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ABSTRACT

Long testing times are usually required for the life testing of very reliable products or materials. The testing process can be hastened by using accelerated life tests. The lifespan of the items that accelerated life tests inspect is reduced since they test products in more severe circumstances than those found in regular use scenarios. Data that was censored and disclosed the precise timings of failure may point to accelerated life tests where all units assigned to test are unknown, or where all units assigned to test have not failed for a few reasons, including challenges with technology, tools, costs, and schedules. The step-stress partially accelerated life test was examined in this work using the type-I progressive hybrid censoring scheme and the type-II progressive censoring scheme. The influence of the stress shift is explained using the tempered random variable model, where the failure times of the items are assumed to follow the alpha power Lomax distribution. The unknown parameters are estimated using the maximum likelihood estimation and Bayesian methods. The asymptotic theory of maximum likelihood estimation is also employed in the construction of the approximate confidence intervals. While the point estimates under two censoring schemes are compared in terms of absolute biases and root mean squared errors, approximate confidence intervals and coverage probabilities are compared in terms of their lengths and coverage probabilities. Additionally, three possible optimal test strategies are investigated using different optimal criteria. The performance of the estimators was evaluated and contrasted with two censoring techniques with various sample sizes using a simulation study. Finally, a numerical example for insulating fluid between electrodes data is presented to illustrate how the methods will work in real-world scenarios.

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1 Introduction

In the contemporary era of technical advancements, product reliability has grown recently due to continued efforts by many organizations to improve their production processes. Using traditional life testing procedures to assess a product's lifetime and predict failures will be costly and time-consuming, especially given the fierce rivalry among companies to quickly launch their products. Therefore, because they can be used to trigger early failures at elevated stress levels of stressors like temperature, voltage (electric field), pressure, and so on, accelerated life tests (ALTs) are frequently employed to explore the lifetime of exceptionally reliable items. After that, life testing and reliability assessments were conducted on the constant-stress and step-stress models in the ALTs, for more information, see [1]. Constant stress ALT (CSALT) is the most widely used and practical form of the ALT experiment. In this type of experiment, independent samples of testing units are subjected to varying stress factor levels and kept constant during the entire experiment. SSALT is a different kind of ALT experiment. A single sample is used in an SSALT experiment, and every test unit is subjected to the same experimental conditions, which change throughout the test process (typically in an increasing way). Specifically, every item is subjected to the same amount of stress, which can be raised once or multiple times, after a set amount of time, or after a predefined number of failures. Compared to CSALT, SSALT trials produce faster results (see [2]). Simple SSALT tests are those that just use one stress-change point. Time and money constraints can lead to the censoring of data in reliability trials and life testing. The hybrid censoring method, which incorporates type-I and type-II censorings as special cases, is widely utilized in reliability research. For those interested in learning more about hybrid censoring, we recommend reading [3]. The hybrid censoring technique does not permit the removal of units during the testing phase due to financial and scheduling constraints. To solve this problem, a progressive censoring technique was created, which permitted the experimental units to be deleted at different points during the test ([3]). Notably, the experiment has no time limit; therefore, it takes a very long time. In contrast, the number of required failures and the number of items that must be eliminated are preset in the increasing type-II hybrid censoring. The type-I progressive hybrid censoring scheme (TIPHCS), which [4] presented as a solution to this problem, has a time and failure constraint that states the experiment will continue until a predefined time point or a certain number of failures, whichever comes first. However, due to the random sample size in TIPHCS, there would only be a small number of failures or none before a predetermined time limit, which would lead to a low efficiency of parameter estimation. A pre-fixed progressive censoring scheme $\rho_1, \rho_2, \ldots, \rho_m$, and a life test with n units subjected to a predetermined number of failures m are the two main components of [5] proposed adaptive type-II PHCS (ATII-PHCS). However, the experimenter is free to alter some of the ρ_{iw}'s during the experiment based on the conditions. When the experiment reaches its first failure time, $x_{1:m:n}$, ρ_1 units are randomly selected from the remaining $n-1$ living items and eliminated. At the second failure time, units $x_{2:m:n}$, ρ_2 are randomly selected among the remaining $n-1-\rho_1$ units, and so forth. The experiment stops at time $x_{m:m:n}$ if the $m-th$ failure time, $x_{m:m:n}$, happens before the preset time T. All leftover $\rho_m = n - m - \sum_{i=1}^{m} \rho_i$ units are then eliminated. The experiment can continue beyond the test termination time limit thanks to the ATII-PHCS. Consequently, the experiment will end fast by putting $\rho_{c+1}, \rho_{c+2}, \ldots, \rho_{m-1} = 0$, if $x_{m:m:n} > T$. This means that until the effective sample of m failures is reached, no surviving item will be removed from the experiment, resulting in the remaining units $\rho_m = n - c - \sum_{i=1}^{c} \rho_i$, if $x_{c:m:n} < T < x_{c+1:m:n}$, where $c+1 < m$ and $x_{c:m:n}$ are the $c-$th failure time occurred before T.

The versatility and flexibility of generalized distribution to simulate real-life data in a variety of practical fields, including survival analysis, engineering, and medicine, is one of its key characteristics. In addition, adding an additional parameter to the baseline distribution to increase its flexibility, for

more information, see [6]. So, increasing the shape parameter to the Lomax distribution in lifetime analysis, it becomes a more flexible model in the actual life applications. In other words, it has been used in a variety of fields, including biological sciences, reliability and life testing, modeling business failure data, and wealth and income data analysis; see [7]. As far as we are aware, not much research has been done on the alpha power Lomax (APL) distribution [8]. We investigate statistical inference of the APL distribution under ATII-PHCS considering this case. The APL distribution's flexibility in terms of its hazard rate function (HRF) and probability density function (PDF) has made it a valuable model for life testing and reliability assessments; see Fig. 1. The cumulative distribution function (CDF), PDF, survival function (SF), and HRF of an APL random variable Y are provided by

$$F(y;\alpha,\beta,\theta) = \frac{\alpha^{1-\left(1+\frac{y}{\beta}\right)^{-\theta}} - 1}{\alpha - 1}, \alpha,\beta,\theta,y > 0, \tag{1}$$

$$f(y;\alpha,\beta,\theta) = \frac{\log(\alpha)\,\theta\left(1+\frac{y}{\beta}\right)^{-\theta-1}\alpha^{1-\left(1+\frac{y}{\beta}\right)^{-\theta}}}{\beta(\alpha-1)}, \alpha,\beta,\theta,y > 0, \tag{2}$$

$$S(y;\alpha,\beta,\theta) = 1 - \frac{\alpha^{1-\left(1+\frac{y}{\beta}\right)^{-\theta}} - 1}{\alpha - 1}, \alpha,\beta,\theta,y > 0, \tag{3}$$

and

$$h(y;\alpha,\beta,\theta) = \frac{\log(\alpha)\,\theta\left(1+\frac{y}{\beta}\right)^{-\theta-1}\alpha^{-\left(1+\frac{y}{\beta}\right)^{-\theta}}}{\beta\left(1-\alpha^{-\left(1+\frac{y}{\beta}\right)^{-\theta}}\right)}, \alpha,\beta,\theta,y > 0, \tag{4}$$

respectively, where $\alpha > 0$ and $\theta > 0$ are the shape parameters and $\beta > 0$ is a scale parameter.

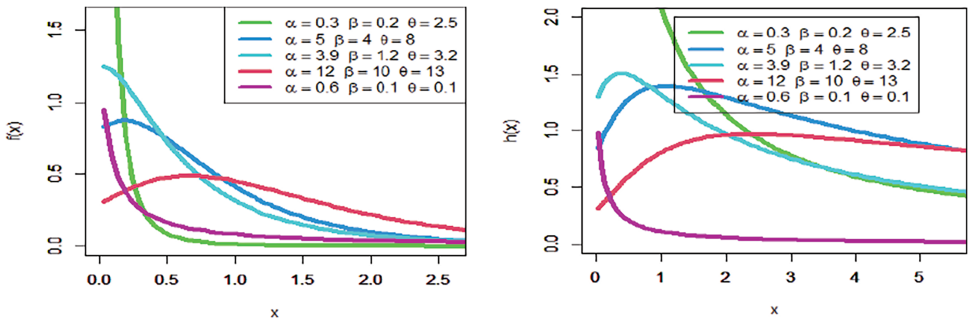

Figure 1: PDF and HRF of APL distribution

It is noted from Fig. 1 that the curves show a wide range of behaviors, such as increasing, decreasing, unimodal, and right skewed which indicate the flexibility of the APL model in survival analysis or reliability engineering. The paper presents several original and valuable contributions, which we summarize below:

This work is among the first to study the APL distribution under a step stress partially accelerated life test (SSPALT) and tampered random variable (TRV) framework. This integration is novel and has not been previously addressed in the literature.

We adopt and implement two advanced censoring mechanisms—TIPHCS and ATII-PHCS—which allow greater flexibility and reflect more realistic testing scenarios. These schemes require careful re-derivation of the likelihood, then calculating the bootstrap and approximate confidence intervals (ACIs) for the model parameters, after that and posterior functions, adding significant methodological value.

The Bayesian analysis is performed not only under standard symmetric loss functions but also under generalized asymmetric loss functions. These are rarely used in reliability studies and provide deeper insight into estimation robustness under various risk preferences.

We introduce and compare multiple optimality criteria (e.g., max-trace, min-trace, min-det) to determine the best progressive censoring strategy, offering practical guidelines for experimental design—a contribution beyond parameter estimation.

The simulation study explores the impact of varying censoring levels, stress times, and prior settings on estimation accuracy, providing a thorough performance comparison of the maximum likelihood estimators (MLEs) and Bayesian methods. Additionally, the real data analysis (on insulating fluid breakdown times) demonstrates the method's practical utility.

In light of these points, we believe that the manuscript makes a substantial contribution both methodologically and practically, going beyond routine parameter estimation, and we hope this clarification addresses the reviewer's concern.

This is how the remainder of the paper is structured. The lifetime model and the test assumptions are explained in Section 2. The MLEs of the APL parameters under the ATII-PHCS are derived in Section 3. The confidence intervals for the unknown parameters are constructed in Section 4. Section 5 illustrates a Bayesian analysis of the unknown parameters. In Section 6, we run simulations to examine the suggested model's performance on a finite sample. An illustrative real-data example is presented in Section 7. In Section 8, concluding remaks is presented, with the model's Fisher information relegated to Appendix A.

2 Testing Assumptions and Procedure

Pretend that there are just two stress levels in a basic SSPALT test: S_{ua} (regular operating conditions) and S_{ac} (accelerated condition). Assume that $S_{ua} < S_{ac}$, where the twins S_{ua} and S_{ac} reside. Every stress level should see at least one failure. At both stress levels in Eq. (2), we assume that the test item failures follow the APL distribution. Then, the lifetime X of a test item is determined and utilizes a TRV model supplied by

$$X = \begin{cases} T, & T < \tau \\ \tau + \dfrac{T-\tau}{\gamma}, & T > \tau \end{cases},$$

where $\gamma > 1$ is an accelerated factor (AF), T indicates an item's lifetime under stress, τ is the time point at which stress is applied, before time τ, the stress level is normal, after time τ,the stress level increases and the item's lifetime is transitioned from *ua* to *ac*. Next, using the TRV model, X's PDF,

CDF, and SF are obtained by

$$f_{ua}(x;\alpha,\beta,\theta) = \frac{\log(\alpha)\,\theta\left(1+\frac{x}{\beta}\right)^{-\theta-1}\alpha^{1-\left(1+\frac{x}{\beta}\right)^{-\theta}}}{\beta(\alpha-1)}, \tag{5}$$

$$F_{ua}(x;\alpha,\beta,\theta) = \frac{\alpha^{1-\left(1+\frac{x}{\beta}\right)^{-\theta}}-1}{\alpha-1}, \tag{6}$$

$$S_{ua}(x;\alpha,\beta,\theta) = 1-\frac{\alpha^{1-\left(1+\frac{x}{\beta}\right)^{-\theta}}-1}{\alpha-1}. \tag{7}$$

Now, the PDF, CDF, and SF of X are as follows when the stress $ST = ac$:

$$f_{ac}(x;\alpha,\beta,\theta,\gamma) = \frac{\log(\alpha)\,\theta\gamma\left(1+\frac{\tau+\gamma(x-\tau)}{\beta}\right)^{-\theta-1}\alpha^{1-\left(1+\frac{\tau+\gamma(x-\tau)}{\beta}\right)^{-\theta}}}{\beta(\alpha-1)}, \alpha,\beta,\theta,x>0,\gamma>1, \tag{8}$$

$$F_{ac}(x;\alpha,\beta,\theta,\gamma) = \frac{\alpha^{1-\left(1+\frac{\tau+\gamma(x-\tau)}{\beta}\right)^{-\theta}}-1}{\alpha-1}, \alpha,\beta,\theta,x>0,\gamma>1, \tag{9}$$

$$S_{ac}(x;\alpha,\beta,\theta,\gamma) = 1-\frac{\alpha^{1-\left(1+\frac{\tau+\gamma(x-\tau)}{\beta}\right)^{-\theta}}-1}{\alpha-1}, \tag{10}$$

where $\alpha,\beta,\theta,x>0,\gamma>1$. Assumed to be tested using SSPALT and a well-known progressive censoring technique $\rho_1,\rho_2,\ldots,\rho_m$, an n-item sample is allocated to the stress level S_{ua}. The items from n that do not fail up to time S_{ua} are placed via S_{ac} to test, and the test will continue until the censorship time is reached. If the m-th failure does not occur within the censoring point T, then no item will be deleted from the test. Testing will go on until the m-th failure is found, at which time it will stop once every item has been ruled out. Thus, the strategy that has been put into practice is $\rho_1,\rho_2,\ldots,\rho_c,0,0,\ldots,0,\rho_m$. Consequently, we derive the observed samples supplied by

$$x_{1:m:n}<x_{2:m:n}<\ldots<x_{m_u:m:n}<\tau<x_{m_u+1:m:n}<\ldots<x_{c:m:n}<T<x_{c+1:m:n}<\ldots<x_{m:m:n},$$

The next part discusses the maximum likelihood approach, and the confidence interval methods used in the classical paradigm to estimate the unknown parameters.

3 The Classical Inference

3.1 Maximum Likelihood Method

Note that the ATII-PHCS was used to produce the likelihood function for the data, which is as follows:

$$L(x;\alpha,\beta,\theta,\gamma) \propto \prod_{i=1}^{m_{ua}} \left\{ f_{ua}(x_i) \left[S_{ua}(x_i) \right]^{\rho_i} \right\} \prod_{i=m_{ua}+1}^{m} \left\{ f_{ac}(x_i) \left[S_{ac}(x_i) \right]^{\rho_i} \left[S_{ac}(x_m) \right]^{\rho m} \right\}, \tag{11}$$

where $x_i = x_{i:m:n}$, $\rho_m = n - c - \sum_{i=1}^{c} \rho_i$, $\left(1 + \frac{\tau + \gamma(x_i - \tau)}{\beta}\right)^{-\theta} = \forall_i$, $\left(1 + \frac{\tau + \gamma(x_m - \tau)}{\beta}\right)^{-\theta} = \forall_m$ and $\left(1 + \frac{x_i}{\beta}\right) = \nabla_i vspace * 3pt$. We may thus rewrite the probability function as X follows the density, as indicated by Eq. (5).

$$L(\mathrm{x};\alpha,\beta,\theta,\gamma) \propto \prod_{i=1}^{m_{ua}} \left\{ \frac{\log(\alpha)\,\theta \nabla_i^{-\theta-1} \alpha^{1-\nabla_i^{-\theta}}}{\beta(\alpha-1)} \left[1 - \frac{\alpha^{1-\nabla_i^{-\theta}} - 1}{\alpha - 1} \right]^{\rho_i} \right\}$$
$$\prod_{i=m_{ua}+1}^{m} \left\{ \frac{\log(\alpha)\,\gamma\theta \forall_i^{-1} \alpha^{1-\forall_i}}{\beta(\alpha-1)} \left[1 - \frac{\alpha^{1-\forall_i} - 1}{\alpha - 1} \right]^{\rho_i} \right\} \left[1 - \frac{\alpha^{1-\forall_m} - 1}{\alpha - 1} \right]^{\rho m}. \tag{12}$$

Since maximum likelihood estimation (MLE) produces estimates with desirable statistical qualities, it is frequently employed to estimate the unknown parameters. The following is the related log-likelihood equation; $\log L(\mathrm{x};\alpha,\beta,\theta,\gamma) = \ell$ of Eq. (12) is shown as follows:

$$\ell = m\log(\log(\alpha)) + m\log(\theta) - (\theta+1)\sum_{i=1}^{m_{ua}} \log \nabla_i + \sum_{i=1}^{m_{ua}} \left[1 - \nabla_i^{-\theta}\right] \log(\alpha) - m\log(\beta) - m\log(\alpha-1)$$
$$+ \sum_{i=1}^{m_{ua}} \rho_i \log\left[\frac{\alpha - \alpha^{1-\nabla_i^{-\theta}}}{\alpha - 1}\right] + (m - m_{ua})\log(\gamma) - (\theta+1)\sum_{i=m_{ua}+1}^{m} \log \forall_i^{\frac{1}{\theta}} + \sum_{i=m_{ua}+1}^{m} (1 - \forall_i)\log(\alpha)$$
$$- \sum_{i=m_{ua}+1}^{m} \rho_i \log\left(\frac{\alpha - \alpha^{1-\forall_i}}{\alpha - 1}\right) + \rho_m \log\left(1 - \frac{\alpha^{1-\forall_m} - 1}{\alpha - 1}\right). \tag{13}$$

The following nonlinear system equations can be solved to determine the MLEs of the parameters α, θ, β and γ:

$$\frac{\partial \ell}{\partial \alpha} = \frac{m}{\alpha \log(\alpha)} + \sum_{i=1}^{m_{ua}} \frac{1 - \nabla_i^{-\theta}}{\alpha} - \frac{m}{\alpha - 1} + \sum_{i=1}^{m_{ua}} \frac{\rho_i \alpha^{1-\nabla_i^{-\theta}} \left(1 - \nabla_i^{-\theta}\right)}{(\alpha-1)\left(\alpha - \alpha^{1-\nabla_i^{-\theta}}\right)}$$
$$+ \sum_{i=m_{ua}+1}^{m} \frac{(1 - \forall_i)}{\alpha} + \sum_{i=m_{ua}+1}^{m} \frac{\rho_i \alpha^{1-\forall_i}(1 - \forall_i)}{(\alpha-1)(\alpha - \alpha^{1-\forall_i})} + \frac{\rho_m \alpha^{1-\forall_m}(1 - \forall_m)}{(\alpha-1)(\alpha - \alpha^{1-\forall_m})}, \tag{14}$$

$$\frac{\partial \ell}{\partial \theta} = \frac{m}{\theta} - \sum_{i=1}^{m_{ua}} \log \nabla_i - \log(\alpha) \sum_{i=1}^{m_{ua}} \nabla_i^{-\theta} \log \nabla_i - \log(\alpha) \sum_{i=1}^{m_{ua}} \left(\frac{\rho_i \alpha^{1-\nabla_i^{-\theta}} (\alpha - 1) \log \nabla_i}{\alpha - \alpha^{1-\nabla_i^{-\theta}}} \right) - \sum_{i=m_{ua}+1}^{m} \log \forall_i^{\frac{1}{\theta}}$$

$$-\log(\alpha)\sum_{i=m_{ua}+1}^{m}\forall_i\log\forall_i^{\frac{1}{\theta}}+\sum_{i=m_{ua}+1}^{m}\frac{\rho_i\forall_i^{\frac{1}{\theta}}\alpha^{1-\forall_i}}{\alpha(\alpha-1)(1-\alpha^{-\forall_i})}+\frac{\rho_m\forall_m^{\frac{1}{\theta}}\alpha^{1-\forall_m}}{(\alpha-1)(\alpha-\alpha^{1-\forall_m})}, \tag{15}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\beta}=&(\theta+1)\sum_{i=1}^{m_{ua}}\frac{x_i}{\beta^2\sum_{i=1}^{m_{ua}}\log\nabla_i-}+\log(\alpha)\sum_{i=1}^{m_{ua}}\sum_{i=1}^{m_{ua}}\log\nabla_i^{-\theta}\log\sum_{i=1}^{m_{ua}}\forall_i^{\frac{1}{\theta}}-+\sum_{i=1}^{m_{ua}}\frac{\rho_i(\alpha-1)\alpha^{1-\nabla_i^{-\theta}}\nabla_i^{-\theta}}{\beta^2\left(\alpha-\alpha^{1-\nabla_i^{-\theta}}\right)}\\&+(\theta+1)\sum_{i=m_{ua}+1}^{m}\frac{\tau+\gamma(x_i-\tau)}{\beta^2\forall_i^{\frac{1}{\theta}}}-\frac{m}{\beta}+\sum_{i=m_{ua}+1}^{m}\log(\alpha)\theta\beta^{-2}\forall_i^{-1}\\&+\sum_{i=m_{ua}+1}^{m}\frac{\rho_i(\alpha-1)\alpha^{1-\forall_i}\forall_i}{\alpha\beta^2(1-\alpha^{-\forall_i})}+\frac{\rho_m(\alpha-1)\alpha^{1-\forall_m}\forall_m}{\alpha\beta^2(1-\alpha^{-\forall_m})},\end{aligned} \tag{16}$$

and

$$\begin{aligned}\frac{\partial\ell}{\partial\gamma}=&\frac{m-m_{ua}}{\gamma}-(\theta+1)\sum_{i=m_{ua}+1}^{m}\frac{(x_i-\tau)}{\forall_i^{\frac{1}{\theta}}}-\theta\log(\alpha)\sum_{i=m_{ua}+1}^{m}\frac{\tau+\gamma(x_i-\tau)}{\beta^2}\forall_i^{-1}\\&+\sum_{i=m_{ua}+1}^{m}\frac{\frac{\theta\rho_i\tau+\gamma(x_i-\tau)}{\beta^2}\forall_i^{-1}\alpha^{-\forall_i}}{\alpha-\alpha^{1-\forall_i}}+\frac{\frac{\theta\rho_m\tau+\gamma(x_m-\tau)}{\beta^2}\forall_m^{-1}\alpha^{-\forall_m}}{\alpha-\alpha^{1-\forall_m}}.\end{aligned} \tag{17}$$

It is observed that the parameters of the nonlinear equations previously described have no easy solution or solutions in closed form. As a result, to obtain the MLEs numerically, we are compelled to employ an iterative method such as the Newton-Raphson method. We look at the confidence interval technique in the following.

3.2 Technique of Confidence Intervals

An integer set that fairly approximates an unknown population attribute is called a confidence interval (CI). We consider the two types of CI listed below while considering the unknown parameters.

3.2.1 Approximate Confidence Intervals

Large sample theory for this distribution is easily established and shows that the MLEs are approximately normally distributed and consistent under certain regularity conditions. Specifically, the variance-covariance matrix is represented as $\left[(\hat{\alpha}-\alpha),\left(\hat{\theta}-\theta\right),\left(\hat{\beta}-\beta\right),(\hat{\gamma}-\gamma)\right]\sim N(0,\sigma)$, where $\sigma=\sigma_{ij},i,j=1,2,3,4$. The observed exact inverse of the Fisher information matrix is a natural estimate of the variance-covariance matrix. The estimated values are shown by the $100(1-\emptyset)\%$ two-sided confidence intervals for the unknown parameter.

$$\left(\hat{\pi}_{Li},\hat{\pi}_{Ui}\right):\hat{\pi}_i\pm Z_{1-\frac{\emptyset}{2}}\sqrt{\hat{\sigma}_{ij}},i,j=1,2,3,4, \tag{18}$$

where $Z_{1-\frac{\emptyset}{2}}$ is the $\emptyset-$th percentile of the conventional normal distribution and $\pi=(\alpha,\theta,\beta,\gamma)$. We address the parametric bootstrap percentile intervals as a corrective measure; refer to [9].

3.2.2 Bootstrap Confidence Intervals

The bootstrap confidence intervals have many appealing qualities; therefore, we investigate them here. The following process is used to calculate the pertinent percentile bootstrap confidence intervals and perform parametric bootstrap sampling.

The Algorithm for Bootstrap

1. Step 0: Assume that $b = 1$ for the fundamental configuration. Next, identify the MLE, $\pi = (\alpha, \theta, \beta, \gamma)$, which can be expressed as $\hat{\pi} = \left(\hat{\alpha}, \hat{\theta}, \hat{\beta}, \hat{\gamma}\right)$.
2. Step 1: Using $F\left(.|\hat{\pi}\right)$, get a sample in the first step. Then, get the $\boldsymbol{B}^{\text{th}}$ bootstrap resample t_p^*, where $\hat{\pi}$ is the MLE is from Step 0.
3. Step 2: Using the t_p^*, resample that was obtained in Step1, bootstrap estimates and calculate the values for $\boldsymbol{B}^{\text{th}}$ and $\hat{\pi}^{*\boldsymbol{B}} = \left(\hat{\alpha}^{*B}, \hat{\theta}^{*B}, \hat{\beta}^{*B}, \hat{\gamma}^{*B}\right)$.
4. Step 4: Repeat steps three and four: Steps 1–3 should be repeated until $\boldsymbol{B}$ equals b after setting $\boldsymbol{B} \leftarrow \boldsymbol{B} + 1$.
5. Step 4: Commence by initiating in ascending order: Sort the estimations in increasing order to get the following:$\{\hat{\alpha}^{*[1]}, \hat{\alpha}^{*[2]}, \ldots, \hat{\alpha}^{*[b]}\}, \left\{\hat{\theta}^{*[1]}, \hat{\theta}^{*[2]}, \ldots, \hat{\theta}^{*[b]}\right\}, \left\{\hat{\beta}^{*[1]}, \hat{\beta}^{*[2]}, \ldots, \hat{\beta}^{*[b]}\right\}, \{\hat{\gamma}^{*[1]}, \hat{\gamma}^{*[2]}, \ldots, \hat{\gamma}^{*[b]}\}$.

Then, for the unknown parameters, the $100\,(1-\emptyset)\,\%$ percentile bootstrap confidence intervals are calculated as $\left(\hat{\pi}_{Li}, \hat{\pi}_{Ui}\right) = \left(\hat{\pi}_i^{*\left[\frac{\emptyset}{2}\right]\text{b}}, \hat{\pi}_i^{*\left[1-\frac{\emptyset}{2}\right]\text{b}}\right)$, where $i = 1, 2, 3, 4$, $\hat{\pi}_1^* = \hat{\alpha}^*$, $\hat{\pi}_2^* = \hat{\theta}^*, \hat{\pi}_3^* = \hat{\beta}^*$ and $\hat{\pi}_4^* = \hat{\gamma}^*$. In the following section, we analyze the estimation using the Bayesian paradigm.

4 Bayesian Estimation

Our primary interest in this case is the Bayes estimate for the unknown parameters. Prior to initiating the Bayesian analysis, the unknown parameters must be specified. This study is predicated on the assumptions that the parameters α, θ, β and γ have independent gamma priors and are statistically independent. The form of the related joint prior is as follows:

$$Pr_1\,(\alpha, \theta, \beta, \gamma) \propto \alpha^{\chi_1 - 1}\theta^{\chi_2 - 1}\beta^{\chi_3 - 1}\gamma^{\chi_4 - 1}e^{-\xi_1\alpha - \xi_2\theta - \xi_3\beta - \xi_4\gamma}, \tag{19}$$

where the pre-established hyperparameters, $\chi_j, \xi_{\text{j}} > 0, j = 1, 2, 3, 4$, denote prior knowledge of the unknown parameters.

Elicitation of Hyper-parameters: The process of determining the hyper-parameters is based on the use of informative prior distributions. Specifically, the informative priors for the parameters α, θ, β, and γ are constructed by matching their means and variances with those of the gamma prior distributions assumed for these parameters in the APL model. Accordingly, and in line with the approach proposed in [10], the hyper-parameters are derived by equating the mean and variance of $\hat{\alpha}, \hat{\theta}, \hat{\beta}$, and $\hat{\gamma}$ to those of the corresponding gamma priors.

$$\frac{1}{R}\sum_{j=1}^{R}\hat{\varpi}_i^j = \frac{\chi_i}{\xi_i}, \frac{1}{R-1}\sum_{j=1}^{R}\left[\hat{\varpi}_i^j - \frac{1}{R}\sum_{j=1}^{R}\hat{\varpi}_i^j\right]^2 = \frac{\chi_i}{\xi_i^2}, i = 1, 2\hat{\varpi} = (\hat{\chi}, \hat{\xi}),$$

R is the number of samples iteration. Now on solving the above two equations, the estimated hyper-parameters can be written as:

$$\chi_i = \frac{\left[R^{-1}\sum_{j=1}^{R}\hat{\varpi}_i^j\right]^2}{(R-1)^{-1}\sum_{j=1}^{R}\left[\hat{\varpi}_i^j - \frac{1}{R}\sum_{j=1}^{R}\hat{\varpi}_i^j\right]^2}, \xi_i = \frac{R^{-1}\sum_{j=1}^{R}\hat{\varpi}_i^j}{(R-1)^{-1}\sum_{j=1}^{R}\left[\hat{\varpi}_i^j - \frac{1}{R}\sum_{j=1}^{R}\hat{\varpi}_i^j\right]^2}.$$

The resulting unknown parameters' cumulative posterior distribution will be:

$$Pr\left(\alpha,\theta,\beta,\gamma|\underline{x}\right) \propto Pr_1\left(\alpha,\theta,\beta,\gamma\right) L\left(\mathrm{x};\alpha,\theta,\beta,\gamma\right). \tag{20}$$

It should be emphasized that inferring posteriors of the model parameter is difficult due to the structure of the joint posterior in Eq. (20). We use Markov chain Monte Carlo (MCMC) techniques to provide posterior samples of the unknown parameters required for posterior inference. For more information, refer to [10]. The collected samples will be utilized to construct the matching greatest posterior density (HPD) credible intervals and approximate Bayes estimates for the unknown parameters. The current analysis produced the following Bayes estimates of the unknown parameters under the symmetric (SLF) and asymmetric (ELF) loss functions:

$$\pounds\left(\alpha,\tilde{\alpha}\right) = \left(\tilde{\alpha}-\alpha\right)^2, \pounds\left(\theta,\tilde{\theta}\right) = \left(\tilde{\theta}-\theta\right)^2, \pounds\left(\beta,\tilde{\beta}\right) = \left(\tilde{\beta}-\beta\right)^2, \pounds\left(\gamma,\tilde{\gamma}\right) = \left(\tilde{\gamma}-\gamma\right)^2. \tag{21}$$

The estimated posterior means of α, θ, β and γ are represented by the variables $\tilde{\alpha}, \tilde{\theta}, \tilde{\beta}$ and $\tilde{\gamma}$. When choosing an asymmetric loss function, we take the following into account. A simple generalization of the entropy loss with the shape parameter ϖ set to 1 yields an asymmetric loss function called the generalized entropy (GE).

$$\pounds\left(\pi,\tilde{\pi}\right) \propto \left(\frac{\tilde{\pi}}{\pi}\right)^{\varpi} - \varpi ln\left(\frac{\tilde{\pi}}{\pi}\right) - 1, \varpi \neq 1, \tag{22}$$

where $\tilde{\pi}$ represents the calculated value of π given by

$$\tilde{\pi}_{GE} = \left[E_\pi\left(\pi^{-\varpi}\right)\right]^{\frac{-1}{\varpi}}. \tag{23}$$

For relevant details, see [11], assuming that $\pi^{-\varpi}$ is real and finite and that E_π represents the expected value.

The following MH-within-Gibbs sampling steps can be used to obtain samples of α, θ, β, and γ

Step 1: Set the initial values $\varsigma^{(0)} = \hat{\varsigma}$ and $\kappa^{(0)} = \hat{\kappa}$.

Step 2: Set I = 1.

Step 3: Generate $\alpha^*, \theta^*, \beta^*$, and γ^* from $N\left(\hat{\alpha}, V_{\hat{\alpha}}\right), N\left(\hat{\theta}, V_{\hat{\theta}}\right), N(\hat{\beta}, V_{\hat{\beta}})$ and $N(\hat{\gamma}, V_{\hat{\gamma}})$, respectively.

Step 4: Obtain $\hbar_\alpha = \min\left[1, \frac{\pi\left(\alpha^*|\theta^{(I-1)},\beta^{(I-1)},\gamma^{(I-1)},t\right)}{\pi\left(\alpha^{(I-1)}|\theta^{(I-1)},\beta^{(I-1)},\gamma^{(I-1)},t\right)}\right], \hbar_\theta = \min\left[1, \frac{\pi\left(\vartheta^*|\alpha^{(I-1)},\beta^{(I-1)},\gamma^{(I-1)},t\right)}{\pi\left(\theta^{(I-1)}|\alpha^{(I-1)},\beta^{(I-1)},\gamma^{(I-1)},t\right)}\right],$
$\hbar_\beta = \min\left[1, \frac{\pi\left(\beta^*|\alpha^{(I-1)},\theta^{(I-1)},\gamma^{(I-1)},t\right)}{\pi\left(\beta^{(I-1)}|\alpha^{(I-1)},\theta^{(I-1)},\gamma^{(I-1)},t\right)}\right]$ and $\hbar_\gamma = \min\left[1, \frac{\pi\left(\gamma^*|\alpha^{(I-1)},\beta^{(I-1)},\theta^{(I-1)},t\right)}{\pi\left(\gamma^{(I-1)}|\alpha^{(I-1)},\beta^{(I-1)},\theta^{(I-1)},t\right)}\right]$.

Step 5: Generate samples $U_j; j = 1, 2, 3, 4$ from the uniform $U(0, 1)$ distribution.

Step 6: If $U_1 \leq \hbar_\alpha, U_2 \leq \hbar_\beta, U_3 \leq \hbar_\theta$ and $U_4 \leq \hbar_\gamma$ then set $\alpha^{(I)} = \alpha^*, \theta^{(I)} = \theta^*, \beta^{(I)} = \beta^*, \gamma^{(I)} = \gamma^*$; otherwise $\alpha^{(I)} = \alpha^{(I-1)}, \theta^{(I)} = \theta^{(I-1)}, \beta^{(I)} = \beta^{(I-1)}$ and $\gamma^{(I)} = \gamma^{(I-1)}$, respectively.

Step 7: Set I = I+ 1.

Step 8: Repeat Steps 3–7 B times and obtain $\alpha^{(I)}, \theta^{(I)}, \beta^{(I)}$ and $\gamma^{(I)}$, for I = 1, 2, …, B.

The Bayesian estimates are obtained via SEL. The 95% two-sided HPD credible interval for the unknown parameters or any function of them is given $[\Psi_{0.025N:N}, \Psi_{0.975N:N}]$ by using the method proposed by [12].

5 Method of Optimization

Recently, there has been a lot of research on the best censoring approach; see [2,13]. For values of n and m determined by the test samples, an acceptable censoring scheme is any possible and allowed combination of $\rho_1, \rho_2, \ldots, \rho_m$. We must first decide which of the many progressive censored approaches yields the most information about the unknown parameters before deciding on the best sampling plan. The first issue might be figuring out how to generate the unknown parameter data using a specific progressive censoring data set. Comparing two different progressive censoring systems' worth of information measurements is the second challenge. The set of optimality criteria we used for this paper is listed in Table 1 below. A variety of popular measures (shown by Opt_1) are provided in Table 1 to assist in selecting the best progressive censoring strategy.

Table 1: Criteria for several optimum censoring schemes

Criterion	Method
Opt_1	Max-trace $[\boldsymbol{I}_{4\times4}(.)]$
Opt_2	Min-trace $[\boldsymbol{I}_{4\times4}(.)]^{-1}$
Opt_3	Min-det $[\boldsymbol{I}_{4\times4}(.)]^{-1}$

It is necessary to maximize the observed Fisher information $[\boldsymbol{I}_{4\times4}(.)]$ for Opt_1. We also minimize the determinant and trace of $[\boldsymbol{I}_{4\times4}(.)]^{-1}$ for Opt_2 and Opt_3 criteria. Comparing several of the previously stated criteria is easy when we only have one parameter. Because Opt_2 is not scale-invariant, comparing the two Fisher information matrices becomes difficult when working with a multiparameter distribution. For the optimal progressive censoring, the highest value of Opt_1 and the lowest value of $Opt_i, i, = 2, 3$.

6 Simulation

This section describes how researchers used computer simulations (Monte Carlo methods) to find the missing pieces (unidentified parameters) of a distribution called APL distribution based on ATII-PHCS. They compared two different estimation methods: a common method called MLE and a Bayesian approach that considers a specific loss function. The researchers assessed these methods by looking at bias, how accurate the estimates were on average mean squared errors (MSEs), and how confident they could be in the range of possible values (LCI). They also evaluated how often the confidence intervals captured the true value (coverage probabilities (CPs)) using a technique called bootstrapping with two different algorithms.

The simulations were run for various combinations of predetermined settings (n, m, τ, etc.) and a data removal process (binomial removal scheme). Finally, they used the simulation results to estimate the APL distribution's parameters based on data samples generated under another method

ATII-PHCS. The entire estimation process followed a specific sequence of steps implemented through computer simulations as follows:

I. Initial sample values: We begin by assigning starting values to variable n, m, τ, and T. These likely represent parameters or settings for simulation.

II. Additional initializations: We also set initial values for parameters of APL based on ATII-PHCS as following cases:

Case: 1	$\alpha = 0.4, \beta = 0.3, \theta = 0.5, \gamma = 1.3$	$\tau = 1.5$	$T = 5$
Case: 2	$\alpha = 0.4, \beta = 0.3, \theta = 0.5, \gamma = 1.3$	$\tau = 1$	$T = 4$
Case: 3	$\alpha = 1.4, \beta = 2, \theta = 0.5, \gamma = 1.3$	$\tau = 1.5$	$T = 5$
Case: 4	$\alpha = 1.4, \beta = 2, \theta = 0.5, \gamma = 1.3$	$\tau = 3$	$T = 5$
Case: 5	$\alpha = 0.4, \beta = 0.5, \theta = 2, \gamma = 2$	$\tau = 1.5$	$T = 3$

III. Dives into generating a specific sample:

ATII-PHCS Sample: We're creating a particular type of sample called ATII-PHCS sample. This likely refers to a specific kind of data point used in the simulation.

Underlying Assumption: The process assumes that the number of units removed after each failure follows a specific pattern.

Binomial Distribution: This pattern is described by a binomial distribution; a common statistical model used for situations with two possible outcomes.

Probability Details: The specific details of this binomial distribution will be provided in the next section (the "pmf" refers to the probability mass function, which defines the probability for each possible outcome).

$$P(\Im_1 = h_1) = \binom{n-h}{h_1} p^{h_1}(1-p)^{n-m-h_1}.$$

While, for $i = 2, 3, \ldots, m-1$,

$$P(\Im_i = h_i \,|\Im_{i-1} = h_{i-1}, \ldots, \Im_1 = h_1 = \binom{n-m-\sum_{j=1}^{i-1} h_j}{h_i} p^{h_i}(1-p)^{n-m-\sum_{j=1}^{i} h_j}.$$

Additionally, assuming that $\Im_i$ is unrelated to X_i for all i, the likelihood function can be articulated as follows:

$$L(x_i, \alpha, \beta, \theta, \gamma) = L(x_i, \alpha, \beta, \theta, \gamma \,|\omega = r) P(\omega = r)\ ,$$

where

$$\begin{aligned} P(\Im = r) &= P(\Im_1 = h_1, \Im_2 = h_2, ..., \Im_{r-1} = h_{m-1}) = P(\Im_{m-1} \\ &= h_{m-1} \,|\Im_{m-2} = h_{m-2}, \ldots, \Im_1 = h_1) \\ &\quad \times P(\Im_{m-2} = h_{m-2} \,|\Im_{m-3} = h_{m-3}, \ldots, \Im_1 = h_1) \ldots P(\Im_2 \\ &= h_2 \,|\Im_1 = h_1) P(\Im_1 = h_1). \end{aligned}$$

Precisely,

$$P(\Im = h) = \frac{(n-m)!}{(n-m-\sum_{i=1}^{m-1} h_i)! \prod_{i=1}^{m-1} h_i} p^{\sum_{i=1}^{m-1} h_i} (1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)h_i}.$$

IV. Generate a random set $(u_1, u_2, \ldots, u_n)$ of size n from a uniform distribution $U(0,1)$ while excluding certain values $\left(\Im_1, \Im_2, \ldots, \Im_{m_u}, \ldots, \Im_c, \ldots, \Im_m\right)$ at stress level τ and time T of ATII-PHCS. Then, the failure data at stress level τ can be acquired from the APL distribution using the inverse CDF method with the given equation:

$$x_i = \begin{cases} \beta \left[\left(1 - \frac{1}{\ln(\alpha)} ln\left((\alpha-1)u_i + 1\right) \right)^{-1/\theta} - 1 \right] & x_i < \tau \\ \beta \left[\left(1 - \frac{1}{\ln(\alpha)} ln\left((\alpha-1)u_i + 1\right) \right)^{-1/\theta} - 1 \right] - \rho & \\ \rho + \frac{}{\gamma} & x_i > \tau \end{cases}$$

Utilize the progressive generator algorithm as introduced by [14].

Acquire ATII-PHCS data under the SSPALT method to validate this sampling technique.

- Arrange the data in ascending order:

$$x_{1:m:n} < x_{2:m:n} < \ldots < x_{m_u:m:n} < \tau < x_{m_u+1:m:n} < \ldots < x_{c:m:n} < T < x_{c+1:m:n} < \ldots < x_{m:m:n}$$

- Obtain ATII-PHCS data following the SSPALT approach by repeating steps 1–3. Stop the test at T, which is the minimum value among $x_{m:m:n}$ and ε.

If $x_{m:m:n} \leq T$, end the test at time $x_{m:m:n}$ by removing all $\Im_m = n - m - \sum_{i=1}^{m-1} \Im_i$ survivals (case I).

If $x_{m:m:n} > T$, conclude the test at time T by eliminating all $\Im_m = n - m - \sum_{i=1}^{c-1} \Im_i$ survivals (case II).

V. Involves obtaining the MLEs and Bayesian estimates of the parameters $(\alpha, \beta, \theta, \gamma)$ using numerical techniques applied to equations. Three suggested R packages, specifically 'CODA', 'maxLik', and 'GoFKernel', are employed for computing MLEs, Bayesian estimates, and generating a series of random variables using MCMC methods.

VI. Involves cycling through steps I to V for a total of 10,000 iterations, acquiring the average MLEs and Bayesian estimates along with their MSEs and absolute Biases (ABiases).

VII. Involves obtaining ACIs along with their corresponding lengths and CPs.

Tables 2–7 consistently demonstrate that the ATII-PHCS estimates closely approximate their true values for censored data. Bayesian estimation generally results in smaller MSEs and Biases compared to MLEs. Increasing sample sizes (n and m) while holding censoring times (τ, and T) constantly reduces MSEs and biases across all scenarios for both censoring schemes ($P = 0.25$ and $P = 0.85$), as expected with larger datasets.

Table 2 Optimal Criteria values for different parameters of binomial removal

Case	n	τ	m	Opt_2	Opt_3	Opt_1	n	p	m	Opt_2	Opt_3	Opt_1
I	40	0.25	30	0.5039	0.088352	12186.953	100	0.2500	75	2.5434	0.0000000022	12610.762
			35	0.4175	0.018622	13829.384			90	1.0264	0.0000000013	14792.714
		0.85	30	2.6579	0.004411	8667.951		0.8500	75	1.4042	0.0000000012	10621.702
			35	0.6589	0.000208	15194.967			90	0.0959	0.0000000002	14336.391
II	40	0.25	30	1.5042	0.003860	7146.267	100	0.2500	75	2.5431	0.0000000062	13883.147
			35	2.3157	0.000005	8412.512			90	2.0089	0.0000000004	17188.915
		0.85	30	2.3087	0.002975	11361.759		0.8500	75	8.1850	0.0000000068	12381.635
			35	0.0206	0.000419	13404.633			90	0.8308	0.0000000003	17609.397
III	40	0.25	30	8.4957	0.000131	5928.9965	100	0.2500	75	7.9470	0.00002553	16449.9343
			35	3.9501	0.000128	7298.4944			90	0.9044	0.00000192	23186.7998
		0.85	30	5.0878	0.000404	6069.7221		0.8500	75	1.7808	0.00000112	17069.3373
			35	1.9876	0.000181	7456.0228			90	0.5711	0.00000016	22868.1729
IV	40	0.25	30	5.1336	0.001170	6309.5312	100	0.2500	75	4.4231	0.00001029	17632.0711
			35	1.9783	0.000727	8078.0444			90	0.2290	0.00000170	23320.2061
		0.85	30	4.0911	0.000295	6269.5702		0.8500	75	3.6769	0.00000806	16562.7931
			35	1.7335	0.000010	7424.3677			90	1.9632	0.00000131	22930.175
V	40	0.25	30	68.7950	183.0628	3820.9734	100	0.2500	75	4.7511	0.55790268	11864.363
			35	17.0372	6.4656	6258.6543			90	4.9960	0.06652824	18657.510
		0.85	30	20.4512	174.1567	4254.4019		0.8500	75	5.0417	0.02944808	11987.025
			35	14.10500784	13.5022	4740.1766			90	10.5849	0.11825564	18545.863

Table 3 AB, MSEs, length of CI for MLE, and different Bayesian estimation based on different LFs: Case 1

Case 1				MLE			Bootstrap			SELF			ELF c = −1.5			ELF c = 1.5		
n	*τ*	*m*		AB	MSEs	LACI	CP	LBP	LBT	AB	MSEs	LCCI	CP	AB	MSEs	LCCI	AB	MSEs	LCCI
40	0.25	30	α	0.3297	0.1200	0.4178	97.6%	0.0383	0.0385	0.0882	0.0339	0.6373	94.6%	0.1062	0.0398	0.6229	0.0156	0.0180	0.6124
			β	0.1188	0.0613	0.8520	95.0%	0.0394	0.0387	0.0233	0.0080	0.3514	96.2%	0.0321	0.0099	0.3105	0.0242	0.0038	0.2292
			θ	0.3463	0.2207	1.2456	95.9%	0.0522	0.0535	0.0816	0.0153	0.3363	92.4%	0.0757	0.0148	0.3618	0.1120	0.0197	0.3300
			γ	0.4097	0.4580	2.1137	95.2%	0.0912	0.0952	0.2987	0.3560	1.2331	93.8%	0.2460	0.3047	1.7312	0.5890	0.7437	1.7570
		35	α	0.3260	0.1147	0.3788	99.0%	0.0331	0.0344	0.0634	0.0192	0.5171	95.8%	0.0754	0.0218	0.4665	0.0059	0.0134	0.5180
			β	0.1048	0.0596	0.7621	96.8%	0.0340	0.0342	0.0058	0.0061	0.2862	95.8%	0.0121	0.0075	0.2691	0.0277	0.0055	0.2246
			θ	0.3156	0.1759	0.8691	98.6%	0.0392	0.0395	0.0792	0.0127	0.2859	94.8%	0.0749	0.0124	0.2943	0.1003	0.0154	0.2683
			γ	0.3252	0.3127	1.7849	96.6%	0.0816	0.0817	0.1902	0.2572	0.9480	68.8%	0.1567	0.2244	1.6362	0.3793	0.5123	1.8357
	0.85	30	α	0.3212	0.1387	0.7398	94.8%	0.0336	0.0333	0.0560	0.0216	0.5628	95.4%	0.0712	0.0256	0.5432	0.0317	0.0146	0.5450
			β	0.1227	0.0650	0.8769	94.9%	0.0381	0.0379	0.0170	0.0119	0.3512	95.8%	0.0261	0.0134	0.3095	0.0318	0.0081	0.2483
			θ	0.3379	0.1941	1.1095	96.2%	0.0496	0.0500	0.0705	0.0155	0.3500	94.0%	0.0644	0.0153	0.3837	0.1021	0.0185	0.3361
			γ	0.3702	0.3886	1.9679	99.2%	0.0848	0.0854	0.3430	0.3609	1.1993	93.4%	0.2865	0.3011	1.6102	0.6587	0.8187	1.6256
		35	α	0.3140	0.1186	0.2016	95.6%	0.0090	0.0092	0.0548	0.0215	0.5171	93.4%	0.0690	0.0249	0.5430	0.0115	0.0141	0.5347
			β	0.1152	0.0467	0.6031	95.0%	0.0261	0.0258	0.0024	0.0039	0.2735	96.4%	0.0035	0.0043	0.2536	0.0303	0.0045	0.2225
			θ	0.3268	0.1474	0.3538	97.2%	0.0155	0.0153	0.0680	0.0126	0.2835	96.2%	0.0623	0.0122	0.3182	0.1015	0.0159	0.3163
			γ	0.3653	0.3813	1.9533	99.8%	0.0810	0.0838	0.2333	0.2538	0.9611	70.4%	0.1995	0.2191	1.5763	0.4366	0.5466	1.5874
100	0.25	75	α	0.3430	0.1180	0.0751	97.2%	0.0032	0.0032	0.0653	0.0129	0.4105	97.0%	0.0725	0.0143	0.3168	0.0279	0.0080	0.3115
			β	0.2057	0.0464	0.2496	96.2%	0.0113	0.0111	0.0064	0.0038	0.1630	96.8%	0.0043	0.0007	0.0940	0.0166	0.0012	0.1057
			θ	0.4006	0.1616	0.1329	94.8%	0.0059	0.0057	0.0749	0.0068	0.1643	97.0%	0.0736	0.0065	0.1225	0.0817	0.0079	0.1250
			γ	0.1967	0.1427	1.2654	94.2%	0.0563	0.0567	0.5110	0.4862	1.1822	92.0%	0.4418	0.3919	1.4989	0.8351	1.0248	1.5790
		90	α	0.3244	0.1108	0.0554	97.4%	0.0025	0.0025	0.0345	0.0056	0.3177	97.4%	0.0405	0.0062	0.2053	0.0092	0.0031	0.1823
			β	0.2003	0.0453	0.1662	96.4%	0.0079	0.0079	0.0052	0.0035	0.1312	98.4%	0.0042	0.0003	0.0556	0.0144	0.0006	0.0603
			θ	0.3941	0.1567	0.1066	95.8%	0.0048	0.0047	0.0637	0.0062	0.1334	97.2%	0.0667	0.0061	0.0962	0.0820	0.0076	0.0939
			γ	0.1439	0.0872	1.0119	94.6%	0.0453	0.0457	0.1402	0.3028	1.0063	95.8%	0.3562	0.2509	1.2728	0.6763	0.7654	1.5347
	0.85	75	α	0.3440	0.1187	0.0777	96.7%	0.0035	0.0034	0.0700	0.0161	0.4135	95.8%	0.0773	0.0180	0.3498	0.0315	0.0083	0.3160
			β	0.2136	0.0489	0.2258	95.6%	0.0106	0.0102	0.0083	0.0053	0.1533	98.0%	0.0065	0.0005	0.0858	0.0173	0.0010	0.0946
			θ	0.4018	0.1625	0.1287	94.6%	0.0058	0.0058	0.0705	0.0059	0.1643	98.0%	0.0692	0.0057	0.1059	0.0772	0.0070	0.1040
			γ	0.1502	0.1046	1.1242	95.0%	0.0522	0.0520	0.5419	0.5006	1.1870	94.2%	0.4709	0.4020	1.4605	0.8851	1.0882	1.5402
		90	α	0.3413	0.1182	0.0532	96.9%	0.0023	0.0024	0.0443	0.0055	0.3268	98.2%	0.0490	0.0061	0.1993	0.0201	0.0032	0.1880
			β	0.2026	0.0452	0.1374	95.8%	0.0063	0.0064	0.0074	0.0024	0.1307	99.6%	0.0062	0.0002	0.0509	0.0133	0.0004	0.0526
			θ	0.3841	0.1566	0.0988	94.8%	0.0045	0.0045	0.0607	0.0054	0.1436	99.2%	0.0572	0.0052	0.0943	0.0718	0.0067	0.0967
			γ	0.1440	0.0812	0.9645	95.6%	0.0424	0.0426	0.1384	0.1322	0.9988	95.2%	0.3373	0.2613	1.4288	0.6518	0.7713	1.4656

Table 4 AB, MSEs, length of CI for MLE, and different Bayesian estimation based on different LFs: Case 2

Case 2			MLE				Bootstrap		SELF				ELF c = −1.5			ELF c = 1.5		
n	*τ*	*m*	AB	MSEs	LACI	CP	LBP	LBT	AB	MSEs	LCCI	CP	AB	MSEs	LCCI	AB	MSEs	LCCI
40	0.25	30	α 0.3178	0.1155	0.4724	97.8%	0.0215	0.0210	0.0757	0.0326	0.5813	94.6%	0.0918	0.0386	0.5874	0.0205	0.0184	0.6101
			β 0.1393	0.0561	0.7514	95.6%	0.0335	0.0334	0.0125	0.0064	0.3576	94.8%	0.0218	0.0076	0.3022	0.0365	0.0060	0.2557
			θ 0.3536	0.1905	1.0037	98.2%	0.0443	0.0450	0.0738	0.0124	0.3310	94.8%	0.0682	0.0121	0.3293	0.1031	0.0155	0.2825
			γ 0.2736	0.2947	1.8398	94.0%	0.0861	0.0859	0.2350	0.3408	1.1283	67.6%	0.1880	0.2955	1.7921	0.4968	0.6798	1.8238
		35	α 0.3027	0.1099	0.2231	95.2%	0.0099	0.0098	0.0457	0.0159	0.4682	95.8%	0.0564	0.0180	0.4755	0.0138	0.0114	0.4597
			β 0.1278	0.0535	0.5790	96.2%	0.0269	0.0265	0.0041	0.0053	0.2666	97.2%	0.0095	0.0060	0.2138	0.0240	0.0037	0.1930
			θ 0.3283	0.1536	0.3309	97.2%	0.0153	0.0150	0.0719	0.0121	0.2679	95.6%	0.0583	0.0116	0.2495	0.1016	0.0156	0.2492
			γ 0.2442	0.2411	1.6714	92.2%	0.0782	0.0766	0.2184	0.2373	0.9596	94.8%	0.1878	0.2077	1.5224	0.3954	0.4736	1.7224
	0.85	30	α 0.3046	0.1321	1.3363	99.4%	0.0551	0.0573	0.0560	0.0216	0.5628	95.4%	0.0712	0.0256	0.5432	0.0317	0.0146	0.5450
			β 0.1716	0.0485	0.5411	95.6%	0.0250	0.0248	0.0039	0.0034	0.3006	96.8%	0.0114	0.0040	0.2475	0.0333	0.0043	0.2335
			θ 0.3724	0.1707	0.7029	99.2%	0.0313	0.0326	0.0790	0.0130	0.3269	94.2%	0.0734	0.0125	0.3508	0.1075	0.0172	0.3018
			γ 0.3090	0.3286	1.8945	90.8%	0.0836	0.0827	0.2628	0.3407	1.1518	71.8%	0.2138	0.2929	1.8466	0.5370	0.7094	1.8854
		35	α 0.2832	0.1082	0.2324	94.8%	0.0103	0.0104	0.0485	0.0192	0.4548	96.8%	0.0586	0.0211	0.4943	0.0091	0.0142	0.5318
			β 0.1682	0.0450	0.3434	94.0%	0.0149	0.0151	0.0036	0.0014	0.2194	96.6%	0.0021	0.0013	0.1730	0.0271	0.0028	0.1862
			θ 0.3196	0.1460	0.2076	94.6%	0.0091	0.0091	0.0718	0.0104	0.2469	93.8%	0.0681	0.0099	0.2555	0.0999	0.0133	0.2527
			γ 0.1999	0.2040	1.5894	91.0%	0.0707	0.0701	0.1581	0.1923	0.8866	84.2%	0.1294	0.1661	1.4602	0.3276	0.4123	1.7017
100	0.25	75	α 0.3293	0.1092	0.1060	96.0%	0.0047	0.0047	0.0356	0.0057	0.3243	98.0%	0.0404	0.0064	0.2102	0.0105	0.0033	0.1984
			β 0.2279	0.0455	0.2030	96.6%	0.0093	0.0089	0.0051	0.0004	0.1419	98.2%	0.0036	0.0004	0.0577	0.0127	0.0008	0.0655
			θ 0.4159	0.1574	0.1203	96.2%	0.0053	0.0054	0.0688	0.0062	0.1738	96.2%	0.0673	0.0060	0.1344	0.0763	0.0074	0.1325
			γ 0.0968	0.0826	1.0620	95.2%	0.0500	0.0486	0.1531	0.3051	1.1702	74.6%	0.4577	0.4119	1.5276	0.8617	1.0818	1.5738
		90	α 0.3305	0.1080	0.1036	96.8%	0.0045	0.0046	0.0219	0.0024	0.2836	99.2%	0.0256	0.0027	0.1425	0.0031	0.0017	0.1313
			β 0.2245	0.0416	0.1129	97.2%	0.0051	0.0051	0.0049	0.0003	0.1059	98.6%	0.0036	0.0003	0.0419	0.0105	0.0005	0.0452
			θ 0.4021	0.1377	0.0915	96.4%	0.0041	0.0041	0.0618	0.0060	0.1433	97.6%	0.0607	0.0059	0.0903	0.0758	0.0072	0.0879
			γ 0.0663	0.0474	0.8136	94.8%	0.0370	0.0366	0.1480	0.2315	1.0257	77.4%	0.3313	0.2539	1.3316	0.6631	0.7893	1.5461
	0.85	75	α 0.3288	0.1090	0.1160	97.0%	0.0051	0.0052	0.0301	0.0048	0.3184	97.6%	0.0350	0.0053	0.2092	0.0052	0.0030	0.1954
			β 0.2346	0.0446	0.1508	96.0%	0.0071	0.0069	0.0047	0.0003	0.1322	98.8%	0.0034	0.0003	0.0647	0.0113	0.0006	0.0695
			θ 0.4168	0.1574	0.1079	96.0%	0.0050	0.0049	0.0673	0.0056	0.1714	97.0%	0.0658	0.0054	0.1298	0.0744	0.0067	0.1238
			γ 0.0804	0.0585	0.8952	94.2%	0.0395	0.0395	0.2745	0.2951	1.1956	80.4%	0.4550	0.4087	1.5473	0.8677	1.0889	1.5895
		90	α 0.3034	0.1012	0.0679	97.2%	0.0032	0.0031	0.0255	0.0030	0.2927	98.6%	0.0293	0.0034	0.1763	0.0051	0.0020	0.1650
			β 0.2046	0.0406	0.1096	98.0%	0.0051	0.0050	0.0041	0.0002	0.1051	99.2%	0.0033	0.0002	0.0336	0.0101	0.0003	0.0409
			θ 0.4122	0.1378	0.0809	96.2%	0.0035	0.0036	0.0668	0.0052	0.1319	98.8%	0.0576	0.0051	0.0814	0.0708	0.0071	0.0811
			γ 0.0536	0.0479	0.8325	95.4%	0.0360	0.0361	0.0389	0.1048	0.8024	97.0%	0.3430	0.2509	0.1243	0.1307	0.1745	0.8005

Table 5 AB, MSEs, length of CI for MLE, and different Bayesian estimation based on different LFs: Case 3

Case 3				MLE			Bootstrap		SELF				ELF c = −1.5			ELF c = 1.5		
n	***τ***	***m***		AB	MSEs	LACI	CP	LBP	LBT	AB	MSEs	LCCI	CP	AB	MSEs	LCCI	AB	MSEs	LCCI
40	0.25	30	α	1.0064	1.1121	1.2361	94.4%	0.0552	0.0556	0.1643	0.2656	1.2348	82.0%	0.1888	0.2777	1.8998	0.0256	0.2378	1.9780
			β	1.5596	2.5510	1.3510	93.0%	0.0610	0.0596	0.0268	0.0728	0.8990	91.0%	0.0162	0.0709	1.1102	0.0854	0.0995	1.1543
			θ	0.4477	0.2006	0.0546	95.8%	0.0024	0.0024	0.1597	0.0360	0.1968	66.0%	0.1570	0.0349	0.3479	0.1731	0.0414	0.3611
			γ	1.3678	2.2442	2.3980	96.2%	0.1114	0.1108	0.2095	0.4587	1.3622	63.6%	0.1491	0.4018	2.1062	0.5223	0.8401	2.0573
		35	α	0.9704	1.0177	1.1679	94.9%	0.0503	0.0511	0.0775	0.1318	0.8913	78.8%	0.0912	0.1337	1.3686	0.0026	0.1395	1.5563
			β	1.3635	2.1776	1.2534	93.8%	0.0563	0.0563	0.0064	0.0577	0.6757	87.0%	0.0006	0.0569	1.0656	0.0358	0.0638	1.0779
			θ	0.4491	0.1620	0.0503	96.8%	0.0023	0.0023	0.1480	0.0358	0.1676	45.0%	0.1516	0.0307	0.3352	0.1675	0.0342	0.3475
			γ	1.2402	2.0367	2.2482	97.4%	0.1091	0.1087	0.1416	0.2605	1.0510	68.0%	0.1070	0.2299	1.6919	0.3430	0.5160	1.9008
	0.85	30	α	1.0417	1.1685	1.1334	95.2%	0.0508	0.0503	0.1396	0.2250	1.1644	82.2%	0.1611	0.2348	1.8805	0.0165	0.2041	1.9994
			β	1.5957	2.6547	1.2917	94.4%	0.0574	0.0566	0.0828	0.0734	0.8866	90.8%	0.0172	0.0729	1.2391	0.0603	0.0840	1.2008
			θ	0.4478	0.2007	0.0585	95.7%	0.0027	0.0027	0.1459	0.0311	0.1918	69.8%	0.1434	0.0302	0.3414	0.1580	0.0357	0.3547
			γ	1.4505	2.5898	2.7353	97.0%	0.1196	0.1208	0.1918	0.4372	1.3658	68.0%	0.1333	0.3872	2.0921	0.5088	0.8048	2.0720
		35	α	1.0326	1.1552	1.1270	95.6%	0.0495	0.0487	0.0679	0.1218	0.8802	82.0%	0.0808	0.1240	1.3615	0.0004	0.1224	1.3710
			β	1.4663	2.5853	1.1557	95.2%	0.0547	0.0534	0.0308	0.0495	0.6571	88.6%	0.0250	0.0478	0.9961	0.0561	0.0612	1.0437
			θ	0.4495	0.1820	0.0519	96.2%	0.0025	0.0025	0.1416	0.0308	0.1708	61.0%	0.1399	0.0301	0.3287	0.1568	0.0348	0.3434
			γ	1.3854	2.3269	2.5052	97.6%	0.1120	0.1103	0.1492	0.2606	1.0462	68.6%	0.1139	0.2290	1.7533	0.3544	0.5255	1.8946
100	0.25	75	α	1.0362	0.9129	0.9219	96.2%	0.0408	0.0410	0.1842	0.1894	1.0432	83.6%	0.2013	0.1962	1.6825	0.0893	0.1730	1.7957
			β	1.7052	1.9521	0.8258	96.0%	0.0364	0.0365	0.0305	0.0392	0.5734	92.6%	0.0258	0.0371	0.7421	0.0543	0.0521	0.8225
			θ	0.4531	0.1621	0.0366	95.4%	0.0016	0.0016	0.1467	0.0330	0.1211	30.8%	0.1657	0.0390	0.3610	0.1729	0.0420	0.3722
			γ	1.3197	1.9481	1.7827	95.7%	0.0816	0.0816	0.2480	0.3932	1.2465	65.8%	0.1918	0.3331	1.9340	0.5207	0.7528	1.9418
		90	α	0.9576	0.8017	0.9257	96.4%	0.0401	0.0402	0.1360	0.1012	0.7644	83.6%	0.1453	0.1041	1.1813	0.0883	0.0898	1.1770
			β	1.5239	1.6123	0.8093	96.8%	0.0341	0.0334	0.0234	0.0248	0.4255	91.0%	0.0206	0.0233	0.6277	0.0367	0.0329	0.6642
			θ	0.3558	0.1421	0.0327	97.0%	0.0014	0.0015	0.1262	0.0316	0.0997	26.0%	0.1614	0.0360	0.3449	0.1665	0.0382	0.3533
			γ	1.2339	1.6985	1.7200	96.5%	0.0767	0.0774	0.1220	0.2166	0.9675	65.8%	0.1829	0.2245	1.5569	0.4174	0.5518	1.7083
	0.85	75	α	1.0707	0.8920	0.8718	95.6%	0.0402	0.0395	0.1681	0.2106	1.0410	77.8%	0.1861	0.2168	1.6860	0.0710	0.1999	1.7893
			β	1.7375	1.0546	0.7416	96.2%	0.0327	0.0330	0.0357	0.0277	0.5466	91.2%	0.0317	0.0263	0.7291	0.0565	0.0366	0.7804
			θ	0.4543	0.1821	0.0350	96.4%	0.0015	0.0016	0.1730	0.0294	0.1232	32.6%	0.1715	0.0420	0.3818	0.1795	0.0454	0.3910
			γ	1.3818	2.1319	1.8507	95.6%	0.0833	0.0828	0.2402	0.4258	1.2391	63.2%	0.1839	0.3661	2.0482	0.4947	0.7389	1.9529
		90	α	1.0466	0.8158	0.8303	96.4%	0.0401	0.0374	0.1234	0.1089	0.7844	81.2%	0.1334	0.1112	1.2510	0.0696	0.1048	1.2736
			β	1.7306	0.9801	0.6118	96.6%	0.0314	0.0309	0.0299	0.0216	0.4270	90.0%	0.0271	0.0203	0.6332	0.0450	0.0310	0.6604
			θ	0.4552	0.1721	0.0347	97.0%	0.0015	0.0015	0.1676	0.0204	0.0977	24.6%	0.1744	0.0416	0.3633	0.1799	0.0443	0.3757
			γ	1.3205	1.9433	1.7536	96.5%	0.0804	0.0805	0.1799	0.2235	0.9685	70.4%	0.1462	0.1904	1.4636	0.3694	0.4848	1.7350

Table 6 AB, MSEs, length of CI for MLE, and different Bayesian estimation based on different LFs: Case 4

Case 4				MLE			Bootstrap		SELF				ELF c = −1.5			ELF c = 1.5		
n	*τ*	*m*		AB	MSEs	LACI	CP	LBP	LBT	AB	MSEs	LCCI	CP	AB	MSEs	LCCI	AB	MSEs	LCCI
40	0.25	30	α	0.9305	0.9755	1.2993	95.0%	0.0577	0.0570	0.1414	0.2316	1.2269	82.2%	0.1650	0.2390	1.9184	0.0632	0.2295	2.1109
			β	1.6065	2.1705	1.3828	92.0%	0.0611	0.0622	0.0032	0.0710	0.8003	88.8%	0.0117	0.0768	0.9381	0.0638	0.0612	0.8440
			θ	0.4482	0.2012	0.0708	95.2%	0.0031	0.0031	0.1550	0.0329	0.1808	69.6%	0.1530	0.0321	0.3239	0.1652	0.0370	0.3346
			γ	0.9093	1.3465	2.8288	95.4%	0.1419	0.1301	0.0652	0.2975	1.3558	80.4%	0.1000	0.2973	1.9194	0.1458	0.3854	2.0726
		35	α	0.9248	0.9698	1.2367	96.0%	0.0564	0.0555	0.0472	0.1061	0.8705	83.4%	0.0597	0.1068	1.2950	0.0190	0.1125	1.4038
			β	1.1676	2.0909	1.2383	93.0%	0.0577	0.0561	0.0012	0.0518	0.7155	87.6%	0.0048	0.0514	0.9190	0.0324	0.0565	0.8279
			θ	0.4351	0.1582	0.0511	96.8%	0.0023	0.0023	0.1448	0.0293	0.1611	67.8%	0.1431	0.0286	0.3054	0.1526	0.0324	0.3257
			γ	0.9002	1.1644	2.2226	95.8%	0.1016	0.1010	0.0228	0.1871	0.9851	72.0%	0.0424	0.1837	1.5582	0.0932	0.2462	1.8706
	0.85	30	α	0.9754	1.0617	1.3032	94.6%	0.0578	0.0581	0.1346	0.2027	1.1994	85.0%	0.1562	0.2106	1.6753	0.0116	0.1932	1.8825
			β	1.1625	2.0755	1.3236	95.8%	0.0592	0.0568	0.0147	0.0718	0.9659	92.0%	0.0058	0.0759	1.2429	0.0768	0.0974	1.3281
			θ	0.4049	0.1620	0.0658	95.8%	0.0030	0.0030	0.1490	0.0315	0.1781	72.2%	0.1470	0.0307	0.3413	0.1587	0.0354	0.3549
			γ	0.9336	1.3462	2.7034	96.4%	0.1174	0.1179	0.1192	0.3236	1.4255	80.8%	0.1559	0.3313	2.1110	0.1003	0.3658	2.2090
		35	α	0.9682	1.0585	1.2000	95.4%	0.0532	0.0539	0.0563	0.1055	0.8453	81.6%	0.0679	0.1065	1.2496	0.0072	0.1108	1.3970
			β	1.0686	2.0093	1.1912	96.8%	0.0548	0.0545	0.0047	0.0481	0.6889	87.0%	0.0011	0.0543	1.0836	0.0342	0.0599	1.1472
			θ	0.3450	0.1520	0.0524	96.2%	0.0024	0.0023	0.1493	0.0311	0.1605	60.6%	0.1476	0.0304	0.3164	0.1572	0.0343	0.3338
			γ	0.9242	1.2864	2.5798	97.8%	0.1108	0.1156	0.0497	0.1932	1.0048	75.4%	0.0691	0.1937	1.6518	0.0585	0.2194	1.7827
100	0.25	75	α	0.8648	0.9097	1.0110	97.0%	0.0490	0.0470	0.1287	0.1454	1.0550	86.8%	0.1468	0.1508	1.4980	0.0294	0.1392	1.6019
			β	0.7376	1.0801	0.9673	96.2%	0.0506	0.0484	0.0563	0.0397	0.6763	94.0%	0.0499	0.0364	0.7159	0.0885	0.0598	0.7711
			θ	0.3549	0.1421	0.0364	95.8%	0.0016	0.0017	0.1705	0.0310	0.1033	30.0%	0.1696	0.0391	0.3468	0.1744	0.0415	0.3638
			γ	0.8954	0.9573	1.5473	94.8%	0.0984	0.0954	0.2489	0.2930	1.2407	78.6%	0.2734	0.3063	1.8657	0.1126	0.2565	1.9370
		90	α	0.7958	0.7807	0.9502	97.5%	0.0447	0.0450	0.1126	0.0802	0.7406	85.4%	0.1214	0.0824	1.0561	0.0266	0.0739	1.0627
			β	0.6757	1.0166	0.9410	96.6%	0.0430	0.0436	0.0236	0.0153	0.4697	93.6%	0.0210	0.0149	0.5300	0.0366	0.0178	0.5432
			θ	0.3046	0.1208	0.0337	96.8%	0.0016	0.0016	0.1464	0.0279	0.0864	29.8%	0.1457	0.0295	0.3290	0.1491	0.0310	0.3423
			γ	0.5907	0.6109	1.4097	98.4%	0.0662	0.0665	0.1084	0.1387	0.8861	79.2%	0.1223	0.1410	1.3946	0.0350	0.1366	1.4715
	0.85	75	α	0.8594	0.8585	1.0361	96.2%	0.0439	0.0455	0.1480	0.1459	1.0163	85.4%	0.1641	0.1528	1.5418	0.0599	0.1293	1.5279
			β	0.7449	0.9112	1.0198	95.6%	0.0438	0.0449	0.0332	0.0221	0.6036	96.0%	0.0288	0.0211	0.4941	0.0553	0.0286	0.5578
			θ	0.3545	0.1121	0.0338	93.8%	0.0015	0.0015	0.1555	0.0303	0.1034	29.8%	0.1547	0.0328	0.3098	0.1591	0.0347	0.3198
			γ	0.8793	0.9743	1.7593	97.0%	0.0767	0.0766	0.1927	0.2592	1.1993	81.8%	0.2163	0.2691	1.9252	0.0663	0.2281	1.9337
		90	α	0.8493	0.7941	1.0364	96.4%	0.0415	0.0405	0.0929	0.0821	0.7558	85.6%	0.1026	0.0842	1.1513	0.0424	0.0786	1.1329
			β	0.7055	0.9017	0.9735	96.6%	0.0430	0.0428	0.0267	0.0205	0.4672	91.8%	0.0238	0.0196	0.4589	0.0416	0.0275	0.5164
			θ	0.3346	0.1021	0.0315	96.8%	0.0014	0.0013	0.1483	0.0283	0.0863	27.2%	0.1476	0.0307	0.2834	0.1511	0.0323	0.3054
			γ	0.8762	0.8174	1.5180	97.2%	0.0679	0.0688	0.1036	0.1360	0.8645	77.8%	0.1168	0.1393	1.3891	0.0339	0.1300	1.3887

Table 7 AB, MSEs, length of CI for MLE, and different Bayesian estimation based on different LFs: Case 5

Case 5		MLE				Bootstrap			SELF				ELF c = −1.5			ELF c = 1.5	
m		AB	MSEs	LACI	CP	LBP	LBT	AB	MSEs	LCCI	CP	AB	MSEs	LCCI	AB	MSEs	LCCI
30	α	0.0840	0.7683	3.4236	94.0%	0.1516	0.1563	0.0558	0.0658	0.5482	95.2%	0.0795	0.0775	0.9526	0.0896	0.0486	0.5345
	β	0.1275	0.2591	1.9336	95.3%	0.0874	0.0849	0.1268	0.1075	0.6325	94.0%	0.1450	0.1208	1.0504	0.0268	0.0542	0.9059
	θ	0.4375	1.8632	5.0736	94.8%	0.2369	0.2320	0.1055	0.2436	0.9842	94.4%	0.0878	0.2388	1.9787	0.1981	0.2867	1.9697
	γ	0.6233	3.0746	7.5335	95.8%	0.3411	0.3348	0.0954	0.2668	1.0215	95.0%	0.0730	0.2510	2.2209	0.2346	0.4464	2.5769
35	α	0.0818	0.7525	3.3887	94.2%	0.1489	0.1536	0.0363	0.0596	0.4646	95.8%	0.0526	0.0643	0.9008	0.0616	0.0450	0.5175
	β	0.0737	0.1812	1.6451	95.8%	0.0735	0.0729	0.0778	0.0454	0.4993	96.2%	0.0893	0.0492	0.7950	0.0156	0.0315	0.7278
	θ	0.4272	1.5723	4.5582	94.9%	0.2038	0.2003	0.1019	0.1382	0.7894	96.8%	0.0780	0.1329	1.4604	0.1757	0.1737	1.6200
	γ	0.6057	2.3809	5.2925	96.6%	0.2310	0.2340	0.0632	0.1459	0.7785	95.9%	0.0514	0.1381	1.5878	0.1266	0.2033	1.8491
30	α	0.0275	0.7279	3.3461	94.8%	0.1472	0.1515	0.0658	0.1017	0.5498	95.8%	0.0892	0.1170	0.9599	0.0747	0.0574	0.4479
	β	0.0862	0.2581	1.9647	93.8%	0.0886	0.0859	0.0860	0.0709	0.5897	95.2%	0.1031	0.0785	0.9517	0.0067	0.0455	0.8090
	θ	0.5291	2.0877	5.2758	95.2%	0.2378	0.2390	0.1316	0.2149	0.9952	95.4%	0.1132	0.2087	1.6670	0.2308	0.2742	1.7992
	γ	0.7263	2.7563	5.8581	95.2%	0.2498	0.2539	0.0666	0.2825	1.0941	95.0%	0.0420	0.2694	2.2673	0.2186	0.4539	2.7047
35	α	0.0024	0.4728	2.6980	95.0%	0.1185	0.1184	0.0433	0.0543	0.4861	96.1%	0.0609	0.0601	0.8570	0.0689	0.0446	0.4268
	β	0.0818	0.1969	1.7111	94.0%	0.0768	0.0750	0.0696	0.0449	0.4965	95.9%	0.0815	0.0493	0.7591	0.0056	0.0296	0.6617
	θ	0.4651	1.4886	4.4262	95.8%	0.1959	0.1945	0.0860	0.1435	0.7742	96.4%	0.0753	0.1391	1.5679	0.1420	0.1744	1.6395
	γ	0.6635	2.6882	5.4883	95.9%	0.2461	0.2467	0.0593	0.1572	0.7966	96.8%	0.0417	0.1507	1.7340	0.1303	0.2247	1.9610
75	α	0.0181	0.4677	2.6825	93.6%	0.1172	0.1206	0.0516	0.0584	0.5164	95.8%	0.0881	0.0973	1.0717	0.0964	0.0587	0.5828
	β	0.1166	0.1167	1.2601	95.6%	0.0561	0.0569	0.0958	0.0707	0.6197	95.6%	0.1146	0.0793	0.9317	0.0006	0.0411	0.7577
	θ	0.3244	1.2875	4.2666	95.6%	0.1939	0.1879	0.1052	0.2347	0.9248	95.4%	0.0844	0.2465	1.9887	0.2135	0.2714	1.9343
	γ	0.9242	2.1094	4.3965	96.4%	0.1991	0.1997	0.1089	0.2496	1.3038	96.6%	0.0818	0.2785	2.0739	0.2666	0.4711	2.6571
90	α	0.0137	0.3728	2.3916	95.0%	0.1034	0.1032	0.0179	0.0375	0.4541	96.4%	0.0332	0.0404	0.7573	0.0696	0.0359	0.5612
	β	0.0919	0.0927	1.1390	96.2%	0.0552	0.0532	0.0482	0.0305	0.5121	96.8%	0.0585	0.0329	0.6010	0.0004	0.0227	0.5489
	θ	0.3090	1.1384	3.3559	95.9%	0.1897	0.1859	0.0926	0.1346	0.8926	96.6%	0.0825	0.1319	1.3909	0.1444	0.1529	1.4264
	γ	0.9127	1.4373	2.9809	96.6%	0.1318	0.1351	0.0550	0.1186	0.9623	96.9%	0.0810	0.1739	1.5060	0.2389	0.2843	1.9030
75	α	0.0241	0.4657	2.6761	95.0%	0.1174	0.1218	0.0769	0.0950	0.6552	95.2%	0.1045	0.1111	1.0789	0.0865	0.0548	0.6881
	β	0.0788	0.1024	1.2170	95.4%	0.0543	0.0539	0.0683	0.0565	0.6449	96.8%	0.0852	0.0629	0.8081	0.0188	0.0353	0.6755
	θ	0.3757	1.1624	3.9654	96.4%	0.1746	0.1752	0.0890	0.2020	1.2016	96.8%	0.0698	0.2191	1.7150	0.1894	0.2436	1.7700
	γ	0.9274	1.6448	3.4760	96.2%	0.1610	0.1587	0.0513	0.1355	1.3026	97.8%	0.1011	0.3329	2.2877	0.2999	0.5718	2.6666
90	α	0.0235	0.2883	2.0995	95.4%	0.0941	0.0928	0.0074	0.0402	0.4648	97.8%	0.0244	0.0422	0.7739	0.0849	0.0432	0.6445
	β	0.0709	0.1012	1.1990	95.8%	0.0538	0.0504	0.0609	0.0423	0.5359	97.2%	0.0806	0.0457	0.7371	0.0115	0.0291	0.6419
	θ	0.3456	1.0400	3.4411	96.9%	0.1700	0.1620	0.0809	0.1350	0.8923	97.6%	0.0682	0.1475	1.4166	0.1460	0.1699	1.4781
	γ	0.9027	0.8146	3.4213	96.6%	0.1578	0.1582	0.0511	0.1248	0.9434	98.4%	0.0957	0.1721	1.6499	0.1875	0.2582	1.9249

Impact of sample size and censoring parameters:

For fixed values of m, τ, and T, MSEs consistently decrease with increasing sample size (n). With fixed values of n, ρ, and m, extending the stress change time ρ leads to lower MSEs and biases, likely due to more failures occurring under normal operating conditions with a longer stress change time.

Impact of censoring time τ:

For ATII-PHCS, increasing the censoring time τ with fixed values of n, m, and τ results in decreasing MSEs and Biases. However, this isn't the case for ATII-PHCS because it has a predetermined number of failures, and surpassing τ doesn't reveal additional failures.

Confidence intervals:

Agresti-Coull Intervals (ACIs), bootstrap, and HPD intervals are reasonably precise for both censoring schemes ($P = 0.25$ and $P = 0.85$). HPD intervals generally have narrower intervals compared to ACIs. Increasing sample sizes (n and m) results in narrower ACIs and bootstrap intervals, with CPs approaching 95% for both censoring schemes, as expected with larger datasets. For fixed values of τ and T, expanding n and m narrows ACIs and bootstrap intervals in all cases for both schemes.

7 Application

This section focuses on analyzing a specific data set to demonstrate how well our proposed model performs. A real data set contained times to breakdown down an insulating fluid between electrodes recorded at different voltages; these data have been discussed by [15]. The following list shows the data as follows: 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.5, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89, are the failure times (in minutes) are presented, which is for an insulating fluid electrode subject to a voltage of 34 kV. Fig. 2 dives into the insulation fluid electrodes dataset using various visualizations and statistical tools to gain a deeper understanding:

Total Time on Test (TTT): Imagine a graph with dots representing individual data points. These dots show how much time elapsed before an event (likely a failure) occurred for each data point.

Estimated Hazard Rate: This visualization is like a risk meter. It shows how likely failure is at any given time, and how that risk increases over time. This will be compared to the TTT plot to see the connection between elapsed time and failure risk. Kernel Density Plots: This creates a smooth curve that depicts the probability of encountering a particular data point at a specific value. Think of it as a landscape, with higher areas showing where data points are more clustered.

Violin and Box Plot: This is a combination of two plots. The violin plot shows the overall spread of the data, with a wider area indicating more data points in that range. The box plot in the middle shows the median (center point), the interquartile range (IQR—middle 50% of the data), and potential outliers. By looking at both plots together, we can see how the data is distributed, where the center is, and if there are any extreme values.

Examining all these visualizations together helps us identify key features and patterns within the data set of insulating fluid electrodes dataset. APL distribution has been estimated of parameters by MLE as $\alpha = 1.5586$, $\beta = 638.6883$, and $\theta = 4.0895$. Also, this model has KS distances which are 0.060773 and p-value is 0.8156. These results confirm that the APL distribution is fitting of this data. Also Fig. 3 confirms this conclusion.

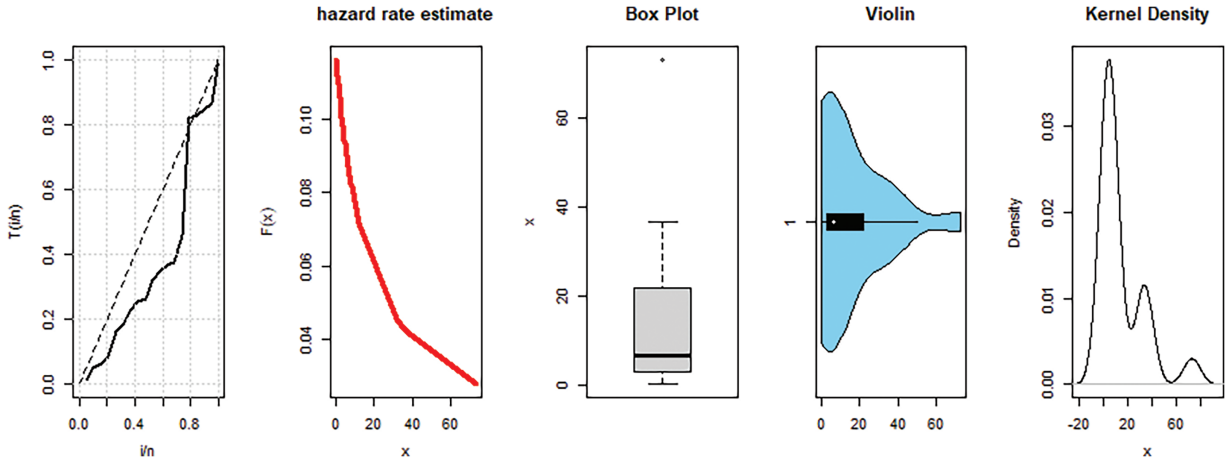

Figure 2: Some plots for insulating fluid electrodes dataset

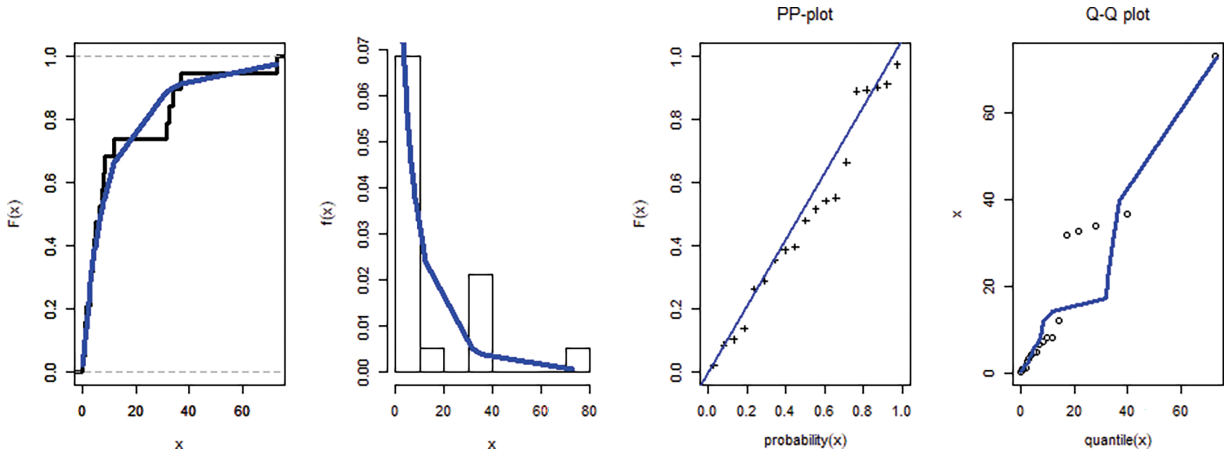

Figure 3: Fitting of the APL model by estimated CDF, PDF, and PP-plot: insulating fluid electrodes dataset

Fig. 3 shows several plots that visually confirm our earlier statement. These plots include P-P Plot: This plot compares the observed data against the theoretical expectations of the APL model. A close match in this plot indicates good agreement. Estimated CDF: This plot shows the probability of observing a value less than or equal to a certain point in the data. The estimated CDF for the data should resemble the expected CDF of the APL model. Histogram with Estimated PDF: This visualization shows the distribution of the data with a smooth curve representing the estimated probability of encountering a specific value. A good fit between the histogram and the estimated PDF suggests the data aligns well with the APL model.

By looking at all these plots together in Fig. 3, we can see a strong correlation between the actual data and what we would expect to see if the data truly follows the APL model.

Fig. 4 helps confirm that the values we estimated are not just the highest (maximal), but also the only ones that reach that maximum (unique). Here's why:

The graph shows the function's derivative, which tells us how the function itself is changing. In this case, the derivative is always decreasing (consistently descends).

The function's curve itself intersects the x-axis (horizontal line at $y = 0$) at only one point.

There's a connection between the derivative and the slope of the function. A constantly decreasing derivative, like we see here, corresponds to a function that's always going down after a certain point.

Since the function's curve only touches the x-axis once, that means it can only reach its maximum value at that single point.

Putting it all together, the continuously decreasing derivative and the unique x-axis intersection strongly suggest that our estimated values are the only maximum points for this function. This guarantees that they are unique.

This part presents the results of applying the proposed method to the insulating fluid electrodes dataset.

Table 8 shows the ATII-PHCS data generated using different sampling schemes for the coal-mining dataset. ATII-PHCS likely refers to a specific data processing method used in the analysis. Table 9 focuses on the estimates obtained using two different approaches: MLE and Bayesian estimation. These estimates are for the parameters of the APL model applied to the ATII-PHCS data. Table 10 discusses the "optimality" of the different sampling schemes used to generate the ATII-PHCS data. Presumably, this table compares the different schemes and identifies the most effective one based on some criteria.

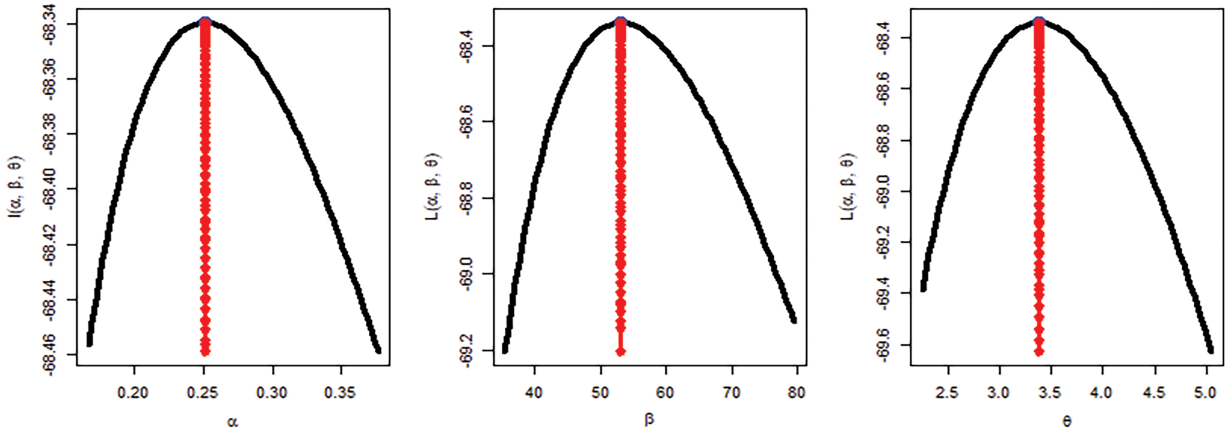

Figure 4: (Continued)

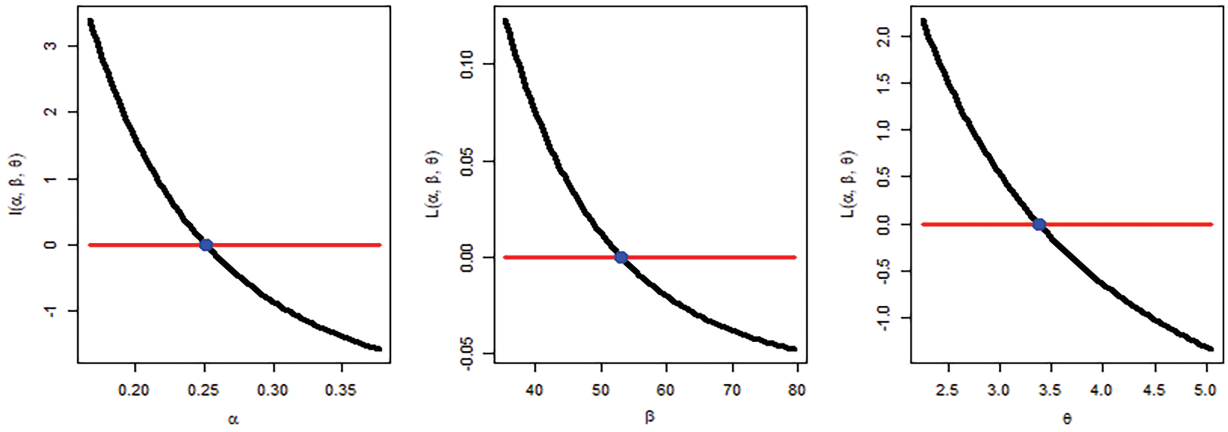

Figure 4: Profile likelihood of the APL parameter: insulating fluid electrodes data

Table 8: Insulating fluid electrodes dataset based on ATII-PHCS data with different scheme measures: T = 40

m	τ	p		Data
15	20	0.25	X1	0.19 0.78 0.96 1.31 2.78 3.16 4.15 6.50 8.27 12.06
			X2	31.75 32.52 33.91 36.71 72.89
			R	2 0 1 1 0 0 0 0 0 0 0 0 0 0 0
	20	0.85	X1	0.19 0.78 0.96 1.31 4.15 4.67 4.85 6.50 7.35 8.27 12.06
			X2	31.75 33.91 36.71 72.89
			R	0 3 1 0 0 0 0 0 0 0 0 0 0 0 0
15	33	0.25	X1	0.19 0.78 0.96 1.31 2.78 3.16 4.15 6.50 8.27 12.06 31.75 32.52
			X2	33.91 36.71 72.89
			R	2 0 1 1 0 0 0 0 0 0 0 0 0 0 0
	33	0.85	X1	0.19 0.78 0.96 1.31 4.15 4.67 4.85 6.50 7.35 8.27 12.06 31.75
			X2	33.91 36.71 72.89
			R	0 3 1 0 0 0 0 0 0 0 0 0 0 0 0
17	20	0.25	X1	0.19 0.78 0.96 2.78 3.16 4.15 4.67 4.85 6.50 7.35 8.01 8.27
			X2	31.75 32.52 33.91 36.71 72.89
			R	1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
	20	0.85	X1	0.19 0.78 0.96 1.31 2.78 4.15 4.67 4.85 6.50 7.35 8.01 8.27 12.06
			X2	31.75 33.91 36.71 72.89
			R	0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
	33	0.25	X1	0.19 0.78 0.96 2.78 3.16 4.15 4.67 4.85 6.50 7.35 8.01 8.27 31.75 32.52
			X2	33.91 36.71 72.89

(Continued)

Table 8 (continued)

m	τ	p		Data
			R	1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
	33	0.85	X1	0.19 0.78 0.96 1.31 2.78 4.15 4.67 4.85 6.50 7.35 8.01 8.27 12.06 31.75
			X2	33.91 36.71 72.89
			R	0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

Table 9: MLE and Bayesian of APL parameters based on ATII-PHCS data for insulating fluid electrodes dataset

m	τ	P		MLE		Bayesian	
				Estimates	StEr	Estimates	StEr
15	20	0.25	α	2.0900	2.9568	1.5947	1.6621
			β	3.2837	266.1142	8.2977	6.2889
			θ	0.7783	0.2353	0.9360	0.2052
			γ	1.6972	2.5826	2.1151	1.7237
	20	0.85	α	0.4920	1.7100	0.9519	0.7453
			β	12.1006	5.5864	15.8101	3.3010
			θ	1.1863	1.6957	1.3756	0.6912
			γ	1.0501	1.1233	1.0519	0.6870
	33	0.25	α	0.0132	0.0083	0.0275	0.0017
			β	4.8723	29.1589	37.6239	5.2151
			θ	0.1926	0.1485	0.5347	0.0825
			γ	19.5058	2.4876	17.2968	1.4017
	33	0.85	α	0.0194	0.0104	0.0442	0.0103
			β	6.6013	4.7836	10.4351	3.2920
			θ	0.2607	0.1829	0.3829	0.1222
			γ	16.5246	1.0311	21.3395	0.5947
17	20	0.25	α	0.4087	8.8981	0.6412	0.6293
			β	13.7586	4.5862	37.3306	2.7049
			θ	1.2026	1.4940	2.2705	1.1531
			γ	1.2048	1.1008	1.3665	0.8385
	20	0.85	α	0.0066	0.0013	0.0087	0.0008
			β	18.9114	161.7310	42.8264	31.3421
			θ	0.5137	1.1891	0.9343	0.6062
			γ	0.9168	0.7878	0.8943	0.5324
	33	0.25	α	0.0161	0.0638	0.0168	0.0018
			β	8.2474	3.6312	9.8217	2.0819
			θ	0.2932	0.2055	0.2920	0.1585
			γ	14.2694	0.9484	27.8584	0.6943

(Continued)

Table 9 (continued)

m	τ	P		MLE		Bayesian	
				Estimates	StEr	Estimates	StEr
	33	0.85	α	0.0167	0.0011	0.0497	0.0004
			β	7.6660	56.6986	7.9837	4.9304
			θ	0.3026	0.1887	0.3683	0.1225
			γ	14.0660	0.0453	43.7248	0.0354

Table 10: Optimal Criteria values for different parameters of binomial removal: insulating fluid electrodes dataset

m	τ	Opt_2	Opt_3	Opt_1	Opt_2	Opt_3	Opt_1
15	20	559.8889	227.9311	29.52902	5741.115	1191.52	15.26482
	33	720.8033	0.455443	3718.039	534.2881	1.285261	1919.388
17	20	1976.079	235.4093	18.84217	1063.709	0.014121	13646.28
	33	416.3275	0.902665	2902.422	396.9384	0.717181	2726.654

Fig. 5 displays the MCMC diagnostic plots for the parameters α, β, θ, and γ based on 10,000 iterations. The trace plots show good mixing and no evident autocorrelation, while the running mean plots indicate rapid stabilization of the posterior means, supporting convergence. The histograms with density curves and boxplots illustrate approximately symmetric and unimodal posterior distributions. Overall, these diagnostics confirm that the MCMC algorithm has converged and produced stable posterior estimates. Nonetheless, a sensitivity analysis to prior choices is recommended to further evaluate the robustness of the Bayesian results.

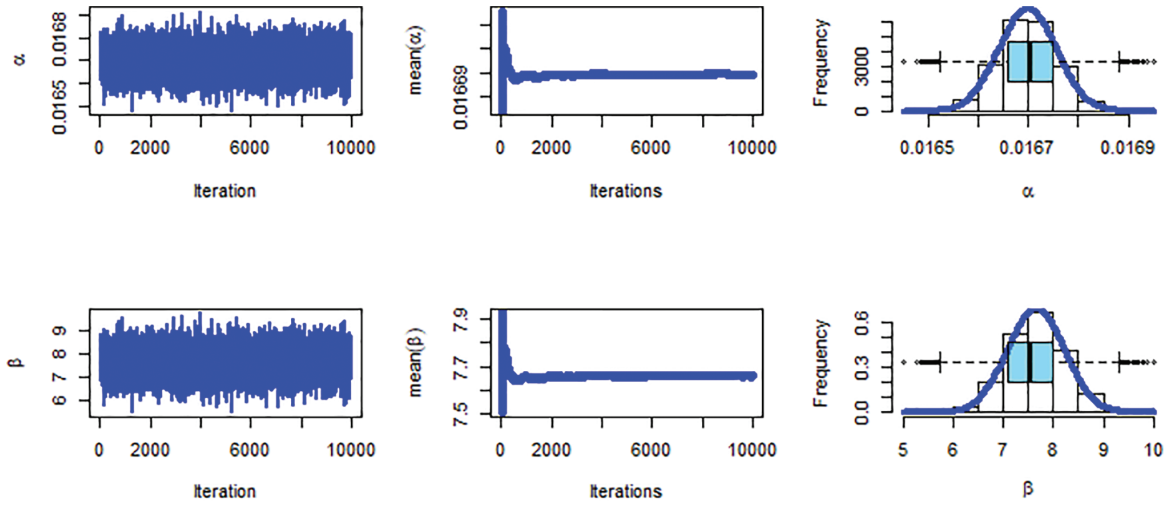

Figure 5: (Continued)

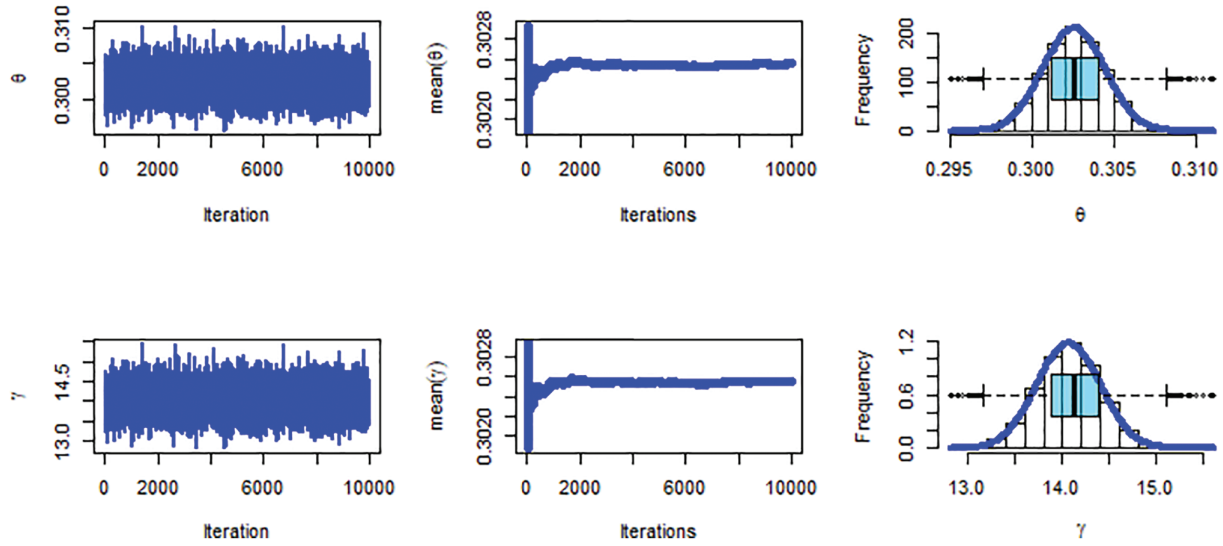

Figure 5: MCMC plots of where m = 17, tau = 33, P = 0.85

8 Concluding Remarks

It is well recognized that a large amount of data from reliability trials and life testing may show inconsistent shapes and are suppressed because of financial and scheduling limitations. As a result, accelerated life tests are frequently performed to investigate the lifespan of dependable objects by exposing them to high stress levels of potential early failure causes. This finding inspired us to use data from the APL distribution under adaptive type-II progressively hybrid censoring to study the step stress partially accelerated life testing model. We considered Bayesian and likelihood-based statistical inferences of the unknown model parameters of the APL distribution. To create two different kinds of approximate confidence intervals for the unknown parameters, we first considered the maximum likelihood estimates for the unknown model parameters. Next, under the symmetric and asymmetric loss functions, we performed Bayesian inference for the unknown parameters with non-informative and informative priors. Additionally, we examined three distinct likely optimal test methods for the suggested model under various ideal standards. The performance of the suggested approach for estimating the APL parameters under various sampling techniques is excellent, as demonstrated by numerical results from simulations and a real-data application. Thus, we may draw the conclusion that the suggested model has a lot of potential for studying life testing and reliability assessments using censored data under the ATII-PHCS.

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Author Contributions: Methodology, writing–original draft and funding, Refah Alotaibi; Methodology and writing–original draft, Hoda Rezk; Software and writing–original draft, Ehab M. Almetwally.

Also, all authors have made a significant scientific contribution to the research in the manuscript, approved its claims, and agreed to be an author. All authors reviewed the results and approved the final version of the manuscript.

Availability of Data and Materials: The data used to support the findings of this study are included within the article.

Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest to report regarding the present study.

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Appendix A Fisher Information Matrix

A fundamental idea in statistics, the Fisher information matrix shows how much knowledge data can provide about an unknown variable. It can be applied to ascertain the maximum-likelihood estimates' behavior close to infinity as well as the variance of an estimator. An estimator of the asymptotic covariance matrix is the inverse of the Fisher information matrix. The expected values of the negative second-partial and mixed-partial derivatives of the log-likelihood function with respect to α, θ, β and γ are used to calculate the Fisher information matrix. Here's an additional justification

$$I_{4\times4}(.) = -\begin{bmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} & \kappa_{14} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} & \kappa_{24} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} & \kappa_{34} \\ \kappa_{41} & \kappa_{42} & \kappa_{43} & \kappa_{44} \end{bmatrix}, \tag{A1}$$

where $\kappa_{11} = E\left(\frac{\partial^2 \ell}{\partial \alpha^2}\right)$, $\kappa_{22} = E\left(\frac{\partial^2 \ell}{\partial \theta^2}\right)$, $\kappa_{33} = E\left(\frac{\partial^2 \ell}{\partial \beta^2}\right)$, $\kappa_{44} = E\left(\frac{\partial^2 \ell}{\partial \gamma^2}\right)$, $\kappa_{12} = \kappa_{21} = E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \theta}\right)$, $\kappa_{13} = \kappa_{31} = E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \beta}\right)$, $\kappa_{14} = \kappa_{41} = E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \gamma}\right)$, $\kappa_{23} = \kappa_{32} = E\left(\frac{\partial^2 \ell}{\partial \theta \partial \beta}\right)$, $\kappa_{24} = \kappa_{42} = E\left(\frac{\partial^2 \ell}{\partial \theta \partial \gamma}\right)$, $\kappa_{34} = \kappa_{43} = E\left(\frac{\partial^2 \ell}{\partial \beta \partial \gamma}\right)$.

The components of the relevant matrices are calculated. Therefore, the following is how the variance-covariance matrix for MLEs can be created.

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{m_{ua}\left(1+\log\alpha\right)}{\alpha^2\left(\log\left(\alpha\right)\right)^2} - \sum_{i=1}^{m_{ua}} \frac{1-\nabla_i^{-\theta}}{\alpha^2} + \frac{m_{ua}}{\left(\alpha-1\right)^2} + -\frac{\mathrm{m}\left(1+\log\alpha\right)}{\alpha^2\left(\log\left(\alpha\right)\right)^2} + \frac{m}{\alpha^2} + \frac{1}{\left(\alpha-1\right)}$$

$$\sum_{i=1}^{m_{ua}} \rho_i \alpha^{-\nabla_i^{-\theta}} \left(\alpha - \alpha^{1-\nabla_i^{-\theta}}\right) - \rho_i \alpha^{1-\nabla_i^{-\theta}} \left[1 + \left(1 - \alpha^{-\nabla_i^{-\theta}}\right)\right]$$

$$- \sum_{i=m_{ua}+1}^{m} \frac{\left(1-\forall_i\right)}{\alpha^2} + \sum_{i=m_{ua}+1}^{\mathrm{m}} \frac{\alpha\rho_i\left(\alpha-1\right)\alpha^{-\forall_i}\left(1-\alpha^{-\forall_i}\right) - \rho_i\alpha^{1-\forall_i}\left[1+\left(\alpha-1\right)\left(1-\alpha^{-\forall_i}\right)\right]}{\left(\alpha-1\right)^2}$$

$$+ \frac{\rho_m\left(\alpha-1\right)\alpha^{-\forall_i}\left(\alpha-\alpha^{1-\forall_i}\right) - \rho_m\alpha^{1-\forall_i}\left[1+\left(\alpha-1\right)\left(1-\alpha^{-\forall_i}\right)\right]}{\left(\alpha-1\right)^2},$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \theta} = \frac{1}{\alpha}\sum_{i=1}^{m_{ua}} \nabla_i^{-\theta}\log\nabla_i - \frac{1}{\left(\alpha-1\right)}\sum_{i=1}^{m_{ua}} \frac{\rho_i \log\nabla_i \alpha^{1-\nabla_i^{-\theta}}\left(\left(1-\nabla_i^{-\theta}\right)\left(2+\left(1-\nabla_i^{-\theta}\right)\right)\right)}{\left(\alpha-\alpha^{1-\nabla_i^{-\theta}}\right)}$$

$$+ \frac{1}{\alpha}\sum_{i=m_{ua}+1}^{\mathrm{m}} \forall_i \log\forall_i^{\frac{1}{\theta}} - \frac{1}{\alpha\left(\alpha-1\right)}\sum_{i=1}^{m_{ua}} \frac{\rho_i\log\forall_i\alpha^{1-\forall_i}\left(\left(1-\forall_i\right)\left(2+\left(1-\forall_i\right)\right)\right)}{\left(-\alpha^{-\forall_i}\right)}$$

$$- \frac{\rho_m \log\left(1+\frac{\tau+\gamma\left(x_m-\tau\right)}{\beta}\right)\alpha^{1-\forall_i}\left(\left(1-\forall_i\right)\left(2+\left(1-\forall_i\right)\right)\right)}{\left(\alpha-\alpha^{1-\forall_i}\right)},$$

$$\frac{\partial^2 \ell}{\partial\alpha\partial\beta} = \frac{\theta}{\alpha}\sum_{i=1}^{m_{ua}} \frac{x_i}{\beta^2}\nabla_i^{-\theta-1} + \frac{1}{(\alpha-1)}\sum_{i=1}^{m_{ua}} \frac{\rho_i\alpha^{1-\nabla_i^{-\theta}}\log\nabla_i\left(\alpha-\alpha^{1-\nabla_i^{-\theta}}\right)\left[\nabla_i^{-\theta}-\alpha^{-\nabla_i^{-\theta}}\nabla_i^{-\theta}-1\right]}{\left(\alpha-\alpha^{1-\nabla_i^{-\theta}}\right)^2}$$
$$+\frac{\theta}{\alpha}\sum_{i=m_{ua}+1}^{\mathrm{m}} \frac{\tau+\gamma\,(x_i-\tau)}{\beta^2}\forall_i^{-1} + \frac{1}{(\alpha-1)}\sum_{i=m_{ua}+1}^{\mathrm{m}} \frac{\rho_i\alpha^{1-\forall_i}\log\forall_i^{\frac{1}{\theta}}\left(-\alpha^{-\forall_i}\right)\left[\forall_i-\alpha^{-\forall_i}\forall_i-1\right]}{\left(1-\alpha^{-\forall_i}\right)^2}$$
$$+\frac{\rho_m\alpha^{1-\forall_m}\log\forall_m^{\frac{1}{\theta}}\left(\alpha-\alpha^{1-\forall_m}\right)\left[\forall_m-\alpha^{-\forall_m}\forall_m-1\right]}{(\alpha-1)\left(\alpha-\alpha^{1-\forall_m}\right)^2},$$

$$\frac{\partial^2 \ell}{\partial\alpha\partial\gamma} = \frac{\theta}{\alpha}\sum_{i=m_{ua}+1}^{m}\nabla_i\forall_i^{-1}$$
$$+\sum_{i=m_{ua}+1}^{m}\frac{\left(\theta\rho_i\theta\rho_i\alpha^{1-\forall_i}\right)}{(\alpha-1)\left(\alpha-\alpha^{1-\forall_i}\right)^2}\left[\left(\alpha-\alpha^{1-\forall_i}\right)\left(\log(\alpha)(1-\forall_i)-\forall_i^{-1}\right)-\alpha^{1-\forall_i}(1-\forall_i)\log(\alpha)\right]$$
$$+\frac{\left(\left(\frac{\theta\rho_m(x_m-\tau)}{\beta}\right)\alpha^{1-\forall_m}\right)}{(\alpha-1)\left(\alpha-\alpha^{1-\forall_m}\right)^2}\left[\left(\alpha-\alpha^{1-\forall_m}\right)\left(\log(\alpha)(1-\forall_m)-\forall_m^{-1}\right)-\alpha^{1-\forall_m}(1-\forall_m)\log(\alpha)\right]$$

$$\frac{\partial^2 \ell}{\partial\theta^2} = -\frac{m_{ua}}{\theta^2}+\log(\alpha)\sum_{i=1}^{m_{ua}}\nabla_i^{-\theta}(\log\nabla_i)^2$$
$$-\theta\log(\alpha)\sum_{i=1}^{m_{ua}}\left(\frac{\rho_i\alpha^{-\nabla_i^{-\theta}}(\log\nabla_i)^2(\alpha-1)\left[\left(\alpha-\alpha^{1-\nabla_i^{-\theta}}\right)-\alpha^{-\nabla_i^{-\theta}}\nabla_i^{-\theta}\right]}{\left(\alpha-\alpha^{1-\nabla_i^{-\theta}}\right)^2}\right)$$
$$-\theta\log(\alpha)\sum_{i=m_{ua}+1}^{\mathrm{m}}\left(\frac{\rho_i\alpha^{-\forall_i}\left(\log\forall_i^{\frac{1}{\theta}}\right)^2(\alpha-1)}{\left(\alpha-\alpha^{1-\forall_i}\right)^2}\left[\left(\alpha-\alpha^{1-\forall_i}\right)-\alpha^{-\forall_i}\forall_i\right]\right)$$
$$-\theta\log(\alpha)\left(\frac{\rho_m\alpha^{-\forall_i}\left(\log\forall_m^{\frac{1}{\theta}}\right)^2(\alpha-1)}{\left(\alpha-\alpha^{1-\forall_m}\right)^2}\left[\left(\alpha-\alpha^{1-\forall_m}\right)-\alpha^{\forall_m}\forall_m\right]\right),$$

$$\frac{\partial^2 \ell}{\partial\theta^2} = -\frac{m_{ua}}{\theta^2}+\log(\alpha)\sum_{i=1}^{m_{ua}}\nabla_i^{-\theta}(\log\nabla_i)^2$$
$$-\theta\log(\alpha)\sum_{i=1}^{m_{ua}}\left(\frac{\rho_i\alpha^{-\nabla_i^{-\theta}}(\log\nabla_i)^2(\alpha-1)\left[\left(\alpha-\alpha^{1-\nabla_i^{-\theta}}\right)-\alpha^{-\nabla_i^{-\theta}}\nabla_i^{-\theta}\right]}{\left(\alpha-\alpha^{1-\nabla_i^{-\theta}}\right)^2}\right)$$

$$- \theta \log(\alpha) \sum_{i=m_{ua}+1}^{\mathrm{m}} \left(\frac{\rho_i \alpha^{-\forall_i} \left(\log \forall_i^{\frac{1}{\theta}} \right)^2 (\alpha - 1)}{(\alpha - \alpha^{1-\forall_i})^2} \left[(\alpha - \alpha^{1-\forall_i}) - \alpha^{-\forall_i} \forall_i \right] \right)$$

$$- \theta \log(\alpha) \left(\frac{\rho_m \alpha^{-\forall_i} \left(\log \forall_m^{\frac{1}{\theta}} \right)^2 (\alpha - 1)}{(\alpha - \alpha^{1-\forall_m})^2} \left[(\alpha - \alpha^{1-\forall_m}) - \alpha^{\forall_m} \forall_m \right] \right),$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \gamma} = \theta \sum_{i=m_{ua}+1}^{m} \frac{\forall_i^{-1} \nabla_i}{\alpha}$$

$$+ \sum_{i=m_{ua}+1}^{m} \frac{\rho_i \alpha^{-\forall_i} \nabla_i \forall_i^{-1} \left((\alpha - \alpha^{1-\forall_i}) [(1 - \forall_i)] - \theta \left(1 - \forall_i^{-1} \right) \right)}{\alpha (\alpha - 1) (1 - \alpha^{-\forall_i})^2}$$

$$+ \frac{\rho_m \alpha^{-\forall_m} \left(\frac{(x_m - \tau)}{\beta} \right) \forall_m^{-1} \left((\alpha - \alpha^{1-\forall_m}) [(1 - \forall_m)] - \theta \left(1 - \forall_m^{-1} \right) \right)}{\alpha (\alpha - 1) (1 - \alpha^{-\forall_m})^2},$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \gamma} = \theta \sum_{i=m_{ua}+1}^{m} \frac{\forall_i^{-1} \nabla_i}{\alpha} + \sum_{i=m_{ua}+1}^{m} \frac{\rho_i \alpha^{-\forall_i} \nabla_i \forall_i^{-1} \left((\alpha - \alpha^{1-\forall_i}) [(1 - \forall_i)] - \theta \left(1 - \forall_i^{-1} \right) \right)}{\alpha (\alpha - 1) (1 - \alpha^{-\forall_i})^2}$$

$$+ \frac{\rho_m \alpha^{-\forall_m} \left(\frac{(x_m - \tau)}{\beta} \right) \forall_m^{-1} \left((\alpha - \alpha^{1-\forall_m}) [(1 - \forall_m)] - \theta \left(1 - \forall_m^{-1} \right) \right)}{\alpha (\alpha - 1) (1 - \alpha^{-\forall_m})^2},$$

$$\frac{\partial^2 \ell}{\partial \gamma^2} = \frac{m}{\gamma^2} + \frac{(\theta + 1)}{\beta} \sum_{i=m_{ua}+1}^{m} \frac{(x_i - \tau)^2}{\forall_i^{\frac{2}{\theta}}} - \theta \log(\alpha) \sum_{i=m_{ua}+1}^{m} \frac{(x_i - \tau)}{\beta^2} \forall_i^{-1} \left(1 - \frac{(\theta + 1)}{\beta} \left(\frac{\tau + \gamma (x_i - \tau)}{\forall_i^{\frac{1}{\theta}}} \right) \right)$$

$$+ \sum_{i=m_{ua}+1}^{m} \frac{\theta^2 \rho_i (x_i - \tau) \forall_i^{-1}}{\alpha \beta^2 (1 - \alpha^{-\forall_i})^2} \left[\left(\alpha^{-\forall_i} - (\theta + 1) \left(\frac{\tau + \gamma (x_i - \tau)}{\beta^2} \right) - \frac{\theta \log(\alpha)}{\beta} \alpha^{-\forall_i} \right) \right.$$

$$\left. + \left(\log(\alpha) \left(\frac{\tau + \gamma (x_i - \tau)}{\beta} \right) \forall_i^{-1} \left(\alpha^{-\forall_i} \right)^2 \right) \right]$$

$$+ \frac{\theta^2 \rho_m (x_m - \tau) \forall_m^{-1}}{\alpha \beta^2 (1 - \alpha^{-\forall_m})^2} \left[\left(\alpha^{-\forall_m} - (\theta + 1) \left(\frac{\tau + \gamma (x_m - \tau)}{\beta^2} \right) - \frac{\theta \log(\alpha)}{\beta} \alpha^{-\forall_m} \right) \right.$$

$$\left. + \left(\log(\alpha) \left(\frac{\tau + \gamma (x_m - \tau)}{\beta} \right) \forall_m^{-1} \left(\alpha^{-\forall_m} \right)^2 \right) \right]$$