Volumetric Constraint Models for Anisotropic Elastic Solids

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Summary

We study three “incompressibility favors” of linearly-elastic anisotropic solids that exhibit volumetric constraints: isochoric, hydroisochoric and rigidotropic. An isochoric material deforms without volume change under any stress system. An hydroisochoric material does not change volume under hydrostatic stress. A rigidotropic material undergoes zero deformations under a certain stress pattern. Whereas the three models coalesce for isotropic materials, important differences appear for anisotropic behavior. We find that isochoric and hydroisochoric models under certain conditions may be hampered by unstable physical behavior. Rigidotropic models can represent semistable physical materials of arbitrary anisotropy while including isochoric and hydroisochoric behavior as special cases.

Keywords: linear elasticity, solids anisotropy, isotropy, rigidity, incompressibility, isochoric, hydroisochoric, volumetric constraints, stability, material, constitutive model, compliance.

1. Introduction

An incompressible linearly-elastic anisotropic solid does not deform under hydrostatic stress. It does not change volume under pressure. Since deviatoric and volumetric deformations uncouple, no volume change occurs under any stress state. The three volumetric constraints just stated coalesce, and it is sufficient to qualify the material as incompressible.

A more useful study is necessary for anisotropic materials. In the present Note we examine three volume constraint models for a linearly elastic anisotropic solid. The following definitions are used for that examination.

A material is called rigidotropic if it does not deform (i.e., experiences zero strain) under a specific stress pattern, which is a null eigenvector of the strain-stress (compliance) matrix. The term “rigidotropic” is used in the sense of “rigidity in a certain way” as defined by that eigenvector.

A material is called isochoric if it does not change volume under any applied stress system [1, Sec. 77]. Alternatively: the volumetric strain is zero under any stress state.

A material is called hydroisochoric if it is isochoric under hydrostatic stress. Isochoric materials are hydroisochoric but the converse is not necessarily true.
As noted the three models coalesce for an isotropic material. For an arbitrary anisotropic solid, however, it will be shown that imposing a isochoric or hydroisochoric constraint may produce a compliance matrix that has at least one negative eigenvalue. This means that under some stress system the material is able to create energy, contradicting the laws of thermodynamics. Such model cannot represent a physically stable material. On the other hand, for rigidtropic behavior it is easier to control material stability for any type of anisotropy because constraints are posed directly on the spectral form.

2. Compliance Relations

We consider a linearly-elastic anisotropic solid in three dimensions referred to axes \( \{x_i\} \). Stresses \( \sigma_{ij} \) and strains \( e_{ij} \) will be arranged as 6-component column vectors constructed from the respective tensors through the usual conventions of structural mechanics:

\[
\sigma = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}^T, \\
e = \begin{bmatrix}
e_{11} & e_{22} & e_{33} & 2e_{12} & 2e_{23} & 2e_{33}
\end{bmatrix}^T.
\]

(1)

The strain-stress constitutive equations in matrix notation are

\[
e = C \sigma,
\]

(2)

Here \( C_{ij} \) are compliance coefficients arranged into the symmetric compliance matrix \( C \). All diagonal entries \( C_{ii} \) are assumed to be nonnegative with a positive sum. The matrix \( C \) is called stable, semistable or unstable if \( C \) is positive definite, positive semidefinite, or indefinite, respectively. In the semistable case it will be assumed that \( C \) is nondegenerate and the rank deficiency of at most one to simplify the analysis.

The eigenvalues of \( C \) are \( \gamma_i \) for \( i = 1, 2, \ldots, 6 \), with \( v_i \) being the corresponding eigenvector normalized to length \( \sqrt{3} \). (This nonstandard normalization simplifies linking up to the hydrostatic stress vector in Section 4ff.) Accordingly the spectral decomposition is

\[
C = \frac{1}{3} \sum_{i=1}^{6} \gamma_i v_i v_i^T, \\
v_i^T v_j = 3 \delta_{ij},
\]

(3)

where \( \delta_{ij} \) is the Kronecker delta. The smallest one and \( \gamma_{max} \) the maximum. For stable or semistable models, \( \gamma_i \geq 0 \) and \( \gamma_j > 0 \) for \( j = 2, \ldots, 6 \).

If \( \gamma_1 = 0 \) the material is rigidtropic according to the definition given in the Introduction, with \( v_1 \) defining the corresponding stress pattern. Volumetric strain is \( e_v = e_{11} + e_{22} + e_{33} \). Isochoric behavior is mathematically characterized by \( e_v = 0 \) under any \( \sigma \). Hydroisochoric behavior means that \( e_v = 0 \) under any \( p \). These constraints are mathematically expressed in terms of \( C \) as follows:

Rigidtropic: \( \gamma_1 = 0, \quad \gamma_i > 0, \quad i = 2, \ldots, 6 \).

Hydroisochoric: \( C_{11} + C_{22} + C_{33} + 2C_{12} + 2C_{13} + 2C_{23} = 0 \).

Isochoric: \( C_{1j} + C_{2j} + C_{3j} = 0, \quad j = 12, 3 \).

(4)
Diagonal compliances are often known reliably from extensional and torsion tests. Off diagonal entries are typically less amenable to accurate measurement. Volumetric constraints, for example on volume change, are checked with triaxial tests. In any case, such constraints may be satisfied only approximately. Reference [2] discusses projection and scaling techniques for finding a “reference model” that satisfies constraints accurately.

3. Examples

The following examples of compliance matrices pertain to an orthotropic material with the \( \{ e_i \} \) aligned with the principal material axes. The diagonal entries are kept the same. The three nonzero off-diagonal entries are adjusted to meet the definitions (4).

Rigid-rotic:

\[
\begin{bmatrix}
  1 & -3/8 & 3/16 \\
  -3/8 & 1/48 & 0 \\
  3/16 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix} = \frac{1}{144}
\begin{bmatrix}
  144 & -54 & -27 & 0 & 0 & 0 \\
  -54 & 36 & -3 & 0 & 0 & 0 \\
  -27 & -3 & 16 & 0 & 0 & 0 \\
  0 & 0 & 0 & 288 & 0 & 0 \\
  0 & 0 & 0 & 0 & 720 & 0 \\
  0 & 0 & 0 & 0 & 0 & 432
\end{bmatrix}
\]

Eigenvalues: \([5, 3, 2.180074, 0, 0.180074, 0]\). The compliance matrix is semistable. The null eigenvector defining the rigid mode is \(v_1 = \sqrt{54/35} [1/2 \ 5/6 \ 1 \ 0 \ 0 \ 0]^T\).

Hydroisochoric:

\[
\begin{bmatrix}
  1 & -11/27 & 95/432 \\
  -11/27 & 1/4 & 23/432 \\
  95/432 & 23/432 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix} = \frac{1}{432}
\begin{bmatrix}
  432 & -176 & -95 & 0 & 0 & 0 \\
  -176 & 108 & -23 & 0 & 0 & 0 \\
  -95 & -23 & 48 & 0 & 0 & 0 \\
  0 & 0 & 0 & 576 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1440 & 0 \\
  0 & 0 & 0 & 0 & 0 & 864
\end{bmatrix}
\]

Eigenvalues: \([5, 3, 2, 1.208659, 0.211580, -0.059158]\). The compliance matrix is unstable.

Isochoric:

\[
\begin{bmatrix}
  1 & -41/72 & 31/72 \\
  -41/72 & 1/4 & 23/72 \\
  31/72 & 23/72 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix} = \frac{1}{144}
\begin{bmatrix}
  144 & -82 & -62 & 0 & 0 & 0 \\
  -82 & 36 & 46 & 0 & 0 & 0 \\
  -62 & 46 & 16 & 0 & 0 & 0 \\
  0 & 0 & 0 & 288 & 0 & 0 \\
  0 & 0 & 0 & 0 & 720 & 0 \\
  0 & 0 & 0 & 0 & 0 & 432
\end{bmatrix}
\]

Eigenvalues: \([5, 3, 2, 1.508769, 0, -0.147669]\). The compliance matrix is unstable.
4. Hydroisochoric Model

Assume that the material modeled by (2) is hydroisochoric. Consequently

\[
\begin{align*}
\sigma_p' &= \begin{bmatrix}
\frac{1}{2}p(C_{11} + C_{12} + C_{13}) & p(C_{12} + C_{22} + C_{33}) & p(C_{13} + C_{23} + C_{33}) \\
C_{12} & C_{12} & C_{13} \\
C_{13} & C_{23} & C_{33} \\
C_{14} & C_{44} & C_{44} \\
C_{55} & C_{55} & C_{55} \\
C_{66} & C_{66} & C_{66}
\end{bmatrix} \begin{bmatrix}
\rho \\
p \\
p \\
0 \\
0 \\
0
\end{bmatrix} &= \begin{bmatrix}
e_{11} \\
e_{22} \\
e_{33} \\
\gamma_{x12} \\
2e_{33} \\
2e_{31}
\end{bmatrix},
\end{align*}
\]

(8)

with \( e_v = e_{11} + e_{22} + e_{33} = p(C_{11} + C_{22} + C_{33} + 2C_{12} + 2C_{13} + 2C_{23}) = 0. \)

(The value of the shear strains is of no interest.) The complementary energy density produced by \( \sigma_p \) is

\[
\mathcal{U}_p' = \frac{1}{2} \sigma_p \cdot \sigma_p = \frac{1}{2} p(e_{11} + e_{22} + e_{33}) = \frac{1}{2} pe_v = 0. \]

(9)

But \( \gamma_p = \mathcal{U}_p'/(\rho c) = \mathcal{U}_p'/(3p) = 0 \) is the Rayleigh quotient of \( \sigma_p \) with \( C \). According to the Courant-Fischer theorem (2), \( \gamma_p \) must lie in the closed interval \([\gamma_{\text{min}}, \gamma_{\text{max}}] \):

\[
\gamma_{\text{min}} \leq \gamma_p \leq 0 \leq \gamma_{\text{max}} \tag{10}
\]

If \( \gamma_p \) is not an eigenvector of \( C \): \( \gamma_{\text{min}} \neq 0 \), the leftmost equality in (10) is not possible. Consequently

\[
\gamma_{\text{min}} < 0, \tag{11}
\]

and the model is unstable.

If \( \sigma_p = 0 \) the sum of the first three columns (or rows) of \( C \) must vanish. The hydroisochoric model then coalesces with the isochoric case, which is analyzed next.

5. Isochoric Model

The model is isochoric if the sum of the first three rows (or columns) of \( C \) is the null 6-vector. Equivalently \( \sigma_p \) is a null eigenvector of \( C \). The Rayleigh quotient test (10) does not offer sufficient information on stability and a deeper look at \( \gamma_p \) is required. Nonetheless a sufficient criterion for instability can be derived by considering the upper 3 principal minor \( \hat{C} \). From the last of (4), \( \hat{C} \) must have the form:

\[
\hat{C} = \begin{bmatrix}
C_{11} & \frac{1}{2}(C_{33} - C_{11} - C_{22}) & \frac{1}{2}(C_{22} - C_{11} - C_{33}) \\
C_{12} & C_{22} & C_{23} \\
\text{symm} & \text{symm} & \text{symm}
\end{bmatrix} \tag{12}
\]

This matrix is singular. Taking \( \alpha = C_{11}/C_{22} \) and \( \beta = C_{11}/C_{33} \) for convenience, an eigenvalue analysis shows that \( \hat{C} \) is indefinite if

\[
1 + \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) < 1 + \left( \frac{1}{\alpha} - \frac{1}{\beta} \right)^2,
\]

(13)

and is positive semidefinite if \( \alpha \) is changed to \( \beta \). If \( \hat{C} \) is indefinite, so is \( C \) and the model is unstable. If \( \hat{C} \) is semidefinite, an eigenvalue analysis of the complete \( C \) is required to decide on stability. The stability conditions of \( \hat{C} \) are shown in Figure 1, where "potentially semistable" indicates that confirmation is required by a analysis of the full \( C \) is required. An exception is an orthotropic material referred to the principal material axes, in which case no further tests are necessary if \( C_{44}, C_{55} \) and \( C_{66} \) are positive.

Figure 1 illustrates that a wide range of diagonal compliances in \( \hat{C} \) is detrimental to stability. For example if \( \alpha = \beta \), instability is guaranteed to happen for \( \alpha > 4 \).
6. Rigidotropic Mode

If $C$ is nonnegative with $\gamma_1 = \beta_1$, and $w = v_1$ is the only null eigenvector the material is rigidotropic under that stress mode. For an isotropic material $w = [1 1 1 0 0 0]^T = \sigma_p$, the hydrostatic stress mode $w$ generally will contain shear stresses. Introducing effective pressure as $p = \frac{1}{2} w^T \sigma$ and effective volumetric strain as $e_v = w^T \sigma$, the volumetric and deviatoric energies can be uncoupled [3].

If the rigid stress mode is $\sigma_p$, rigidotropic reduces to isochoric. This inclusion is pictured in Figure 2.

7. Isotropic Material

If the solid is isotropic with elastic modulus $E > 0$ and Poisson’s ratio $\nu$,

$$
C = \frac{1}{E} \begin{bmatrix}
-\nu & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
2(1+\nu) & \frac{1}{2} & 0 & 2(1+\nu) \\
2(1+\nu) & \frac{1}{2} & 0 & 2(1+\nu) \\
\end{bmatrix} \text{ symm}
$$

(14)

Under hydrostatic stress $\sigma_p$, $\sigma_p = 3(1 - 2\nu)p/E$, which vanishes for $\nu = \frac{1}{2}$. It is easy to verify that if $\nu = \frac{1}{2}$, $\gamma_1 = 0$ for any $\sigma$ and the material is isochoric. Furthermore $\sigma_p$ is the only null eigenvector of $C$. Consequently $\gamma_1 = \gamma_2 = 0$ and $C$ has no negative eigenvalues. The definitions of rigidotropic, incompressible and isochoric behavior coalesce for this model.

8. Conclusion

It remains to pin down the label “incompressible.” In continuum mechanics this term means that the stress is determined by the deformation history only up to a hydrostatic pressure or “extra stress” $p$ [4, Sec. 30]. This is equivalent to what we call here the hydrosochoric model, which as previously shown for semistable material energies with the isochoric model. Restricting attention to the semistable
case, the model nesting is:

\[ \text{Isotropic semistable} = \text{Isochoric semistable} = \text{Incompressible} \subseteq \text{Rigidotropic}. \]  \hspace{1cm} (15)

These and related model inclusions are sketched in Figure 2. From a mathematical standpoint, the splitting techniques used for the rigidotropic model by Felippa and Oñate [3] apply equally to isochoric behavior, and no special distinction is made for the incompressible case needs to be made.

We do not consider here the comparatively rare case of a multiple deficient C possessing two or more zero eigenvalues. For these the analysis is complicated by the appearance of a multidimensional null space. Such "multi-rigidotropic" models require separate treatment.

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References


