** Ranking results of all estimates for different combinations of $\boldsymbol{\varphi}$

| $\boldsymbol{\varphi}^{\top}$ | $n$ | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|
| $(\hat{\lambda} = 0.15, \hat{\rho} = 0.15, \hat{\eta} = 0.25)$ | 20 | 2 | 1 | 6 | 3 | 4 | 5 |
| | 50 | 5.5 | 2 | 4 | 1 | 3 | 5.5 |
| | 150 | 6 | 2.5 | 4 | 1 | 2.5 | 5 |
| | 300 | 6 | 3 | 4 | 1 | 2 | 5 |
| | 500 | 6 | 3 | 4 | 1 | 2 | 5 |
| $(\hat{\lambda} = 0.15, \hat{\rho} = 0.15, \hat{\eta} = 0.15)$ | 20 | 2 | 3 | 6 | 1 | 4 | 5 |
| | 50 | 2 | 3.5 | 5.5 | 1 | 3.5 | 5.5 |
| | 150 | 4 | 3 | 5 | 1 | 2 | 6 |
| | 300 | 4 | 3 | 5 | 1 | 2 | 6 |
| | 500 | 4 | 3 | 5 | 1 | 2 | 6 |
| $(\hat{\lambda} = 0.25, \hat{\rho} = 0.15, \hat{\eta} = 0.5)$ | 20 | 3 | 1 | 5.5 | 4 | 2 | 5.5 |
| | 50 | 6 | 2 | 4 | 1 | 3 | 5 |
| | 150 | 6 | 2 | 4 | 1 | 3 | 5 |
| | 300 | 5 | 2 | 4 | 1 | 3 | 6 |
| | 500 | 4 | 2 | 5 | 1 | 3 | 6 |

(Continued)

**Table 10 (continued)**

| $\varphi^\top$ | $n$ | MLEs | WLSEs | CRVMEs | MPSEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|
| $(\hat{\lambda}=0.15, \hat{\rho}=0.75, \hat{\eta}=0.15)$ | 20 | 1 | 2 | 6 | 3 | 4 | 5 |
| | 50 | 2 | 4 | 6 | 1 | 3 | 5 |
| | 150 | 4 | 3 | 6 | 1 | 2 | 5 |
| | 300 | 4 | 3 | 6 | 1 | 2 | 5 |
| | 500 | 3 | 4 | 6 | 1 | 2 | 5 |
| $(\hat{\lambda}=0.5, \hat{\rho}=0.5, \hat{\eta}=0.5)$ | 20 | 6 | 1 | 3.5 | 5 | 2 | 3.5 |
| | 50 | 4 | 1 | 5.5 | 3 | 2 | 5.5 |
| | 150 | 4 | 1 | 5 | 2 | 3 | 6 |
| | 300 | 4 | 1 | 5 | 2 | 3 | 6 |
| | 500 | 4 | 1 | 5 | 2 | 3 | 6 |
| $(\hat{\lambda}=0.75, \hat{\rho}=0.75, \hat{\eta}=0.25)$ | 20 | 3.5 | 1 | 5 | 6 | 2 | 3.5 |
| | 50 | 1 | 2 | 5.5 | 4 | 3 | 5.5 |
| | 150 | 2 | 3 | 5 | 1 | 4 | 6 |
| | 300 | 2 | 4 | 6 | 1 | 3 | 5 |
| | 500 | 3 | 4 | 5 | 1 | 2 | 6 |
| $(\hat{\lambda}=0.25, \hat{\rho}=1.25, \hat{\eta}=0.5)$ | 20 | 5 | 1 | 3 | 6 | 2 | 4 |
| | 50 | 4 | 1 | 5.5 | 2 | 3 | 5.5 |
| | 150 | 1 | 2 | 6 | 3 | 4 | 5 |
| | 300 | 1 | 2.5 | 6 | 2.5 | 4 | 5 |
| | 500 | 2 | 3 | 6 | 1 | 4 | 5 |
| $(\hat{\lambda}=1.25, \hat{\rho}=0.75, \hat{\eta}=0.15)$ | 20 | 1 | 4 | 6 | 2 | 5 | 3 |
| | 50 | 1.5 | 4 | 6 | 1.5 | 3 | 5 |
| | 150 | 1 | 4 | 5 | 2 | 3 | 6 |
| | 300 | 2 | 4 | 5 | 1 | 3 | 6 |
| | 500 | 1 | 4 | 5 | 2 | 3 | 6 |
| $\sum$ *Ranks* | | 132.5 | 100.5 | 205 | 77 | 115 | 210 |
| Overall rank | | 4 | 2 | 5 | 1 | 3 | 6 |

From Tables 2–9, it is evident that the values of $|BIAS|$, MSE, and MRE decrease as the sample size $n$ increases, confirming the consistency of the proposed estimators. Based on the partial and overall rankings across all simulation scenarios, the ordering of the estimation methods from best to worst performance is: MPSE, WLSE, ADE, MLE, LSE, CRVME, and RADE. Notably, the MPSE method achieves the lowest overall rank score (74 points), indicating that it is the most accurate and reliable estimator for the parameters of the NExLL distribution. Thus, we recommend MPSE as the preferred estimation technique in practical applications.

In addition to accuracy, we also assessed the computational effort required by each estimation method. All estimators rely on iterative numerical optimization; however, the complexity varies depending on the smoothness and dimensionality of the corresponding objective functions. The MLE and MPSE estimators typically require more iterations and function evaluations due to their likelihood-based nature, leading to slightly higher computational time. In contrast, the LSE, WLSE, ADE, CRVME, and RADE methods generally converge faster because their objective functions are simpler to evaluate. Overall, the MPSE method offers the best trade-off between computational cost and estimation precision, maintaining high efficiency even for moderate to large sample sizes.

## 5 Applications in Medicine and Electronics

In this section, we demonstrate the effectiveness of the MGEx distribution in modeling skewed data by applying it to two real datasets from medicine and electronics. The performance of the MGEx distribution is compared with several alternative models using a range of evaluation criteria, including the Kolmogorov–Smirnov (KS) statistic (along with its associated p-value), the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), the Anderson–Darling statistic ($A^*$), and the Cramér–Von Mises statistic ($W^*$).

The MGEx model is assessed alongside various competing models, such as the modified exponential (MEx) by Rasekhi et al. [2], the Marshall–Olkin alpha power exponential (MOAPEx) by Nassar et al. [4], the odd log-logistic Lindley exponential (OLLLiEx) by Alizadeh et al. [15], the transmuted generalized exponential (TGEx) by Khan et al. [16], the alpha power exponential (APEx) by Mahdavi and Kundu [17], the logistic exponential (LEx) by Lan and Leemis [18], the exponentiated exponential (EEx) by Gupta and Kundu [19], and the standard exponential distributions.

The first dataset comprises the remission times (in months) of 128 bladder cancer patients, as reported by Lee and Wang [22]. The observations are as follows: 8.26, 11.98, 19.13, 7.93, 11.79, 18.10, 2.75, 4.26, 5.41, 43.01, 1.19, 7.63, 5.71, 1.46, 5.34, 7.59, 10.66, 17.12, 46.12, 1.26, 5.06, 7.09, 9.22, 13.80, 8.37, 12.02, 2.02, 3.31, 17.36, 1.40, 3.02, 4.34, 7.26, 9.47, 14.24, 5.49, 7.66, 11.25, 4.40, 5.85, 36.66, 1.05, 2.69, 4.23, 5.41, 0.08, 2.09, 3.36, 6.76, 12.07, 8.65, 12.63, 22.69 3.48, 25.74, 0.50, 2.46, 4.87, 1.76, 3.25, 4.50, 6.25, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 12.03, 20.28, 2.02, 4.51, 6.54, 8.53, 21.73, 2.07, 3.36, 6.93, 25.82, 0.51, 2.54, 3.70, 5.17, 7.62, 10.75, 16.62, 32.15, 2.64, 0.81, 2.62, 3.82, 7.28, 9.74, 14.76, 7.87, 11.64, 6.97, 9.02, 1.35, 2.87, 5.62, 3.88, 5.32, 7.39, 0.90, 2.69, 4.18, 5.32, 7.32, 10.06, 3.64, 5.09, 14.77, 15.96, 2.83, 4.33, 17.14, 79.05, 10.34, 14.83, 34.26, 0.40, 2.26, 3.57, 13.29, 26.31.

In the second dataset, we examine the lifetimes (in days) of 30 electronic devices, as reported by Rahman et al. [23]. The data are as follows: 0.029, 0.020, 0.044, 0.034, 0.057, 0.106, 0.096, 0.139, 0.164, 0.156, 0.167, 0.250, 0.177, 0.326, 0.607, 0.406, 0.650, 0.676, 0.672, 0.736, 0.838, 0.817, 0.910, 0.946, 0.931, 0.953, 0.961, 0.982, 0.990, 0.981.

This section employs R programming to generate numerical results. Tables 11 and 12 display the ML estimates, along with their corresponding standard errors (SEs), and the values of various fit criteria such as AIC, CAIC, BIC, HQIC, $W^*$, $A^*$, KS, and the KS $p$-value for the MGEx distribution. These results are compared to those for the MEx, MOAPEx, OLLLiEx, TGEx, APEx, LEx, EEx, and exponential distributions.

**Table 11:** The ML estimates, SEs and goodness-of-fit measures for bladder cancer dataset

| Model | Par. | Estimates | SEs | AIC | CAIC | BIC | HQIC | $W^*$ | $A^*$ | KS | KS $p$-value |
|---|---|---|---|---|---|---|---|---|---|---|---|
| MGEx | $\hat{\lambda}$ | 0.1740741 | (0.05625669) | 824.9107 | 825.1042 | 833.4668 | 828.3871 | 0.01566231 | 0.09384645 | 0.03652122 | 0.9955879 |
| | $\hat{\rho}$ | 0.3811662 | (0.14859343) | | | | | | | | |
| | $\hat{\eta}$ | 3.0788553 | (1.15382649) | | | | | | | | |
| MEx | $\hat{\alpha}$ | 4.85624192 | (3.93904340) | 835.1997 | 835.5249 | 846.6079 | 839.8349 | 0.1220125 | 0.7305448 | 0.07070481 | 0.5442505 |
| | $\hat{\beta}$ | 40.29137929 | (35.77323927) | | | | | | | | |
| | $\hat{\gamma}$ | 0.71628373 | (0.09611378) | | | | | | | | |
| | $\hat{\lambda}$ | 0.02657186 | (0.02469445) | | | | | | | | |

(Continued)

**Table 11 (continued)**

| Model | Par. | Estimates | SEs | AIC | CAIC | BIC | HQIC | $W^*$ | $A^*$ | KS | KS $p$-value |
|---|---|---|---|---|---|---|---|---|---|---|---|
| MOAPEx | $\hat{\alpha}$ | 0.007273391 | (0.007243386) | 826.5749 | 826.7685 | 835.131 | 830.0513 | 0.05026704 | 0.2858003 | 0.04553301 | 0.9534219 |
| | $\hat{\lambda}$ | 0.064179222 | (0.022669026) | | | | | | | | |
| | $\hat{\theta}$ | 4.531621133 | (2.064911401) | | | | | | | | |
| OLLLiEx | $\hat{\alpha}$ | 0.02292192 | (0.04493293) | 825.5056 | 825.6991 | 834.0617 | 828.9819 | 0.02564318 | 0.164197 | 0.03805873 | 0.992498 |
| | $\hat{\lambda}$ | 0.05507419 | (0.02674479) | | | | | | | | |
| | $\hat{\theta}$ | 1.41681607 | (0.15593533) | | | | | | | | |
| TGEx | $\hat{\alpha}$ | 1.30275033 | (0.14893158) | 827.8933 | 828.0869 | 836.4494 | 831.3697 | 0.0608215 | 0.3664275 | 0.05210095 | 0.8779299 |
| | $\hat{\lambda}$ | 0.08777606 | (0.01828324) | | | | | | | | |
| | $\hat{\theta}$ | 0.69832738 | (0.22974386) | | | | | | | | |
| APEx | $\hat{\alpha}$ | 1.1716160 | ( 0.84168056) | 832.6364 | 832.7324 | 838.3404 | 834.954 | 0.12816 | 0.7663824 | 0.07940018 | 0.3950706 |
| | $\hat{\lambda}$ | 0.1112739 | (0.02262234) | | | | | | | | |
| LEx | $\hat{\lambda}$ | 0.100694 | (0.008481586) | 829.2507 | 829.3467 | 834.9548 | 831.5683 | 0.1010008 | 0.6050901 | 0.06909518 | 0.5741454 |
| | $\hat{\theta}$ | 1.163382 | (0.091502303) | | | | | | | | |
| EEx | $\hat{\alpha}$ | 1.217971 | (0.14883390) | 830.1552 | 830.2512 | 835.8592 | 832.4728 | 0.1122072 | 0.6741157 | 0.07251524 | 0.5113106 |
| | $\hat{\lambda}$ | 0.121168 | (0.01357185) | | | | | | | | |
| Ex | $\hat{\lambda}$ | 0.106774 | (0.009436746) | 830.6838 | 830.7155 | 833.5358 | 831.8426 | 0.1192913 | 0.7159815 | 0.08463457 | 0.3183423 |

**Table 12:** The ML estimates, SEs and goodness-of-fit measures for electronic devices dataset

| Model | Par. | Estimates | SEs | AIC | CAIC | BIC | HQIC | $W^*$ | $A^*$ | KS | KS $p$-value |
|---|---|---|---|---|---|---|---|---|---|---|---|
| MGEx | $\hat{\lambda}$ | 20.86077 | (0.3835859) | 4.770589 | 5.693666 | 8.974181 | 6.115354 | 0.1401965 | 0.8371264 | 0.1535515 | 0.4356301 |
| | $\hat{\rho}$ | 0.0766334 | (0.01636357) | | | | | | | | |
| | $\hat{\eta}$ | 569957800 | (109588.3) | | | | | | | | |
| MEx | $\hat{\alpha}$ | 0.5139658 | (1.5727675) | 24.74802 | 26.34802 | 30.35281 | 26.54104 | 0.2194284 | 1.339615 | 0.195456 | 0.1766703 |
| | $\hat{\beta}$ | 0.4956883 | (1.7541497) | | | | | | | | |
| | $\hat{\gamma}$ | 0.8962775 | (0.4431781) | | | | | | | | |
| | $\hat{\lambda}$ | 5.6880142 | (17.5434020) | | | | | | | | |
| MOAPEx | $\hat{\alpha}$ | 1.000002 | (1.632437) | 22.66499 | 23.58807 | 26.86858 | 24.00976 | 0.2184728 | 1.333118 | 0.1803265 | 0.251524 |
| | $\hat{\lambda}$ | 2.710378 | (0.776356) | | | | | | | | |
| | $\hat{\theta}$ | 1.965066 | (2.046472) | | | | | | | | |
| OLLLiEx | $\hat{\alpha}$ | 2.9391148 | (4.2729675) | 22.98003 | 23.9031 | 27.18362 | 24.32479 | 0.2211404 | 1.348956 | 0.1904917 | 0.1990343 |
| | $\hat{\lambda}$ | 2.4128795 | (0.5775431) | | | | | | | | |
| | $\hat{\theta}$ | 0.9056628 | (0.3714769) | | | | | | | | |
| TGEx | $\hat{\alpha}$ | 0.9929090 | (0.3236882) | 23.29842 | 24.22149 | 27.50201 | 24.64318 | 0.2212886 | 1.351685 | 0.1989204 | 0.1622635 |
| | $\hat{\lambda}$ | 2.2520989 | (0.5215658) | | | | | | | | |
| | $\hat{\theta}$ | −0.2525669 | (0.4719531) | | | | | | | | |
| APEx | $\hat{\alpha}$ | 2.272690 | (2.1976612) | 21.04641 | 21.49086 | 23.84881 | 21.94292 | 0.2205111 | 1.346077 | 0.192386 | 0.1902562 |
| | $\hat{\lambda}$ | 2.419151 | (0.5923636) | | | | | | | | |
| LEx | $\hat{\lambda}$ | 2.024056 | (0.4334621) | 21.69086 | 22.13531 | 24.49326 | 22.58737 | 0.2210701 | 1.352842 | 0.2073121 | 0.1312102 |
| | $\hat{\theta}$ | 1.000075 | (0.1707643) | | | | | | | | |
| EEx | $\hat{\alpha}$ | 1.078672 | (0.2556855) | 21.59032 | 22.03476 | 24.39271 | 22.48683 | 0.2213578 | 1.354566 | 0.2059523 | 0.1358883 |
| | $\hat{\lambda}$ | 2.121508 | (0.4863517) | | | | | | | | |
| Ex | $\hat{\lambda}$ | 2.024155 | (0.3695584) | 19.69086 | 19.83372 | 21.09206 | 20.13912 | 0.2210675 | 1.352828 | 0.2073161 | 0.1311968 |

The analysis in Tables 11 and 12 reveal that the MGEx distribution yields the lowest values for all goodness-of-fit criteria among the models considered. This suggests that the MGEx distribution is the optimal choice for modeling the datasets. The results consistently show that the MGEx distribution outperforms the other fitted models across all goodness-of-fit measures.

Furthermore, Figs. 3 and 4 present the fitted functions for the MGEx model, including the PDF, CDF, SF, and probability-probability (PP) plots for both datasets. The PP plots for all the fitted models are provided in Figs. 5 and 6 for each dataset. These plots further support the findings from Tables 11 and 12, offering strong evidence that the MGEx distribution is a robust alternative to existing exponential extensions.

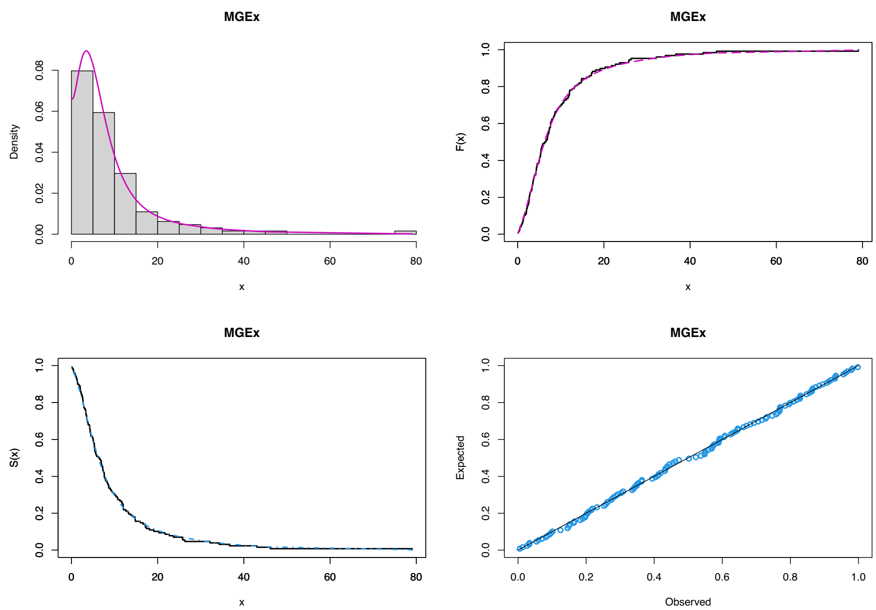


**Figure 3:** The fitted MGEx PDF, CDF, SF, and PP plots for the bladder cancer dataset

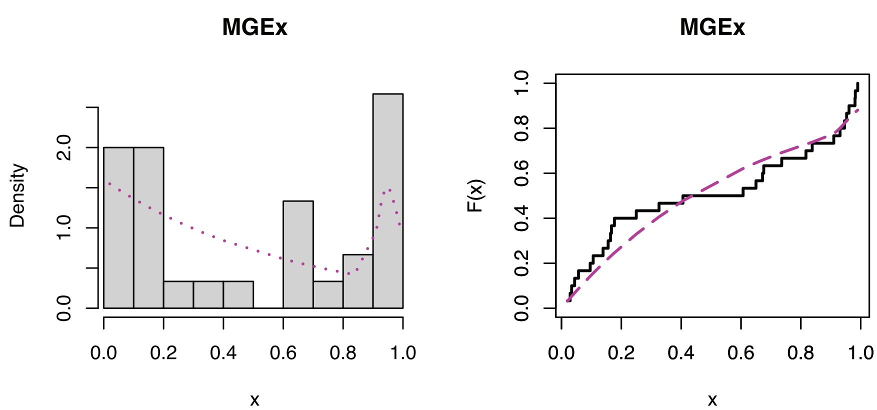


**Figure 4:** (Continued)

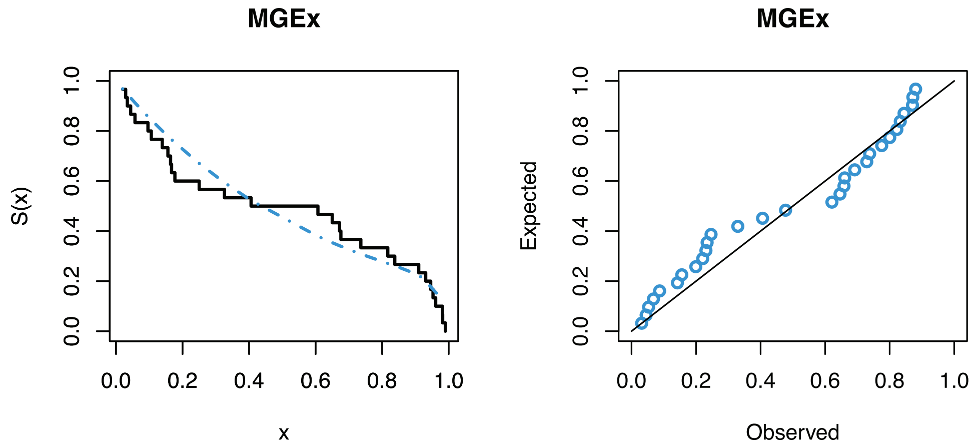


**Figure 4:** The fitted MGEx PDF, CDF, SF, and PP plots for the electronic devices dataset

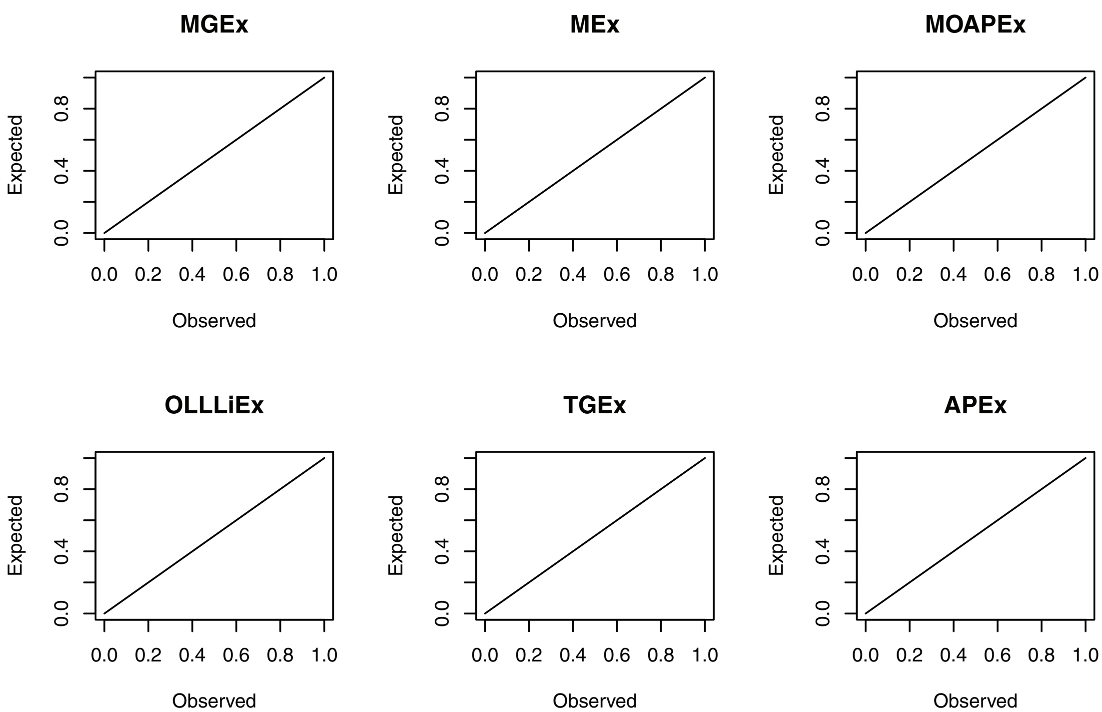


**Figure 5:** (Continued)

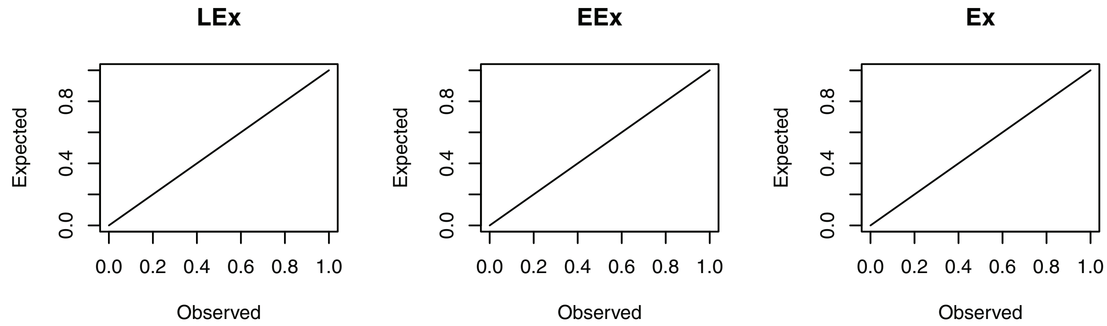


**Figure 5:** PP plots of the MGEx distribution with other models for balder cancer dataset

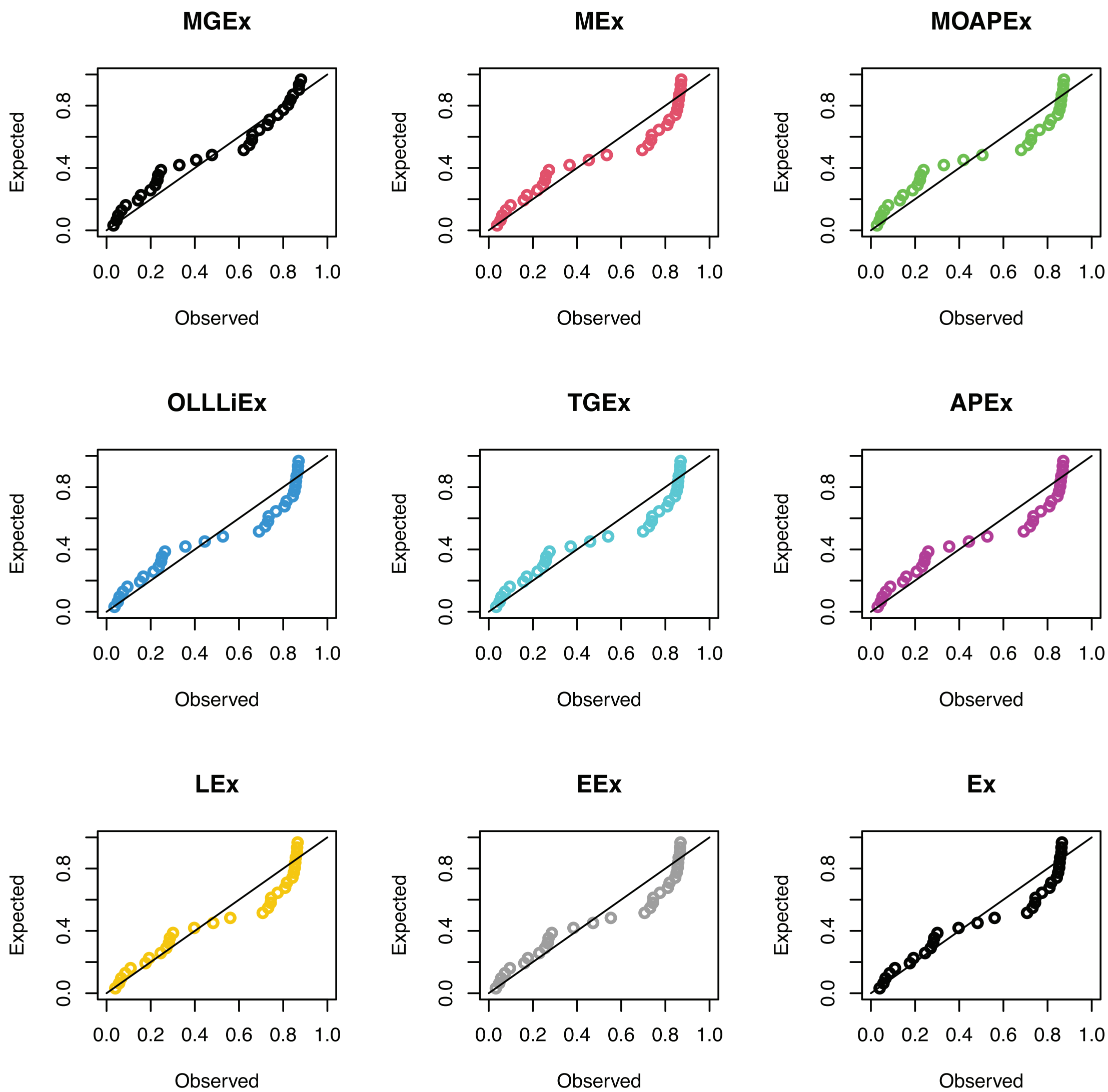


**Figure 6:** PP plots of the MGEx distribution with other models for electronic devices dataset

## 6 Conclusions and Future Scope

This paper introduces a new flexible extension of the exponential distribution, called the modified generalized-exponential (MGEx) distribution. The MGEx density is expressed as a mixture of exponential densities, and its failure rate can exhibit a variety of shapes, including increasing, unimodal, and decreasing patterns. We derive the mathematical properties of the MGEx model and employ six different estimation methods to determine its parameters. Extensive simulation studies are conducted to evaluate the accuracy of these estimation techniques. The results show that the maximum product of spacing (MPS) method outperforms all other approaches, achieving the best overall performance. Based on these findings, we recommend the MPS method for estimating the parameters of the MGEx distribution.

Additionally, we explore the practical applications of the MGEx distribution by modeling two real-life datasets. Goodness-of-fit statistics indicate that the MGEx distribution provides a better fit compared to competing exponential models.

The proposed MGEx distribution opens several avenues for further investigation. Future work could explore Bayesian and Bayesian-nonparametric estimation frameworks to incorporate prior information and uncertainty quantification. Another promising extension lies in regression-based modeling, where covariate effects can be integrated to analyze lifetime or reliability data with explanatory variables. Moreover, multivariate or copula-based generalizations of the MGEx model could provide flexible tools for modeling dependent lifetimes. Further, the analysis of censored or truncated data within the MGEx framework represents an important practical direction.

In addition, the development of a Double MGEx distribution constitutes a natural extension of the present work. Such a model, incorporating an additional shape parameter, is expected to offer enhanced flexibility for modeling complex data patterns. This potential extension will be examined in a separate future study, including its mathematical properties, estimation procedures, and empirical performance.

In terms of applications, the flexibility of the MGEx distribution makes it suitable for a wide range of fields, including hydrology, environmental sciences, actuarial modeling, biomedical survival analysis, and financial risk management. These domains often involve non-negative, skewed, or heavy-tailed data, where the MGEx model's adaptable structure can yield improved inference and prediction performance.

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**Conflicts of Interest:** The author declares no conflicts of interest to report regarding the present study.

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