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Stress, strain and energy splittings for anisotropic elastic solids under volumetric constraints

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Abstract

We define stress and strain splittings appropriate to linearly elastic anisotropic materials with volumetric constraints. The treatment includes rigidtropic materials, which develop no strains under a stress pattern that is a null eigenvector of the compliance matrix. This model includes as special case incompressible materials, for which the eigenvector is hydrostatic stress. The main finding is that pressure and volumetric strain must be redefined as effective quantities. Using this idea, an energy decomposition that exactly separates deviatoric and volumetric energy follows.

Keywords: Elasticity; Anisotropy; Rigidtropy; Incompressibility; Isochoricity; Volumetric constraints; Stability; Constitutive equation; Compliance; Deviatoric; Splitting

1. Introduction

An isotropic solid is called incompressible if it is susceptible only of isochoric motions [1, Sec. 77]. If the material is linearly elastic the stress is specified by the strains only up to an arbitrary hydrostatic stress (pressure). This uncoupling forms the basis of splittings into volumetric and deviatoric strains long used in the analytical and numerical modeling of those materials.

The concept loses transparency for non-isotropic behavior because changes of shape and volume are generally coupled. A generalization aimed at linearly elastic anisotropic solids is the *rigidtropic* model defined in Appendix A, which summarizes key results from [2]. A rigidtropic material is infinitely rigid under a specific nonzero stress pattern. The pattern is defined by the eigenvector that renders the compliance (strain–stress) matrix singular. If this eigenvector happens to be hydrostatic stress, it is shown in Appendix A that the material is isochoric (volume preserving) under any stress state. This specialization defines an *incompressible* anisotropic material.

With the generalization in place, useful splittings of stresses, strains and internal energy can be obtained for the numerical treatment of anisotropic solids with a singular compliance. An auxiliary tool used in this development is the concept of free–free flexibility and stiffness presented in [3,4]. These two singular matrices are the Moore–Penrose generalized inverses of each other. They can be expressed as projected ordinary inverses of modified matrices.

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The splittings are worked out first for a general anisotropic material. They are then specialized to the orthotropic and isotropic cases. The treatment of the isotropic case is included to illustrate the connection to existing splitting methods.

2. Anisotropic material

We consider a linearly elastic anisotropic solid in three dimensions referred to axes $\{x_i\}$. Stresses σ_{ij} and strains e_{ij} will be arranged as six-component column vectors constructed from the tensors through the usual conventions of structural mechanics:

$$\boldsymbol{\sigma} = [\,\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{23} \quad \sigma_{31} \quad \sigma_{12}\,]^{\mathrm{T}}, \quad \mathbf{e} = [\,e_{11} \quad e_{22} \quad e_{33} \quad 2e_{23} \quad 2e_{31} \quad 2e_{12}\,]^{\mathrm{T}}. \tag{1}$$

The mean normal stress is $\sigma_m = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$. A hydrostatic stress state is defined by $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_m$, others zero. The volumetric strain is $e_v = e_{11} + e_{22} + e_{33}$.

2.1. Constitutive equations

The strain–stress constitutive equation is

$$\mathbf{e} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{13} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ \text{symm} & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \mathbf{C}\boldsymbol{\sigma}. \tag{2}$$

Here C_{ij} are compliance coefficients arranged into the symmetric compliance matrix $\mathbf{C}$. All diagonal entries C_{ii} are assumed to be nonnegative with a positive sum. The compliance is called *stable*, *semistable* or *unstable* if $\mathbf{C}$ is positive definite, positive semidefinite, or indefinite, respectively. In the semistable case it will be assumed that $\mathbf{C}$ has a rank deficiency of at most one to simplify the analysis.

The eigenvalues of $\mathbf{C}$ are γ_i for $i = 1, 2, \ldots, 6$, with $\mathbf{v}_i$ the corresponding eigenvector normalized to length $\sqrt{3}$. (This nonstandard normalization simplifies linkage to the incompressible case.) Accordingly the spectral decomposition is

$$\mathbf{C} = \frac{1}{3}\sum_{i=1}^{6} \gamma_i \mathbf{v}_i \mathbf{v}_i^{\mathrm{T}}, \quad \mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j = 3\delta_{ij}, \tag{3}$$

where δ_{ij} is the Kronecker delta. The eigenvalues are arranged so that $\gamma_1 = \gamma_{\min}$ and $\gamma_6 = \gamma_{\max}$ are the algebraically smallest and largest, respectively. The eigenvector $\mathbf{v}_1$ associated with γ_1 is often renamed $\mathbf{w}$. For stable models, $\gamma_i > 0$ for $i = 1, \ldots, 6$. For semistable models $\gamma_1 = 0$. Unstable models are not physically acceptable.

If $\mathbf{C}$ is stable its ordinary inverse is called $\mathbf{E} = \mathbf{C}^{-1}$. This matrix collects the moduli E_{ij} and relates stresses to strains: $\boldsymbol{\sigma} = \mathbf{E}\mathbf{e}$. It is called the elasticity or rigidity matrix in the continuum mechanics literature. If $\mathbf{C}$ is semistable (singular) a generalized inverse is defined in Section 2.7.

The Rayleigh quotients of the $\mathbf{v}_i$ are noted for later use:

$$\gamma_i = \frac{\mathbf{v}_i^{\mathrm{T}}\mathbf{C}\mathbf{v}_i}{\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_i} = \frac{1}{3}\mathbf{v}_i^{\mathrm{T}}\mathbf{C}\mathbf{v}_i. \tag{4}$$

2.2. Volumetric constraints

The volumetric constraints defined in Section 1 are mathematically expressed in terms of $\mathbf{C}$ as

$$\text{Rigidtropic:} \quad \gamma_1 = 0, \quad \gamma_i > 0, \ \ i = 2, \ldots, 6. \tag{5}$$

$$\text{Incompressible:} \quad \text{rigidtropic and } C_{1j} + C_{2j} + C_{3j} = 0, \ \ j = 1, 2, 3. \tag{6}$$

In both cases

$$\det(\mathbf{C}) = 0, \quad \mathbf{C}\mathbf{w} = \mathbf{0}, \quad \text{with } \mathbf{w} \equiv \mathbf{v}_1, \ \ \mathbf{w}^{\mathrm{T}}\mathbf{w} = 3. \tag{7}$$

If $\mathbf{w}$ is the hydrostatic stress mode $[1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$ the zero-column-sum condition (6) is verified and the material is incompressible. In the sequel no particular attention is placed on incompressibility because it is a subset of the more general rigidtropic model.

The following example matrix, referred to principal material axes, has the entry pattern appropriate to an hexagonal (trigonal-pyramidal, rhombohedral) material governed by the $\mathscr{C}_8$ symmetry group [5, p. 89]

$$\mathbf{C}_{\mathrm{hex}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & 0 \\ & C_{11} & C_{13} & -C_{14} & -C_{15} & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & -2C_{15} \\ & & & & C_{55} & 2C_{14} \\ \text{symm} & & & & & 2(C_{11}-C_{12}) \end{bmatrix} = \begin{bmatrix} 12 & -6 & -12 & -7 & 10 & 0 \\ -6 & 12 & -12 & 7 & -10 & 0 \\ -12 & -12 & 48 & 0 & 0 & 0 \\ -7 & 7 & 0 & 20 & 0 & -20 \\ 10 & -10 & 0 & 0 & 20 & -14 \\ 0 & 0 & 0 & -20 & -14 & 36 \end{bmatrix}. \tag{8}$$

The set of numerical entries are exact integers. The eigenvalues (listed to six places if not integer) are $\gamma_1 = 0$, $\gamma_2 = 1.70838$, $\gamma_3 = 2.30953$, $\gamma_4 = 36.2916$, $\gamma_5 = 53.6905$ and $\gamma_6 = 54$. Thus (8) is semistable and rigidtropic. The null eigenvector is $\mathbf{v}_1 \equiv \mathbf{w} = (1/\sqrt{3})[2 \quad 2 \quad 1 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$, $\mathbf{w}^{\mathrm{T}}\mathbf{w} = 3$.

2.3. Quasi-rigidtropic materials and the reference model

Models that satisfy (7) exactly or to high numerical accuracy are rare if entries come from experimental data. It is far more common to find "quasi-rigidtropic" (QRT) behavior in the sense that $\mathbf{C}$ has a tiny positive compliance in a stress eigenmode. Mathematically:

$$\gamma_1 > 0, \quad \gamma_1 < \epsilon_r \gamma_6, \quad \epsilon_r \ll 1. \tag{9}$$

Here ϵ_r is a preset tolerance, for example 10^{-3}, which triggers treatment of the material model as QRT. Note that the reciprocal $1/\epsilon_r$ is a spectral condition number for $\mathbf{C}$. A tiny but negative γ_1 may also occur due to experimental noise; stabilization of such models is discussed in Section 2.6.

If (9) is verified, one proceeds to compute two auxiliary quantities: the *reference rigidtropic model* $\overline{\mathbf{C}}$ and an *effective bulk modulus* K. These are used to split the constitutive equations and internal energy in Sections 2.8ff.

The reference rigidtropic model $\overline{\mathbf{C}}$ is that nearest $\mathbf{C}$ satisfying (5). But how to define "nearness"? The answer is not unique. Two methods for constructing $\overline{\mathbf{C}}$ are described below: projection and scaling. The key relations of these two methods are summarized in Fig. 1.

2.4. $\overline{\mathbf{C}}$ by projection

To apply the projection method, compute the eigensystem of $\mathbf{C}$. Pick the algebraically smallest eigenvalue γ_1, and check whether the model classifies as QRT as per (9). (For a tiny but negative γ_1 see Section 2.6.) Call the associated eigenvector $\mathbf{w} \equiv \mathbf{v}_1$, $\mathbf{w}^{\mathrm{T}}\mathbf{w} = 3$. Redo the spectral decomposition (3) subtracting off γ_1:

$$\overline{\mathbf{C}} = \mathbf{C} - \frac{1}{3}\gamma_1 \mathbf{w}\mathbf{w}^{\mathrm{T}} = \sum_{i=2}^{6} \frac{1}{3}\gamma_i \mathbf{v}_i \mathbf{v}_i^{\mathrm{T}}. \tag{10}$$

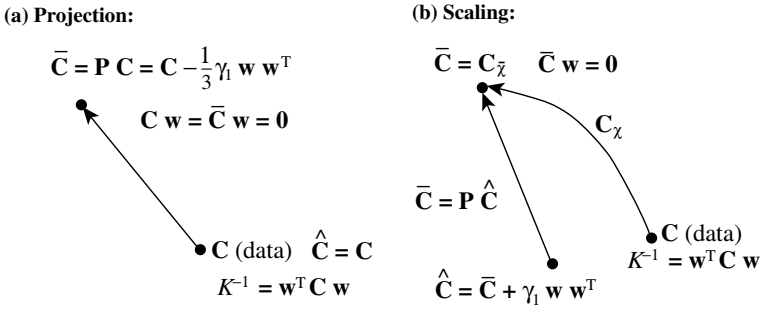

Fig. 1. Two methods for handling a QRT compliance: (a) projection and (b) scaling. Starting from the same given $\mathbf{C}$ they generally produce different $\overline{\mathbf{C}}$s.

This is called a projection method since it is equivalent to pre- or post-multiplying $\mathbf{C}$ by an orthogonal projector $\mathbf{P}$ defined by $\mathbf{w}$:

$$\overline{\mathbf{C}} = \mathbf{PC} = \mathbf{CP}, \quad \mathbf{P} = \mathbf{I} - \tfrac{1}{3}\mathbf{w}\mathbf{w}^{\mathrm{T}}, \quad \mathbf{P}^2 = \mathbf{P}, \tag{11}$$

where $\mathbf{I}$ is the 6×6 identity matrix. The effective bulk modulus K is computed as

$$K^{-1} = 3\gamma_1 = \mathbf{w}^{\mathrm{T}}\mathbf{C}\mathbf{w}. \tag{12}$$

The constitutive meaning of K as ratio of effective pressure to effective volumetric strain is discussed later after those quantities are introduced.

To build an example QRT matrix, the nonzero entries of the example matrix (8) are perturbed on the order of 1% by a random number generator. The results of the process is

$$\mathbf{C} = \begin{bmatrix} 11.78029 & -5.90183 & -11.79849 & -6.96337 & 10.14722 & 0 \\ -5.90183 & 11.93375 & -11.82934 & 7.10696 & -9.92362 & 0 \\ -11.79849 & -11.82934 & 47.38653 & 0 & 0 & 0 \\ -6.96337 & 7.10696 & 0 & 20.01094 & 0 & -19.85809 \\ 10.14722 & -9.92362 & 0 & 0 & 20.04357 & -13.97715 \\ 0 & 0 & 0 & -19.85809 & -13.97715 & 36.16219 \end{bmatrix}. \tag{13}$$

The eigenvalues are now $\gamma_1 = 0.04409$, $\gamma_2 = 1.53218$, $\gamma_3 = 2.50946$, $\gamma_4 = 36.2653$, $\gamma_5 = 53.2816$ and $\gamma_6 = 53.6847$. The eigenvector corresponding to $\gamma_1 = 0.04409$ is

$$\mathbf{w} = \mathbf{v}_1 = [\,1.20725 \quad 1.09464 \quad 0.57438 \quad -0.02075 \quad -0.10602 \quad -0.05244\,]^{\mathrm{T}}. \tag{14}$$

The projected reference model is

$$\overline{\mathbf{C}} = \mathbf{C} - \frac{1}{3}\gamma_1\mathbf{w}\mathbf{w}^{\mathrm{T}} = \begin{bmatrix} 11.75887 & -5.92125 & -11.80868 & -6.96300 & 10.14910 & 0.00093 \\ -5.92125 & 11.91614 & -11.83858 & 7.10729 & -9.92191 & 0.00084 \\ -11.80868 & -11.83858 & 47.38169 & 0.00018 & 0.00089 & 0.00044 \\ -6.96300 & 7.10729 & 0.00018 & 20.01094 & -0.00003 & -19.85810 \\ 10.14910 & -9.92191 & 0.00089 & -0.00003 & 20.04340 & -13.97723 \\ 0.00093 & 0.00084 & 0.00044 & -19.85810 & -13.97723 & 36.16215 \end{bmatrix}. \tag{15}$$

The eigenvalues of (15) are the same as those of $\mathbf{C}$ except for $\gamma_1 = 0$. The eigenvectors do not change. The effective bulk modulus is $K = 1/(3\gamma_1) = 1/(\mathbf{w}^{\mathrm{T}}\mathbf{C}\mathbf{w}) = 7.56067$.

2.5. $\overline{\mathbf{C}}$ by scaling

Projection preserves eigenvectors (and all eigenvalues but γ_1) but generally changes all entries of $\mathbf{C}$ as can be observed comparing (15) to (13). Occasionally it is desirable to keep the same diagonal entries or to preserve zero off-diagonal entries resulting from material symmetries known as *a priori*.

In the scaling procedure the off-diagonal entries of the upper 3×3 minor are multiplied by $(1 + \chi)$, where χ is a real parameter:

$$\mathbf{C}_\chi = \begin{bmatrix} C_{11} & (1+\chi)C_{12} & (1+\chi)C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & (1+\chi)C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ \text{symm} & & & & & C_{66} \end{bmatrix}. \tag{16}$$

Expand $\det(\mathbf{C}_\chi)$ as a polynomial in χ. Solve $\det \mathbf{C}_\chi = 0$ for the smallest real root $\overline{\chi}$. Apply $\overline{\chi}$ to $\mathbf{C}_\chi$ to get $\overline{\mathbf{C}} = \mathbf{C}_{\overline{\chi}}$. Although scaling keeps diagonal entries invariant and preserves zero off-diagonal ones, it will generally change all eigenvalues and eigenvectors. Hence it is necessary to redo the eigensystem analysis to get the spectral decomposition

$$\overline{\mathbf{C}} = \frac{1}{3}\sum_{i=1}^{6} \overline{\gamma}_i \overline{\mathbf{v}}_i \overline{\mathbf{v}}_i^{\mathrm{T}}, \quad \overline{\mathbf{v}}_i^{\mathrm{T}}\overline{\mathbf{v}}_i = 3. \tag{17}$$

Verify the rigidtropy and stability conditions: $\overline{\gamma}_1 = 0$, $\overline{\gamma}_i > 0$ for $i = 2, \ldots, 6$. If $\overline{\mathbf{C}}$ is unstable the projection method should be used as fall-back. If $\overline{\mathbf{C}}$ is semistable, pick $\mathbf{w} \equiv \mathbf{v}_1$. The effective bulk modulus is computed via the Rayleigh quotient of $\mathbf{w}$ on the original $\mathbf{C}$:

$$K^{-1} = 3\frac{\mathbf{w}^{\mathrm{T}}\mathbf{C}\mathbf{w}}{\mathbf{w}^{\mathrm{T}}\mathbf{w}} = \mathbf{w}^{\mathrm{T}}\mathbf{C}\mathbf{w}. \tag{18}$$

Why not scale all off-diagonal entries? The answer is that scaling is only recommended when rigidtropy is associated with lateral contraction (Poisson) effects controlled by C_{12}, C_{13} and C_{23}, so it is applied where it is most effective. For instance the null eigenvector $\mathbf{w}$ of the exact rigidtropic example compliance (8) has zeros in the last three components, while in the QRT perturbed matrix (13) those are very small.

To scale (13), entries C_{12}, C_{13} and C_{23} are multiplied by $(1+\chi)$. The determinant equation $\det \mathbf{C}_\chi = 0$ has three roots: $\chi_1 = -3.00511$, $\chi_2 = -0.467716$ and $\chi_3 = 0.002822$. Pick $\overline{\chi} = \chi_3$ as that closest to zero, and substitute $\overline{\chi}$ to get

$$\overline{\mathbf{C}} = \begin{bmatrix} 11.78029 & -5.91849 & -11.83179 & -6.96337 & 10.14722 & 0 \\ -5.91849 & 11.93375 & -11.86272 & 7.10696 & -9.92362 & 0 \\ -11.83179 & -11.86272 & 47.38653 & 0 & 0 & 0 \\ -6.96337 & 7.10696 & 0 & 20.01094 & 0 & -19.85809 \\ 10.14722 & -9.92362 & 0 & 0 & 20.04357 & -13.97715 \\ 0 & 0 & 0 & -19.85809 & -13.97715 & 36.16219 \end{bmatrix}. \tag{19}$$

This has eigenvalues $\overline{\gamma}_1 = 0$, $\overline{\gamma}_2 = 1.54084$, $\overline{\gamma}_3 = 2.50934$, $\overline{\gamma}_4 = 36.273$, $\overline{\gamma}_5 = 53.3091$ and $\overline{\gamma}_6 = 53.685$, which is acceptable as regards stability. The null eigenvector $\mathbf{w} \equiv \bar{\mathbf{v}}_1$ is

$$\mathbf{w} = \bar{\mathbf{v}}_1 = [\,1.20524 \quad 1.09659 \quad 0.57545 \quad -0.02113 \quad -0.10313 \quad -0.05146\,]^{\mathrm{T}}. \tag{20}$$

The effective bulk modulus is $K = 1/(\mathbf{w}^{\mathrm{T}}\mathbf{C}\mathbf{w}) = 7.55588$. Comparing with the results of the projection method the answers are seen to be very similar. The main visible difference between the reference models (15) and (19) is that zero entries are exactly preserved in the latter.

2.6. Recommendations on method selection

Fig. 1 summarizes both methods in graphic form. Projection has the important advantage of being observer invariant since it works with the spectrum of $\mathbf{C}$. If $\mathbf{C}$ is stable, $\overline{\mathbf{C}}$ is guaranteed to be semistable since γ_2 through γ_6 are unchanged. Scaling would produce different results as axes $\{x_i\}$ are rotated; moreover it might be necessary to extend the $(1+\chi)$ factors to all off-diagonal components in some cases. For this reason scaling should be restricted to models satisfying two conditions:

- The projection method would disturb zeros entries that are to be preserved to express material symmetries, as in the example (13).
- Rigidtropy is due to lateral contraction (Poisson) effects governed by C_{12}, C_{13} and C_{23}.

Although the QRT presumption (9) excludes QRT unstable models, in practice discovering a tiny but negative γ_1 is as likely as having a positive value. Experimental noise coupled to pre-selection of symmetry groups may cause that to happen. The good news is that both projection and scaling are unaffected by the sign of γ_1: if negative the computation of $\overline{\mathbf{C}}$ can be viewed as a model stabilization step. For example, consider the unstable QRT compliance:

$$\mathbf{C} = \begin{bmatrix} 12.15638 & -6.00271 & -12.09760 & -6.95476 & 10.05482 & 0 \\ -6.00271 & 11.81220 & -11.77289 & 7.11267 & -10.14806 & 0 \\ -12.09760 & -11.77289 & 47.27442 & 0 & 0 & 0 \\ -6.95476 & 7.11267 & 0 & 19.67066 & 0 & -20.27321 \\ 10.05482 & -10.14806 & 0 & 0 & 20.32512 & -13.90944 \\ 0 & 0 & 0 & -20.27321 & -13.90944 & 35.68183 \end{bmatrix}, \tag{21}$$

with eigenvalues $\gamma_1 = -0.05323$, $\gamma_2 = 1.5004$, $\gamma_3 = 2.0792$, $\gamma_4 = 36.489$, $\gamma_5 = 53.298$ and $\gamma_6 = 53.607$. The projected matrix is

$$\overline{\mathbf{C}} = \mathbf{C} - \frac{1}{3}\gamma_1 \mathbf{w}\mathbf{w}^{\mathrm{T}} = \begin{bmatrix} 12.17677 & -5.97940 & -12.08659 & -6.95657 & 10.05598 & -0.00058 \\ -5.97940 & 11.83887 & -11.76030 & 7.11059 & -10.14674 & -0.00066 \\ -12.08659 & -11.76030 & 47.28036 & -0.00098 & 0.00062 & -0.00031 \\ -6.95657 & 7.11059 & -0.00098 & 19.67082 & -0.00010 & -20.27316 \\ 10.05598 & -10.14674 & 0.00062 & -0.00010 & 20.32519 & -13.90948 \\ -0.00058 & -0.00066 & -0.00031 & -20.27316 & -13.90948 & 35.68185 \end{bmatrix}. \tag{22}$$

The eigenvalues of (22) are the same as those of (21) except for $\overline{\gamma}_1 = 0$. The scaling method gives

$$\overline{\mathbf{C}} = \begin{bmatrix} 12.15638 & -5.98251 & -12.05688 & -6.95476 & 10.05482 & 0 \\ -5.98251 & 11.81220 & -11.73326 & 7.11267 & -10.14806 & 0 \\ -12.05688 & -11.73326 & 47.27442 & 0 & 0 & 0 \\ -6.95476 & 7.11267 & 0 & 19.67066 & 0 & -20.27321 \\ 10.05482 & -10.14806 & 0 & 0 & 20.32512 & -13.90944 \\ 0 & 0 & 0 & -20.27321 & -13.90944 & 35.68183 \end{bmatrix}, \tag{23}$$

with eigenvalues $\overline{\gamma}_1 = 0$, $\overline{\gamma}_2 = 1.4925$, $\overline{\gamma}_3 = 2.0770$, $\overline{\gamma}_4 = 36.480$, $\overline{\gamma}_5 = 53.264$ and $\overline{\gamma}_6 = 53.607$. This compliance matrix is acceptable from the standpoint of stability.

2.7. The stress–strain matrices

From now on a compliance that exactly satisfies (5) will be called $\overline{\mathbf{C}}$. This has the spectral decomposition (17). This matrix generally comes from adjusting a QRT compliance $\mathbf{C}$ by one of the methods discussed in the previous sections. The null eigenvector of $\overline{\mathbf{C}}$ is always called $\mathbf{w}$ for brevity:

$$\overline{\mathbf{C}}\mathbf{w} = \mathbf{0}, \quad \mathbf{w} = [\,w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6\,]^{\mathrm{T}}, \quad \mathbf{w}^{\mathrm{T}}\mathbf{w} = 3. \tag{24}$$

The projected elasticity matrix is defined as the Moore–Penrose generalized inverse of $\overline{\mathbf{C}}$:

$$\overline{\mathbf{E}} = \mathbf{P}\left(\overline{\mathbf{C}} + \frac{1}{3}\mathbf{w}\mathbf{w}^{\mathrm{T}}\right)^{-1} = \frac{1}{3}\sum_{i=2}^{6} \frac{1}{\overline{\gamma}_i}\overline{\mathbf{v}}_i\overline{\mathbf{v}}_i^{\mathrm{T}}, \quad \mathbf{P} = \mathbf{I} - \frac{1}{3}\mathbf{w}\mathbf{w}^{\mathrm{T}}. \tag{25}$$

These expressions follow from the theory of the free–free flexibility developed in [3,4]. Matrix $\overline{\mathbf{E}}$ is symmetric, singular and positive semidefinite with null eigenvector $\mathbf{w}$: $\overline{\mathbf{E}}\mathbf{w} = \mathbf{0}$. It verifies $\overline{\mathbf{E}}\overline{\mathbf{C}} = \overline{\mathbf{C}}\overline{\mathbf{E}} = \mathbf{P}$ as well as other relations catalogued in those references. Relation (25) is dual: it holds if $\overline{\mathbf{C}}$ and $\overline{\mathbf{E}}$ are switched and $1/\overline{\gamma}_i$ replaced by $\overline{\gamma}_i$.

As an example let us compute the $\overline{\mathbf{E}}$ of (8), for which $\mathbf{w}^{\mathrm{T}} = [\,2 \quad 2 \quad 1 \quad 0 \quad 0 \quad 0\,]/\sqrt{3}$:

$$\overline{\mathbf{E}}_{\mathrm{hex}} = \left(\mathbf{I} - \frac{1}{3}\mathbf{w}\mathbf{w}^{\mathrm{T}}\right)\left(\mathbf{C}_{\mathrm{hex}} + \frac{1}{3}\mathbf{w}\mathbf{w}^{\mathrm{T}}\right)^{-1} = \frac{1}{30,132}\begin{bmatrix} 4891 & -4829 & -124 & 3402 & -4860 & 0 \\ -4829 & 4891 & -124 & -3402 & 4860 & 0 \\ -124 & -124 & 496 & 0 & 0 & 0 \\ 3402 & -3402 & 0 & 8748 & 0 & 4860 \\ -4860 & 4860 & 0 & 0 & 8748 & 3402 \\ 0 & 0 & 0 & 4860 & 3402 & 4860 \end{bmatrix}. \tag{26}$$

As a visual check, the pattern of zero entries is exactly that of $\mathbf{C}_{\mathrm{hex}}$. Furthermore $\overline{E}_{44} = \frac{1}{2}(\overline{E}_{11} - \overline{E}_{12})$, $\overline{E}_{46} = -\overline{E}_{15}$, etc., as appropriate for the hexagonal symmetry group [5, p. 89]. The ordinary inverse of $\mathbf{C}_{\mathrm{hex}}$ does not exist.

For future use define

$$\widehat{\mathbf{C}} = \overline{\mathbf{C}} + \tfrac{1}{9}K^{-1}\mathbf{w}\mathbf{w}^{\mathrm{T}}, \quad \widehat{\mathbf{E}} = \overline{\mathbf{E}} + K\mathbf{w}\mathbf{w}^{\mathrm{T}}. \tag{27}$$

These matrices have the useful properties

$$\widehat{\mathbf{E}}\widehat{\mathbf{C}} = \widehat{\mathbf{C}}\widehat{\mathbf{E}} = \mathbf{I}, \quad \widehat{\mathbf{C}}\mathbf{w} = \tfrac{1}{3}K^{-1}\mathbf{w}, \quad \widehat{\mathbf{E}}\mathbf{w} = 3K\mathbf{w}, \quad \overline{\mathbf{C}} = \mathbf{P}\widehat{\mathbf{C}} = \widehat{\mathbf{C}}\mathbf{P}, \quad \overline{\mathbf{E}} = \mathbf{P}\widehat{\mathbf{E}} = \widehat{\mathbf{E}}\mathbf{P}. \tag{28}$$

Of course $\widehat{\mathbf{E}}$ blows up as $K \to \infty$ but this matrix is never assembled as such; it is a convenient "splitting tool" in the spirit of the delta function, which works behind the scenes and disappears once done.

If the projection method is used to get $\overline{\mathbf{C}}$, matrices $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{E}}$ reduce to $\mathbf{C}$ and $\mathbf{E}$, respectively.

2.8. Effective pressure and volumetric strain

Define effective pressure p and effective volumetric strain θ as

$$\begin{aligned} p &= \tfrac{1}{3}\mathbf{w}^{\mathrm{T}}\boldsymbol{\sigma} = \tfrac{1}{3}(w_1\sigma_{11} + w_2\sigma_{22} + w_3\sigma_{33} + w_4\sigma_{23} + w_5\sigma_{31} + w_6\sigma_{12}),\\ \theta &= \mathbf{w}^{\mathrm{T}}\mathbf{e} = w_1e_{11} + w_2e_{22} + w_3e_{33} + 2w_4e_{23} + 2w_5e_{31} + 2w_6e_{12}. \end{aligned} \tag{29}$$

The ratio of p to θ is called the effective bulk modulus:

$$p = K\theta, \quad \theta = K^{-1}p. \tag{30}$$

The proof that $K^{-1} = \mathbf{w}^{\mathrm{T}}\mathbf{C}\mathbf{w}$ is given in Section 2.10.

For the incompressible specialization $\mathbf{w} = [\,1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0\,]^{\mathrm{T}}$. In this case p and θ reduce to the mean normal stress and usual volumetric strain, respectively:

$$p = \tfrac{1}{3}\mathbf{w}^{\mathrm{T}}\boldsymbol{\sigma} = \tfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \sigma_m, \quad \theta = \mathbf{w}^{\mathrm{T}}\mathbf{e} = e_{11} + e_{22} + e_{33} = e_v. \tag{31}$$

Incompressibility may be also characterized by $\theta = \operatorname{div}\mathbf{u} = \partial u_i/\partial x_i = 0$ for any admissible motion, where $\mathbf{u} = [\,u_1 \quad u_2 \quad u_3\,]^{\mathrm{T}}$ is the displacement field.

2.9. Stress–strain splitting

The definitions (29) are used to introduce effective deviatoric stresses s_{ij} and deviatoric strains g_{ij} by

$$\mathbf{s} = \boldsymbol{\sigma} - \mathbf{w}p, \quad \mathbf{g} = \mathbf{e} - \tfrac{1}{3}\mathbf{w}\theta, \tag{32}$$

where vector arrangements mimic those of (1):

$$\mathbf{s} = [\,s_{11} \quad s_{22} \quad s_{33} \quad s_{23} \quad s_{31} \quad s_{12}\,]^{\mathrm{T}}, \quad \mathbf{g} = [\,g_{11} \quad g_{22} \quad g_{33} \quad 2g_{23} \quad 2g_{31} \quad 2g_{12}\,]^{\mathrm{T}}. \tag{33}$$

The key decoupling relations that follow from this definition are

$$\mathbf{w}^{\mathrm{T}}\mathbf{s} = 0, \quad \mathbf{w}^{\mathrm{T}}\mathbf{g} = 0. \tag{34}$$

From these easily follow the projective properties

$$\mathbf{s} = \mathbf{P}\boldsymbol{\sigma}, \quad \mathbf{g} = \mathbf{P}\mathbf{e}, \quad \mathbf{s}^{\mathrm{T}}\mathbf{e} = \mathbf{s}^{\mathrm{T}}\mathbf{g}, \quad \mathbf{g}^{\mathrm{T}}\mathbf{s} = \mathbf{g}^{\mathrm{T}}\boldsymbol{\sigma}, \tag{35}$$

in which $\mathbf{P} = \mathbf{I} - \frac{1}{3}\mathbf{w}\mathbf{w}^{\mathrm{T}}$.

2.10. QRT constitutive equations

To derive the split constitutive equations of a QRT material, $\mathbf{e} = \mathbf{C}\boldsymbol{\sigma}$ is replaced by $\mathbf{e} = \widehat{\mathbf{C}}\boldsymbol{\sigma}$. The splitting (32) and the bulk equation (30) are introduced, and the decoupling relations (34) and (35) used:

$$\mathbf{e} = \widehat{\mathbf{C}}\boldsymbol{\sigma} = \left(\overline{\mathbf{C}} + \tfrac{1}{9}K^{-1}\mathbf{w}\mathbf{w}^{\mathrm{T}}\right)(\mathbf{s} + \mathbf{w}p) = \overline{\mathbf{C}}\mathbf{s} + \tfrac{1}{3}\mathbf{w}K^{-1}p = \mathbf{g} + \tfrac{1}{3}\mathbf{w}\theta. \tag{36}$$

Identifying gives $\mathbf{g} = \overline{\mathbf{C}}\mathbf{s}$. To work out the inverse relation, $\boldsymbol{\sigma} = \mathbf{E}\mathbf{e}$ is replaced by $\boldsymbol{\sigma} = \widehat{\mathbf{E}}\mathbf{e}$ and use made of (28):

$$\boldsymbol{\sigma} = \widehat{\mathbf{E}}\mathbf{e} = (\overline{\mathbf{E}} + K\mathbf{w}\mathbf{w}^{\mathrm{T}})\left(\mathbf{g} + \tfrac{1}{3}\mathbf{w}\theta\right) = \overline{\mathbf{E}}\mathbf{g} + \mathbf{w}K\theta = \mathbf{s} + \mathbf{w}p. \tag{37}$$

Identifying gives $\mathbf{s} = \overline{\mathbf{E}}\mathbf{g}$. Summarizing, the split constitutive equations for finite K are

$$\mathbf{g} = \overline{\mathbf{C}}\mathbf{s} = \overline{\mathbf{C}}\boldsymbol{\sigma}, \quad \mathbf{s} = \overline{\mathbf{E}}\mathbf{g} = \overline{\mathbf{E}}\mathbf{e}, \quad \theta = K^{-1}p, \quad p = K\theta \tag{38}$$

and in terms of total strains and stresses

$$\mathbf{e} = \overline{\mathbf{C}}\mathbf{s} + \tfrac{1}{3}\mathbf{w}K^{-1}p = \overline{\mathbf{C}}\boldsymbol{\sigma} + \tfrac{1}{3}\mathbf{w}K^{-1}p, \quad \boldsymbol{\sigma} = \overline{\mathbf{E}}\mathbf{g} + \mathbf{w}K\theta = \overline{\mathbf{E}}\mathbf{e} + \mathbf{w}K\theta. \tag{39}$$

If the material is exactly rigidtropic, $K \to \infty$, in which case (38) collapse to $\mathbf{e} = \overline{\mathbf{C}}\mathbf{s}$, $\mathbf{s} = \overline{\mathbf{E}}\mathbf{e}$, $\theta = 0$, and the effective pressure p is completely decoupled from deformations.

We can now show that $K^{-1} = \mathbf{w}^{\mathrm{T}}\mathbf{C}\mathbf{w}$ as follows. For the projection treatment of a QRT model:

$$K = \frac{p}{\theta} = \frac{\frac{1}{3}\mathbf{w}^{\mathrm{T}}\boldsymbol{\sigma}}{\mathbf{w}^{\mathrm{T}}\mathbf{e}} = \frac{\frac{1}{3}\mathbf{w}^{\mathrm{T}}\boldsymbol{\sigma}}{\mathbf{w}^{\mathrm{T}}\mathbf{C}\boldsymbol{\sigma}} = \frac{\frac{1}{3}\mathbf{w}^{\mathrm{T}}(\mathbf{s}+\mathbf{w}p)}{\mathbf{w}^{\mathrm{T}}\left(\overline{\mathbf{C}}+\frac{1}{3}\gamma_1\mathbf{w}\mathbf{w}^{\mathrm{T}}\right)(\mathbf{s}+\mathbf{w}p)} = \frac{p}{3\gamma_1 p} = \frac{1}{3\gamma_1} = \frac{1}{\mathbf{w}^{\mathrm{T}}\mathbf{C}\mathbf{w}}. \tag{40}$$

If $\overline{\mathbf{C}}$ is obtained by scaling, $K^{-1} = \mathbf{w}^{\mathrm{T}}\mathbf{C}\mathbf{w}$ is simply a definition suggested by the Rayleigh quotient approximation properties. It is easily verified that for an isotropic material of elastic modulus E and Poisson's ratio ν, K reduces to the ordinary bulk modulus $\frac{1}{3}E/(1-2\nu)$ for any ν.

One final observation: the deviatoric relations $\mathbf{g} = \overline{\mathbf{C}}\mathbf{s}$ and $\mathbf{s} = \overline{\mathbf{E}}\mathbf{g}$ remain unchanged if any multiple of $\mathbf{w}\mathbf{w}^{\mathrm{T}}$ is added to either $\overline{\mathbf{C}}$ or $\overline{\mathbf{E}}$. This device can be occasionally used to advantage to simplify their forms. For example the modified $\overline{\mathbf{E}}$ can be diagonalized for isotropic materials to derive the Lamé deviatoric–volumetric split, as done in Section 3.

2.11. Energy splitting

Insertion of (32) into the internal energy density $\mathscr{U} = \frac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{e}$ produces an exact split into deviatoric and volumetric energies

$$2\mathscr{U} = 2\mathscr{U}_{\mathrm{d}} + 2\mathscr{U}_{\mathrm{v}} = \mathbf{s}^{\mathrm{T}}\mathbf{g} + p\theta = \mathbf{s}^{\mathrm{T}}\mathbf{e} + p\theta = \boldsymbol{\sigma}^{\mathrm{T}}\mathbf{g} + p\theta, \tag{41}$$

in which the last two transformations follow from (35). Substitution of the constitutive equations (38) or (39) and use of orthogonality properties permits selective replacements. For example

$$2\mathscr{U}_{\mathrm{d}} = \mathbf{s}^{\mathrm{T}}\mathbf{e} = \mathbf{s}^{\mathrm{T}}\overline{\mathbf{C}}\mathbf{s} = \boldsymbol{\sigma}^{\mathrm{T}}\overline{\mathbf{C}}\boldsymbol{\sigma} = \mathbf{g}^{\mathrm{T}}\overline{\mathbf{E}}\mathbf{g} = \mathbf{e}^{\mathrm{T}}\overline{\mathbf{E}}\mathbf{e}, \quad 2\mathscr{U}_{\mathrm{v}} = p\theta = \frac{p^2}{K} = K\theta^2. \tag{42}$$

In $\mathscr{U}_{\mathrm{d}}$, deviators $\mathbf{g}$ may be replaced by $\mathbf{e}$ as in $\mathbf{s}^{\mathrm{T}}\mathbf{g} = \mathbf{s}^{\mathrm{T}}\mathbf{e}$, etc., according (35). Likewise $\mathbf{s}$ can be replaced by $\boldsymbol{\sigma}$ as in $\mathbf{s}^{\mathrm{T}}\mathbf{g} = \boldsymbol{\sigma}^{\mathrm{T}}\mathbf{g}$, etc. In displacement FEM formulations it is often convenient to use total strains.

In exact rigidtropy the volumetric energy vanishes.

3. Isotropic material

The foregoing derivations are specialized to isotropic linear elasticity to illustrate connections to conventional splitting methods. The strain–stress relations are

$$\mathbf{e} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\nu) & 0 & 0 \\ & & & & 2(1+\nu) & 0 \\ \text{symm} & & & & & 2(1+\nu) \end{bmatrix}\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \mathbf{C}\boldsymbol{\sigma}, \tag{43}$$

where E is the elastic modulus and ν is Poisson's ratio. The determinant of $\mathbf{C}$ is $8(1+\nu)^5(1-2\nu)/E^6$. For nonnegative ν this vanishes for $\nu = \frac{1}{2}$. Ordinary inversion of (43) yields

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = E_\nu\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{1}{2}(1-2\nu) & 0 & 0 \\ & & & & \frac{1}{2}(1-2\nu) & 0 \\ \text{symm} & & & & & \frac{1}{2}(1-2\nu) \end{bmatrix}\begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \end{bmatrix} = \mathbf{E}\mathbf{e}, \tag{44}$$

where $E_\nu = E/((1-2\nu)(1+\nu))$. Coefficient E_ν "blows up" if $\nu \to \frac{1}{2}$. This is a well known problem in the numerical analysis of such models by displacement-based finite element methods. For this application it has been addressed by splittings into deviatoric stresses and pressure since the mid 1960s [6].

Let $\overline{\mathbf{C}}$ be $\mathbf{C}$ evaluated at $\nu = \frac{1}{2}$. This matrix projects any $\boldsymbol{\sigma}$ onto the space of traceless (divergence free) strain tensors. The null eigenvector of $\overline{\mathbf{C}}$ is

$$\mathbf{w} = [\,1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0\,]^{\mathrm{T}}. \tag{45}$$

This represents hydrostatic pressure; consequently rigidtropic and incompressible models coalesce. In fact, (45) is an an eigenvector of the compliance matrix (43) for any ν. It is easily verified that

$$\mathbf{w}^{\mathrm{T}}\mathbf{C}\mathbf{w} = \frac{3(1-2\nu)}{E} = K^{-1}, \tag{46}$$

where K is the conventional bulk modulus. Hence $\gamma_1 = \frac{1}{3}K^{-1}$ exactly for any ν. The orthogonal projector $\mathbf{P}$ is

$$\mathbf{P} = \mathbf{I} - \frac{1}{3}\mathbf{w}\mathbf{w}^{\mathrm{T}} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 & -1 \\ -1 & -1 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & -1 & -1 \\ -1 & -1 & -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & -1 & 2 \end{bmatrix}. \tag{47}$$

The generalized inverse of $\overline{\mathbf{C}}$ defined by (22) becomes

$$\overline{\mathbf{E}} = \mathbf{P}(\overline{\mathbf{C}} + \mathbf{w}\mathbf{w}^{\mathrm{T}})^{-1} = \frac{E}{9}\begin{bmatrix} 4 & -2 & -2 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 \\ -2 & -2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} = \frac{2(1+\nu)}{3}\mathbf{P}\mathbf{E} = \frac{2(1+\nu)}{3}\mathbf{E}\mathbf{P}, \tag{48}$$

$\overline{\mathbf{E}}$ projects any strain $\mathbf{e}$ onto the space of traceless stress tensors. Since all relations survive if $\overline{\mathbf{C}}$ and $\overline{\mathbf{E}}$ are interchanged, they are *dual* of each other. In particular, exchanging $\overline{\mathbf{C}}$ and $\overline{\mathbf{E}}$ in (48) recovers $\overline{\mathbf{C}}$ as expected:

$$\overline{\mathbf{C}} = \mathbf{P}(\overline{\mathbf{E}} + \mathbf{w}\mathbf{w}^{\mathrm{T}})^{-1} = \frac{1}{2E}\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} = \frac{3}{2(1+\nu)}\mathbf{P}\mathbf{C} = \frac{3}{2(1+\nu)}\mathbf{C}\mathbf{P}. \tag{49}$$

The mean-deviatoric splitting of strains and stresses is $\boldsymbol{\sigma} = \mathbf{s} + \mathbf{w}p$ and $\mathbf{e} = \mathbf{g} + \frac{1}{3}\mathbf{w}\theta$, with

$$p = \tfrac{1}{3}\mathbf{w}^{\mathrm{T}}\boldsymbol{\sigma} = \tfrac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}), \quad \theta = \mathbf{w}^{\mathrm{T}}\mathbf{e} = e_{11}+e_{22}+e_{33}, \quad p = K\theta. \tag{50}$$

Here $\mathbf{s}$ and $\mathbf{g}$ are the deviatoric stresses and strains arranged as per (30), p is the mean normal stress and θ the conventional volumetric strain. The induced decomposition of the constitutive relations reads

$$\mathbf{e} = \mathbf{C}\boldsymbol{\sigma} = \mathbf{C}\mathbf{s} + \mathbf{w}\tfrac{1}{3}K^{-1}p = \mathbf{C}\mathbf{s} + \tfrac{1}{3}\mathbf{w}\theta, \quad \boldsymbol{\sigma} = \mathbf{E}\mathbf{e} = \mathbf{E}\mathbf{g} + \mathbf{w}K\theta = \mathbf{E}\mathbf{g} + \mathbf{w}p. \tag{51}$$

The deviatoric constitutive equations reduce to

$$\mathbf{g} = \mathbf{C}\mathbf{s} = \frac{2(1+\nu)}{3}\overline{\mathbf{C}}\boldsymbol{\sigma} = \frac{2(1+\nu)}{3}\overline{\mathbf{C}}\mathbf{s}, \quad \mathbf{s} = \mathbf{E}\mathbf{g} = \frac{3}{2(1+\nu)}\overline{\mathbf{E}}\mathbf{e} = \frac{3}{2(1+\nu)}\overline{\mathbf{E}}\mathbf{g}, \tag{52}$$

where those involving $\overline{\mathbf{E}}$ stay bounded as $\nu \to \frac{1}{2}$.

As observed in Section 2.10, one can add a multiple of $\mathbf{w}\mathbf{w}^{\mathrm{T}}$, say $\beta\mathbf{w}\mathbf{w}^{\mathrm{T}}$, to $\overline{\mathbf{E}}$ while still verifying $\mathbf{s} = (\overline{\mathbf{E}} + \beta\mathbf{w}\mathbf{w}^{\mathrm{T}})\mathbf{g} = \mathbf{E}_\beta\mathbf{g}$. Taking $\beta = 2E/3 = 2G$, where $G = E/3$ is the shear modulus, reduces $\mathbf{E}_\beta$ to the diagonal form: $\mathbf{diag}(2G, 2G, 2G, G, G, G)$. This yields the Lamé deviatoric split:

$$s_{ij} = 2Gg_{ij}, \tag{53}$$

which in fact is valid for any Poisson's ratio [5, p. 210].

The strain energy density splits exactly into deviatoric and volumetric parts for any ν:

$$2\mathscr{U} = \boldsymbol{\sigma}^{\mathrm{T}}\mathbf{e} = \mathbf{s}^{\mathrm{T}}\mathbf{g} + p\theta = \mathbf{s}^{\mathrm{T}}\mathbf{e} + p\theta = \boldsymbol{\sigma}^{\mathrm{T}}\mathbf{g} + p\theta. \tag{54}$$

This decomposition is the basis of energy methods (in particular finite element methods) for incompressible and near-incompressible materials. For example, in a displacement–pressure formulation

$$2\mathscr{U} = \frac{3}{2(1+\nu)}\mathbf{e}^{\mathrm{T}}\overline{\mathbf{E}}\mathbf{e} + \frac{p^2}{K}. \tag{55}$$

Here **e** comes from displacements. Displacement–pressure coupling terms, such as $p\,\mathrm{div}\,\mathbf{u}$, are added to establish mixed variational principles [7]. For an incompressible material $K \to \infty$ and the p^2/K term vanishes.

4. Orthotropic materials

In this section we study an orthotropic material with the x_i aligned with the principal material axes. This case is worth separate consideration on account of the technical importance of such materials and because several important results can be obtained in closed form.

4.1. Rigidtropy and incompressibility conditions

The principal elastic moduli are E_1, E_2 and E_3, whereas the principal shear moduli are G_1, G_2 and G_3. The six moduli are assumed to be positive. Three Poisson ratios are defined symmetrically with respect to the geometric means $\sqrt{E_1E_2}$, $\sqrt{E_2E_3}$ and $\sqrt{E_3E_1}$, a device that circumvents the widespread but confusing practice of carrying six Poisson ratios linked by three symmetry constraints. Accordingly the compliance matrix is written

$$\mathbf{C} = \begin{bmatrix} 1/E_1 & -\nu_{12}/\sqrt{E_1E_2} & -\nu_{13}/\sqrt{E_1E_3} & 0 & 0 & 0 \\ & 1/E_2 & -\nu_{23}/\sqrt{E_2E_3} & 0 & 0 & 0 \\ & & 1/E_3 & 0 & 0 & 0 \\ & & & 1/G_1 & 0 & 0 \\ & & & & 1/G_2 & 0 \\ \text{symm} & & & & & 1/G_3 \end{bmatrix}. \tag{56}$$

The determinant of **C** is $(1-(\nu_{12}^2+\nu_{23}^2+\nu_{13}^2)+2\nu_{12}\nu_{23}\nu_{13})/(E_1E_2E_3G_1G_2G_3)$. This vanishes if

$$\nu_{12}^2+\nu_{23}^2+\nu_{13}^2+2\nu_{12}\nu_{23}\nu_{13}=1. \tag{57}$$

If $\nu_{12}=\nu_{23}=\nu_{13}=\nu$ condition (57) reduces to $\left(\nu-\frac{1}{2}\right)(\nu+1)^2=0$, which has the positive root $\nu=\frac{1}{2}$ and the negative double root -1. This result is more general than the isotropic material studied in Section 3 because it holds for arbitrary E_1, E_2 and E_3. Many rational solutions of (57) exist. The simplest one with three different positive ratios is $\nu_{12}=9/16$, $\nu_{23}=3/4$, $\nu_{13}=1/8$, and cyclic permutations thereof; furthermore the signs of two Poisson ratios may be simultaneously switched.

Eq. (57) is only a necessary condition for rigidtropy. It remains to check stability. One eigenvalue of **C** is 0 and three others are simply $1/G_i>0$. Two roots remain. An inertia analysis of the uppermost 3×3 principal minor shows that both roots are positive if and only if

$$-1<\nu_{12}<1, \quad -1<\nu_{13}<1, \quad -1<\nu_{23}<1, \tag{58}$$

for any $E_1>0$, $E_2>0$ and $E_3>0$. Consequently (57) and (58) are necessary and sufficient conditions for rigidtropy. Poisson's ratios over that range may be realized in composite materials.

The incompressibility conditions are far more restrictive: $C_{11}+C_{12}+C_{13}=C_{12}+C_{22}+C_{23}=C_{13}+C_{23}+C_{33}=0$. Given E_1, E_2, and E_3 as data, these can only be satisfied if the Poisson ratios become $\nu_{12}=\frac{1}{2}(E_1^{-1}+E_2^{-1}-E_3^{-1})\sqrt{E_1E_2}$, $\nu_{13}=\frac{1}{2}(E_3^{-1}+E_1^{-1}-E_2^{-1})\sqrt{E_1E_3}$ and $\nu_{23}=\frac{1}{2}(E_2^{-1}+E_3^{-1}-E_1^{-1})\sqrt{E_2E_3}$. If $E_1=E_2=E_3=E$ this gives $\nu_{12}=\nu_{13}=\nu_{23}=\frac{1}{2}$ as can be expected. But if the moduli are widely different, at least one of the Poisson ratios may stray out of the stable region (58). As a consequence the compliance matrix becomes unstable, meaning that incompressible orthotropic models are physically impossible if the ratios E_1/E_2 and E_1/E_3 depart sufficiently away from unity. Appendix A provides the appropriate analysis along with a stability chart.

4.2. Spectral analysis

Suppose the rigidtropy conditions (57) and (58) exactly hold so that **C** is renamed $\overline{\mathbf{C}}$. The null eigenvector of squared-length 3 is as usual called **w**, which has the configuration

$$\mathbf{w}=[\,w_1 \quad w_2 \quad w_3 \quad 0 \quad 0 \quad 0\,]^{\mathrm{T}}, \quad \mathbf{w}^{\mathrm{T}}\mathbf{w}=w_1^2+w_2^2+w_3^2=3. \tag{59}$$

A closed form solution for the first three entries is:

$$
\begin{aligned}
&w_1 = v_1\sqrt{3E_1/S}, \quad w_2 = v_2\sqrt{3E_2/S}, \quad w_3 = v_3\sqrt{3E_3/S},\\
&v_1 = (1+v_{23})(1+v_{12}+v_{13}-v_{23}), \quad v_2 = (1+v_{13})(1+v_{12}-v_{13}+v_{23}),\\
&v_3 = (1+v_{12})(1-v_{12}+v_{13}+v_{23}), \quad S = E_1v_1^2 + E_2v_2^2 + E_3v_3^2.
\end{aligned} \tag{60}
$$

The effective pressure and volumetric strain are defined as explained in Section 2.8:

$$
p = \tfrac{1}{3}\mathbf{w}^{\mathrm{T}}\boldsymbol{\sigma} = \tfrac{1}{3}(w_1\sigma_{11} + w_2\sigma_{22} + w_3\sigma_{33}), \quad \theta = \mathbf{w}^{\mathrm{T}}\mathbf{e} = w_1e_{11} + w_2e_{22} + w_3e_{33}. \tag{61}
$$

The projected elasticity matrix, split constitutive equations and split energies are obtained as described previously.

For the QRT case the reference model $\overline{\mathbf{C}}$ can be obtained by projection or scaling. Since the last three components of the eigenvector $\mathbf{v}_1$ should be zeros, both methods preserve the off-diagonal zero entries and consequently the orthotropy pattern. Hence as discussed in Section 2.6 the projection method is preferable.

4.3. Example

As an example, consider the QRT orthotropic compliance:

$$
\mathbf{C} = \frac{1}{144}\begin{bmatrix}
144.00396 & -53.99340 & -26.99208 & 0 & 0 & 0\\
-53.99340 & 36.01100 & -2.98680 & 0 & 0 & 0\\
-26.99208 & -2.98680 & 16.01584 & 0 & 0 & 0\\
0 & 0 & 0 & 288 & 0 & 0\\
0 & 0 & 0 & 0 & 720 & 0\\
0 & 0 & 0 & 0 & 0 & 432
\end{bmatrix}. \tag{62}
$$

The reference model is obtained by the projection method. The eigenvalues of $\mathbf{C}$ are $\gamma_1 = 0.03080$, $\gamma_2 = 25.93059$, $\gamma_3 = 170.06941$, $\gamma_4 = 288$, $\gamma_5 = 432$ and $\gamma_6 = 720$. The eigenvector corresponding to γ_1, normalized to squared-length 3 is

$$
\mathbf{w} = \mathbf{v}_1 = [\,0.621059 \quad 1.035098 \quad 1.242118 \quad 0 \quad 0 \quad 0\,]^{\mathrm{T}}. \tag{63}
$$

from which $\overline{\mathbf{C}} = \mathbf{C} - \frac{1}{3}\gamma_1\mathbf{w}\mathbf{w}^{\mathrm{T}}$ giving

$$
\overline{\mathbf{C}} = \frac{1}{144}\begin{bmatrix}
144 & -54 & -27 & 0 & 0 & 0\\
-54 & 36 & -3 & 0 & 0 & 0\\
-27 & -4 & 16 & 0 & 0 & 0\\
0 & 0 & 0 & 288 & 0 & 0\\
0 & 0 & 0 & 0 & 720 & 0\\
0 & 0 & 0 & 0 & 0 & 432
\end{bmatrix} = \begin{bmatrix}
1 & -3/8 & -3/16 & 0 & 0 & 0\\
-3/8 & 1/4 & -1/48 & 0 & 0 & 0\\
-3/16 & -1/48 & 1/9 & 0 & 0 & 0\\
0 & 0 & 0 & 2 & 0 & 0\\
0 & 0 & 0 & 0 & 5 & 0\\
0 & 0 & 0 & 0 & 0 & 3
\end{bmatrix}. \tag{64}
$$

The eigenvalues are the same as $\mathbf{C}$ except for $\overline{\gamma}_1 = 0$. The bulk modulus is $K = 1/(\mathbf{w}^{\mathrm{T}}\mathbf{C}\mathbf{w}) = 1/(3\gamma_1) = 10.8225$.

The nice result (64) should come as no surprise since (62) was "cooked" by a spectral shift of $\gamma_1 = 77/2500$ of the $\overline{\mathbf{C}}$ matrix fabricated with $E_1 = 1$, $E_2 = 4$, $E_3 = 9$, $G_1 = 1/2$, $G_2 = 1/5$, $G_3 = 1/3$, $v_{12} = 3/4$, $v_{13} = 9/16$ and $v_{23} = 1/8$. The exact null eigenvector is $\mathbf{w}^{\mathrm{T}} = [\,3 \quad 5 \quad 6 \quad 0 \quad 0 \quad 0\,]\sqrt{3/70}$.

The projected elasticity matrix is

$$
\overline{\mathbf{E}} = \mathbf{P}\left(\overline{\mathbf{C}} + \frac{1}{3}\mathbf{w}\mathbf{w}^{\mathrm{T}}\right)^{-1} = \frac{1}{7350}\begin{bmatrix}
6432 & 2880 & -5616 & 0 & 0 & 0\\
2880 & 21,600 & -19,440 & 0 & 0 & 0\\
-5616 & -19,440 & 19,008 & 0 & 0 & 0\\
0 & 0 & 0 & 3675 & 0 & 0\\
0 & 0 & 0 & 0 & 1470 & 0\\
0 & 0 & 0 & 0 & 0 & 2450
\end{bmatrix}. \tag{65}
$$

The eigenvalues of $\overline{\mathbf{E}}$ are $\overline{\mu}_1 = 0$, $\overline{\mu}_2 = 5.55329$, $\overline{\mu}_3 = 0.846713$, $\overline{\mu}_4 = 0.5$, $\overline{\mu}_5 = 0.333333$, and $\overline{\mu}_6 = 0.2$. These are the reciprocals of the corresponding eigenvalues of $\overline{\mathbf{C}}$, except for $\gamma_1 = 0$, which maps to $\mu_1 = 0$.

The effective pressure and volumetric strain are

$$
p = \frac{1}{3}\mathbf{w}^{\mathrm{T}}\boldsymbol{\sigma} = \sqrt{\frac{1}{210}}(3\sigma_{11} + 5\sigma_{22} + 6\sigma_{33}), \quad \theta = \mathbf{w}^{\mathrm{T}}\mathbf{e} = \sqrt{\frac{3}{70}}(3e_{11} + 5e_{22} + 6e_{33}). \tag{66}
$$

The calculation of a similar $\overline{\mathbf{C}}$ by scaling is left as an exercise.

5. Conclusions

We have examined a wide class of volumetric constraints applicable to linearly elastic anisotropic solids. Our main objective is to obtain splittings appropriate for downstream numerical treatment.

The constraints are subsumed under the "umbrella" of rigidtropic behavior, in which the material does not deform under a stress pattern characterized by the null eigenvector of the compliance matrix. This model includes incompressible behavior as special case when that pattern is hydrostatic stress. Anisotropic incompressibility, however, is relatively rare in comparison to the isotropic case. Further, if enforced for highly anisotropic materials, incompressibility may lead to unstable models as shown in Appendix A.

The main findings of this study are:

1. Rigidtropic behavior can be characterized by a spectral analysis of the compliance matrix. The incompressibility subset is defined by a specialization of the null eigenvector.
2. QRT models, which represent the extension of the quasi-incompressible models of isotropic materials, can be handled by decomposition into a exactly rigidtropic reference model and the bulk behavior. Two methods: projection and scaling, for effecting this decomposition have been described. Both methods can also be used for fixing slightly unstable compliances due to noisy experimental data.
3. Correct splitting of the constitutive equations and internal energy requires a redefinition of pressure and volumetric strain as effective quantities.
4. The concept of bulk modulus can be generalized as a ratio between effective pressure and effective volumetric strain.

Using the splittings derived here, mixed variational principles appropriate for constructing finite element models have been derived in a separate paper [8].

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Appendix A. Three flavors of anisotropic incompressibility

This Appendix studies three volumetric constraint models of a linearly elastic anisotropic solid, summarizing results of [2]. The following terminology is introduced.

A material is called *rigidtropic* if it does not deform (i.e., experiences zero strains) under a specific stress pattern, which is a null eigenvector of the strain–stress (compliance) matrix. The term "rigidtropic" is used in the sense of "rigidity in a certain way" as defined by that eigenvector.

A material is called *isochoric* if it does not change volume under any applied stress system [1, Sec. 77]. Alternatively: the volumetric strain is zero under any stress state.

A material is called *hydroisochoric* if it is isochoric under hydrostatic stress. Isochoric materials are hydroisochoric but the converse is not necessarily true.

The three models coalesce for an isotropic material. For an arbitrary anisotropic solid, however, it will be shown below that imposing a isochoric or hydroisochoric constraint may produce a compliance matrix that has at least one negative eigenvalue. This means that under some stress system the material is able to create energy, contradicting the laws of thermodynamics. Such model cannot represent a physically stable material. On the other hand, for rigidtropic behavior it is easier to control material stability for any type of anisotropy because constraints are posed directly on the spectral form.

The mathematical expressions of the foregoing constraints on the strain–stress matrix relation (2) are

$$\begin{aligned}&\text{Rigidtropic: } \gamma_1 = 0, \quad \gamma_i > 0, \ \ i = 2, \ldots, 6.\\ &\text{Hydroisochoric: } C_{11} + C_{22} + C_{33} + 2C_{12} + 2C_{13} + 2C_{23} = 0.\\ &\text{Isochoric: } C_{1j} + C_{2j} + C_{3j} = 0, \quad j = 1, 2, 3.\end{aligned} \tag{A.1}$$

We now examine some key properties of these models.

A.1. Hydroisochoric model

Assume that the material modeled by (5) is hydroisochoric. Let $\boldsymbol{\sigma}_p$ represent a hydrostatic pressure stress system with $\sigma_{ij} = \delta_{ij} p$. According to the foregoing definition

$$\mathbf{C}\boldsymbol{\sigma}_p = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ \text{symm} & & & & & C_{66} \end{bmatrix} \begin{bmatrix} p \\ p \\ p \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p(C_{11}+C_{12}+C_{13}) \\ p(C_{12}+C_{22}+C_{23}) \\ p(C_{13}+C_{23}+C_{33}) \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \end{bmatrix} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \end{bmatrix}, \tag{A.2}$$

with $e_v = e_{11} + e_{22} + e_{33} = p(C_{11} + C_{22} + C_{33} + 2C_{12} + 2C_{13} + 2C_{23}) = 0$.

(The value of the shear strains is of no interest.) The complementary energy density produced by $\boldsymbol{\sigma}_p$ is

$$\mathscr{U}_p^* = \tfrac{1}{2}\boldsymbol{\sigma}_p^{\mathrm{T}}\mathbf{C}\boldsymbol{\sigma}_p = \tfrac{1}{2}p(e_{11}+e_{22}+e_{33}) = \tfrac{1}{2}pe_v = 0. \tag{A.3}$$

But $\gamma_p = \mathscr{U}_p^*/(\boldsymbol{\sigma}_p^{\mathrm{T}}\boldsymbol{\sigma}) = \mathscr{U}_p^*/(3p^2) = 0$ is the Rayleigh quotient of $\boldsymbol{\sigma}_p$ with $\mathbf{C}$. According to the Courant–Fisher theorem [9], γ_p must lie in the closed interval $[\gamma_{\min}, \gamma_{\max}]$:

$$\gamma_1 \leqslant \gamma_p = 0 \leqslant \gamma_6. \tag{A.4}$$

If $\boldsymbol{\sigma}_p$ is not an eigenvector of $\mathbf{C}$: $\mathbf{C}\boldsymbol{\sigma}_p \neq \mathbf{0}$, the leftmost equality in (A.4) is not possible. Consequently

$$\gamma_1 < 0, \tag{A.5}$$

and the model is unstable.

If $\mathbf{C}\boldsymbol{\sigma}_p = \mathbf{0}$ the sum of the first three columns (or rows) of $\mathbf{C}$ must vanish. The hydroisochoric model then coalesces with the isochoric one, which is analyzed next.

A.2. Isochoric model

The model is isochoric if the sum of the first three rows (or columns) of $\mathbf{C}$ is the null 6-vector. Equivalently $\boldsymbol{\sigma}_p$ is a null eigenvector of $\mathbf{C}$. The Rayleigh quotient test (A.4) does not offer sufficient information on stability and a deeper look at $\mathbf{C}$ is required. A *sufficient* criterion for instability can be derived by considering the upper 3×3 principal minor $\widetilde{\mathbf{C}}$. From the last of (A.1), $\widetilde{\mathbf{C}}$ must have the form:

$$\widetilde{\mathbf{C}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ \text{symm} & & C_{33} \end{bmatrix} = \begin{bmatrix} C_{11} & \frac{1}{2}(C_{33}-C_{11}-C_{22}) & \frac{1}{2}(C_{22}-C_{11}-C_{33}) \\ & C_{22} & \frac{1}{2}(C_{11}-C_{22}-C_{33}) \\ \text{symm} & & C_{33} \end{bmatrix}. \tag{A.6}$$

This matrix is singular. Taking $\alpha = C_{11}/C_{22}$ and $\beta = C_{11}/C_{33}$ for convenience, an eigenvalue analysis shows that $\widetilde{\mathbf{C}}$ is indefinite if

$$2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) < 1 + \left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2, \tag{A.7}$$

and is positive semidefinite if the inequality is reversed. If $\widetilde{\mathbf{C}}$ is indefinite, so is $\mathbf{C}$ and the model is unstable. If $\widetilde{\mathbf{C}}$ is semidefinite, an eigenvalue analysis of the complete $\mathbf{C}$ is required to decide on stability. The stability regions of $\widetilde{\mathbf{C}}$ are shown in Fig. 2, where "potentially semistable" indicates that confirmation by a analysis of the full $\mathbf{C}$ is required. An exception is an orthotropic material referred to principal material axes, in which case no further tests are necessary if C_{44}, C_{55} and C_{66} are positive.

Fig. 2 illustrates that a wide range of diagonal compliances in $\widetilde{\mathbf{C}}$ is detrimental to stability. For example if $\alpha = \beta$, instability is guaranteed to happen for $\alpha > 4$.

A.3. Rigidtropic and isotropic models

If $\mathbf{C}$ is nonnegative with $\gamma_1 = 0$ and $\mathbf{w} \equiv \mathbf{v}_1$ is the only null eigenvector, the material is rigidtropic under that stress mode. For an isotropic material $\mathbf{w} = [\,1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0\,]^{\mathrm{T}} = \boldsymbol{\sigma}_p/p$, which is the hydrostatic stress mode. For an anisotropic material, however, mode $\mathbf{w}$ generally will contain shear stresses. Introducing effective pressure as $p = \frac{1}{3}\mathbf{w}^{\mathrm{T}}\boldsymbol{\sigma}$

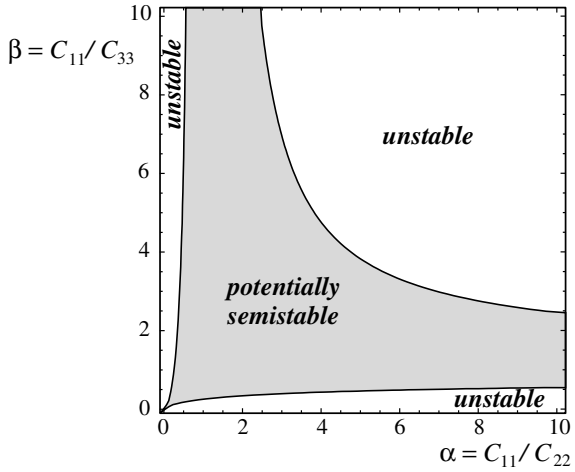

Fig. 2. Stability chart for the principal minor (A.6) of an isochoric material as function of the ratios C_{11}/C_{22} and C_{11}/C_{33}.

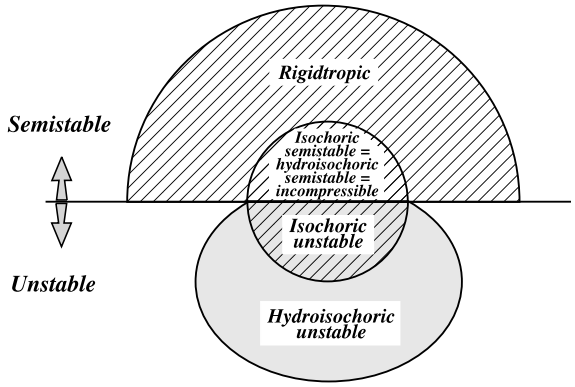

Fig. 3. Schematic of inclusions between rigidtropic, isochoric and hydroisochoric models. The crosshatched area marks a singular **C** matrix.

and effective volumetric strain as $e_v = \mathbf{w}^{\mathrm{T}}\boldsymbol{\sigma}$, the volumetric and deviatoric energies can be uncoupled as discussed in Sections 2.9 and 2.11.

If the rigid stress mode is $\boldsymbol{\sigma}_p$, rigidtropic behavior reduces to isochoric behavior. This inclusion is depicted in Fig. 3.

An isotropic material is defined by the strain–stress relation (43). Under hydrostatic stress $\boldsymbol{\sigma}_p$, $e_v = 3(1-2\nu)p/E$, which vanishes for $\nu = \frac{1}{2}$. It is easy to verify that if $\nu = \frac{1}{2}$, $e_v = 0$ for any $\boldsymbol{\sigma}$ and the material is isochoric. Furthermore $\boldsymbol{\sigma}_p/p$ is the only null eigenvector of **C**. Consequently $\gamma_p = \gamma_1 = 0$ and **C** has no negative eigenvalues. The definitions of rigidtropic, incompressible and isochoric behavior coalesce for this model.

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