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ADAPTIVE FINITE ELEMENTS WITH LARGE ASPECT RATIO FOR ALUMINIUM ELECTROLYSIS: A CONTINUATION ALGORITHM.

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Summary. A continuation anisotropic adaptive algorithm to solve elliptic PDEs is presented. The p-laplacian problem and the Stokes equation are considered. The algorithm is based on an a posteriori error indicator justified in [7] and [10]. The goal is to produce an anisotropic mesh such that the relative estimated error is close to a preset tolerance TOL. A continuation method is used to decrease TOL. Numerical results show that the computational time is considerably reduced when using such a continuation algorithm.

1 INTRODUCTION

The use of adaptive algorithm to solve PDEs can reduce considerably the computational cost for a given accuracy. Often the adaptive algorithm is based on a posteriori error estimates. In [9, 11, 2] an adaptive algorithm based on different error indicators can be observed.

The goal of this paper is to present in detail the adaptive algorithm and in particular a strategy which has the goal to reduce the computational cost. Error indicators, discussed in [7, 10] are presented for the p-Laplacian like problem and Stokes problem respectively. Two different approaches of our adaptive algorithm are discussed.

The outline is the following. In Section 2 the algorithm is presented. In section 3 the two model problems are introduced together with the respective error indicators. Section 4 is devoted to numerical experiments. Finally in Section 5 a conclusion is presented.

2 A Continuation Anisotropic Adaptive Algorithm

We present a Continuation Anisotropic Adaptive Algorithm in dimension $d = 2, 3$ to solve elliptics PDEs. First we introduce an Anisotropic Adaptive Algorithm, applications of which can be found in [1, 8, 11, 2]. Let $\Omega \subset \mathbb{R}^d$ be a polygonal domain, for any $0 < h < 1$ let $\mathcal{T}_h$ be a conforming triangulation of $\bar{\Omega}$. We denote, for $p \geq d$ and for

$s = 1, \ldots, d$, $u \in \left(W_0^{1,p}(\Omega)\right)^s$ a solution of the considered partial differential equation and u_h the continuous piecewise linear approximation. In what follows when $s = 1$ $u : \Omega \to \mathbb{R}$ is solution of a p-Laplacian problem and when $s = d$, $u : \Omega \to \mathbb{R}^s$ is a solution of a Stokes problem. We note the local error indicator

$$\eta_K^2 = \left(\eta_{1,K}^2 + \cdots + \eta_{d,K}^2\right)^{1/2}, \text{ with } \eta_{i,K}^2 = \rho_K^2 \lambda_{i,K}^2 \mathbf{r}_{i,K} G_K(u - u_h) \mathbf{r}_{i,K} \tag{1}$$

so that $\sum_{K \in \mathcal{T}_h} \eta_K^2$ is an error indicator for the error in a norm (or quasi-norm) $|| \cdot ||$. Vectors $\mathbf{r}_{i,K}$ represent the stretching directions of triangle $K \in \mathcal{T}_h$ and $\lambda_{i,K}$ the respective magnitudes ($\lambda_{1,K} \geq \cdots \geq \lambda_{d,K}$), see details in [3, 4]. The residual quantity ρ_K depends on the considered equation and finally

$$(G_K(u - u_h))_{ij} = \sum_{l=1}^{s} \int_{\Delta K} \frac{\partial(u_l - (u_h)_l)}{\partial x_i} \frac{\partial(u_l - (u_h)_l)}{\partial x_j} \quad i, j = 1, \ldots, d.$$

In practice post-processing techniques, as Zienkiewicz−Zhu (ZZ), can be applied to approximate $\frac{\partial u_l}{\partial x_i}$ [14, 15, 16]. Note that, for $i = 1, \ldots, d$, the quantity $\eta_{i,K}^2$ represents the error in direction $\mathbf{r}_{i,K}$.

The goal of the adaptive algorithm is to build sequence of meshes possibly having large aspect ratio such that a relative estimated error is close to a given, preset, tolerance TOL, i.e.

$$0.75\text{TOL} \leq \left(\frac{\sum_{K \in \mathcal{T}_h} \eta_K^2}{||u_h||^p} \right)^{1/p} \leq 1.25\text{TOL}. \tag{2}$$

While adapting, the following main goals are considered

- Equidistribute the error in all stretching directions of each triangle. In numerical experiments performed for example in [9, 2] this approach is suggested.

- Align the stretching directions $\mathbf{r}_{i,K}$ for $i = 1, \cdots, d$ with the eigenvectors of $G_K(u - u_h)$. In Lemma 4.1 of [3], this choice is justified.

A sufficient condition for (2) to hold is to require that for each $K \in \mathcal{T}_h$ the local error is equidistributed

$$\frac{\mathcal{L}}{N_K} \leq \eta_K^2 \leq \frac{\mathcal{R}}{N_K}, \tag{3}$$

where we define N_K the number of triangle $K \in \mathcal{T}_h$, $\mathcal{L} = 0.75^p \text{TOL}^p ||u_h||^p$ and $\mathcal{R} = 1.25^p \text{TOL}^p ||u_h||^p$. In order to insure (3), we require for each $K \in \mathcal{T}_h$ and for $i = 1, \ldots, d$

$$\frac{\mathcal{L}^2}{dN_K^2} \leq \eta_{i,K}^2 \leq \frac{\mathcal{R}^2}{dN_K^2}. \tag{4}$$

To update the mesh, a mesh generator (BL2D [5] if $d = 2$ and MeshGems [12] if $d = 3$) is used. Below an Anisotropic Adaptive Algorithm is presented.

Algorithm 1 : Anisotropic Adaptive Algorithm

Data: TOL, starting mesh $\mathcal{T}_h^1$

$k \leftarrow 1$
while (2) not satisfied **do**
  Solve problem on $\mathcal{T}_h^k$
  Compute error indicator
  for $i = 1, \cdots, d$ **do**
    if $\eta_{i,K}^2 \leq \frac{\mathcal{L}^2}{dN_K^2}$ **then**
      $\lambda_{i,K} \leftarrow 1.5\lambda_{i,K}$
    else if $\eta_{i,K}^2 \geq \frac{\mathcal{R}^2}{dN_K^2}$ **then**
      $\lambda_{i,K} \leftarrow \lambda_{i,K}/1.5$
    End
    Align direction $\mathbf{r}_{i,K}$ with ith eigenvector of $G_K(u - u_h)$
  End
  Update the mesh with an anisotropic mesh generator: $\mathcal{T}_h^{k+1}$
  $k \leftarrow k + 1$
End
$\mathcal{T}_h^{final} \leftarrow \mathcal{T}_h^k$
Output: Final mesh $\mathcal{T}_h^{final}$

In the spirit of the Anisotropic Adaptive Algorithm introduced, we present now the Continuation Anisotropic Adaptive Algorithm. The idea is to set TOL $= 2^N \text{TOL}_{goal}$, where $N \geq 1$, run the Anisotropic Adaptive Algorithm and decrease N by 1 until the desired tolerance TOL_{goal} is reached. Hereafter the Continuation Anisotropic Adaptive Algorithm is presented.

Algorithm 2 : Continuation Anisotropic Adaptive Algorithm

Data: TOL_{goal}, $N >= 1$, starting mesh $\mathcal{T}_h^1$

$\mathcal{T}_h^{final} \leftarrow \mathcal{T}_h^1$
for n=N,...,0 **do**
  $TOL = 2^n \text{TOL}_{goal}$
  $\mathcal{T}_h^{final} \leftarrow$ Anisotropic Adaptive Algorithm(TOL,$\mathcal{T}_h^{final}$)
End
Output: Final mesh $\mathcal{T}_h^{final}$

3 Two model problems

We present two model problems on which we test our adaptive strategy: a p-Laplacian like problem ($s = 1$, $d = 2$, $p = 3$) and Stokes problem ($s = 3$, $d = 3$, $p = 2$). Let $\Omega \subset \mathbb{R}^d$ for $d = 2, 3$ be a polygonal domain. Given $\mu \geq 0$ and $f : \Omega \to \mathbb{R}$, we are looking for $u : \Omega \to \mathbb{R}$ solution of the following problem

$$\begin{cases} -\nabla \cdot ((\mu + |\nabla u|^{p-2})\nabla u) = f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $|\cdot|$ denotes the Euclidian norm in $\mathbb{R}^d$. For any $0 < h < 1$, a conforming triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ having triangles K of diameter $h_K \leq h$ is considered. Let u_h be the continuous piecewise linear approximation on triangles $K \in \mathcal{T}_h$ with zero value on the boundary $\partial\Omega$ obtained by finite element method. We consider the following quasi norm error [13, 7]

$$||u - u_h||^p := \int_\Omega |\nabla(u - u_h)|^2(\mu + |\nabla(u - u_h)| + |\nabla u|)^{p-2}.$$

Consider $u \in W_0^{1,p}(\Omega)$ solution of the variational problem arising from (5). As stated in [7] the following error indicator, where the constant C is independent of μ, u, the mesh size and the aspect ratio, can be derived

$$||u - u_h||^p \leq C \sum_{K \in \mathcal{T}_h} \eta_K^2 \tag{6}$$

and where η_K^2 is given by (1) with

$$\rho_K = ||\nabla \cdot \big((\mu + |\nabla u_h|^{p-2})\nabla u_h\big) + f||_{L^2(K)} + \frac{1}{2\lambda_{2,K}^{1/2}} ||[(\mu + |\nabla u_h|^{p-2})\nabla u_h \cdot \mathbf{n}]||_{L^2(\partial K)}. \tag{7}$$

Here $\mathbf{n}$ stands for the unit outer normal to triangle K, $[\cdot]$ denotes the jump across the edges of K ($[\cdot] = 0$ if $\partial K \subset \partial\Omega$).

As a second model problem, Stokes equation is considered. Given $\mu > 0$ and $f : \Omega \to \mathbb{R}^d$, we are looking for $u : \Omega \to \mathbb{R}^d$ and $r : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\mu\Delta u + \nabla r = f & \text{in } \Omega, \\ \operatorname{div}(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

Consider again for any $0 < h < 1$, a conforming triangulation $\mathcal{T}_h$ of $\bar{\Omega}$. Let u_h and r_h be the continuous piecewise linear approximation on triangles $K \in \mathcal{T}_h$ with zero value on the boundary $\partial\Omega$ obtained by finite element method. Let $p = 2$, under the same notations previously discussed, in [10] the error indicator $\sum_{K \in \mathcal{T}_h} \eta_K^2$ for the H^1 semi-norm

$$||u - u_h||^2 := ||\nabla(u - u_h)||_{L^2(\Omega)}^2 \tag{9}$$

defined by (1) with

$$\rho_K = ||\frac{1}{\mu}(f - \nabla r_h) + \Delta u_h||_{L^2(K)} + \frac{1}{2\lambda_{2,K}^{1/2}} ||[\nabla u_h \cdot \mathbf{n}]||_{L^2(\partial K)}.$$

is discussed.

4 Adaptive algorithm experiments

Results showing the sharpness of the error indicators presented can be observed in [7, 10]. Our goal is to demonstrate the benefits of the continuation adaptive algorithm. In particular we will compare algorithms 1 and 2. Consider problem (5), $\Omega = (0,1)^2$ and let f be such that the exact solution is given by $u(x_1, x_2) = \tanh(\frac{x_1 - 0.5}{0.1})$ and $\mu = 0$. Set $\text{TOL}_{goal} = 0.0078125$ and choose a starting mesh of size $0.1 - 0.1$. We run algorithm 1 for 140 iterations. Choosing $N = 7$ we run also algorithm 2. The tolerance is divided by two each 20 iterations for a total of 140 iterations. In Figure 1 we present the obtained results. Both algorithms give similar final meshes, the number of vertices is considerably close and the solution obtained have comparable accuracy. However algorithm 1 requires an higher number of vertices along first iterations. For this nonlinear problem we reported the total number of Conjugate gradient iterations (sum for each Newton method step). The first approach increases considerably the CPU time. This is due to the higher refinement of the mesh at initial iterations. Algorithm 2 is clearly the fastest and best option.

Motivated by Aluminium Electrolysis [6] Stokes problem (8), is solved in a flat domain. Let $\Omega = (0,1) \times (0,1) \times (0,0.1)$ and f be such that $u(x,y,z) = [x^3(1-x)^3y^2(1-y)^2(1-2y), -x^2(1-x)^2y^3(1-y)^3(1-2x), 0]^T$ and $r(x,y,z) = xy - 0.25$. We set $\text{TOL}_{goal} = 0.125$ and consider a structured starting mesh of size $0.1 - 0.1 - 0.05$. We run algorithm 1 for 140 iterations. We set $N = 7$ and perform 140 iterations of algorithm 2, dividing the tolerance each 20 iterations. In Figure 2 the obtained results can be observed. Similar conclusion as the previous two dimensional experiment can be done. In Figure 3 a cut of the obtained mesh, when algorithm 2 is applied can be observed. In the industrial problem the necessity of the continuation algorithm is even more clear. When algorithm 1 is applied, an increase in computational time can be observed and adaptation is not possible.

5 Conclusion

We presented two possible versions of adaptive algorithm based on an a posteriori error indicator for a p-Laplacian like problem and a Stokes problem. The goal of the algorithms is to construct a mesh having a normalized error indicator near to a given tolerance. Numerical experiments show the benefit of a continuation algorithm on the preset tolerance parameter. The CPU time is considerably reduced and the benefits in industrial problems with complex geometries are important.

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Figure 1: For both algorithms we report the obtained results solving problem (5) with $d = 2$, $p = 3$, $s = 1$, exact solution $u(x_1, x_2) = \tanh(\frac{x_1 - 0.5}{0.1})$, $\mathrm{TOL}_{goal} = 0.0078125$ and $N = 7$. Top left: quasi-norm error at each iteration. Top right: Number of vertices at each iteration. Bottom left: CPU time at each iteration. Bottom right: sum of conjugate gradient iterations for Newton method at each iteration of the adaptive algorithm.

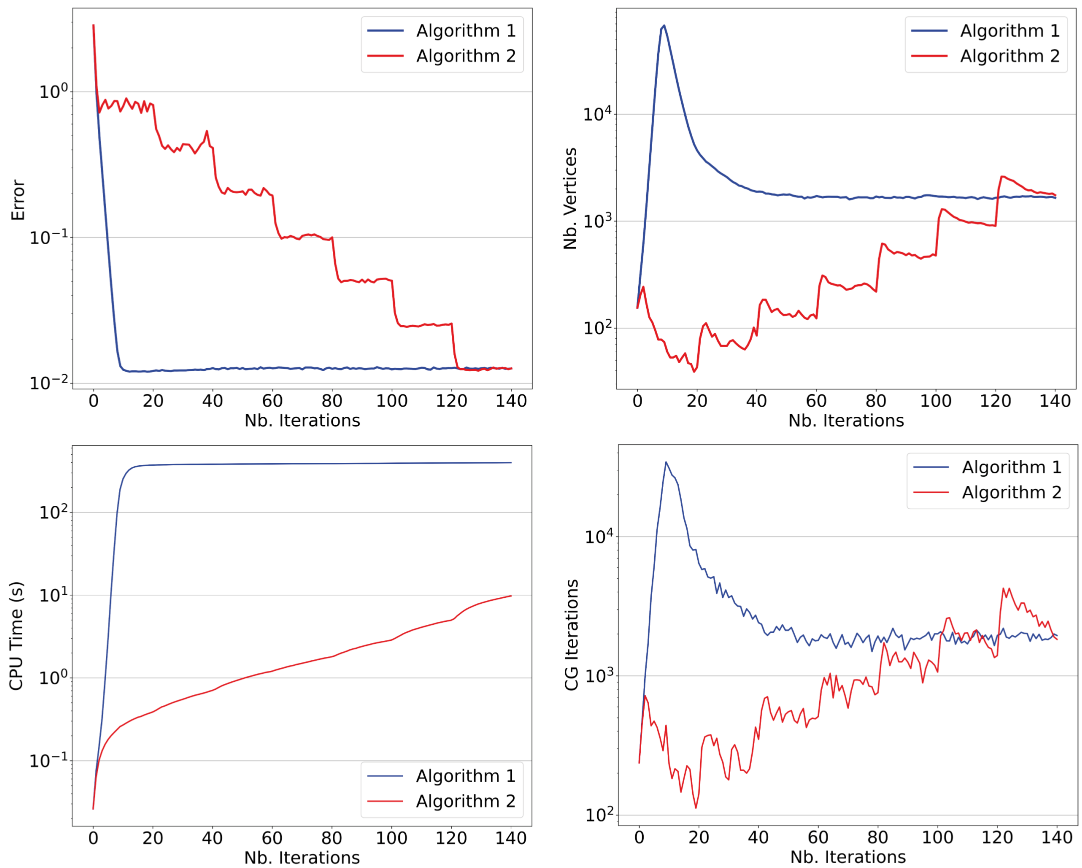

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Figure 2: For both algorithms we report the obtained results when solving problem (8) with $d = 3$, $p = 2$, $s = 3$, exact solution $u(x, y, z) = [x^3(1-x)^3y^2(1-y)^2(1-2y), -x^2(1-x)^2y^3(1-y)^3(1-2x), 0]^T$ and $r(x, y, z) = xy - 0.25$, $\text{TOL}_{goal} = 0.125$ and $N = 7$. Top left: quasi-norm error at each iteration. Top right: Number of vertices at each iteration. Bottom left: CPU time at each iteration. Bottom right: GMRES iterations at each iteration of the adaptive algorithm.

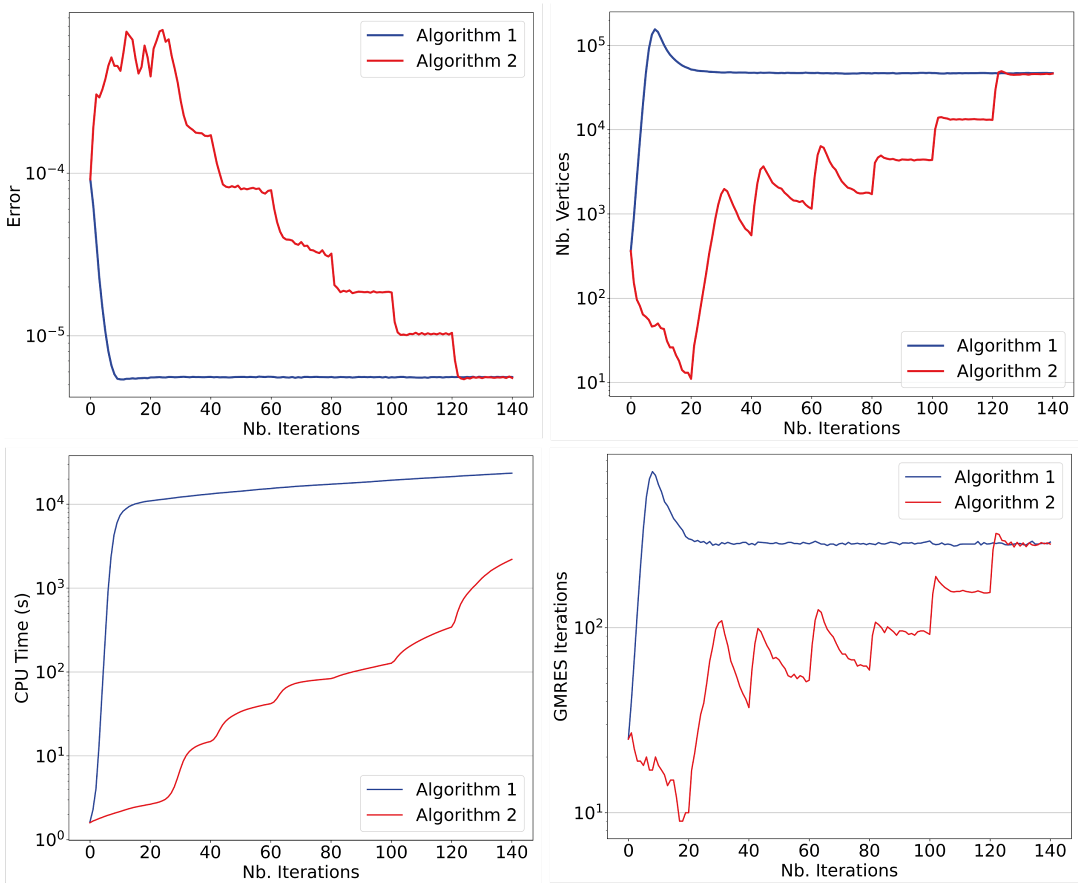

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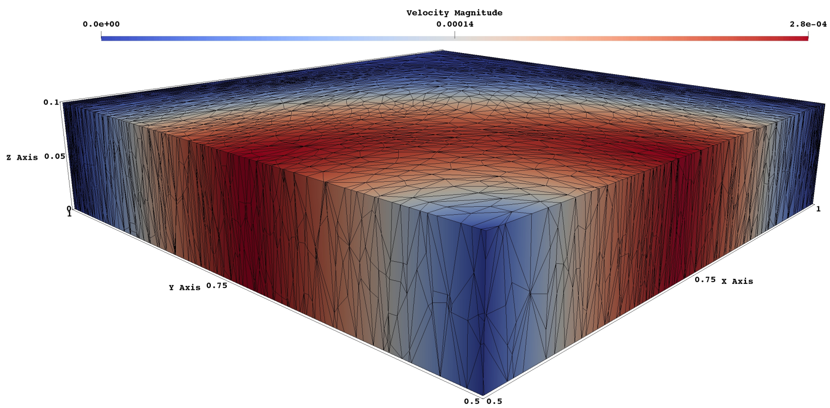
Figure 3: Cut at $x = 0.5$ and $y = 0.5$ of the final mesh obtained by algorithm 2 with respective velocity magnitude, when solving problem (8) with $d = 3$, $p = 2$, $s = 3$, exact solution $u(x, y, z) = [x^3(1-x)^3y^2(1-y)^2(1-2y), -x^2(1-x)^2y^3(1-y)^3(1-2x), 0]^T$, $r(x, y, z) = xy-0.25$, $\text{TOL}_{goal} = 0.125$ and $N = 7$.



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