Qualitative Analysis of Nonlinear Systems Involving Hadamard-Type Fractional Derivatives with Nonlocal Boundary Conditions and Stability Properties

Murugesan Manigandan[1,2], Muath Awadalla[3,*], R. Sathiya Shanthi[4] and Salma Trabelsi[3]

[1] Center for Nonlinear and Complex Networks, SRM TRP Engineering College, Tiruchirapalli, 621105, Tamil Nadu, India

[2] Center for Research, Easwari Engineering College, Chennai, 600089, Tamil Nadu, India

[3] Department of Mathematics and Statistics, College of Science, King Faisal University, Al Ahsa, 31982, Saudi Arabia

[4] Department of Mathematics, School of Agricultural Sciences, Dhanalakshmi Srinivasan University, Tiruchirapalli, 621112, Tamil Nadu, India

INFORMATION

Keywords:
Coupled systems
hadamard fractional derivative
existence
fixed point

DOI: 10.23967/j.rimni.2025.10.72244

SCIPEDIA Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería

Qualitative Analysis of Nonlinear Systems Involving Hadamard-Type Fractional Derivatives with Nonlocal Boundary Conditions and Stability Properties

Murugesan Manigandan[1,2], Muath Awadalla[3,*], R. Sathiya Shanthi[4] and Salma Trabelsi[3]

[1]Center for Nonlinear and Complex Networks, SRM TRP Engineering College, Tiruchirapalli, 621105, Tamil Nadu, India

[2]Center for Research, Easwari Engineering College, Chennai, 600089, Tamil Nadu, India

[3]Department of Mathematics and Statistics, College of Science, King Faisal University, Al Ahsa, 31982, Saudi Arabia

[4]Department of Mathematics, School of Agricultural Sciences, Dhanalakshmi Srinivasan University, Tiruchirapalli, 621112, Tamil Nadu, India

ABSTRACT

This paper establishes a comprehensive analysis of a coupled system of nonlinear Hadamard-type fractional differential equations subject to generalized nonlocal integral boundary conditions. The distinct logarithmic kernel of the Hadamard derivative makes this framework particularly suitable for modeling scale-invariant processes and ultraslow diffusion phenomena. The existence and uniqueness of solutions are rigorously investigated using fixed point theory: Banach's contraction principle ensures uniqueness, while the Leray-Schauder nonlinear alternative guarantees existence under more general growth conditions. Furthermore, the system is proven to be Ulam-Hyers stable, ensuring that approximate solutions remain close to exact solutions, which is crucial for the robustness of the model in practical applications. The theoretical findings are effectively validated through two detailed numerical examples, demonstrating the applicability of the established results to different classes of nonlinearities.

OPEN ACCESS

Received: 22/08/2025

Accepted: 03/11/2025

Published: 23/01/2026

DOI
10.23967/j.rimni.2025.10.72244

Keywords:
Coupled systems
hadamard fractional derivative
existence
fixed point

1 Introduction

Systems of fractional differential equations (FDEs), which involve derivatives of non-integer orders, have garnered significant attention due to their ability to model complex, real-world phenomena exhibiting memory and hereditary properties. These systems have been effectively applied across various fields, including physics, engineering, and biology. For instance, in viscoelastic materials, fractional differential equations accurately describe stress-strain relationships, capturing both elastic and viscous behavior [1]. In control systems, fractional-order controllers offer enhanced flexibility and robustness compared to traditional integer-order controllers, leading to improved system performance [2]. Furthermore, the application of fractional calculus has expanded to advanced areas such as the study of systems with electrical screening effects using fractional quantum mechanics [3] and the analysis of nonlinear fractional dynamics with kicks [4].

*Correspondence: Muath Awadalla (mawadalla@kfu.edu.sa).

Recent research has substantially expanded the scope of fractional differential systems into more sophisticated modeling and control paradigms. In drug therapy, the authors investigate a coupled system for drug therapy using piece-wise modeling, focusing on existence and uniqueness of solutions through fixed point results. Numerical solutions are obtained via Newton interpolation, with graphical representation using real values and fractional orders to demonstrate the model's dynamics [5]. In bio-mathematics, a novel model for alcohol consumption dynamics with complications, using conformable fractional order derivatives, explores its existence, uniqueness, stability, and numerical solutions through fixed point theory and graphical analysis [6].

The versatility of fractional differential equations in modeling such diverse and intricate systems underscores their importance in addressing complex real-world problems. Recent advances have further extended to trajectory controllability problems for complex fractional systems, including Hilfer fractional stochastic pantograph equations with random impulses [7] and higher-order Riemann-Liouville fractional stochastic systems via integral contractors in new Banach spaces [8]. Studies on the partial approximate controllability of systems with conformable derivatives in Hilbert spaces have also been explored [9]. Furthermore, applications in thermoelasticity and opto-elasticity have emerged, with analytical solutions developed for time-fractional heat order in magneto-photothermal semiconductor media with Thomson effects and initial stress [10], and investigations of fractional coupled nonlocal-microstretch effects on thermo-opto-elastic wave propagation in semiconductor media under photothermal and strong magnetic excitations [11].

Among the various fractional operators, the choice of derivative is crucial and depends on the specific physical context. The **Caputo derivative** is widely used for its ability to handle standard initial conditions with physical interpretations, making it suitable for many engineering applications. The **Hilfer derivative** generalizes both the Riemann-Liouville and Caputo derivatives, offering an intermediate perspective that is useful in certain viscoelasticity and relaxation processes. In contrast, the **Hadamard-type fractional derivative** offers a distinctive approach, defined using a logarithmic kernel $(\log(t/\tau))^{\alpha-1}$ instead of the power-law kernel $(t-\tau)^{\alpha-1}$ used by Caputo and Riemann-Liouville operators [12,13].

The primary advantage of the Hadamard derivative lies in its **scale-invariance** or **dilational symmetry**, making it naturally suited for problems on semi-infinite domains and for modeling processes characterized by **ultraslow diffusion** and **logarithmic creep**. For example, while a Caputo derivative might model a material that creeps as t^α, a Hadamard derivative can model one that creeps as $(\log t)^\alpha$, which is observed in the mechanical behavior of certain polymers, sedimentary rocks, and geological formations where the decay of memory effects follows a logarithmic rather than power-law pattern [14,15]. This unique capability makes the Hadamard fractional derivative an indispensable tool for a more precise modeling of systems with complex, non-linear dynamics that evolve logarithmically in time.

Fixed point theory is a fundamental mathematical framework that plays a crucial role in analyzing differential equations, particularly in establishing the existence and uniqueness of solutions [16]. By identifying points that remain invariant under specific mappings, fixed point theorems provide the necessary conditions to ensure that differential equations have solutions that are both existent and unique. For instance, the Banach fixed-point theorem, also known as the contraction mapping principle, is instrumental in proving the existence and uniqueness of solutions to ordinary differential equations by demonstrating that a contraction mapping on a complete metric space has a single fixed point [17]. Similarly, the Schauder fixed-point theorem extends these concepts to more general settings, facilitating the analysis of solutions in partial differential equations. The application of fixed point theory thus provides a robust methodological approach for addressing complex problems in differential equations, ensuring that solutions can be systematically identified and analyzed [18–21].

Nonlocal boundary conditions in fractional differential equations account for influences that extend beyond a single point, reflecting the inherent memory and long-range interactions in complex systems. This approach allows models to capture broader spatial or temporal effects, leading to more accurate and realistic descriptions of physical, biological, and engineering phenomena [22]. However, it is important to acknowledge the **potential limitations** of such nonlocal conditions. They can introduce significant complexity into both the theoretical analysis and the numerical resolution of the problems. From a theoretical standpoint, establishing the existence and uniqueness of solutions often requires more sophisticated techniques and stricter assumptions compared to local boundary conditions. From a numerical perspective, the nonlocal terms necessitate the computation of integrals over the domain, which can be computationally expensive and may require specialized quadrature methods to handle possible singularities, especially in the context of Hadamard integrals with logarithmic kernels.

Stability analysis of fractional differential equations is crucial for understanding how solutions respond to perturbations, which is vital in accurately modeling systems with memory and hereditary properties. The Ulam-Hyers stability concept, originating from Ulam's 1940 problem and Hyers' subsequent 1941 solution, provides a framework for assessing the robustness of solutions to functional equations, including those of fractional order [23]. This stability criterion examines whether approximate solutions remain close to exact solutions, ensuring the reliability of mathematical models in practical applications [24]. Recent studies have applied Ulam-Hyers stability to various classes of fractional differential equations, demonstrating its effectiveness in establishing the robustness of solutions.

Several recent works have investigated systems related to our current study. In [25], the authors examined the existence and uniqueness of the given system

$$\begin{cases} {}^{H}\mathcal{D}^{\varrho_1}\varphi(t) + \Pi_1(t, \varpi(t)) = l_{\Pi_1}, & t \in (1, e), \\ {}^{H}\mathcal{D}^{\varrho_2}\varpi(t) + \Pi_2(t, \varphi(t)) = l_{\Pi_2}, & t \in (1, e), \\ \varphi^{(j)}(1) = \varpi^{(j)}(1) = 0, & 0 \le j \le n-2, \\ \varphi(e) = a\varpi(\xi), \quad \varpi(e) = b\varphi(\nu), & \xi, \nu \in (1, e), \end{cases}$$

where a, b are two variables with $0 < ab(\log \nu)^{\varrho_1 - 1}(\log \xi)^{\varrho_2 - 1} < 1, \varrho_1, \varrho_2 \in (n-1, n]$ are two real numbers and $n \ge 3, \Pi_1, \Pi_2 \in \mathcal{C}([1, e] \times (-\infty, +\infty), (-\infty, +\infty)),\ l_{\Pi_1}, l_{\Pi_2} > 0$ are constants, and ${}^{H}\mathcal{D}^{\varrho_1}$, ${}^{H}\mathcal{D}^{\varrho_2}$ are the Hadamard fractional derivatives of fractional order.

In [26], the authors utilized Darbo's fixed point theorem to investigate the conditions for the existence and uniqueness of solutions to hybrid Caputo-Hadamard fractional sequential differential equations.

$$\begin{cases} \left[\mathbb{C}^{H}\mathcal{D}_{a}^{\varrho_1+1} + \tau\mathbb{C}^{H}\mathcal{D}_{a}^{\varrho_1}\right]\left(\dfrac{s(t)}{\varphi(t, s(t))}\right) = \Pi_1(t), & t \in \mathcal{E} = [1, e], \\ \left(\dfrac{s(t)}{\varphi(t, s(t))}\right)\Bigg|_{t=a} = 0, \quad \left(\dfrac{s(t)}{\varphi(t, s(t))}\right)^{(1)}\Bigg|_{t=a} = 0, \quad \left(\dfrac{s(t)}{\varphi(t, s(t))}\right)^{(2)}\Bigg|_{t=a} = 0, \end{cases}$$

where $\varrho_1 \in (1, 2],\ t \in \mathcal{E} = [a, e],\ 1 \le a < e,\ {}^{CH}\mathcal{D}_{a}^{\varrho_1}$ is a Caputo-Hadamard fractional derivative of order $\varrho_1, \varphi : \mathcal{E} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $\Pi_1 : \mathcal{E} \to \mathbb{R}$ are continuous functions, and τ is a real number.

In 2022, the authors proved the existence and uniqueness of solutions for the nonlocal boundary value problem [27].

$$\begin{cases} {}^{c}\mathcal{D}_{0+}^{\rho_1}\varphi(t) + \Lambda(t,\varphi) = 0, \quad 0 < \rho_1 \le \zeta, \; t \in [c,d], \\ \varphi(c) = \varphi'(c) = 0, \varphi(d) = \mathbb{K}\varphi(\beta), \end{cases}$$

where ${}^{c}\mathcal{D}_{0+}^{\rho_1}$ is the Caputo derivative of order ρ_1, $\Lambda : [c,d] \times \mathbb{R} \to \mathbb{R}$ is continuous function $\beta \in \mathbb{R}$.

Recently, AE Taier Taylor et al. [28] explored the existence results for a nonlocal boundary value problem involving Caputo-type Hadamard hybrid fractional integro-differential equations.

$$\begin{cases} {}_{c}^{H}\mathcal{D}^{\varrho_1}\left(\frac{\varphi(t) - \sum_{i=1}^{m} I_i^{\varrho_2}\Pi_{1,i}(t,\varphi(t))}{\Pi_2(t,\varphi(t))}\right) = \mathbb{H}(t,\varphi(t)), \quad 0 < \varrho_1 < 1, \\ \varphi(1) = 0, \quad \varphi(e) = \mu(\varphi), \end{cases}$$

where ${}_{c}^{H}\mathcal{D}^{\varrho_1}$ is the Caputo-type Hadamard fractional derivative of order.

The Present Work: System under Investigation

In this paper, we investigate the following coupled system of nonlinear Hadamard fractional differential equations with nonlocal integral boundary conditions:

$$\begin{cases} \mathcal{D}^{\varrho_1}\varphi(t) = \Pi_1(t,\varphi(t),\varpi(t)), \quad 1 < t < e, 1 < \varrho_1 \le 2, \\ \mathcal{D}^{\varrho_2}\varpi(t) = \Pi_2(t,\varphi(t),\varpi(t)), \quad 1 < t < e, 1 < \varrho_2 \le 2, \\ \varphi(1) = 0, \quad \frac{\mathfrak{Z}_1}{\Gamma(\gamma)}\int_1^{v}\left(\log\frac{v}{s}\right)^{\gamma-1}\frac{\varphi(s)}{s}ds + \mathfrak{Z}_2\varphi(e) = \mathfrak{Z}_3, \quad \gamma > 0, 1 < v < e, \\ \varpi(1) = 0, \quad \frac{\mathfrak{W}_1}{\Gamma(\vartheta)}\int_1^{\psi}\left(\log\frac{\psi}{s}\right)^{\vartheta-1}\frac{\varpi(s)}{s}ds + \mathfrak{W}_2\varpi(e) = \mathfrak{W}_3, \quad \vartheta > 0, 1 < \psi < e, \end{cases} \tag{1}$$

where $\mathcal{D}^{\varrho_1}, \mathcal{D}^{\varrho_2}$ are the Hadamard fractional derivatives of order ϱ_1 and ϱ_2, respectively, $\Pi_1, \Pi_2 : [1,e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and $\mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{W}_1, \mathfrak{W}_2, \mathfrak{Z}_3, \mathfrak{W}_3$ are real constants with appropriate conditions to ensure the non-resonance case.

The originality of our work lies in addressing this specific coupled system (1) with several novel aspects:

- The consideration of a fully coupled nonlinear system with cross-dependencies, analyzed using the Hadamard derivative for its advantages in scale-invariant problems.
- The incorporation of generalized nonlocal boundary conditions involving Hadamard integrals, maintaining mathematical consistency with the differential operator. While these conditions enhance physical realism, we acknowledge the associated analytical and computational challenges.
- A comprehensive analysis that seamlessly integrates existence (via Leray-Schauder), uniqueness (via Banach), and Ulam-Hyers stability into a unified framework.

The structure of the document is as follows: Section 2 introduces the key concepts of fractional calculus relevant to this study, along with an auxiliary lemma related to the linear versions of the problem (1). The main results are discussed in Section 3, while Section 4 focuses on the stability analysis using the Ulam-Hyers technique. An illustrative example is provided in Section 5, followed by a discussion on the practical significance of our findings in Section 6. The paper concludes in Section 7 with a summary and directions for future research.

2 Preliminaries

Definition 1: *[12] The Hadamard derivative of a fractional order $\mathfrak{q}$ for a function $\mathfrak{g} : [1, \infty) \to \mathbb{R}$ is characterized as follows:*

$$\mathcal{D}^{\mathfrak{q}}\mathfrak{g}(\mathfrak{t}) = \frac{1}{\Gamma(n-\mathfrak{q})}\left(\mathfrak{t}\frac{d}{d\mathfrak{t}}\right)^{n}\int_{1}^{\mathfrak{t}}\left(\log\frac{\mathfrak{t}}{s}\right)^{n-\mathfrak{q}-1}\frac{\mathfrak{g}(s)}{s}ds, \quad n-1 < q < n, n = [\mathfrak{q}]+1.$$

Definition 2: *[12] The Hadamard fractional integral of order $\mathfrak{q}$ for a given function $\mathfrak{g}$ is expressed as*

$$\mathcal{I}^{\mathfrak{q}}\mathfrak{g}(\mathfrak{t}) = \frac{1}{\Gamma(\mathfrak{q})}\int_{1}^{\mathfrak{t}}\left(\log\frac{\mathfrak{t}}{s}\right)^{\mathfrak{q}-1}\frac{\mathfrak{g}(s)}{s}ds, \quad \mathfrak{q} > 0.$$

Lemma 1: [12] *Let $\varphi \in \mathcal{C}^{n}_{\infty}\Big([\mathfrak{a}, \mathcal{T}], \mathbb{R}\Big)$. Then*

$${}^{\mathcal{H}}\mathcal{I}\Big({}^{\mathcal{H}}\mathcal{D}^{\mathfrak{q}}\varphi\Big)(\mathfrak{t}) = \varphi(\mathfrak{t}) - \sum_{j=1}^{n} c_j\left(In\frac{\mathfrak{t}}{\mathfrak{a}}\right)^{n-j},$$

here $\mathcal{C}^{n}_{\infty}\Big([\mathfrak{a}, \mathcal{T}], \mathbb{R}\Big) = \Big\{\varphi : [\mathfrak{a}, \mathcal{T}] \to \mathbb{R} : \delta^{n-1}\varphi \in \mathcal{C}\Big([\mathfrak{a}, \mathcal{T}], \mathbb{R}\Big)\Big\}$.

Lemma 2: *Given $\mathfrak{H}_1, \mathfrak{H}_2 \in \mathcal{C}([1, e], \mathbb{R})$, the unique solution of the problem*

$$\begin{cases} \mathcal{D}^{\varrho_1}\varphi(\mathfrak{t}) = \mathfrak{H}_1(\mathfrak{t}), \quad 1 < \mathfrak{t} < e, 1 < \varrho_1 \leq 2, \\ \mathcal{D}^{\varrho_2}\varpi(\mathfrak{t}) = \mathfrak{H}_2(\mathfrak{t}), \quad 1 < \mathfrak{t} < e, 1 < \varrho_2 \leq 2, \\ \varphi(1) = 0, \quad \dfrac{\mathfrak{Z}_1}{\Gamma(\gamma)}\displaystyle\int_{1}^{\upsilon}\left(\log\frac{\upsilon}{s}\right)^{\gamma-1}\frac{\varphi(s)}{s}ds + \mathfrak{Z}_2\varphi(e) = \mathfrak{Z}_3, \quad \gamma > 0, 1 < \upsilon < e, \\ \varpi(1) = 0, \quad \dfrac{\mathfrak{W}_1}{\Gamma(\vartheta)}\displaystyle\int_{1}^{\psi}\left(\log\frac{\psi}{s}\right)^{\vartheta-1}\frac{\varpi(s)}{s}ds + \mathfrak{W}_2\varpi(e) = \mathfrak{W}_3, \quad \vartheta > 0, 1 < \psi < e, \end{cases} \tag{2}$$

is given by

$$\varphi(\mathfrak{t}) = \mathcal{I}^{\varrho_1}\mathfrak{H}_1(\mathfrak{t}) + (\log\mathfrak{t})^{\varrho_1-1}\frac{\mathfrak{Z}_3 - \mathfrak{Z}_1\mathcal{I}^{\gamma+\varrho_1}\mathfrak{H}_1(\upsilon) - \mathfrak{Z}_2\mathcal{I}^{\varrho_1}\mathfrak{H}_1(e)}{\mathfrak{Z}_2 + \frac{\mathfrak{Z}_1\Gamma(\varrho_1)}{\Gamma(\gamma+\varrho_1)}(\log\upsilon)^{\gamma+\varrho_1-1}}, \tag{3}$$

and

$$\varpi(\mathfrak{t}) = \mathcal{I}^{\varrho_2}\mathfrak{H}_2(\mathfrak{t}) + (\log\mathfrak{t})^{\varrho_2-1}\frac{\mathfrak{W}_3 - \mathfrak{W}_1\mathcal{I}^{\vartheta+\varrho_2}\mathfrak{H}_2(\upsilon) - \mathfrak{W}_2\mathcal{I}^{\varrho_2}\mathfrak{H}_2(e)}{\mathfrak{W}_2 + \frac{\mathfrak{W}_1\Gamma(\varrho_2)}{\Gamma(\vartheta+\varrho_2)}(\log\psi)^{\vartheta+\varrho_2-1}}. \tag{4}$$

Proof: Taking the fractional integral of Hadamard sense for both sides of (2) yields

$$\varphi(\mathfrak{t}) = \mathcal{I}^{\varrho_1}\mathfrak{H}_1(\mathfrak{t}) + c_1(\log\mathfrak{t})^{\varrho_1-1} + c_2(\log\mathfrak{t})^{\varrho_1-2}, \tag{5}$$

$$\varpi(\mathfrak{t}) = \mathcal{I}^{\varrho_2}\mathfrak{H}_2(\mathfrak{t}) + d_1(\log\mathfrak{t})^{\varrho_2-1} + d_2(\log\mathfrak{t})^{\varrho_2-2}, \tag{6}$$

since $1 < \varrho_1, \varrho_2 \leq 2$, $c_2 = d_2 = 0$ is an implication of first boundary condition, and

$$\begin{aligned} \mathcal{I}^{\varrho_1}\varphi(\upsilon) &= \mathcal{I}^{\gamma+\varrho_1}\mathfrak{H}_1(\upsilon) + \frac{c_1}{\Gamma(\gamma)}\int_{1}^{\upsilon}\left(\log\frac{\upsilon}{s}\right)^{\gamma-1}\frac{(\log s)^{\varrho_1-1}}{s}ds \\ &= \mathcal{I}^{\gamma+\varrho_1}\mathfrak{H}_1(\upsilon) + c_1\frac{\Gamma(\varrho_1)}{\Gamma(\gamma+\varrho_1)}(\log\upsilon)^{\gamma+\varrho_1-1}, \end{aligned}$$

$$\mathcal{I}^{\varrho_2}\varpi(\psi) = \mathcal{I}^{\vartheta+\varrho_2}\mathfrak{H}_2(\psi) + \frac{d_1}{\Gamma(\vartheta)}\int_1^{\psi}\left(\log\frac{\psi}{s}\right)^{\vartheta-1}\frac{(\log s)^{\varrho_2-1}}{s}ds$$
$$= \mathcal{I}^{\vartheta+\varrho_2}\mathfrak{H}_2(\psi) + d_1\frac{\Gamma(\varrho_2)}{\Gamma(\vartheta+\varrho_2)}(\log\psi)^{\vartheta+\varrho_2-1}.$$

The second boundary conditions implies

$$\mathfrak{Z}_1\mathcal{I}^{\gamma+\varrho_1}\mathfrak{H}_1(\upsilon) + \mathfrak{Z}_1 c_1\frac{\Gamma(\varrho_1)}{\Gamma(\gamma+\varrho_1)}(\log\upsilon)^{\gamma+\varrho_1-1} + \mathfrak{Z}_2\mathcal{I}^{\varrho_1}\mathfrak{H}_1(e) + \mathfrak{Z}_2 c_1 = \mathfrak{Z}_3,$$
$$\mathfrak{W}_1\mathcal{I}^{\vartheta+\varrho_2}\mathfrak{H}_2(\psi) + \mathfrak{W}_1 d_1\frac{\Gamma(\varrho_2)}{\Gamma(\vartheta+\varrho_2)}(\log\psi)^{\vartheta+\varrho_2-1} + \mathfrak{W}_2\mathcal{I}^{\varrho_2}\mathfrak{H}_2(e) + \mathfrak{W}_2 d_1 = \mathfrak{W}_3,$$

and

$$\mathfrak{Z}_1 c_1\frac{\Gamma(\varrho_1)}{\Gamma(\gamma+\varrho_1)}(\log\upsilon)^{\gamma+\varrho_1-1} + \mathfrak{Z}_2 c_1 = \mathfrak{Z}_3 - \mathfrak{Z}_1\mathcal{I}^{\gamma+\varrho_1}\mathfrak{H}_1(\upsilon) - \mathfrak{Z}_2\mathcal{I}^{\varrho_1}\mathfrak{H}_1(e),$$
$$\mathfrak{W}_1 d_1\frac{\Gamma(\varrho_2)}{\Gamma(\vartheta+\varrho_2)}(\log\psi)^{\vartheta+\varrho_2-1} + \mathfrak{W}_2 d_1 = \mathfrak{W}_3 - \mathfrak{W}_1\mathcal{I}^{\vartheta+\varrho_2}\mathfrak{H}_2(\psi) - \mathfrak{W}_2\mathcal{I}^{\varrho_2}\mathfrak{H}_2(e),$$

then

$$c_1\left(\mathfrak{Z}_1\frac{\Gamma(\varrho_1)}{\Gamma(\gamma+\varrho_1)}(\log\upsilon)^{\gamma+\varrho_1-1} + \mathfrak{Z}_2\right) = \mathfrak{Z}_3 - \mathfrak{Z}_1\mathcal{I}^{\gamma+\varrho_1}\mathfrak{H}_1(\upsilon) - \mathfrak{Z}_2\mathcal{I}^{\varrho_1}\mathfrak{H}_1(e),$$
$$d_1\left(\mathfrak{W}_1\frac{\Gamma(\varrho_2)}{\Gamma(\vartheta+\varrho_2)}(\log\psi)^{\vartheta+\varrho_2-1} + \mathfrak{W}_2\right) = \mathfrak{W}_3 - \mathfrak{W}_1\mathcal{I}^{\vartheta+\varrho_2}\mathfrak{H}_2(\psi) - \mathfrak{W}_2\mathcal{I}^{\varrho_2}\mathfrak{H}_2(e),$$

which gives

$$c_1 = \frac{\mathfrak{Z}_3 - \mathfrak{Z}_1\mathcal{I}^{\gamma+\varrho_1}\mathfrak{H}_1(\upsilon) - \mathfrak{Z}_2\mathcal{I}^{\varrho_1}\mathfrak{H}_1(e)}{\mathfrak{Z}_2 + \frac{\mathfrak{Z}_1\Gamma(\varrho_1)}{\Gamma(\gamma+\varrho_1)}(\log\upsilon)^{\gamma+\varrho_1-1}}.$$

and

$$d_1 = \frac{\mathfrak{W}_3 - \mathfrak{W}_1\mathcal{I}^{\vartheta+\varrho_2}\mathfrak{H}_2(\psi) - \mathfrak{W}_2\mathcal{I}^{\varrho_2}\mathfrak{H}_2(e)}{\mathfrak{W}_2 + \frac{\mathfrak{W}_1\Gamma(\varrho_2)}{\Gamma(\vartheta+\varrho_2)}(\log\psi)^{\vartheta+\varrho_2-1}}.$$

By replacing the values of c_1, c_2, d_1, and d_2 into Eqs. (5) and (6), we derive Eqs. (3) and (4), thereby concluding the proof.

Based on Lemma 2, the integral solution for the problem described in Eq. (1) can be expressed as,

$$\begin{aligned}\varphi(\mathfrak{t}) &= \frac{1}{\Gamma(\varrho_1)}\int_1^{\mathfrak{t}}\left(\log\frac{\mathfrak{t}}{s}\right)^{\varrho_1-1}\frac{\Pi_1(s,\varphi(s),\varpi(s))}{s}ds\\ &+ \frac{(\log\mathfrak{t})^{\varrho_1-1}}{\Delta_1}\left\{\mathfrak{Z}_3 - \frac{\mathfrak{Z}_1}{\Gamma(\gamma+\varrho_1)}\int_1^{\upsilon}\left(\log\frac{\upsilon}{s}\right)^{\gamma+\varrho_1-1}\frac{\Pi_1(s,\varphi(s),\varpi(s))}{s}ds\right.\\ &\left.- \frac{\mathfrak{Z}_2}{\Gamma(\varrho_1)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\varrho_1-1}\frac{\Pi_1(s,\varphi(s),\varpi(s))}{s}ds\right\},\quad \mathfrak{t}\in[1,e],\end{aligned} \tag{7}$$

and

$$\begin{aligned}\varpi(\mathrm{t}) &= \frac{1}{\Gamma(\varrho_2)}\int_1^{\mathrm{t}}\left(\log\frac{\mathrm{t}}{s}\right)^{\varrho_2-1}\frac{\Pi_2(s,\varphi(s),\varpi(s))}{s}ds \\ &+ \frac{(\log \mathrm{t})^{\varrho_2-1}}{\Delta_2}\left\{\mathfrak{W}_3 - \frac{\mathfrak{W}_1}{\Gamma(\vartheta+\varrho_2)}\int_1^{\psi}\left(\log\frac{\psi}{s}\right)^{\vartheta+\varrho_2-1}\frac{\Pi_2(s,\varphi(s),\varpi(s))}{s}ds\right. \\ &\left.- \frac{\mathfrak{W}_2}{\Gamma(\varrho_2)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\varrho_2-1}\frac{\Pi_2(s,\varphi(s),\varpi(s))}{s}ds\right\}, \quad \mathrm{t}\in[1,e], \end{aligned} \tag{8}$$

where

$$\Delta_1 = \mathfrak{Z}_2 + \frac{\mathfrak{Z}_1\Gamma(\varrho_1)}{\Gamma(\gamma+\varrho_1)}(\log\upsilon)^{\gamma+\varrho_1-1},$$

and

$$\Delta_2 = \mathfrak{W}_2 + \frac{\mathfrak{W}_1\Gamma(\varrho_2)}{\Gamma(\vartheta+\varrho_2)}(\log\psi)^{\vartheta+\varrho_2-1}. \ \square$$

3 Main Results

We define an operator $\mathcal{Q} = \{\varphi(\mathrm{t})|\varphi(\mathrm{t}) \in \mathcal{C}([1,e])\}$ endowed with the norm $||\varphi|| = \max\{|\varphi(\mathrm{t})|, \mathrm{t} \in [1,e]\}$. Obviously $(\mathcal{Q}, ||\cdot||)$ is a Banach space. Also let $\mathcal{P} = \{\varpi(\mathrm{t})|\varpi(\mathrm{t}) \in \mathcal{C}([1,e])\}$ endowed with the norm $||\varpi|| = \max\{|\varpi(\mathrm{t})|, \mathrm{t} \in [1,e]\}$. The product space $(\mathcal{Q}\times\mathcal{P}, ||(\varphi,\varpi)||)$ is also Banach space with norm $||(\varphi,\varpi)|| = ||\varphi|| + ||\varpi||$.

Based on Lemma 2, we introduce the operator $\mathcal{T} : \mathcal{Q}\times\mathcal{P} \to \mathcal{Q}\times\mathcal{P}$ defined by

$$\mathcal{T}(\varphi,\varpi)(\mathrm{t}) = \begin{pmatrix}\mathcal{T}_1(\varphi,\varpi)(\mathrm{t})\\ \mathcal{T}_2(\varphi,\varpi)(\mathrm{t})\end{pmatrix},$$

where

$$\begin{aligned}\mathcal{T}_1(\varphi,\varpi)(\mathrm{t}) &= \frac{1}{\Gamma(\varrho_1)}\int_1^{\mathrm{t}}\left(\log\frac{\mathrm{t}}{s}\right)^{\varrho_1-1}\frac{\Pi_1(s,\varphi(s),\varpi(s))}{s}ds \\ &+ \frac{(\log \mathrm{t})^{\varrho_1-1}}{\Delta_1}\left\{\mathfrak{Z}_3 - \frac{\mathfrak{Z}_1}{\Gamma(\gamma+\varrho_1)}\int_1^{\upsilon}\left(\log\frac{\upsilon}{s}\right)^{\gamma+\varrho_1-1}\frac{\Pi_1(s,\varphi(s),\varpi(s))}{s}ds\right. \\ &\left.- \frac{\mathfrak{Z}_2}{\Gamma(\varrho_1)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\varrho_1-1}\frac{\Pi_1(s,\varphi(s),\varpi(s))}{s}ds\right\}, \quad \mathrm{t}\in[1,e], \end{aligned} \tag{9}$$

and

$$\begin{aligned}\mathcal{T}_2(\varphi,\varpi)(\mathrm{t}) &= \frac{1}{\Gamma(\varrho_2)}\int_1^{\mathrm{t}}\left(\log\frac{\mathrm{t}}{s}\right)^{\varrho_2-1}\frac{\Pi_2(s,\varphi(s),\varpi(s))}{s}ds \\ &+ \frac{(\log \mathrm{t})^{\varrho_2-1}}{\Delta_2}\left\{\mathfrak{W}_3 - \frac{\mathfrak{W}_1}{\Gamma(\vartheta+\varrho_2)}\int_1^{\psi}\left(\log\frac{\psi}{s}\right)^{\vartheta+\varrho_2-1}\frac{\Pi_2(s,\varphi(s),\varpi(s))}{s}ds\right. \\ &\left.- \frac{\mathfrak{W}_2}{\Gamma(\varrho_2)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\varrho_2-1}\frac{\Pi_2(s,\varphi(s),\varpi(s))}{s}ds\right\}, \quad t\in[1,e]. \end{aligned} \tag{10}$$

For simplicity, we let

$$\mathbb{M}_1 = \frac{1}{\Gamma(\varrho_1+1)} + \frac{1}{\Delta_1}\left\{\frac{\mathfrak{Z}_1(\log\upsilon)^{\gamma+\varrho_1}}{\Gamma(\gamma+\varrho_1+1)} + \frac{\mathfrak{Z}_2}{\Gamma(\varrho_1+1)}\right\} + \frac{\mathfrak{Z}_3}{\Delta_1} \tag{11}$$

and

$$\mathbb{M}_2 = \frac{1}{\Gamma(\varrho_2+1)} + \frac{1}{\Delta_2}\left\{\frac{\mathfrak{W}_1(\log\psi)^{\vartheta+\varrho_2}}{\Gamma(\vartheta+\varrho_2+1)} + \frac{\mathfrak{W}_2}{\Gamma(\varrho_2+1)}\right\} + \frac{\mathfrak{W}_3}{\Delta_2}, \tag{12}$$

$$\mathbb{M}_0 = \min\{1-(\mathbb{M}_1\varkappa_1+\mathbb{M}_2\rho_1), 1-(\mathbb{M}_1\varkappa_2+\mathbb{M}_2\rho_2)\}, \varkappa_i, \rho_i \geq 0 \ \ (i=1,2). \tag{13}$$

In our first main result, we employ Banach's contraction mapping principle to establish the uniqueness of the solution. This theorem is particularly suited for our purpose as it provides a direct and constructive method for proving the existence of a *unique* solution under Lipschitz continuity conditions. The Lipschitz condition, while a standard assumption, is a natural and manageable constraint for a wide class of nonlinear functions, and it ensures that the solution not only exists but is also stable with respect to the initial data and parameters.

Theorem 1: *Let $\Pi_1, \Pi_2 : [1,e]\times\mathbb{R}^2 \to \mathbb{R}$ be continuous functions defined on $[1,e]\times\mathbb{R}^2 \to \mathbb{R}$ and suppose that there exist constants $m_i, n_i, i=1,2$ such that for all $\mathrm{t}\in[1,e]$ and $\varphi_i, \varpi_i \in \mathbb{R}, i=1,2$,*

$$|\Pi_1(\mathrm{t},\varphi_1,\varphi_2) - \Pi_1(\mathrm{t},\varpi_1,\varpi_2)| \leq m_1|\varphi_1-\varpi_1| + m_2|\varphi_2-\varpi_2|$$

and

$$|\Pi_2(\mathrm{t},\varphi_1,\varphi_2) - \Pi_2(\mathrm{t},\varpi_1,\varpi_2)| \leq n_1|\varphi_1-\varpi_1| + n_2|\varphi_2-\varpi_2|.$$

Furthermore, assume that the condition

$$\mathbb{M}_1(m_1+m_2) + \mathbb{M}_2(n_1+n_2) < 1,$$

holds, where $\mathbb{M}_1$ and $\mathbb{M}_2$ are defined by (11) and (12), respectively. Under this assumption, the boundary value problem (1) has a unique solution.

The Lipschitz conditions are central to this analysis. They allow us to control the growth of the nonlinearities Π_1 and Π_2 and are fundamental for ensuring that the operator $\mathcal{T}$ defined later is a contraction on the Banach space. The subsequent condition $\mathbb{M}_1(m_1+m_2)+\mathbb{M}_2(n_1+n_2) < 1$ is a sufficient criterion that guarantees the contraction mapping principle is applicable.

Proof of Theorem 1: Define $\sup_{t\in[0,1]} \Pi_1(\mathrm{t},0,0) = N_1 < \infty$ and $\sup_{t\in[0,1]} \Pi_2(\mathrm{t},0,0) = N_2 < \infty$ Next we show that $\mathcal{T}B_r \subset B_r$, where $B_r = \{(\varphi,\varpi)\in\mathcal{Q}\times\mathcal{P} : \|(\varphi,\varpi)\| \leq r\}$, with

$$r \geq \frac{N_1\mathbb{M}_1 + N_2\mathbb{M}_2}{1-\mathbb{M}_1(m_1+m_2)-\mathbb{M}_2(n_1+n_2)}.$$

For $(\varphi, \varpi) \in B_r$, we have

$$\mathcal{T}_1(\varphi,\varpi)(\mathrm{t}) \le \max_{\mathrm{t}\in[1,e]}\Bigg[\frac{1}{\Gamma(\varrho_1)}\int_1^{\mathrm{t}}\left(\log\frac{\mathrm{t}}{s}\right)^{\varrho_1-1}\frac{\Pi_1(s,\varphi(s),\varpi(s))}{s}\,ds$$
$$+\frac{(\log \mathrm{t})^{\varrho_1-1}}{\Delta_1}\Bigg\{\mathfrak{Z}_3-\frac{\mathfrak{Z}_1}{\Gamma(\gamma+\varrho_1)}\int_1^{\upsilon}\left(\log\frac{\upsilon}{s}\right)^{\gamma+\varrho_1-1}\frac{\Pi_1(s,\varphi(s),\varpi(s))}{s}\,ds$$
$$+\frac{\mathfrak{Z}_2}{\Gamma(\varrho_1)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\varrho_1-1}\frac{\Pi_1(s,\varphi(s),\varpi(s))}{s}\,ds\Bigg\}\Bigg],$$
$$\le \max_{\mathrm{t}\in[1,e]}\Bigg[\frac{1}{\Gamma(\varrho_1)}\int_1^{\mathrm{t}}\left(\log\frac{\mathrm{t}}{s}\right)^{\varrho_1-1}\frac{(\Pi_1(s,\varphi(s),\varpi(s))-\Pi_1(s,0,0)|+|\Pi_1(s,0,0)|)}{s}\,ds$$
$$+\frac{(\log \mathrm{t})^{\varrho_1-1}}{\Delta_1}\Bigg\{\mathfrak{Z}_3-\frac{\mathfrak{Z}_1}{\Gamma(\gamma+\varrho_1)}\int_1^{\upsilon}\left(\log\frac{\upsilon}{s}\right)^{\gamma+\varrho_1-1}$$
$$\times\frac{(\Pi_1(s,\varphi(s),\varpi(s))-\Pi_1(s,0,0)|+|\Pi_1(s,0,0)|)}{s}\,ds$$
$$+\frac{\mathfrak{Z}_2}{\Gamma(\varrho_1)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\varrho_1-1}\frac{(\Pi_1(s,\varphi(s),\varpi(s))-\Pi_1(s,0,0)|+|\Pi_1(s,0,0)|)}{s}\,ds\Bigg\}\Bigg],$$
$$\le\left\{\frac{1}{\Gamma(\varrho_1+1)}+\frac{1}{\Delta_1}\left\{\frac{\mathfrak{Z}_1(\log\upsilon)^{\gamma+\varrho_1}}{\Gamma(\gamma+\varrho_1+1)}+\frac{\mathfrak{Z}_2}{\Gamma(\varrho_1+1)}\right\}+\frac{\mathfrak{Z}_3}{\Delta_1}\right\}(m_1||\varphi||+m_2||\varpi||+N_1)$$
$$\le \mathbb{M}_1\Big[(m_1+m_2)r+N_1\Big].$$

Hence,

$$||\mathcal{T}_1(\varphi,\varpi)(\mathrm{t})|| \le \mathbb{M}_1\Big[(m_1+m_2)r+N_1\Big],$$

similarly,

$$||\mathcal{T}_2(\varphi,\varpi)(\mathrm{t})|| \le \mathbb{M}_2\Big[(n_1+n_2)r+N_2\Big].$$

As a result,

$$||\mathcal{T}(\varphi,\varpi)|| \le r.$$

Now for $(\varphi_2,\varpi_2),(\varphi_1,\varpi_1)\in\mathcal{Q}\times\mathcal{P}$, and for any $\mathrm{t}\in[1,e]$, we get

$$|\mathcal{T}_1(\varphi_2,\varpi_2)(\mathrm{t})-\mathcal{T}_1(\varphi_1,\varpi_1)(\mathrm{t})|$$
$$\le \max_{\mathrm{t}\in[1,e]}\Bigg[\frac{1}{\Gamma(\varrho_1)}\int_1^{\mathrm{t}}\left(\log\frac{\mathrm{t}}{s}\right)^{\varrho_1-1}\frac{|\Pi_1(s,\varphi_2(s),\varpi_2(s))-\Pi_1(s,\varphi_1(s),\varpi_1(s))|}{s}\,ds$$
$$+\frac{(\log \mathrm{t})^{\varrho_1-1}}{\Delta_1}\Bigg\{\mathfrak{Z}_3-\frac{\mathfrak{Z}_1}{\Gamma(\gamma+\varrho_1)}\int_1^{\upsilon}\left(\log\frac{\upsilon}{s}\right)^{\gamma+\varrho_1-1}\frac{|\Pi_1(s,\varphi_2(s),\varpi_2(s))-\Pi_1(s,\varphi_1(s),\varpi_1(s))|}{s}\,ds$$
$$+\frac{\mathfrak{Z}_2}{\Gamma(\varrho_1)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\varrho_1-1}\frac{|\Pi_1(s,\varphi_2(s),\varpi_2(s))-\Pi_1(s,\varphi_1(s),\varpi_1(s))|}{s}\,ds\Bigg\}\Bigg],$$

$$\le \left\{ \frac{1}{\Gamma(\varrho_1+1)} + \frac{1}{\Delta_1}\left\{ \frac{\mathfrak{Z}_1(\log \upsilon)^{\gamma+\varrho_1}}{\Gamma(\gamma+\varrho_1+1)} + \frac{\mathfrak{Z}_2}{\Gamma(\varrho_1+1)} \right\} + \frac{\mathfrak{Z}_3}{\Delta_1} \right\} \Big[m_1|\varphi_2-\varphi_1| + m_2|\varpi_2-\varpi_1| \Big]$$

$$\le \mathbb{M}_1\Big[m_1||\varphi_2-\varphi_1|| + m_2||\varpi_2-\varpi_1|| \Big],$$

$$\le \mathbb{M}_1(m_1+m_2)\Big[||\varphi_2-\varphi_1|| + ||\varpi_2-\varpi_1|| \Big].$$

and consequently, we obtain

$$\|\mathcal{T}_1(\varphi_2,\varpi_2)(t) - \mathcal{T}_1(\varphi_1,\varpi_1)(t)\| \le \mathbb{M}_1(m_1+m_2)(\|\varphi_2-\varphi_1\| + \|\varpi_2-\varpi_1\|). \tag{14}$$

Similarly,

$$\|\mathcal{T}_2(\varphi_2,\varpi_2)(t) - \mathcal{T}_2(\varphi_1,\varpi_1)(t)\| \le \mathbb{M}_2(n_1+n_2)(\|\varphi_2-\varphi_1\| + \|\varpi_2-\varpi_1\|). \tag{15}$$

It follows from (14) and (15) that

$$\|\mathcal{T}(\varphi_2,\varpi_2)(t) - \mathcal{T}(\varphi_1,\varpi_1)(t)\| \le [\mathbb{M}_1(m_1+m_2) + \mathbb{M}_2(n_1+n_2)](\|\varphi_2-\varphi_1\| + \|\varpi_2-\varpi_1\|). \tag{16}$$

Since $\mathbb{M}_1(m_1+m_2)+\mathbb{M}_2(n_1+n_2) < 1$, thus, $\mathcal{T}$ is identified as a contraction operator. Consequently, by applying Banach's fixed point theorem, $\mathcal{T}$ is guaranteed to have a unique fixed point, which corresponds to the unique solution of problem (1). This concludes the proof. □

While Banach's theorem is powerful for establishing uniqueness, it requires the nonlinearities to be Lipschitz. To address problems where the nonlinearities may have more general growth, we now utilize the nonlinear alternative of Leray-Schauder. This topological method is well-suited for establishing the existence of solutions without requiring the operator to be a contraction. It relies on establishing a priori bounds for all possible solutions, thus providing a more general framework for existence when the stricter conditions of Banach's theorem are not met. Other topological methods, such as Krasnoselskii's fixed point theorem, could be considered for problems where the nonlinearity can be decomposed into contractive and compact parts, but the Leray-Schauder alternative offers a direct path for our coupled system under growth conditions.

Theorem 2: *Suppose there are real constants $\varkappa_i, \rho_i \ge 0$ for $(i=1,2)$ and $\varkappa_0 > 0$, $\rho_0 > 0$, such that $\forall \varphi_i \in \mathbb{R}$ where $(i=1,2)$, the following holds:*

$$|\Pi_1(t,\varphi_1,\varphi_2)| \le \varkappa_0 + \varkappa_1|\varphi_1| + \varkappa_2|\varphi_2|,$$

$$|\Pi_2(t,\varphi_1,\varphi_2)| \le \rho_0 + \rho_1|\varphi_1| + \rho_2|\varphi_2|.$$

In addition, it is assumed that

$$\mathbb{M}_1\varkappa_1 + \mathbb{M}_2\rho_1 < 1 \quad \text{and} \quad \mathbb{M}_1\varkappa_2 + \mathbb{M}_2\rho_2 < 1,$$

where $\mathbb{M}_1$ and $\mathbb{M}_2$ are defined by equations by (11) and (12), respectively. It follows that the boundary value problem (1) possesses at least one solution.

The growth conditions in Theorem 2 are less restrictive than the Lipschitz conditions of Theorem 1. They allow the nonlinear functions Π_1 and Π_2 to grow linearly with the state variables φ and ϖ, which encompasses a broader class of potential applications. The subsequent inequalities $\mathbb{M}_1\varkappa_1 + \mathbb{M}_2\rho_1 < 1$ and $\mathbb{M}_1\varkappa_2 + \mathbb{M}_2\rho_2 < 1$ are crucial as they ensure the necessary a priori bounds can be established for the application of the Leray-Schauder alternative.

Proof of Theorem 2: First, we demonstrate that the operator $\mathcal{T} : \mathcal{Q} \times \mathcal{P} \to \mathcal{Q} \times \mathcal{P}$ is completely continuous. Using the continuity of the functions Π_1 and Π_2, it follows that the operator $\mathcal{T}$ is continuous. Then, there exist positive constant $\mathcal{L}_1$ and $\mathcal{L}_2$ such that,

$$|\Pi_1(t, \varphi(t), \varpi(t))| \leq \mathcal{L}_1,$$

$$|\Pi_2(t, \varphi(t), \varpi(t))| \leq \mathcal{L}_2.$$

Then for any $(\varphi, \varpi) \in \mathcal{Q} \times \mathcal{P}$, we get

$$\begin{aligned}\mathcal{T}_1(\varphi, \varpi)(t) &= \frac{1}{\Gamma(\varrho_1)} \int_1^t \left(\log \frac{t}{s}\right)^{\varrho_1 - 1} \frac{|\Pi_1(s, \varphi(s), \varpi(s))|}{s} ds \\ &+ \frac{(\log t)^{\varrho_1 - 1}}{|\Delta_1|} \left\{ \mathfrak{Z}_3 - \frac{\mathfrak{Z}_1}{\Gamma(\gamma + \varrho_1)} \int_1^{\upsilon} \left(\log \frac{\upsilon}{s}\right)^{\gamma + \varrho_1 - 1} \frac{|\Pi_1(s, \varphi(s), \varpi(s))|}{s} ds \right. \\ &\left. - \frac{\mathfrak{Z}_2}{\Gamma(\varrho_1)} \int_1^e \left(\log \frac{e}{s}\right)^{\varrho_1 - 1} \frac{|\Pi_1(s, \varphi(s), \varpi(s))|}{s} ds \right\}, \quad t \in [1, e], \qquad (17)\\ &\leq \frac{1}{\Gamma(\varrho_1)} \int_1^t \left(\log \frac{t}{s}\right)^{\varrho_1 - 1} \frac{1}{s} ds \\ &+ \frac{(\log t)^{\varrho_1 - 1}}{|\Delta_1|} \left\{ \mathfrak{Z}_3 - \frac{\mathfrak{Z}_1}{\Gamma(\gamma + \varrho_1)} \int_1^{\upsilon} \left(\log \frac{\upsilon}{s}\right)^{\gamma + \varrho_1 - 1} \frac{1}{s} ds \right. \\ &\left. - \frac{\mathfrak{Z}_2}{\Gamma(\varrho_1)} \int_1^e \left(\log \frac{e}{s}\right)^{\varrho_1 - 1} \frac{1}{s} ds \right\},\end{aligned}$$

which implies that

$$||\mathcal{T}_1(\varphi, \varpi)|| \leq \mathcal{L}_1 \left\{ \frac{1}{\Gamma(\varrho_1 + 1)} + \frac{1}{\Delta_1} \left\{ \frac{\mathfrak{Z}_1 (\log \upsilon)^{\gamma + \varrho_1}}{\Gamma(\gamma + \varrho_1 + 1)} + \frac{\mathfrak{Z}_2}{\Gamma(\varrho_1 + 1)} \right\} + \frac{\mathfrak{Z}_3}{\Delta_1} \right\} = \mathcal{L}_1 \mathbb{M}_1.$$

similarly

$$||\mathcal{T}_2(\varphi, \varpi)|| \leq \mathcal{L}_2 \left\{ \frac{1}{\Gamma(\varrho_2 + 1)} + \frac{1}{\Delta_2} \left\{ \frac{\mathfrak{W}_1 (\log \psi)^{\vartheta + \varrho_2}}{\Gamma(\vartheta + \varrho_2 + 1)} + \frac{\mathfrak{W}_2}{\Gamma(\varrho_2 + 1)} \right\} + \frac{\mathfrak{W}_3}{\Delta_2} \right\} = \mathcal{L}_2 \mathbb{M}_2.$$

From the inequalities above, it can be concluded that the operator $\mathcal{T}$ is uniformly bounded. Next, we prove that $\mathcal{T}$ is equicontinuous. Let $t_1, t_2 \in [1, e]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}&\|\mathcal{T}_1(\varphi_2, \varpi_2)(t_1) - \mathcal{T}_1(\varphi_1, \varpi_1)(t_2)\| \\ &\leq \frac{1}{\Gamma(\varrho_1)} \left| \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{\varrho_1 - 1} \frac{1}{s} ds - \int_1^{t_2} \left(\log \frac{t_2}{s}\right)^{\varrho_1 - 1} \frac{1}{s} ds \right| \\ &+ \left| \frac{(\log t_2)^{\varrho_1 - 1} - (\log t_1)^{\varrho_1 - 1}}{|\Delta_1|} \left\{ \mathfrak{Z}_3 - \frac{\mathfrak{Z}_1}{\Gamma(\gamma + \varrho_1)} \int_1^{\upsilon} \left(\log \frac{\upsilon}{s}\right)^{\gamma + \varrho_1 - 1} \frac{1}{s} ds \right.\right. \\ &\left.\left. + \frac{\mathfrak{Z}_2}{\Gamma(\varrho_1)} \int_1^e \left(\log \frac{e}{s}\right)^{\varrho_1 - 1} \frac{1}{s} ds \right\} \right|,\end{aligned}$$

$$\leq \frac{1}{\Gamma(\varrho_1)}\left|\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\varrho_1-1}\frac{1}{s}ds - \int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{\varrho_1-1}\frac{1}{s}ds\right| + \frac{1}{\Gamma(\varrho_1)}\left|\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\varrho_1-1}\frac{1}{s}ds\right|$$
$$+\left|\frac{(\log t_2)^{\varrho_1-1}-(\log t_1)^{\varrho_1-1}}{|\Delta_1|}\left\{\mathfrak{Z}_3 - \frac{\mathfrak{Z}_1}{\Gamma(\gamma+\varrho_1)}\int_1^{\upsilon}\left(\log\frac{\upsilon}{s}\right)^{\gamma+\varrho_1-1}\frac{1}{s}\,ds\right.\right.$$
$$\left.\left.+\frac{\mathfrak{Z}_2}{\Gamma(\varrho_1)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\varrho_1-1}\frac{1}{s}ds\right\}\right|.$$

Analogously, we can obtain

$$\|\mathcal{T}_2(\varphi_2,\varpi_2)(t_2)-\mathcal{T}_2(\varphi_1,\varpi_1)(t_1)\|$$
$$\leq \frac{1}{\Gamma(\varrho_2)}\left|\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\varrho_2-1}\frac{1}{s}ds - \int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{\varrho_2-1}\frac{1}{s}ds\right|$$
$$+\frac{1}{\Gamma(\varrho_2)}\left|\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\varrho_2-1}\frac{1}{s}ds\right|$$
$$+\left|\frac{(\log t_2)^{\varrho_2-1}-(\log t_1)^{\varrho_2-1}}{|\Delta_2|}\left\{\mathfrak{W}_3-\frac{\mathfrak{W}_1}{\Gamma(\vartheta+\varrho_2)}\int_1^{\psi}\left(\log\frac{\psi}{s}\right)^{\vartheta+\varrho_2-1}\frac{1}{s}\,ds\right.\right.$$
$$\left.\left.+\frac{\mathfrak{W}_2}{\Gamma(\varrho_2)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\varrho_2-1}\frac{1}{s}ds\right\}\right|.$$

Note that the right hand side of the above two inequalities tends to zero independently of $(\varphi,\varpi)\in\mathcal{Q}\times\mathcal{P}$ as $t_2\to t_1$. Therefore, it follows by the Arzela-Ascoli theorem that t is completely continuous.

$$\mathcal{E}=\{(\varphi,\varpi)\in\mathcal{Q}\times\mathcal{P}\mid(\varphi,\varpi)=\lambda\mathcal{T}(\varphi,\varpi),0\leq\lambda\leq 1\},$$

is bounded. Let $(\varphi,\varpi)\in\mathcal{E}$, then $(\varphi,\varpi)=\lambda\mathcal{T}(\varphi,\varpi)$. For any $t\in[1,e]$, we have

$$\varphi(t)=\lambda\mathcal{T}_1(\varphi,\varpi)(t),\quad \varpi(t)=\lambda\mathcal{T}_2(\varphi,\varpi)(t).$$

Then

$$|\varphi(t)|\leq\left\{\frac{1}{\Gamma(\varrho_1+1)}+\frac{1}{\Delta_1}\left\{\frac{\mathfrak{Z}_1(\log\upsilon)^{\gamma+\varrho_1}}{\Gamma(\gamma+\varrho_1+1)}+\frac{\mathfrak{Z}_2}{\Gamma(\varrho_1+1)}\right\}+\frac{\mathfrak{Z}_3}{\Delta_1}\right\}(\varkappa_0+\varkappa_1\|\varphi\|+\varkappa_2\|\varpi\|)$$

and

$$|\varpi(t)|\leq\left\{\frac{1}{\Gamma(\varrho_2+1)}+\frac{1}{\Delta_2}\left\{\frac{\mathfrak{W}_1(\log\psi)^{\vartheta+\varrho_2}}{\Gamma(\vartheta+\varrho_2+1)}+\frac{\mathfrak{W}_2}{\Gamma(\varrho_2+1)}\right\}+\frac{\mathfrak{W}_3}{\Delta_2}\right\}(\rho_0+\rho_1\|\varphi\|+\rho_2\|\varpi\|).$$

Hence we have

$$\|\varphi\|\leq\mathbb{M}_1(\varkappa_0+\varkappa_1\|\varphi\|+\varkappa_2\|\varpi\|),$$

and

$$\|\varpi\|\leq\mathbb{M}_2(\rho_0+\rho_1\|\varphi\|+\rho_2\|\varpi\|),$$

which imply that

$$\|\varphi\| + \|\varpi\| \le (\mathbb{M}_1\varkappa_0 + \mathbb{M}_2\rho_0) + (\mathbb{M}_1\varkappa_1 + \mathbb{M}_2\rho_1)\|\varphi\| + (\mathbb{M}_1\varkappa_2 + \mathbb{M}_2\rho_2)\|\varpi\|.$$

Consequently,

$$\|(\varphi, \varpi)\| \le \frac{\mathbb{M}_1\varkappa_0 + \mathbb{M}_2\rho_0}{\mathbb{M}_0}.$$

For any $t \in [0, 1]$, where $\mathbb{M}_0$ is defined by Eq. (13), it follows that $\mathcal{E}$ is bounded. Therefore, by Leray-Schauder's nonlinear alternative, the operator $\mathcal{T}$ is shown to have at least one fixed point. As a result, the boundary value problem (1) is guaranteed to have at least one solution. The concludes the proof. □

4 Ulam-Hyers Stability

This section is dedicated to the stability analysis of the proposed system. We begin by stating a precise definition of Ulam-Hyers stability for our coupled system (1).

Definition 3: *[23] The coupled system of Hadamard fractional differential Eq. (1) is said to be Ulam-Hyers stable if there exist real constants $C_1 > 0, C_2 > 0$ such that for every $\epsilon_1 > 0, \epsilon_2 > 0$ and for every pair of functions $(\varphi, \varpi) \in \mathcal{C}([1, e], \mathbb{R}) \times \mathcal{C}([1, e], \mathbb{R})$ satisfying the following inequalities:*

$$\begin{cases} |\mathcal{D}^{\varrho_1}\varphi(t) - \Pi_1(t, \varphi(t), \varpi(t))| \le \epsilon_1, \\ |\mathcal{D}^{\varrho_2}\varpi(t) - \Pi_2(t, \varphi(t), \varpi(t))| \le \epsilon_2, \end{cases} \quad \text{for all } t \in [1, e], \tag{18}$$

there exists a solution (φ^*, ϖ^*) of the original system (1) such that

$$\begin{cases} \|\varphi - \varphi^*\| \le C_1\epsilon_1, \\ \|\varpi - \varpi^*\| \le C_2\epsilon_2. \end{cases} \tag{19}$$

The constants C_1, C_2 are called the Ulam-Hyers stability constants for the system.

This definition formalizes the concept that any "approximate solution" (i.e., a pair of functions satisfying the system to within a small error ϵ) must be close to an "exact solution" of the system. The stability constants C_1, C_2 quantify the robustness of the system, indicating how much an approximate solution can deviate from an exact one.

To prove the Ulam-Hyers stability of our system, we first establish an important remark that connects the inequality (18) to a perturbed differential system.

Remark 1: *A pair $(\varphi, \varpi) \in \mathcal{C}([1, e], \mathbb{R}) \times \mathcal{C}([1, e], \mathbb{R})$ satisfies the inequalities (18) if and only if there exist continuous functions $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{C}([1, e], \mathbb{R})$, which depend on φ and ϖ, respectively, such that*

$$|\mathcal{Q}_1(t)| \le \epsilon_1, \quad |\mathcal{Q}_2(t)| \le \epsilon_2, \quad \text{for all } t \in [1, e],$$

and the following perturbed system holds:

$$\begin{cases} \mathcal{D}^{\varrho_1}\varphi(t) = \Pi_1(t, \varphi(t), \varpi(t)) + \mathcal{Q}_1(t), \\ \mathcal{D}^{\varrho_2}\varpi(t) = \Pi_2(t, \varphi(t), \varpi(t)) + \mathcal{Q}_2(t). \end{cases} \tag{20}$$

Theorem 3: *Assume that the conditions of Theorem 1 hold. Then, the coupled system of nonlinear Hadamard fractional differential Eq. (1) is Ulam-Hyers stable.*

Proof: Let $\epsilon_1, \epsilon_2 > 0$ be given, and let (φ, ϖ) be a pair of functions satisfying the inequalities (18). By Remark 1, there exist functions $\mathcal{Q}_1(t), \mathcal{Q}_2(t)$ with $|\mathcal{Q}_1(t)| \leq \epsilon_1$ and $|\mathcal{Q}_2(t)| \leq \epsilon_2$ for $t \in [1, e]$, such that the perturbed system (20) is satisfied.

Following the same methodology used to derive the solution (7) and (8) for the original system, the solution of the perturbed system (20) can be expressed as:

$$\begin{aligned}\varphi(t) &= \frac{1}{\Gamma(\varrho_1)}\int_1^t \left(\log\frac{t}{s}\right)^{\varrho_1-1}\frac{\Pi_1(s,\varphi(s),\varpi(s)) + \mathcal{Q}_1(s)}{s}ds \\ &+ \frac{(\log t)^{\varrho_1-1}}{\Delta_1}\left\{\mathfrak{Z}_3 - \frac{\mathfrak{Z}_1}{\Gamma(\gamma+\varrho_1)}\int_1^v \left(\log\frac{v}{s}\right)^{\gamma+\varrho_1-1}\frac{\Pi_1(s,\varphi(s),\varpi(s)) + \mathcal{Q}_1(s)}{s}ds\right. \\ &\left. - \frac{\mathfrak{Z}_2}{\Gamma(\varrho_1)}\int_1^e \left(\log\frac{e}{s}\right)^{\varrho_1-1}\frac{\Pi_1(s,\varphi(s),\varpi(s)) + \mathcal{Q}_1(s)}{s}ds\right\},\end{aligned}$$

and

$$\begin{aligned}\varpi(t) &= \frac{1}{\Gamma(\varrho_2)}\int_1^t \left(\log\frac{t}{s}\right)^{\varrho_2-1}\frac{\Pi_2(s,\varphi(s),\varpi(s)) + \mathcal{Q}_2(s)}{s}ds \\ &+ \frac{(\log t)^{\varrho_2-1}}{\Delta_2}\left\{\mathfrak{W}_3 - \frac{\mathfrak{W}_1}{\Gamma(\vartheta+\varrho_2)}\int_1^\psi \left(\log\frac{\psi}{s}\right)^{\vartheta+\varrho_2-1}\frac{\Pi_2(s,\varphi(s),\varpi(s)) + \mathcal{Q}_2(s)}{s}ds\right. \\ &\left. - \frac{\mathfrak{W}_2}{\Gamma(\varrho_2)}\int_1^e \left(\log\frac{e}{s}\right)^{\varrho_2-1}\frac{\Pi_2(s,\varphi(s),\varpi(s)) + \mathcal{Q}_2(s)}{s}ds\right\}.\end{aligned}$$

Let (φ^*, ϖ^*) be the unique solution of the original system (1) guaranteed by Theorem 1. Then, by the linearity of the solution operator and using the bounds $|\mathcal{Q}_1(t)| \leq \epsilon_1$ and $|\mathcal{Q}_2(t)| \leq \epsilon_2$, we obtain the following estimates:

$$|\varphi(t) - \varphi^*(t)| \leq \left[\frac{1}{\Gamma(\varrho_1+1)} + \frac{1}{\Delta_1}\left(\frac{\mathfrak{Z}_1(\log v)^{\gamma+\varrho_1}}{\Gamma(\gamma+\varrho_1+1)} + \frac{\mathfrak{Z}_2}{\Gamma(\varrho_1+1)}\right)\right]\epsilon_1 = \mathbb{M}_1\epsilon_1,$$

$$|\varpi(t) - \varpi^*(t)| \leq \left[\frac{1}{\Gamma(\varrho_2+1)} + \frac{1}{\Delta_2}\left(\frac{\mathfrak{W}_1(\log \psi)^{\vartheta+\varrho_2}}{\Gamma(\vartheta+\varrho_2+1)} + \frac{\mathfrak{W}_2}{\Gamma(\varrho_2+1)}\right)\right]\epsilon_2 = \mathbb{M}_2\epsilon_2.$$

Taking the supremum over $t \in [1, e]$, we conclude that

$$\|\varphi - \varphi^*\| \leq \mathbb{M}_1\epsilon_1 \quad \text{and} \quad \|\varpi - \varpi^*\| \leq \mathbb{M}_2\epsilon_2.$$

Therefore, by Definition 3, the system (1) is Ulam-Hyers stable with explicit stability constants $C_1 = \mathbb{M}_1$ and $C_2 = \mathbb{M}_2$. This completes the proof. □

5 Illustrative Examples

In this section, we provide two comprehensive examples to validate the theoretical results established in Theorems 1 and 2. The first example demonstrates the application of the uniqueness result (Theorem 1), while the second illustrates the existence result (Theorem 2) under different growth conditions.

Example 1

Consider the coupled system (1) with the following specific parameters:

$$\varrho_1 = \frac{3}{2}, \quad \varrho_2 = \frac{3}{2}, \quad \gamma = \frac{1}{2}, \quad \mathfrak{z}_1 = 1, \quad \mathfrak{z}_2 = 1, \quad \mathfrak{z}_3 = 4, \quad v = 2,$$

$$\vartheta = \frac{3}{2}, \quad \mathfrak{W}_1 = 1, \quad \mathfrak{W}_2 = 1, \quad \mathfrak{W}_3 = 2, \quad \psi = \frac{5}{2}.$$

Let the nonlinear functions be defined as:

$$\Pi_1(t, \varphi, \varpi) = \frac{1}{4(t+2)^2} \frac{|\varphi|}{1+|\varphi|} + 1 + \frac{1}{32} \sin^2 \varpi,$$

$$\Pi_2(t, \varphi, \varpi) = \frac{1}{32\pi} \sin(2\pi\varphi) + \frac{|\varpi|}{16(1+|\varpi|)} + \frac{1}{2}.$$

We will verify all conditions of Theorem 1 (the Banach contraction principle) step by step.

Step 1: Continuity.

The functions Π_1 and Π_2 are compositions of continuous functions (polynomial, trigonometric, and rational functions where the denominator is never zero on $[1, e] \times \mathbb{R}^2$). Therefore, $\Pi_1, \Pi_2 \in \mathcal{C}([1, e] \times \mathbb{R}^2, \mathbb{R})$.

Step 2: Lipschitz Conditions.

For any $\varphi_1, \varphi_2, \varpi_1, \varpi_2 \in \mathbb{R}$ and $t \in [1, e]$, we have:

$$\begin{aligned} |\Pi_1(t, \varphi_1, \varpi_1) - \Pi_1(t, \varphi_2, \varpi_2)| &\leq \frac{1}{4(t+2)^2} \left| \frac{|\varphi_1|}{1+|\varphi_1|} - \frac{|\varphi_2|}{1+|\varphi_2|} \right| + \frac{1}{32} |\sin^2 \varpi_1 - \sin^2 \varpi_2| \\ &\leq \frac{1}{4 \cdot 3^2} |\varphi_1 - \varphi_2| + \frac{1}{32} \cdot 2|\varpi_1 - \varpi_2| \\ &\leq \frac{1}{36} |\varphi_1 - \varphi_2| + \frac{1}{16} |\varpi_1 - \varpi_2|. \end{aligned}$$

The inequality $\left| \frac{|x|}{1+|x|} - \frac{|y|}{1+|y|} \right| \leq |x - y|$ was used. Similarly,

$$\begin{aligned} |\Pi_2(t, \varphi_1, \varpi_1) - \Pi_2(t, \varphi_2, \varpi_2)| &\leq \frac{1}{32\pi} |\sin(2\pi\varphi_1) - \sin(2\pi\varphi_2)| + \frac{1}{16} \left| \frac{|\varpi_1|}{1+|\varpi_1|} - \frac{|\varpi_2|}{1+|\varpi_2|} \right| \\ &\leq \frac{1}{32\pi} \cdot 2\pi |\varphi_1 - \varphi_2| + \frac{1}{16} |\varpi_1 - \varpi_2| \\ &\leq \frac{1}{16} |\varphi_1 - \varphi_2| + \frac{1}{16} |\varpi_1 - \varpi_2|. \end{aligned}$$

Thus, the Lipschitz constants are:

$$m_1 = \frac{1}{36}, \quad m_2 = \frac{1}{16}, \quad n_1 = \frac{1}{16}, \quad n_2 = \frac{1}{16}.$$

Step 3: Calculation of Constants.

We now calculate the constants $\Delta_1, \Delta_2, \mathbb{M}_1$, and $\mathbb{M}_2$ defined in (11) and (12).

$$\Delta_1 = \mathfrak{Z}_2 + \frac{\mathfrak{Z}_1\Gamma(\varrho_1)}{\Gamma(\gamma+\varrho_1)}(\log \nu)^{\gamma+\varrho_1-1} = 1 + \frac{\Gamma(1.5)}{\Gamma(2)}(\log 2)^1 \approx 1 + \frac{0.886227}{1}\cdot 0.693147 \approx 1.6142,$$

$$\Delta_2 = \mathfrak{W}_2 + \frac{\mathfrak{W}_1\Gamma(\varrho_2)}{\Gamma(\vartheta+\varrho_2)}(\log \psi)^{\vartheta+\varrho_2-1} = 1 + \frac{\Gamma(1.5)}{\Gamma(3)}(\log 2.5)^2 \approx 1 + \frac{0.886227}{2}\cdot (0.916291)^2 \approx 1.3720.$$

$$\begin{aligned}\mathbb{M}_1 &= \frac{1}{\Gamma(\varrho_1+1)} + \frac{1}{\Delta_1}\left(\frac{\mathfrak{Z}_1(\log \nu)^{\gamma+\varrho_1}}{\Gamma(\gamma+\varrho_1+1)} + \frac{\mathfrak{Z}_2}{\Gamma(\varrho_1+1)}\right) + \frac{\mathfrak{Z}_3}{\Delta_1}\\
&= \frac{1}{\Gamma(2.5)} + \frac{1}{1.6142}\left(\frac{1\cdot(\log 2)^2}{\Gamma(3)} + \frac{1}{\Gamma(2.5)}\right) + \frac{4}{1.6142}\\
&\approx 0.886227 + 0.6195\left(\frac{0.480453}{2} + 0.886227\right) + 2.4780 \approx 1.1739.\end{aligned}$$

$$\begin{aligned}\mathbb{M}_2 &= \frac{1}{\Gamma(\varrho_2+1)} + \frac{1}{\Delta_2}\left(\frac{\mathfrak{W}_1(\log \psi)^{\vartheta+\varrho_2}}{\Gamma(\vartheta+\varrho_2+1)} + \frac{\mathfrak{W}_2}{\Gamma(\varrho_2+1)}\right) + \frac{\mathfrak{W}_3}{\Delta_2}\\
&= \frac{1}{\Gamma(2.5)} + \frac{1}{1.3720}\left(\frac{1\cdot(\log 2.5)^3}{\Gamma(4.5)} + \frac{1}{\Gamma(2.5)}\right) + \frac{2}{1.3720}\\
&\approx 0.886227 + 0.7289\left(\frac{0.769031}{11.6317} + 0.886227\right) + 1.4578 \approx 1.2384.\end{aligned}$$

Step 4: Verification of the Contraction Condition.

We now check the core condition of Theorem 1:

$$\begin{aligned}\mathbb{M}_1(m_1+m_2) + \mathbb{M}_2(n_1+n_2) &= 1.1739\left(\frac{1}{36}+\frac{1}{16}\right) + 1.2384\left(\frac{1}{16}+\frac{1}{16}\right)\\
&= 1.1739\,(0.02778 + 0.0625) + 1.2384\,(0.125)\\
&= 1.1739\cdot 0.09028 + 0.1548 \approx 0.1059 + 0.1548 = 0.2607 < 1.\end{aligned}$$

Since the contraction condition is satisfied, all hypotheses of Theorem 1 are fulfilled. Therefore, the boundary value problem (1) with the given data has a **unique solution** on $[1, e]$.

Example 2

Consider the same system (1) and parameters as in Example 1, but with different nonlinearities that exhibit polynomial growth:

$$\Pi_1(t,\varphi,\varpi) = \frac{1}{10}\varphi + \frac{1}{25}\cos(\varpi) + e^{-t}, \quad \Pi_2(t,\varphi,\varpi) = \frac{1}{30}\sin(\varphi) + \frac{1}{20}\varpi + \frac{1}{5}.$$

We will show that Theorem 2 (the Leray-Schauder alternative) applies.

Step 1: Growth Conditions.

For any $\varphi, \varpi \in \mathbb{R}$ and $t \in [1, e]$, we have:

$$|\Pi_1(t, \varphi, \varpi)| \leq \frac{1}{10}|\varphi| + \frac{1}{25} \cdot 1 + 1 = \frac{1}{10}|\varphi| + 0 + 1.04,$$

$$|\Pi_2(t, \varphi, \varpi)| \leq \frac{1}{30} \cdot 1 + \frac{1}{20}|\varpi| + \frac{1}{5} = 0 + \frac{1}{20}|\varpi| + 0.2333.$$

Thus, the growth constants are:

$$\varkappa_0 = 1.04, \quad \varkappa_1 = \frac{1}{10}, \quad \varkappa_2 = 0; \qquad \rho_0 = 0.2333, \quad \rho_1 = 0, \quad \rho_2 = \frac{1}{20}.$$

Step 2: Verification of Theorem 2 Conditions.

Using the constants $\mathbb{M}_1 \approx 1.1739$ and $\mathbb{M}_2 \approx 1.2384$ from Example 1, we check the conditions:

$$\mathbb{M}_1\varkappa_1 + \mathbb{M}_2\rho_1 = 1.1739 \cdot 0.1 + 1.2384 \cdot 0 = 0.11739 < 1,$$

$$\mathbb{M}_1\varkappa_2 + \mathbb{M}_2\rho_2 = 1.1739 \cdot 0 + 1.2384 \cdot 0.05 = 0.06192 < 1.$$

Since both conditions are satisfied, it follows from Theorem 2 that the boundary value problem (1) with this second set of nonlinearities has **at least one solution** on $[1, e]$. These examples conclusively demonstrate the applicability of our main theorems to distinct classes of nonlinearities, validating our theoretical framework.

6 Discussion and Practical Significance

The theoretical findings of this work—existence, uniqueness, and Ulam-Hyers stability—carry significant implications for both mathematical theory and practical applications. This section discusses the broader context, practical relevance, and limitations of our results.

6.1 Practical Significance of Theoretical Guarantees

The primary practical contribution of this research lies in the rigorous guarantees it provides for the coupled system (1). The proof of a **unique solution** ensures that the mathematical model is well-posed. For scientists and engineers, this means that for a given set of inputs, parameters, and boundary conditions, the outcome is predictable and deterministic. This is a fundamental prerequisite for using the model for simulation, control, or design purposes in applied fields.

Furthermore, the demonstration of **Ulam-Hyers stability** is equally crucial for practical applications. This property guarantees that small perturbations in the model's formulation—which inevitably arise from measurement errors, parameter uncertainties, or numerical approximations—lead to only proportionally small deviations in the solution. This robustness validates the use of numerical methods, as it ensures that computed solutions will remain close to the true, unknown solution of the model, thereby justifying the computational effort.

6.2 Advantages of the Hadamard Framework and Nonlocal Conditions

Our choice of the Hadamard fractional derivative, defined with a logarithmic kernel, is not merely a mathematical generalization. It is particularly advantageous for modeling physical processes exhibiting **scale-invariance** or **ultraslow diffusion**. Such behaviors are observed in the transport through fractal and porous media [14] and the long-term rheological response of complex materials like polymers and geological formations [15], where memory effects decay logarithmically in time.

The nonlocal boundary conditions, incorporating Hadamard-type integrals, enhance the model's physical realism. They are adept at representing scenarios where the state at a boundary point is influenced by the cumulative effect of the system's state distributed over an internal region. This is relevant for modeling phenomena with long-range interactions or hereditary properties, such as in certain heat conduction problems with memory effects or population dynamics with nonlocal interactions.

6.3 Limitations and Numerical Challenges

While the nonlocal conditions and the Hadamard operators increase the model's sophistication, they also introduce significant complexities. A primary limitation is the increased **analytical and computational cost** associated with handling the logarithmic kernels and integral boundary conditions. The evaluation of Hadamard integrals, which possess weak singularities, requires careful numerical treatment, often necessitating specialized quadrature rules that can be computationally intensive compared to methods for local problems.

Furthermore, the sufficient conditions for uniqueness and existence derived in our theorems, while general, may not be necessary. There could be a class of problems outside the scope of our Lipschitz or growth conditions that still admit solutions. The potential for **multiple solutions** under different conditions, or when our contraction condition is violated, remains an open question for future investigation.

6.4 Broader Applications and Future Outlook

The framework established here is not confined to a single physical context. It can be adapted to model various real-world phenomena, including but not limited to:

- **Anomalous Transport:** Ultraslow diffusion in highly heterogeneous or fractal porous media.
- **Viscoelasticity:** Stress relaxation in complex materials where the creep compliance follows a logarithmic law.
- **Systems Biology:** Dynamics of biological systems with memory and nonlocal interaction effects.

Future research will focus on overcoming the current limitations. This includes developing **efficient numerical algorithms** tailored for Hadamard fractional operators and nonlocal conditions, extending the analysis to include **more general fractional derivatives** (e.g., the Hilfer-Hadamard derivative), and investigating the **controllability** of such systems for potential applications in engineering control theory. By addressing these challenges, the theoretical foundation laid in this work can be fully leveraged for solving complex problems in science and engineering.

7 Conclusion

This research has established a comprehensive theoretical framework for analyzing a coupled system of nonlinear Hadamard fractional differential equations subject to nonlocal boundary conditions. The primary objectives were to investigate the existence and uniqueness of solutions and to examine the stability of the system.

Our main contributions can be summarized as follows:

- We successfully derived sufficient criteria for the **existence and uniqueness** of solutions. The uniqueness was established using **Banach's contraction principle**, while the existence result was obtained via the **Leray-Schauder nonlinear alternative**, thus covering a broad class of nonlinearities.

- We proved that the system is **Ulam-Hyers stable**. This crucial finding ensures that approximate solutions remain close to exact solutions, guaranteeing the robustness of the model against small perturbations and validating future numerical simulations.
- The theoretical results were substantiated through **detailed illustrative examples** that verified all the assumptions of our main theorems, demonstrating their practical applicability.

The **novelty** of this work is multifaceted. Firstly, we analyzed a **coupled system** with fully nonlinear interaction terms, which is more general than many previously studied single-equation models. Secondly, we incorporated **generalized nonlocal boundary conditions** involving Hadamard-type integral operators, which maintain mathematical consistency with the derivative operators and model more complex physical scenarios than standard local conditions. Finally, we provided a **unified analysis** that seamlessly integrates existence, uniqueness, and stability within a single framework for this class of problems.

Building upon this foundation, several avenues for **future research** present themselves:

- **Generalized Operators:** Extending the analysis to include more complex fractional operators, such as the **Hilfer-Hadamard** or ψ**-Hadamard** fractional derivatives, to model an even wider range of processes.
- **Advanced Numerical Schemes:** Developing and implementing efficient computational methods, such as spectral methods or adaptive quadrature rules, to solve these challenging integro-differential equations numerically.
- **Controllability Analysis:** Investigating the trajectory controllability of fractional systems involving Hadamard derivatives, which would have significant implications for engineering control theory.
- **Application-Specific Models:** Applying this theoretical framework to develop and analyze concrete models in fields like thermodynamics of complex materials, anomalous transport in porous media, or systems biology.

In conclusion, this study provides a rigorous and robust mathematical foundation for a significant class of fractional differential systems. The results not only advance the theoretical landscape of fractional calculus but also pave the way for reliable applications in modeling complex physical phenomena characterized by memory, nonlocality, and scale invariance.

Acknowledgement: The authors extend their appreciation to the Deanship of Research and Graduate Studies at King Faisal University.

Funding Statement: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. KFU253922].

Author Contributions: The authors confirm contribution to the paper as follows: Conceptualization, Murugesan Manigandan and Muath Awadalla; methodology, Muath Awadalla; software, Murugesan Manigandan; validation, R. Sathiya Shanthi and Salma Trabelsi; formal analysis, Muath Awadalla; investigation, Muath Awadalla; resources, Salma Trabelsi; writing—original draft preparation, Murugesan Manigandan; writing—review and editing, Murugesan Manigandan and Muath

Awadalla; visualization, Muath Awadalla and R. Sathiya Shanthi; supervision, Muath Awadalla. All authors reviewed the results and approved the final version of the manuscript.

Availability of Data and Materials: Not applicable.

Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest to report regarding the present study.

References

1. Podlubny I. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, some methods of their solution and some of their applications. Cambridge, MA, USA: Academic Press; 1999.
2. Monje CA, Chen Y, Vinagre BM, Xue D, Feliu-Batlle V. Fractional-order systems and controls: fundamentals and applications. Berlin/Heidelberg, Germany: Springer Science & Business Media; 2010.
3. Al-Raeei M. Applying fractional quantum mechanics to systems with electrical screening effects. Chaos Solitons Fract. 2021;150:111209.
4. Tarasov VE. Nonlinear fractional dynamics with kicks. Chaos Solitons Fract. 2021;151:111259.
5. Shah K, Abdeljawad T, Alrabaiah H. On coupled system of drug therapy via piecewise equations. Fractals. 2022;30(8):2240206. doi:10.1142/s0218348x2240206x.
6. Sher M, Shah K, Sarwar M, Alqudah MA, Abdeljawad T. Mathematical analysis of fractional order alcoholism model. Alex Eng J. 2023;78(11):281–91. doi:10.1016/j.aej.2023.07.010.
7. Kasinathan R, Kasinathan R, Chalishajar D, Kasinathan D, Rautaray M. Trajectory controllability of hilfer fractional stochastic pantograph equations with random impulses. Complex Anal Operator Theory. 2025;19(7):176. doi:10.1007/s11785-025-01801-8.
8. Chalishajar D, Kasinathan D, Kasinathan R, Kasinathan R. Trajectory controllability of higher-order Riemann-Liouville fractional stochastic systems via integral contractors in a new Banach space. Bulletin des Sciences Mathématiques. 2025;204(11):103659. doi:10.1016/j.bulsci.2025.103659.
9. Jneid M. Results on partial approximate controllability of fractional control systems in Hilbert spaces with conformable derivatives. AIP Adv. 2024;14(2):25128. doi:10.1063/5.0190334.
10. Mahdy AMS, Lotfy K, Ismail EA, El-Bary A, Ahmed M, El-Dahdouh AA. Analytical solutions of time-fractional heat order for a magneto-photothermal semiconductor medium with Thomson effects and initial stress. Results Phys. 2020;18(8):103174. doi:10.1016/j.rinp.2020.103174.
11. Kamel A, Mahdy A, El-bary AA, Ibrahim E, Lotfy K, Erman S. Investigation of fractional coupled nonlocal-microstretch effects on thermo-opto-elastic wave propagation in semiconductor media under photothermal and strong magnetic excitations. Arch Appl Mech. 2025;95(9):1–25. doi:10.1007/s00419-025-02921-5.
12. Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. In: North-Holland mathematics studies. Vol. 204. Amsterdam, The Netherlands: Elsevier Science B.V.; 2006.
13. Alsaedi A, Ntouyas SK, Ahmad B, Hobiny A. Nonlinear Hadamard fractional differential equations with Hadamard type nonlocal non-conserved conditions. Adv Differ Equ. 2015;2015(1):1–13. doi:10.1186/s13662-015-0589-1.
14. Tarasov VE. Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media. Berlin/Heidelberg, Germany: Springer Science & Business Media; 2011.
15. Mainardi F. Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models. London, UK: Imperial College Press; 2010.
16. Granas A, Dugundji J. Fixed point theory. New York, NY, USA: Springer-Verlag; 2003.

17. Smart DR. Fixed point theorems. Cambridge, UK: Cambridge University Press; 1980.
18. Abbas S, Benchohra M, Lazreg JE, Zhou Y. A survey on Hadamard and Hilfer fractional differential equations: analysis and stability. Chaos Solitons Fract. 2017;102:47–71.
19. Ahmad B, Alsaedi A, Ntouyas SK, Tariboon J. Hadamard-type fractional differential equations, inclusions and inequalities. Berlin/Heidelberg, Germany: Springer International Publishing; 2017.
20. Awadalla M, Murugesan M, Muthaiah S, Unyong B, Egami RH. Existence results for a system of sequential differential equations with varying fractional orders via Hilfer-Hadamard sense. AIMS Math. 2024;9(4):9926–50. doi:10.3934/math.2024486.
21. Manigandan M, Subramanian M, Duraisamy P, Gopal TN. On Caputo-Hadamard type fractional differential equations with nonlocal discrete boundary conditions. Discontinuity Nonlinear Complex. 2021;10(2):185–94. doi:10.1088/1742-6596/1139/1/012020.
22. Mahmudov NI, Awadalla M, Abuassba K. Hadamard and caputo-hadamard FDE's with three point integral boundary conditions. Nonlinear Anal Differ Equ. 2017;5(6):271–82. doi:10.12988/nade.2017.7916.
23. Rus IA. Ulam stabilities of ordinary differential equations in a Banach space. Carpathian J Math. 2010;26:103–7.
24. Koksal ME. Stability analysis of fractional differential equations with unknown parameters. arXiv: 1709.05402. 2017.
25. Zhai C, Wang W, Li H. A uniqueness method to a new Hadamard fractional differential system with four-point boundary conditions. J Inequal Appl. 2018;2018(1):1–16. doi:10.1186/s13660-018-1801-0.
26. Yaseen M, Mumtaz S, George R, Hussain A, Nabwey HA. Darbo's fixed-point theorem: establishing existence and uniqueness results for hybrid caputo-hadamard fractional sequential differential equations. Fractal Fract. 2024;8(6):326. doi:10.3390/fractalfract8060326.
27. Erturk VS, Ali A, Shah K, Kumar P, Abdeljawad T. Existence and stability results for nonlocal boundary value problems of fractional order. Boundary Value Probl. 2022;2022(1):25. doi:10.1186/s13661-022-01606-0.
28. Taier AE, Wu R, Benyoub FZ. The existence and uniqueness results for a nonlocal boundary value problem of caputo-type hadamard hybrid fractional integro-differential equations. Am J Appl Math. 2024;12(6):246–57. doi:10.11648/j.ajam.20241206.14.