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C.A. Felippa
E. Oñate

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International Center for Numerical Methods in Engineering
Gran Capitán s/n, 08034 Barcelona, Spain

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CARLOS A. FELIPPA* AND EUGENIO OÑATE†

** Department of Aerospace Engineering Sciences*
and Center for Aerospace Structures
University of Colorado
Boulder, Colorado 80309-0429, USA

† International Center for Numerical Methods in Engineering (CIMNE)
Edificio C-1, c. Gran Capitán s/n
Universidad Politécnica de Cataluña
08034 Barcelona, Spain

Summary

We study three "incompressibility flavors" of linearly-elastic anisotropic solids that exhibit volumetric constraints: isochoric, hydroisochoric and rigidtropic. An isochoric material deforms without volume change under any stress system. An hydroisochoric material does not change volume under hydrostatic stress. A rigidtropic material undergoes zero deformations under a certain stress pattern. Whereas the three models coalesce for isotropic materials, important differences appear for anisotropic behavior. We find that isochoric and hydroisochoric models under certain conditions may be hampered by unstable physical behavior. Rigidtropic models can represent semistable physical materials of arbitrary anisotropy while including isochoric and hydroisochoric behavior as special cases.

Keywords: linear elasticity, solid, anisotropy, isotropy, rigidtropy, incompressibility, isochoric, hydroisochoric, volumetric constraints, stability, material, constitutive model, compliance.

1. Introduction

An incompressible linearly-elastic isotropic solid does not deform under hydrostatic stress. It does not change volume under pressure. Since deviatoric and volumetric deformations uncouple, no volume change occurs under any stress state. The three volumetric constraints just stated coalesce, and it is sufficient to qualify the material as incompressible.

A more careful study is necessary for anisotropic materials. In the present Note we examine three volumetric constraint models of a linearly elastic anisotropic solid. The following definitions are used for that examination.

A material is called *rigidtropic* if it does not deform (i.e., experiences zero strains) under a specific stress pattern, which is a null eigenvector of the strain-stress (compliance) matrix. The term "rigidtropic" is used in the sense of "rigidity in a certain way" as defined by that eigenvector.

A material is called *isochoric* if it does not change volume under any applied stress system [1, Sec. 77]. Alternatively: the volumetric strain is zero under any stress state.

A material is called *hydroisochoric* if it is isochoric under hydrostatic stress. Isochoric materials are hydroisochoric but the converse is not necessarily true.

As noted the three models coalesce for an isotropic material. For an arbitrary anisotropic solid, however, it will be shown that imposing a isochoric or hydroisochoric constraint may produce a compliance matrix that has at least one negative eigenvalue. This means that under some stress system the material is able to create energy, contradicting the laws of thermodynamics. Such model cannot represent a physically stable material. On the other hand, for rigidtropic behavior it is easier to control material stability for any type of anisotropy because constraints are posed directly on the spectral form.

2. Compliance Relations

We consider a linearly-elastic anisotropic solid in three dimensions referred to axes $\{x_i\}$. Stresses σ_{ij} and strains e_{ij} will be arranged as 6-component column vectors constructed from the respective tensors through the usual conventions of structural mechanics:

$$\sigma = [\,\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{12} \quad \sigma_{23} \quad \sigma_{31}\,]^T, \qquad \mathbf{e} = [\,e_{11} \quad e_{22} \quad e_{33} \quad 2e_{12} \quad 2e_{23} \quad 2e_{31}\,]^T. \tag{1}$$

The strain-stress constitutive equations in matrix notation are

$$\mathbf{e} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ \text{symm} & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \mathbf{C}\sigma, \tag{2}$$

Here C_{ij} are compliance coefficients arranged into the symmetric compliance matrix $\mathbf{C}$. All diagonal entries C_{ii} are assumed to be nonnegative with a positive sum. The matrix $\mathbf{C}$ is called *stable*, *semistable* or *unstable* if $\mathbf{C}$ is positive definite, positive semidefinite, or indefinite, respectively. In the semistable case it will be assumed that $\mathbf{C}$ is has a rank deficiency of at most one to simplify the analysis.

The eigenvalues of $\mathbf{C}$ are γ_i for $i = 1, 2 \ldots 6$, with $\mathbf{v}_i$ being the corresponding eigenvector normalized to length $\sqrt{3}$. (This nonstandard normalization simplifies linking up to the hydrostatic stress vector in Sections 4ff.) Accordingly the spectral decomposition is

$$\mathbf{C} = \tfrac{1}{3} \sum_{i=1}^{6} \gamma_i \, \mathbf{v}_i \, \mathbf{v}_i^T, \qquad \mathbf{v}_i^T \mathbf{v}_j = 3\delta_{ij}, \tag{3}$$

where δ_{ij} is the Kronecker delta. The eigenvalues will be arranged so that $\gamma_1 = \gamma_{min}$ is the algebraically smallest one and $\gamma_6 = \gamma_{max}$ the maximum. For stable or semistable models, $\gamma_1 \geq 0$ and $\gamma_j > 0$ for $j = 2, \ldots 6$.

If $\gamma_1 = 0$ the material is rigidtropic according to the definition given in the Introduction, with $\mathbf{v}_1$ defining the corresponding stress pattern. The volumetric strain is $e_v = e_{11} + e_{22} + e_{33}$. Isochoric behavior is mathematically characterized by $e_v = 0$ under any σ. Hydroisochoric behavior means that $e_v = 0$ under $\sigma_p = p\,[1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0]^T$ for any p. These constraints are mathematically expressed in terms of $\mathbf{C}$ as follows.

$$\begin{aligned} \text{Rigidtropic:} \quad & \gamma_1 = 0, \;\; \gamma_i > 0, \;\; i = 2, \ldots 6. \\ \text{hydroisochoric:} \quad & C_{11} + C_{22} + C_{33} + 2C_{12} + 2C_{13} + 2C_{23} = 0. \\ \text{Isochoric:} \quad & C_{1j} + C_{2j} + C_{3j} = 0, \;\; j = 12\ ,3. \end{aligned} \tag{4}$$

Diagonal compliances are often known reliably from extensional and torsion tests. Off diagonal entries are typically less amenable to accurate measurement. Volumetric constraints, for example on volume change, are checked with triaxial tests. In any case, such constraints may be satisfied only approximately. Reference [2] discusses projection and scaling techniques for finding a "reference model" that satisfies constraints accurately.

3. Examples

The following examples of compliance matrices pertain to an orthotropic material with the $\{x_i\}$ aligned with the principal material axes. The diagonal entries are kept the same. The three nonzero off-diagonal entries are adjusted to meet the definitions (4).

Rigidtropic:

$$\mathbf{C}_{rig} = \begin{bmatrix} 1 & -3/8 & -3/16 & 0 & 0 & 0 \\ -3/8 & 1/4 & -1/48 & 0 & 0 & 0 \\ -3/16 & -1/48 & 1/9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} = \frac{1}{144} \begin{bmatrix} 144 & -54 & -27 & 0 & 0 & 0 \\ -54 & 36 & -3 & 0 & 0 & 0 \\ -27 & -3 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 288 & 0 & 0 \\ 0 & 0 & 0 & 0 & 720 & 0 \\ 0 & 0 & 0 & 0 & 0 & 432 \end{bmatrix} \tag{5}$$

Eigenvalues: $[\,5 \quad 3 \quad 2 \quad 1.181038 \quad 0.180074 \quad 0\,]$. The compliance matrix is semistable. The null eigenvector defining the rigid mode is $\mathbf{v}_1 = \sqrt{54/35}\,[\,1/2 \quad 5/6 \quad 1 \quad 0 \quad 0 \quad 0\,]^T$.

Hydroisochoric:

$$\mathbf{C}_{hyd} = \begin{bmatrix} 1 & -11/27 & -95/432 & 0 & 0 & 0 \\ -11/27 & 1/4 & -23/432 & 0 & 0 & 0 \\ -95/432 & -23/432 & 1/9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} = \frac{1}{432} \begin{bmatrix} 432 & -176 & -95 & 0 & 0 & 0 \\ -176 & 108 & -23 & 0 & 0 & 0 \\ -95 & -23 & 48 & 0 & 0 & 0 \\ 0 & 0 & 0 & 576 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1440 & 0 \\ 0 & 0 & 0 & 0 & 0 & 864 \end{bmatrix} \tag{6}$$

Eigenvalues: $[\,5 \quad 3 \quad 2 \quad 1.208689 \quad 0.211580 \quad -0.059158\,]$. The compliance matrix is unstable.

Isochoric:

$$\mathbf{C}_{iso} = \begin{bmatrix} 1 & -41/72 & -31/72 & 0 & 0 & 0 \\ -41/72 & 1/4 & 23/72 & 0 & 0 & 0 \\ -31/72 & 23/72 & 1/9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} = \frac{1}{144} \begin{bmatrix} 144 & -82 & -62 & 0 & 0 & 0 \\ -82 & 36 & 46 & 0 & 0 & 0 \\ -62 & 46 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 288 & 0 & 0 \\ 0 & 0 & 0 & 0 & 720 & 0 \\ 0 & 0 & 0 & 0 & 0 & 432 \end{bmatrix} \tag{7}$$

Eigenvalues: $[\,5 \quad 3 \quad 2 \quad 1.508781 \quad 0 \quad -0.147669\,]$. The compliance matrix is unstable.

4. Hydroisochoric Model

Assume that the material modeled by (2) is hydroisochoric. Consequently

$$\mathbf{C}\sigma_p = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ \text{symm} & & & & & C_{66} \end{bmatrix} \begin{bmatrix} p \\ p \\ p \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p(C_{11}+C_{12}+C_{13}) \\ p(C_{12}+C_{22}+C_{23}) \\ p(C_{13}+C_{23}+C_{33}) \\ 2e_{12} \\ 2e_{23} \\ 2e_{31} \end{bmatrix} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{12} \\ 2e_{23} \\ 2e_{31} \end{bmatrix}, \tag{8}$$

with $e_v = e_{11} + e_{22} + e_{33} = p(C_{11} + C_{22} + C_{33} + 2C_{12} + 2C_{13} + 2C_{23}) = 0.$

(The value of the shear strains is of no interest.) The complementary energy density produced by σ_p is

$$\mathcal{U}_p^* = \tfrac{1}{2}\sigma_p^T \mathbf{C}\sigma_p = \tfrac{1}{2}p(e_{11} + e_{22} + e_{33}) = \tfrac{1}{2}pe_v = 0. \tag{9}$$

But $\gamma_p = \mathcal{U}_p^*/(\sigma_p^T\sigma) = \mathcal{U}_p^*/(3p^2) = 0$ is the Rayleigh quotient of σ_p with $\mathbf{C}$. According to the Courant-Fisher theorem [2], γ_p must lie in the closed interval $[\gamma_{min}, \gamma_{max}]$:

$$\gamma_1 \le \gamma_p = 0 \le \gamma_6 \tag{10}$$

If σ_p is not an eigenvector of $\mathbf{C}$: $\mathbf{C}\sigma_p \neq \mathbf{0}$, the leftmost equality in (10) is not possible. Consequently

$$\gamma_1 < 0, \tag{11}$$

and the model is unstable.

If $\mathbf{C}\sigma_p = \mathbf{0}$ the sum of the first three columns (or rows) of $\mathbf{C}$ must vanish. The hydroisochoric model then coalesces with the isochoric one, which is analyzed next.

5. Isochoric Model

The model is isochoric if the sum of the first three rows (or columns) of $\mathbf{C}$ is the null 6-vector. Equivalently σ_p is a null eigenvector of $\mathbf{C}$. The Rayleigh quotient test (10) does not offer sufficient information on stability and a deeper look at $\mathbf{C}$ is required. Nonetheless a *sufficient* criterion for instability can be derived by considering the upper 3 × 3 principal minor $\tilde{\mathbf{C}}$. From the last of (4), $\tilde{\mathbf{C}}$ must have the form:

$$\tilde{\mathbf{C}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ \text{symm} & & C_{33} \end{bmatrix} = \begin{bmatrix} C_{11} & \tfrac{1}{2}(C_{33} - C_{11} - C_{22}) & \tfrac{1}{2}(C_{22} - C_{11} - C_{33}) \\ & C_{22} & \tfrac{1}{2}(C_{11} - C_{22} - C_{33}) \\ & \text{symm} & C_{33} \end{bmatrix}. \tag{12}$$

This matrix is singular. Taking $\alpha = C_{11}/C_{22}$ and $\beta = C_{11}/C_{33}$ for convenience, an eigenvalue analysis shows that $\tilde{\mathbf{C}}$ is indefinite if

$$2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) < 1 + \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2, \tag{13}$$

and is positive semidefinite if $<$ is changed to $>$. If $\tilde{\mathbf{C}}$ is indefinite, so is $\mathbf{C}$ and the model is unstable. If $\tilde{\mathbf{C}}$ is semidefinite, an eigenvalue analysis of the complete $\mathbf{C}$ is required to decide on stability. The stability regions of $\tilde{\mathbf{C}}$ are shown in Figure 1, where "potentially semistable" indicates that confirmation by a analysis of the full $\mathbf{C}$ is required. An exception is an orthotropic material referred to the principal material axes, in which case no further tests are necessary if C_{44}, C_{55} and C_{66} are positive.

Figure 1 illustrates that a wide range of diagonal compliances in $\tilde{\mathbf{C}}$ is detrimental to stability. For example if $\alpha = \beta$, instability is guaranteed to happen for $\alpha > 4$.

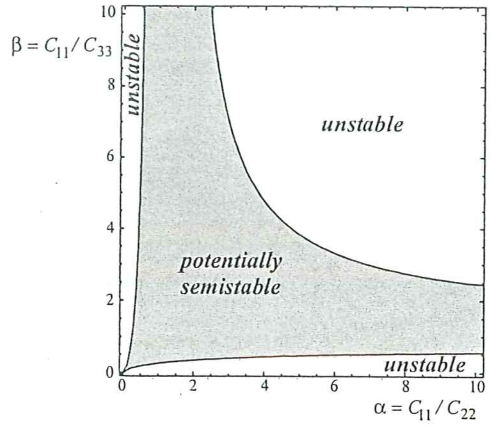

Figure 1. Stability chart for the principal minor (12) of an isochoric material as function of the ratios C_{11}/C_{22} and C_{11}/C_{33}.

6. Rigidtropic Model

If $\mathbf{C}$ is nonnegative with $\gamma_1 = 0$ and $\mathbf{w} \equiv \mathbf{v}_1$ is the only null eigenvector the material is rigidtropic under that stress mode. For an isotropic material $\mathbf{w} = [1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0]^T = \sigma_p$, the hydrostatic stress mode. For an anisotropic material mode $\mathbf{w}$ generally will contain shear stresses. Introducing effective pressure as $p = \frac{1}{3}\mathbf{w}^T\sigma$ and effective volumetric strain as $e_v = \mathbf{w}^T\sigma$, the volumetric and deviatoric energies can be uncoupled [3].

If the rigid stress mode is σ_p, rigidtropic reduces to isochoric. This inclusion is pictured in Figure 2.

7. Isotropic Material

If the solid is isotropic with elastic modulus $E > 0$ and Poisson's ratio ν,

$$\mathbf{C} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\nu) & 0 & 0 \\ & & & & 2(1+\nu) & 0 \\ \text{symm} & & & & & 2(1+\nu) \end{bmatrix}. \tag{14}$$

Under hydrostatic stress σ_p, $e_v = 3(1-2\nu)p/E$, which vanishes for $\nu = \frac{1}{2}$. It is easy to verify that if $\nu = \frac{1}{2}$, $e_v = 0$ for any σ and the material is isochoric. Furthermore σ_p is the only null eigenvector of $\mathbf{C}$. Consequently $\gamma_p = \gamma_1 = 0$ and $\mathbf{C}$ has no negative eigenvalues. The definitions of rigidtropic, incompressible and isochoric behavior coalesce for this model.

8. Conclusion

It remains to pin down the label "incompressible." In continuum mechanics this term means that the stress is determined by the deformation history only up to a hydrostatic pressure or "extra stress" p [4, Sec. 30]. This is equivalent to what we call here the hydroisochoric model, which as previously shown for semistable materials merges with the isochoric model. Restricting attention to the semistable

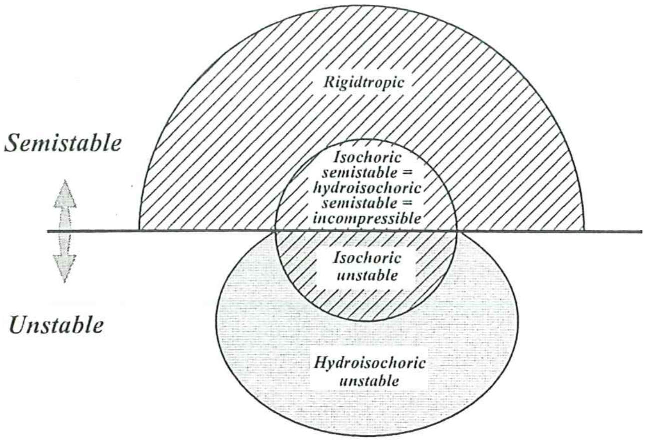

Figure 2. Sketch of inclusions between rigidtropic, isochoric and hydroisochoric models. The crosshatched area marks a singular **C** matrix.

case, the model nesting is:

$$\text{Isotropic semistable} \equiv \text{Hydroisochoric semistable} \equiv \text{Incompressible} \in \text{Rigidtropic}. \tag{15}$$

These and related model inclusions are sketched in Figure 2. From a mathematical standpoint, the splitting techniques used for the rigidtropic model by Felippa and Oñate [3] apply equally to isochoric behavior, and no special distinction for the incompressible case needs to be made.

We do not consider here the comparatively rare case of a multiple deficient **C** possessing two or more zero eigenvalues. For those the analysis is complicated by the appearance of a multidimensional null space. Such "multi-rigidtropic" models require separate treatment.

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