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Goal-oriented adaptivity for shell structures

Error assessment and remeshing criteria

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ABSTRACT. *The reliable computation of shell structures requires a tool to assess and control the quality of the finite element solution. For practical purposes, the quality of the numerical solution must be measured using a quantity of engineering interest rather than in the standard energy norm. However, the assessment of the error in an output of interest is based on a standard energy norm error estimator. The standard error estimator has to be applied to both the original problem (primal) and a dual problem related with the selected engineering quantity. In shells with assumed-strain models, the combination of primal and dual error estimation is performed differently than in the continuum mechanics case. Moreover, a part from the goal-oriented error estimator, the adaptive process requires a remeshing criterion. This work introduces an specific remeshing criterion for goal-oriented adaptivity and its particularization to the context of shell elements.*

RÉSUMÉ. *Pour mettre en oeuvre des calculs fiables pour les structures coques, il est nécessaire d'évaluer la qualité de la solution élément finis. Pour des raisons pratiques, la qualité de la solution numérique doit être mesurée à partir d'une grandeur qui intéresse l'ingénieur, une grandeur d'intérêt (*output of interest*). La mesure standard, en norme énergétique, n'étant pas satisfaisante. Néanmoins, l'analyse de l'erreur dans la grandeur d'intérêt est basée sur un estimateur d'erreur de la norme énergétique standard qui doit être appliqué à la fois sur le problème original (primal) et un problème dual défini à partir de la grandeur d'intérêt. Dans les modèles de coques qui utilisent la technique des déformations prédéterminées (*assumed-strains*), la combinaison des estimations d'erreur primale et duale est différente des cas standards. Le reste du travail est consacré à introduire un critère de remaillage pour l'adaptivité orienté au résultat, c'est-à-dire une formule permettant de traduire l'erreur estimée localement en la taille d'élément souhaitée dans le nouveau maillage du procès adaptatif. Le critère est introduit de façon générale et particularisé aux problèmes de coques.*

KEYWORDS: *goal-oriented adaptivity, remeshing criteria, shells, outputs of interest.*

MOTS-CLÉS : *adaptivité orientée à l'objectif, critères de remaillage, coques, grandeurs d'intérêt.*

1. Introduction

Shell models are extensively used in the engineering practice for structural analysis. It is therefore extremely important to assess and control the quality of the numerical computation of shell structures. The previous work on error assessment techniques for shells [CIR 98, DÍE 00, DÍE , LEE 99, LAC 02, HAN 00] focus on the evaluation of energetic error quantities. An additional difficulty must be accounted for because, for practical purposes, the quality of the numerical solution has to be measured using a quantity of engineering interest rather than the standard energy norm.

The usual approach to estimate the error in a quantity of interest is to solve a dual problem. The dual problem describes the influence of every zone of the domain in the specific output of the solution. In fact, the solution of the dual problem contains information on the pollution error that affects the quantity of interest. The dual problem has the same structure as the original problem (primal) but with a different right-hand-side term, related with the considered output. Then, the error in the quantity of interest is assessed combining the energy norm of the errors in the primal and the dual problems [PAR 97, PRU 99].

2. Error in outputs of interest

The most standard technique to assess the error in outputs of interests introduces a dual problem and combines the errors in energy associated with the primal and dual problems. The essential concepts of this approach are briefly revisited in this section.

2.1. *Primal and dual problems*

We use, for the sake of a simple presentation, the linear mechanical problem. The strong form of the mechanical equilibrium equation is:

$$-\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(u) = f \qquad \text{in } \Omega \tag{1a}$$

$$\boldsymbol{\sigma}(u) \cdot n = t \qquad \text{on } \Gamma_N \tag{1b}$$

$$u = u_D \qquad \text{on } \Gamma_D \tag{1c}$$

In the following, the problem described in its strong form by Equations (1) is denoted as the *primal* problem.

The weak form of the primal problem is stated as follows: find $u \in \mathcal{S}$ such that

$$a\big(u, v\big) = l\big(v\big), \text{ for all } v \in \mathcal{V}, \tag{2}$$

where $a\big(\cdot,\cdot\big)$ is the standard bilinear form associated with the internal energy,

$$a\big(u, v\big) := \int_\Omega \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(v)\, d\Omega, \tag{3}$$

$l(\cdot)$ is a linear form representing the energy of the external loads,

$$l(v) := \int_\Omega v \cdot f \, d\Omega + \int_{\Gamma_N} v \cdot t \, d\Gamma, \tag{4}$$

$\mathcal{S}$ is an affine functional space verifying the Dirichlet boundary conditions (1c), i.e. $\mathcal{S}$ contains the solution, and $\mathcal{V}$ is is a functional space verifying the homogeneous Dirichlet boundary conditions (test functions).

The finite element approximation u^H to u is taken in a finite dimensional space $\mathcal{S}^H \subset \mathcal{S}$ such that

$$a(u^H, v) = l(v), \text{ for all } v \in \mathcal{V}^H \subset \mathcal{V}. \tag{5}$$

The goal of this analysis is to assess the error of a linear functional of the solution. Let $J(u)$ be a magnitude of engineering interest of the solution. The functional $J(\cdot)$ is assumed to be linear. Thus, the quantity to assess is the output of the error, $e := u - u^H$, that is, $J(e) = J(u) - J(u^H)$. In the standard applications $J(u)$ may be the value of the displacements or the stresses at some points. Other possible definitions for $J(u)$ are averaged displacements or stresses in parts of the domain where the solution is interesting for the structural analyst.

In order to assess this quantity, a new problem is introduced, in which the output $J(\cdot)$ is the right-hand-side term of the weak form. This problem is denoted *dual* problem and it is stated as follows. Find $\varphi \in \mathcal{V}$ such that

$$a(v, \varphi) = J(v), \text{ for all } v \in \mathcal{V}. \tag{6}$$

The solution of the dual problem, φ, describes how the residue in every part of the domain affects the error in the output. The function φ accounts for the influence in the local magnitude $J(e)$ of any perturbation in the solution, even if located at distant zones. The function φ is often denoted *extractor* and contains useful information to study the effects of *pollution* on the output. The dual problem is also solved with the mesh of characteristic size H. An approximation to φ, $\varphi^H \in \mathcal{V}^H$ is obtained such that

$$a(v, \varphi^H) = J(v), \text{ for all } v \in \mathcal{V}^H. \tag{7}$$

The error in the approximation of the dual problem is denoted by $\varepsilon := \varphi - \varphi^H$.

2.2. *Residual error equations*

The errors of the primal and dual problems, e and ε respectively, are the solution of the following residual equations:

$$a(e, v) = l(v) - a(u^H, v) =: R^P(v), \text{ for all } v \in \mathcal{V} \tag{8}$$

$$a(v, \varepsilon) = J(v) - a(v, \varphi^H) =: R^D(v), \text{ for all } v \in \mathcal{V} \tag{9}$$

where the primal and dual residues, $R^P(\cdot)$ and $R^D(\cdot)$, have been introduced.

In order to assess the error in energy norm, the residual equations (8) and (9) are solved approximately. For all practical purposes, the error estimators may be analyzed considering a reference error associated with much finer mesh, a "truth mesh". Let be $\mathcal{S}^h$ and $\mathcal{V}^h$ the interpolation and test spaces associated with a reference mesh of characteristic element size h ($h \ll H$). Thus, the solutions u^h and φ^h of the primal and dual problems are much more accurate than u^H and φ^H and the corresponding reference errors are fair approximations of the exact errors, that is $e \approx e^h := u^h - u^H$ and $\varepsilon \approx \varepsilon^h := \varphi^h - \varphi^H$. The equations for the reference errors are the following:

$$a\big(e^h, v\big) = R^P\big(v\big), \text{ for all } v \in \mathcal{V}^h \tag{10}$$

$$a\big(v, \varepsilon^h\big) = R^D\big(v\big), \text{ for all } v \in \mathcal{V}^h. \tag{11}$$

The error estimator procedure obtains a proper approximation to the reference error solving only local problems, that is local restrictions of (10) and (11).

2.3. *Representations of the error in the quantity of interest*

Recall that our goal is to assess the reference error in the quantity of interest, $J\big(e^h\big)$. In order to drive the adaptive process, the contribution from every element Ω_k of the mesh to $J\big(e^h\big)$ is also required. That is, we need a representation of $J\big(e^h\big)$ as a sum of elementary contributions. Moreover, each of the local contributions to $J\big(e^h\big)$ must be a function of the (local) energy norm of e^h and ε^h, which are the magnitudes we are able to evaluate with standard error estimators.

The following expression is derived replacing $v = e^h$ in (11):

$$J\big(e^h\big) = a\big(e^h, \varepsilon^h\big) + a\big(e^h, \varphi^H\big) = a\big(e^h, \varphi^h\big). \tag{12}$$

Using Galerkin orthogonality, the previous equation results in

$$J\big(e^h\big) = a\big(e^h, \varepsilon^h\big). \tag{13}$$

Equations (12) and (13) allow to identify the contributions to the error in the quantity of interest, $J\big(e^h\big)$, of every element of the mesh of characteristic size H. Let $a_{\Omega_k}\big(\cdot,\cdot\big)$ be the restriction of $a\big(\cdot,\cdot\big)$ to element Ω_k of the mesh. Then, $a\big(\cdot,\cdot\big) = \sum_k a_{\Omega_k}\big(\cdot,\cdot\big)$ and, consequently

$$J\big(e^h\big) = \sum_k a_{\Omega_k}\big(e^h, \varepsilon^h\big) = \frac{1}{4}\|e^h + \varepsilon^h\|_{\Omega_k} - \frac{1}{4}\|e^h - \varepsilon^h\|_{\Omega_k}. \tag{14}$$

The aim of goal oriented adaptivity is to design a mesh such that the local magnitudes $a_{\Omega_k}\big(e^h, \varepsilon^h\big)$ are small enough to keep $J\big(e^h\big)$ under a prescribed tolerance.

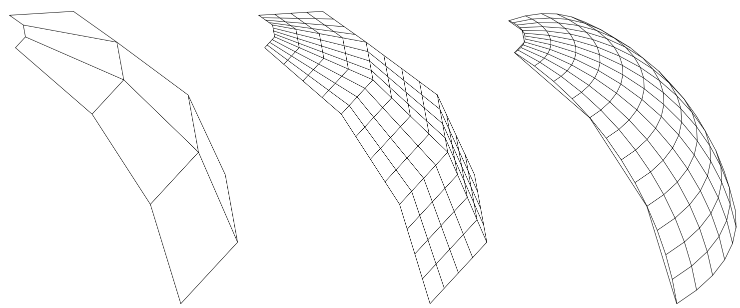

Figure 1. *Computational H-mesh (left), refined mesh (h) build up without any additional geometry data (center, obviously a bad option) and refined mesh (h) build up using real geometry description (right)*

3. Error assessment in the shells context

Basically, the additional difficulties associated with the use of a Reissner-Midlin shell formulation come from 1) the geometrical approximation of the curved geometry of the shell and 2) the assumed-strain model. In [DÍE 00, DÍE] a residual type error estimator is presented that overcomes these two difficulties. This estimator is a modification of the estimator introduced in [DÍE 98] and it is used to assess the error measured with the energy norm. Two relevant features of this energy norm estimator in the shell context are recalled (a general outline is given in [DÍE 00]):

– The reference discretization must be adapted to the real (curved) geometry of the mesh. The CAD information describing the geometry of the mesh is used in the error estimation strategy in a very simple manner. It suffices to locate the nodes of the reference mesh (or h-mesh) in the proper positions, see figure 1 for an illustration.

– The transfer of the solution from the computational mesh (or H-mesh) to the reference h-mesh has to be done carefully. The stresses associated with the solution u^H must be transferred instead of the displacements. This is due to the assumed-strain model: a direct interpolation of the generalized displacements would not preserve the physical quantities (e.g. energy). This is discussed in more detail in the subsections 3.1 and 3.2.

Additional difficulties are encountered when this energy norm estimator is used to assess the error in some quantity of interest. The main difficulty is that the standard representation of the error in the quantity of interest does not stand in the shells context. A new error representation is derived in the subsection 3.3. That also holds for the assumed strain shell models. In subsection 3.4, the assessment of the error in the quantity of interest for shells is discussed.

3.1. *Assumed-strain models*

The assumed-strain model is used in order to avoid shear and membrane locking [AYA 98, BAT 85, DON 87, DVO 84, HUG 87, LEE 99]. It introduces a correction in the strains and, hence, in the stresses at the element level. The correction suppresses the polynomial terms of higher degree in the expression of shear and membrane strains. Thus, these terms of the strain tensor are not derived directly from the displacement by the usual kinematic relation. In some sense, the strain operator, mapping the displacement vector into the strain tensor, depends intrinsically on the mesh. Obviously, the same remark stands for the stresses. Thus, instead of writing $\boldsymbol{\epsilon}(u^H)$ to design the strain tensor associated with the approximate solution u^H, in the shell context the notation $\boldsymbol{\epsilon}^H(u^H)$ is preferred (viz. $\boldsymbol{\sigma}^H(u^H)$).

Consequently, the bilinear form $a(\cdot,\cdot)$ depends also intrinsically on the mesh and the notation $a_H(\cdot,\cdot)$ is introduced to denote

$$a_H(u,v) := \int_\Omega \boldsymbol{\sigma}^H(u) : \boldsymbol{\epsilon}^H(v)\, d\Omega. \tag{15}$$

Note that the bilinear form associated with the reference mesh of characteristic size h, $a_h(\cdot,\cdot)$, is defined in the same fashion and now $a_H(\cdot,\cdot) \neq a_h(\cdot,\cdot)$.

Thus, the equations giving the solutions of the primal and dual problems, associated with the H and h meshes are

$$a_H(u^H,v) = l(v), \text{ for all } v \in \mathcal{V}^H \tag{16a}$$

$$a_h(u^h,v) = l(v), \text{ for all } v \in \mathcal{V}^h \tag{16b}$$

$$a_H(v,\varphi^H) = J(v), \text{ for all } v \in \mathcal{V}^H \text{ and, finally,} \tag{16c}$$

$$a_h(v,\varphi^h) = J(v), \text{ for all } v \in \mathcal{V}^h. \tag{16d}$$

Obviously, the equations giving u^h and φ^h, (16b) and (16d), cannot be solved globally due to their prohibitive computational cost. The error estimator procedure solves local, usually element by element, restrictions of (16b) and (16d) to obtain the error estimates, that is approximations to e^h and ε^h. The error estimator used here is described in [DÍE 00, DÍE].

3.2. *Proper transfer from mesh H to mesh h*

Once the approximate solutions, u^H and φ^H, are computed, the goal is to obtain fair approximations of the reference errors, $e^h := u^h - u^H$ and $\varepsilon^h := \varphi^h - \varphi^H$. In the shells context, a proper definition of the errors e^h and ε^h requires to transfer the approximate solutions u^H and φ^H to the fine h-mesh. For instance, in order to evaluate the error norm $\|e^h\|$, that is $\sqrt{a_h(u^h - u^H, u^h - u^H)}$, the solution u^H must be transferred from mesh H to mesh h.

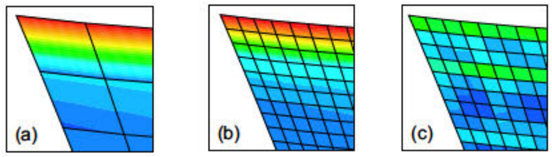

Figure 2. *Detail of Von Mises stress distribution for the example 2 (subsection 5.2). Von Mises stress distribution in the computational H-mesh,* $\boldsymbol{\sigma}^H(u^H)$, ***(a)****. Von Mises stress distribution in the refined h-mesh, corresponding to* $\left[\boldsymbol{\sigma}^H(u^H)\right]_h$, ***(b)****, and corresponding to* $\boldsymbol{\sigma}^h(\left[u^H\right]_h)$, ***(c)****. In* ***(c)*** *both the shape of the distribution and the values are different: the values of the color scale used in* ***(c)*** *are 10 times larger than in* ***(a)*** *and* ***(b)***

The use of an assumed-strain model induces an additional problem. Even if the reference solutions u^h and φ^h were computed, the proper definition of e^h and ε^h is non-trivial. The first idea is to assume that e^h is in $\mathcal{V}^h$ and, then, to transfer u^H from $\mathcal{S}^H$ to $\mathcal{S}^h$. Thus, the interpolated displacements, denoted by $\left[u^H\right]_h$, would allow to compute $e^h = u^h - \left[u^H\right]_h$. However, the interpolated displacements $\left[u^H\right]_h$ do not represent the physical features of u^H. For instance, in the cooling tower example presented in the subsection 5.2, using the mesh 0, the global energy computed from $\left[u^H\right]_h$ is 0.3, which is more than 6.6 times the energy computed from u^H (equal to 0.045). This difference is of the same order in all the adapted meshes. This is due to the assumed-strain approach. It has already been noticed that the strain and stress operators are intrinsically related with the mesh. Thus, the stresses associated with u^H are denoted by $\boldsymbol{\sigma}^H(u^H)$ and the stresses associated with $\left[u^H\right]_h$ are denoted by $\boldsymbol{\sigma}^h(\left[u^H\right]_h)$. The stress fields $\boldsymbol{\sigma}^H(u^H)$ and $\boldsymbol{\sigma}^h(\left[u^H\right]_h)$ are very different, both in value and distribution, see figure 2.

A proper transfer procedure in this context must account for the particularities of the assumed-strain model and preserve the mechanical properties of the solution. The solution adopted in [DÍE 00, DÍE] is to directly transfer the stresses, $\boldsymbol{\sigma}^H(u^H)$ and obtain then $\left[\boldsymbol{\sigma}^H(u^H)\right]_h$. The right transfer strategy is illustrated in figure 3. The same must be done for the dual problem.

This transfer strategy preserves the energetic properties of the solution u^H. This is due to the fact that the energy is computed directly from the stresses and strains and the proper transfer strategy is, in fact, a simple stress and strain interpolation. In particular, the energy computed from $\boldsymbol{\sigma}^H(u^H)$ and the energy computed from $\left[\boldsymbol{\sigma}^H(u^H)\right]_h$ are practically identical. As a consequence of the adopted transfer procedure, the error e^h cannot be expressed in terms of displacements. The only representation of the error that makes sense is in terms of stresses. The same stands for the solution of the dual problem, φ^H, and the corresponding error, ε^h.

In all the numerical tests, the adopted transfer procedure does preserve every energetic quantity. In the following, the integral operator in the fine mesh, $a_h(\cdot,\cdot)$ is

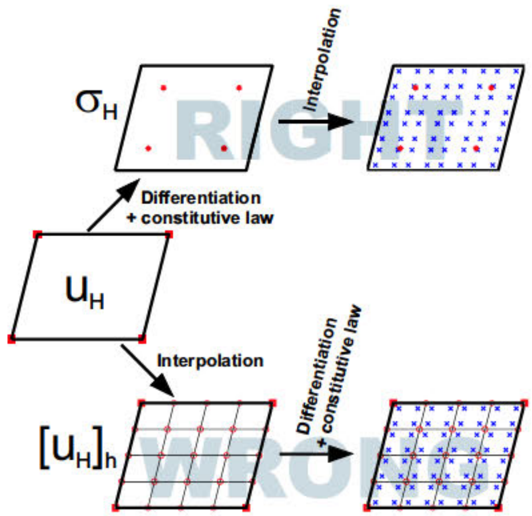

Figure 3. *Transfer of the solution from the computational (coarse, H) mesh to the reference (refined, h) mesh. The solution must be transferred to the truth mesh by interpolation of the generalized stresses (top). Stresses on the fine mesh cannot be properly computed from the interpolated displacements (bottom)*

also applied to the quantities expressed in the coarse H-mesh. It is assumed that the quantities are transferred from mesh H to mesh h using the above described strategy. For instance, the notation $a_h(u^H, u^H)$ is used and means

$$a_h(u^H, u^H) = \int_\Omega [\boldsymbol{\sigma}^H(u^H)]_h : [\boldsymbol{\epsilon}^H(u^H)]_h \, d\Omega$$

Using this notation, the following equalities are verified

$$a_h(u^H, u^H) = a_H(u^H, u^H), \tag{17a}$$

$$a_h(\varphi^H, \varphi^H) = a_H(\varphi^H, \varphi^H), \tag{17b}$$

$$\text{and} \quad a_h(u^H, \varphi^H) = a_H(u^H, \varphi^H) = J(u^H) = l(\varphi^H). \tag{17c}$$

These results are verified in the numerical tests and, therefore, they do demonstrate the efficiency of the transfer procedure.

Remark 1 *Note that the above introduced notation is used also to combine in $a_h(\cdot,\cdot)$ arguments from different functional spaces, namely $\mathcal{V}^H$ and $\mathcal{V}^h$. In this case $a_h(\cdot,\cdot)$ is acting over each argument in a different manner. For instance, if we write $a_h(u^h, \varphi^H)$ we must understand*

$$a_h(u^h, \varphi^H) = \int_\Omega \boldsymbol{\sigma}^h(u^h) : [\boldsymbol{\epsilon}^H(\varphi^H)]_h \, d\Omega.$$

3.3. ***Representation of the error in the context of shells***

In the shell context, due to the assumed-strain model, the Galerkin orthogonality does not stand anymore, that is for every $v \in \mathcal{V}^H$

$$a_h\big(e^h, \varphi^H\big) \neq 0 \ \text{ and } \ a_h\big(u^H, \varepsilon^h\big) \neq 0. \tag{18}$$

This is due to the fact that, in general, for $v \in \mathcal{V}^H$,

$$\begin{aligned} a_h\big(u^h, v\big) &= \int_\Omega \boldsymbol{\sigma}^h(u^h) : \big[\boldsymbol{\epsilon}^H(v)\big]_h \, d\Omega \\ &\neq \int_\Omega \boldsymbol{\sigma}^h(u^h) : \boldsymbol{\epsilon}^h(v) \, d\Omega = l\big(v\big), \end{aligned}$$

that is, for $v = \varphi^H$,

$$a_h\big(u^h, \varphi^H\big) \neq l\big(\varphi^H\big)$$

and

$$a_h\big(e^h, \varphi^H\big) = a_h\big(u^h, \varphi^H\big) - a_h\big(u^H, \varphi^H\big) = a_h\big(u^h, \varphi^H\big) - l\big(\varphi^H\big) \neq 0.$$

The same rationale is followed to derive $a_h\big(u^H, \varepsilon^h\big) \neq 0$.

Moreover, we have already mentioned that the error e^h can only be expressed in terms of stresses, never in displacements. Then, in the general case, the transferred stresses $\big[\boldsymbol{\sigma}^H(u^H)\big]_h$ do not derive from any displacement field in the fine h-mesh. In other words, it does not exist any $v \in \mathcal{V}^h$ such that $\boldsymbol{\sigma}^h(v) = \big[\boldsymbol{\sigma}^H(u^H)\big]_h$. Note that this is due mainly to the boundary conditions: the transferred stress field does not represent a kinematically admissible solution with the resolution of the fine mesh. A direct consequence of this is that the error e^h cannot replace the test function v in the weak residual equation (11).

Thus, the representation of the error in the output of interest, $J\big(e^h\big)$, given by equations (12) and (13) is therefore no longer valid in the context of shells. This is due to two factors: 1) Galerkin orthogonality does not stand and 2) the error e^h cannot be fairly introduced as a test function in the residual equation.

In the context of shells, an analogous but different representation of the error in the output of interest must be used. It follows from equations (17) that

$$a_h\big(e^h, \varepsilon^h\big) = a_h\big(u^h - u^H, \varphi^h - \varphi^H\big) = J\big(e^h\big) - a_h\big(e^h, \varphi^H\big) - a_h\big(u^H, \varepsilon^h\big). \tag{19}$$

That is,

$$J\big(e^h\big) = a_h\big(e^h, \varepsilon^h\big) + a_h\big(e^h, \varphi^H\big) + a_h\big(u^H, \varepsilon^h\big). \tag{20}$$

Equation (20) shows that, in the context of shells, the standard error representation used in the literature on goal oriented adaptivity, see equation (13), must be modified by adding the terms $a_h\big(e^h, \varphi^H\big)$ and $a_h\big(u^H, \varepsilon^h\big)$. Note that these terms vanish in the standard case due to Galerkin orthogonality and that equation (13) is therefore recovered. In the case of shells, the terms $a_h(e^h, \varphi^H)$ and $a_h(u^H, \varepsilon^h)$ are not negligible in front of $a_h(e^h, \varepsilon^h)$: the numerical tests show that in all the examples they are at least of the same order of magnitude.

3.4. *Error assessment in quantities of interest for shells*

In the standard case (continuum mechanics), the error assessment in energy norm of the primal and dual problems suffices to estimate the error in the quantity of interest. This is shown by equation (14), once the local approximation to e^h and ε^h (standard error estimates) are computed, an estimate for $J(e^h)$ follows easily.

In the shell context, in order to evaluate $J(e^h)$, the extra terms $a_h(e^h, \varphi^H)$ and $a_h(u^H, \varepsilon^h)$, must also be accounted for. The standard error estimates for e^h and ε^h are designed to evaluate norms, and hence the estimates for $\|e^h\|$ and $\|\varepsilon^h\|$ are reliable and accurate. These terms may be therefore bounded using the Schwarz inequality, that is

$$|a_h(e^h, \varphi^H)| \leq \|e^h\|\|\varphi^H\| \ \text{ and } \ |a_h(u^H, \varepsilon^h)| \leq \|u^H\|\|\varepsilon^h\| \qquad (21)$$

Unfortunately, the *angle* between e^h and φ^H and the angle between u^H and ε^h is almost straight and, consequently, the inequalities in equation (21) are not sharp. The values for the ratio $\|e^h\|\|\varphi^H\|/|a_h(e^h, \varphi^H)|$ obtained in the numerical examples are of the order of 100. That means that the orthogonality condition between e^h and φ^H is not rigorously fulfilled but that the angle is close to 90^o ($\arccos 0.01 = 89.4^o$). Thus, if the value of $a_h(e^h, \varphi^H)$ and $a_h(u^H, \varepsilon^h)$ is estimated using the quantities $\|e^h\|\|\varphi^H\|$ and $\|u^H\|\|\varepsilon^h\|$, the overestimation is extremely large. If these values were used in an adaptive process, the obtained meshes would be excessively refined.

When the error estimates are used as functions to directly compute $a_h(e^h, \varphi^H)$ and $a_h(u^H, \varepsilon^h)$, the results are very poor in the coarse meshes but quite accurate in the adapted meshes. In the coarse meshes, compared with the reference solution, the numerical tests yield effectivity indices from -1 (estimate of opposite sign with respect to the reference) up to 6 (estimate six time larger than the reference). In the adapted meshes, the quality of the assessment of theses error quantities is much better (effectivities from 80% to 170%).

As already noted, in all the studied examples we observe that the terms $a_h(e^h, \varphi^H)$ and $a_h(u^H, \varepsilon^h)$ in equation (20) are of the same order of the remainder term of the right-hand-side, $a_h(e^h, \varepsilon^h)$. This is also verified along the remeshing process: when $a_h(e^h, \varepsilon^h)$ is reduced, the term $a_h(e^h, \varphi^H) + a_h(u^H, \varepsilon^h)$ that completes $J(e^h)$ is reduced proportionally. Note that the analysis of the behavior of the term $a_h(e^h, \varphi^H) + a_h(u^H, \varepsilon^h)$ along a refining process (when H decreases) is not easy. In order to obtain a priori estimates, one has to account for the reduction of the errors e^h and ε^h with H, which is standard, but also for the dependence on H of the orthogonality defaults. The combination of the two effects yields likely an expression of $a_h(e^h, \varphi^H) + a_h(u^H, \varepsilon^h)$ as a function of H similar to the expression for $a_h(e^h, \varepsilon^h)$.

Thus, in the following, the remeshing strategy is derived assuming that the behavior of the error in the quantity of interest, $J(e^h)$, is the same as the behavior of $a_h(e^h, \varepsilon^h)$. That is, we assume that the term $a_h(e^h, \varphi^H) + a_h(u^H, \varepsilon^h)$ does not modify the dependence of $J(e^h)$ with respect to the mesh parameter H. The remeshing strategy is therefore the same that has to be used in the continuum mechanics case. In other words, it is assumed that it exists a constant factor that maps $J(e^h)$ into $a_h(e^h, \varepsilon^h)$. Of course, in the shells context, $a_h(e^h, \varepsilon^h)$ does not coincide with the error in the output of interest, $J(e^h)$. Thus, in the remeshing criterion derived in the next section, one has to replace $a_h(e^h, \varepsilon^h)$ by the complete expression of $J(e^h)$. The numerical evidence shows that this assumption is fair and that the adaptive process reduces and controls $J(e^h)$ efficiently.

4. Remeshing criterion

An important part of the adaptive loop is the definition of the new mesh from the error estimate. The remeshing criterion is an expression allowing to compute the desired element size as a function of the error in the previous mesh. The desired element size is then the input for a mesh generator. Two ingredients are needed in order to derive a remeshing criterion, 1) a priori error estimates and 2) an optimality criterion, see [DÍE 99] for details.

4.1. *A priori estimates*

The a priori error estimates are used to assess the convergence rates of the finite element approximations. They indicate the evolution of the error during a refinement process.

The usual expressions for the a priori error estimates are for the primal and dual problem

$$\|e\| \approx CH^p \text{ and } \|\varepsilon\| \approx C^\star H^p, \tag{22}$$

where H is the characteristic size of the mesh, p is the complete degree of interpolation of the finite elements and C and $C^\star$ are constants independent of H and p.

The local (element by element) counterparts of the previous equation are

$$\|e\|_{\Omega_k} \approx CH_k^{p+d/2} \text{ and } \|\varepsilon\|_{\Omega_k} \approx C^\star H_k^{p+d/2}, \tag{23}$$

where $\|\cdot\|_{\Omega_k}$ is the restriction of the energy norm to element Ω_k, H_k is the size of Ω_k and d is the dimension of the manifold in which Ω is included ($d = 2$ for 3D shells or standard plane problems). To derive equation (23) from (22) it is assumed that $\Omega_k \approx H_k^d$.

The same estimate is assumed to hold in the new mesh of the remeshing process, that is

$$\|\widetilde{e}\|_{\widetilde{\Omega}_k} \approx C\widetilde{H}_k^{p+d/2} \text{ and } \|\widetilde{\varepsilon}\|_{\widetilde{\Omega}_k} \approx C^{\star}\widetilde{H}_k^{p+d/2}, \tag{24}$$

where $\widetilde{\Omega}_k \approx \widetilde{H}_k^d$ and it is assumed that the element $\widetilde{\Omega}_k$ of the new mesh is included in Ω_k. The parameter $\widetilde{H}_k$ stands for the element size in the new mesh in the zone occupied by Ω_k, $\widetilde{e}$ and $\widetilde{\varepsilon}$ are the errors associated with the solutions of the primal and dual problems in the new mesh.

4.2. *Optimality criterion*

In order to derive a remeshing criterion the a priori error estimates given above are not sufficient and further hypothesis are required. The optimality criterion is an additional condition that prescribes some equidistribution of the error in the domain. Different optimality criteria can be defined based on different rationale [DÍE 99, LI 95b, LI 95a, OÑA 93].

The simplest hypothesis is to assume that, in the optimal mesh, the contribution to the error of every zone of the domain is uniform, that is

$$a_{\Omega_k}\left(\widetilde{e}, \widetilde{\varepsilon}\right) = \underbrace{J\left(\widetilde{e}\right)}_{\text{prescribed}} \frac{\Omega_k}{\Omega}. \tag{25}$$

This criterion is derived recalling the expression

$$J\left(\widetilde{e}\right) = \sum_k a_{\Omega_k}\left(\widetilde{e}, \widetilde{\varepsilon}\right), \tag{26}$$

which stands in the standard (continuum mechanics) case and is also used here in the adaptive procedure. The k-th term of the sum in the right-hand-side of (26) is imposed to be proportional to the measure (area) of Ω_k.

An additional assumption is used in the next developments. The scalar product $a_{\Omega_k}\left(e^h, \varepsilon^h\right)$ is assumed to verify

$$|a_{\Omega_k}\left(e^h, \varepsilon^h\right)| \approx \widetilde{C}\|e\|_{\Omega_k}\|\varepsilon\|_{\Omega_k} \approx \widetilde{C}CC^{\star}H_k^{2p+d}, \tag{27}$$

where $\widetilde{C} \leq 1$ is a constant accounting for the “cosine” of the angle between e^h and ε^h.

4.3. *Derivation of a remeshing criterion*

The remeshing criterion gives $\widetilde{H}_k$ as a function of H_k, and the corresponding local errors. The expression of the remeshing criterion is derived from the a priori estimates and the optimality criterion. The a priori estimates describe the evolution of the error as a function of the element size and the optimality criterion sets the local distribution of the error that has to be attained with the new mesh [DÍE 99].

Using the previous assumptions, after some algebra, the following remeshing criterion is obtained

$$\widetilde{H}_k^{2p} = \frac{a\big(e,\varepsilon\big)}{\Omega} H_k^{2p+d} \frac{1}{|a_{\Omega_k}\big(\widetilde{e},\widetilde{\varepsilon}\big)|}. \tag{28}$$

A slight variation based on the assumption introduced by equation (27) may also be used

$$\widetilde{H}_k^{2p} = \frac{\|e\|\|\varepsilon\|}{\Omega} H_k^{2p+d} \frac{1}{\|\widetilde{e}\|_{\Omega_k}\|\widetilde{\varepsilon}\|_{\Omega_k}}. \tag{29}$$

This expression is more consistant with the nature of the energy norm error estimator we are applying to the primal and dual problem: the estimates are assumed to properly approximate the error norms, $\|e\|\|\varepsilon\|$, and not necessarily the product $a\big(e,\varepsilon\big)$.

In the numerical test we use both the expressions given by equations (28) and (29). We refer as *Criterion 1* to the first and *Criterion 2* to the latter.

In the adaptive loop, the remeshing criterion plays an important role. Once the error is estimated (in this case the primal and dual errors), the mesh is designed following the element size distribution prescribed by the remeshing criterion. It is worth noting that in the examples shown in the next section, the use of the different criteria lead to different meshes. Even if the two expressions of equations (28) and (29) are very similar, the resulting meshes are quite different.

As already mentioned, the remeshing criteria must be adapted to the shells case by replacing $a\big(e,\varepsilon\big)$ by the complete expression for $J\big(e\big)$ given by equation (20).

5. Numerical examples

5.1. *Example 1: bending plane shell*

Let us consider the uniformly loaded plate of figure 4. The selected output of interest is the vertical displacement of the point located at the extreme of the center cross section. Consequently, the dual solution results of applying a concentrated vertical force at this point. The numerical solution is computed with a mesh of four-noded quadrilateral elements.

The distributions of error, for both the primal and dual problems, are depicted in figure 5. The quality of the error assessment in energy norm is analyzed in figures 5 and 6. Figure 5 shows that the estimated error map is very similar to the reference ("exact") error map. The values displayed in figure 5 correspond to the local contributions to the squared energy norms of the error ($\|e\|^2$ and $\|\varepsilon\|^2$). The use of squared norms leads to very small values (from 10^{-13} to 10^{-11} for the dual problem). However, in the dual problem (mesh 0), the global relative error is 5% (see table 1),

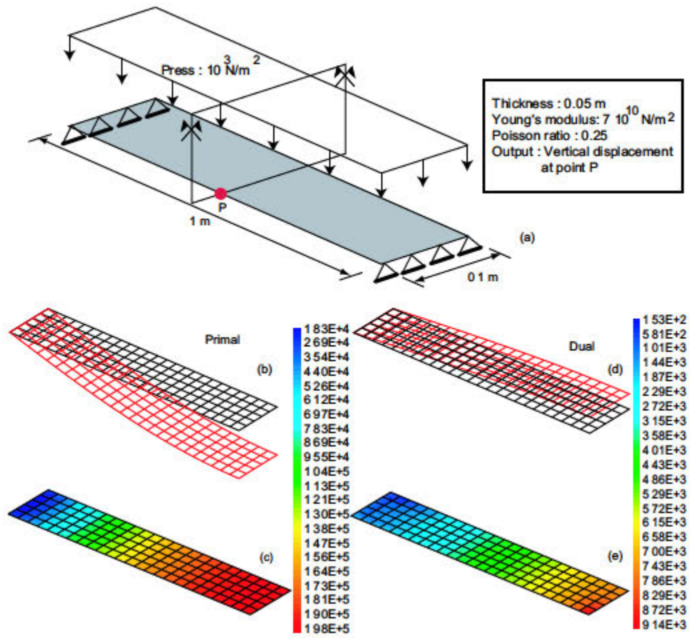

Figure 4. *Problem statement for example 1 (**a**). Deformed shape, (**b**) and (**d**), and Von Mises stress distribution, (**c**) and (**e**), for the primal problem (left) and the dual problem (right)*

much larger than the roundoff error of the computation. The good quality of the local error estimate is analyzed in figure 6, where the distributions of the local effectivity indices in both problems are shown. The effectivity index is found to be uniformly distributed in space with a large number of elements having the same local values (see histograms). As expected, the assessment of the error (both of the primal and dual problems) in energy norm is accurate.

In the goal oriented adaptive process, $a_{\Omega_k}(e^h, \varepsilon^h)$ and $\|e^h\|_{\Omega_k}\|\varepsilon^h\|_{\Omega_k}$ are the relevant local error quantities, because they are the input for the remeshing criteria 1 and 2 (described by (28) and (29) respectively). The effectivity on the assessment of these error quantities is described in figures 7 and 8. These figures show that the estimates for both $a_{\Omega_k}(e^h, \varepsilon^h)$ and $\|e^h\|_{\Omega_k}\|\varepsilon^h\|_{\Omega_k}$ are sharp. It is worth noting that the estimate for $a_{\Omega_k}(e^h, \varepsilon^h)$ is not anymore a lower bound while the estimate for $\|e^h\|_{\Omega_k}\|\varepsilon^h\|_{\Omega_k}$ obviously preserves the lower bound character of the original error estimate.

The distributions of $a_{\Omega_k}(e^h, \varepsilon^h)$ and $\|e^h\|_{\Omega_k}\|\varepsilon^h\|_{\Omega_k}$ are used in the remeshing criteria 1 and 2 to obtain two series of adapted meshes. The prescribed accuracy (relative error) in the output is set to 0.025% (2.5×10^{-4}). The meshes obtained with the two criteria are shown in figure 9. In order to evaluate the remeshing strategy, the error associated with every mesh in figure 9, $J(e^h)/J(u^h)$, is computed from

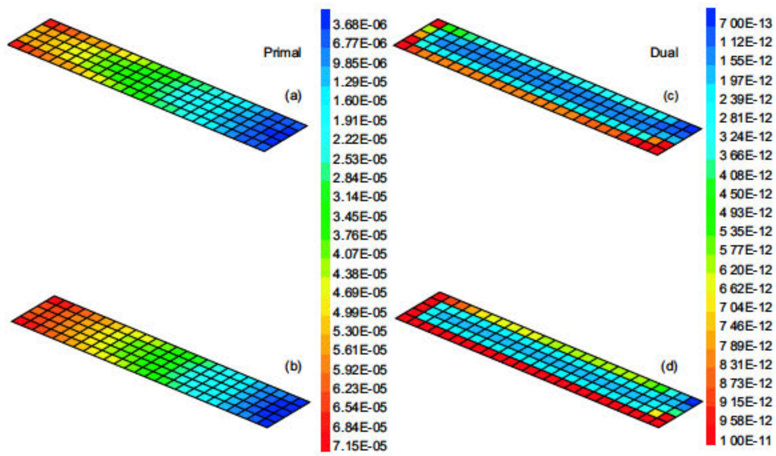

Figure 5. *Error distribution for the primal (left) and dual (right) problems. The magnitudes represented are the local contributions to the squared energy norms:* $\|e^h\|^2_{\Omega_k}$ *in* ***b***, $\|\varepsilon^h\|^2_{\Omega_k}$ *in* ***d***, *and their estimated counterparts (plots* ***a*** *and* ***c***, *respectively). The estimated error (top, plots* ***a*** *and* ***c***) *is in good agreement with the reference error (bottom, plots* ***b*** *and* ***d***)

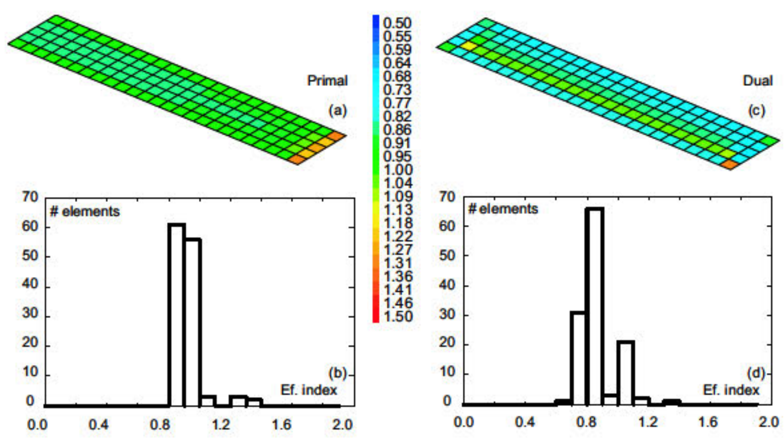

Figure 6. *Distribution of the local effectivity index for the primal (****a****) and dual (****c****) problems. Histogram showing the number of elements with a given local effectivity index for the primal (****b****) and dual (****d****) problems. The spatial distribution is quite uniform and, consequently, the histogram is narrow*

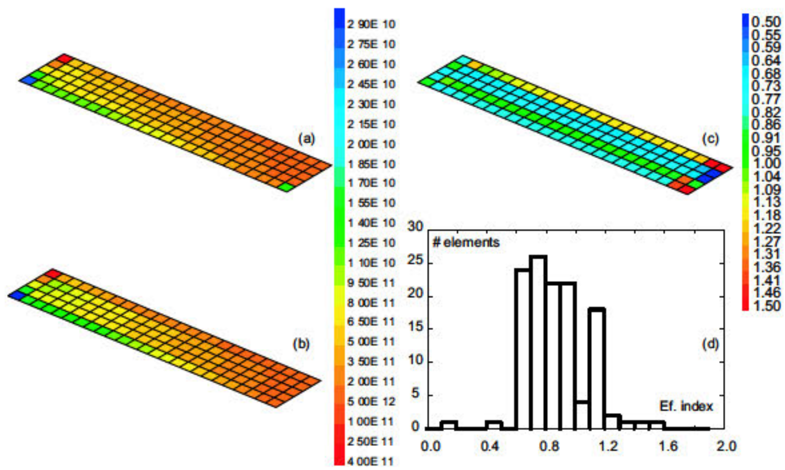

Figure 7. *Distribution of the local quantity* $a_{\Omega_k}\left(e^h, \varepsilon^h\right)$. *The estimated values (**a**) are fair approximations of the reference values (**b**). The effectivity index associated with this quantity: spatial distribution (**c**) and histogram showing the number of elements with a given local effectivity index (**d**)*

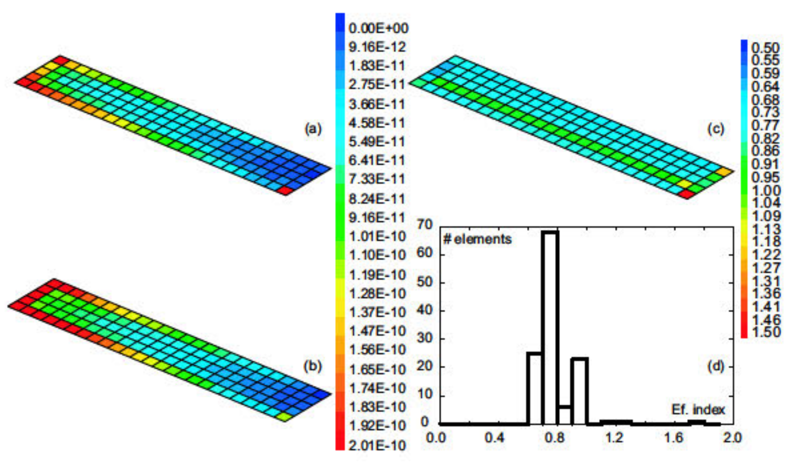

Figure 8. *Distribution of the local quantity* $\|e^h\|_{\Omega_k}\|\varepsilon^h\|_{\Omega_k}$. *The estimated values (**a**) are fair approximations of the reference values (**b**). The effectivity index associated with this quantity: spatial distribution (**c**) and histogram (**d**)*

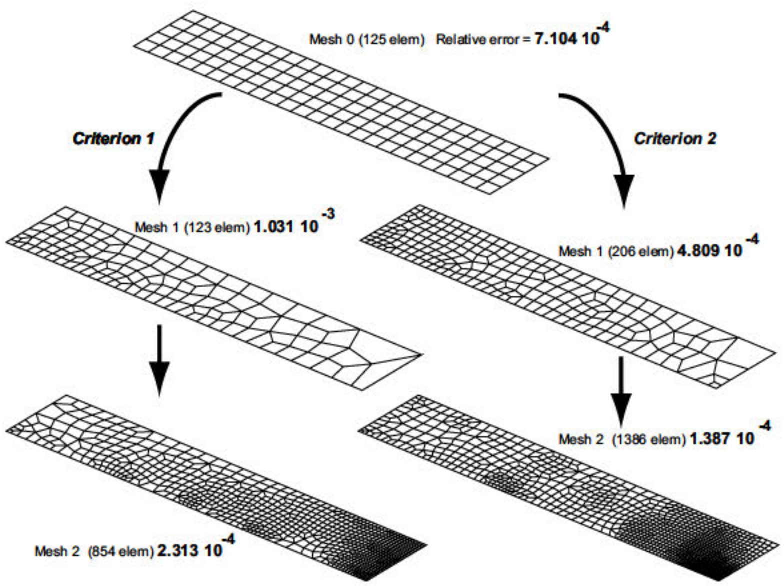

Figure 9. *Meshes obtained in the adaptive process for the two criteria. The prescribed relative error in the output of interest is* 2.5×10^{-4}

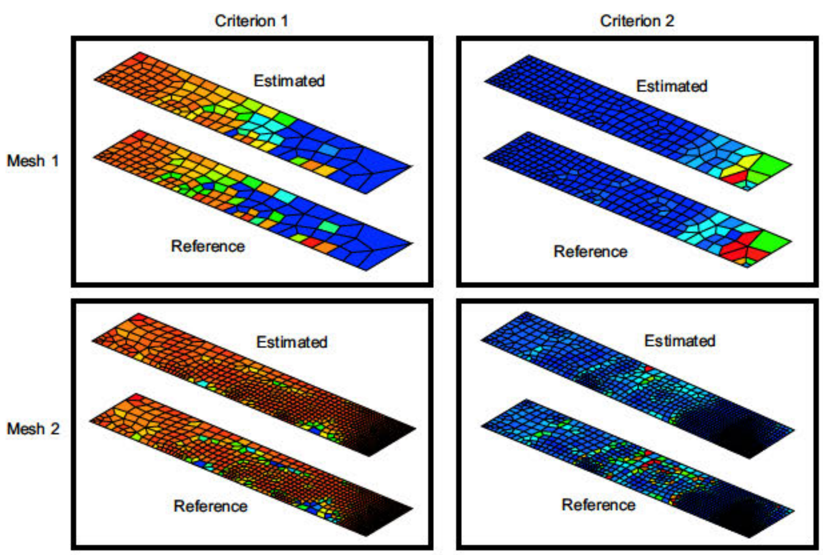

Figure 10. *Distribution of the error magnitudes used in the remeshing criterion:* $a_{\Omega_k}\left(e^h, \varepsilon^h\right)$ *for criterion 1 and* $\|e^h\|_{\Omega_k}\|\varepsilon^h\|_{\Omega_k}$ *for criterion 2*

the reference solution. Note that the reference solution is obviously not used in the adaptive process. As already mentioned in the previous section, the remeshing criteria accounts for an estimated error quantity, $a_h(e^h, \varepsilon^h)$, that does not coincide exactly with the output of the error, $J(e^h)$, that is the two additional terms in equation (20) are neglected. Nevertheless, the adaptive process converges to a mesh with the prescribed value also for the actual output. In the adaptive process following criterion 1, it can be noted that the accuracy in mesh 1 is lower than in mesh 0. This is due to the fact that the number of elements has been slightly reduced (from 125 to 123) and, more important, to the fact that in mesh 1 there are very distorted elements in the zone where the output is evaluated. The final meshes (mesh 2 for both the criteria) display an error lower that the targeted value of 2.5×10^{-4}. The error quantity associated with criterion 2, $a_h(e^h, \varphi^H)$, is larger than the quantity associated with criterion 1,$a_h(u^H, \varepsilon^h)$, and, consequently, the number of elements of the mesh obtained with criterion 2 is also larger.

Table 1 displays the values of all the relevant error quantities, both estimated and computed using the reference solutions. It can be observed that the estimates of the error norms are quite sharp and that the estimates of the cross products with the error functions are less accurate. In particular, the estimates for $a_h(e^h, \varphi^H)$ and $a_h(u^H, \varepsilon^h)$ are very bad for the initial mesh, mesh 0, but they improve significantly in the following.

Finally, figure 10 shows the error distributions used in every remeshing criterion. For every mesh (meshes 1 and 2 of the two criteria) the corresponding error quantity is displayed, both the estimated and the reference error distributions. The plots show that all along the remeshing process the assessment of the relevant error quantities is accurate both in global value and spatial distribution.

5.2. *Example 2: cooling tower*

The following example is introduced in [CIR 98]. A cooling tower with periodic supports in the bottom is loaded with a vertical force uniformly distributed along the top edge, see figure 11 for a complete problem statement. In this example, the geometry is a curved surface and therefore the error estimation procedure must account for it. Thus, the reference mesh is build up using the exact description of the geometry, in this case the surface is a hyperboloid. In this example the used element type is the same as in the previous example (four-noded quadrilateral element).

The results are shown following the same structure of example 1: the energy norm estimates for the primal and dual problems are shown in figure 12. The analysis of the quality of these estimates is performed in figure 13. The distributions of the local values for $a(e, \varepsilon)$ and $\|e\|\|\varepsilon\|$ are shown in figures 14 and 15. Then, the adapted meshes obtained with both criteria are shown in figure 16. Here, the prescribed relative error in the output of interest is set to 5×10^{-2}. In this case, the output exhibits a singularity. The error associated with the the mesh obtained with criterion 2 in the first

	Reference	Estimated	Effectivity
	mesh0: $J(e^h) = -1.278 \times 10^{-8} = 0.071\%$ of $J(u^h)$		
$\|e^h\|/\|u^h\|$	1.885×10^{-2}	1.721×10^{-2}	91.3%
$\|\varepsilon^h\|/\|\varphi^h\|$	5.027×10^{-2}	5.554×10^{-2}	110.4%
$a_h(e^h, \varepsilon^h)$	-6.101×10^{-9}	-5.223×10^{-9}	85.6%
$a_h(e^h, \varphi^H)$	-5.355×10^{-9}	5.234×10^{-9}	-97.7%
$a_h(u^H, \varepsilon^h)$	-1.327×10^{-9}	5.265×10^{-9}	-396.7%
	mesh1 (crit 1): $J(e^h) = -1.856 \times 10^{-8} = 0.103\%$ of $J(u^h)$		
$\|e^h\|/\|u^h\|$	1.20×10^{-1}	8.297×10^{-2}	69.1%
$\|\varepsilon^h\|/\|\varphi^h\|$	1.424×10^{-1}	9.895×10^{-2}	69.5%
$a_h(e^h, \varepsilon^h)$	-2.919×10^{-7}	-1.378×10^{-7}	47%
$a_h(e^h, \varphi^H)$	1.292×10^{-7}	1.366×10^{-7}	105.6%
$a_h(u^H, \varepsilon^h)$	1.441×10^{-7}	1.375×10^{-7}	95.4%
	mesh2 (crit 1): $J(e^h) = -4.162 \times 10^{-9} = 0.023\%$ of $J(u^h)$		
$\|e^h\|/\|u^h\|$	5.444×10^{-2}	3.642×10^{-2}	66.9%
$\|\varepsilon^h\|/\|\varphi^h\|$	6.577×10^{-2}	5.999×10^{-2}	91.2%
$a_h(e^h, \varepsilon^h)$	-4.988×10^{-8}	-2.190×10^{-8}	43.9%
$a_h(e^h, \varphi^H)$	2.221×10^{-8}	2.188×10^{-8}	98.5%
$a_h(u^H, \varepsilon^h)$	2.351×10^{-8}	2.201×10^{-8}	93.6%
	mesh1 (crit 2): $J(e^h) = -8.653 \times 10^{-9} = 0.048\%$ of $J(u^h)$		
$\|e^h\|/\|u^h\|$	9.901×10^{-2}	6.384×10^{-2}	64.5%
$\|\varepsilon^h\|/\|\varphi^h\|$	1.21×10^{-1}	8.940×10^{-2}	73.8%
$a_h(e^h, \varepsilon^h)$	-2.023×10^{-7}	-8.523×10^{-8}	42.1%
$a_h(e^h, \varphi^H)$	9.577×10^{-8}	8.451×10^{-8}	88.2%
$a_h(u^H, \varepsilon^h)$	9.783×10^{-8}	8.609×10^{-8}	88%
	mesh2 (crit 2): $J(e^h) = -2.496 \times 10^{-9} = 0.013\%$ of $J(u^h)$		
$\|e^h\|$	5.122×10^{-2}	3.334×10^{-2}	65.1%
$\|\varepsilon^h\|$	6.155×10^{-2}	5.745×10^{-2}	93.3%
$a_h(e^h, \varepsilon^h)$	-4.381×10^{-8}	-1.847×10^{-8}	42.2%
$a_h(e^h, \varphi^H)$	2.026×10^{-8}	1.845×10^{-8}	91.1%
$a_h(u^H, \varepsilon^h)$	2.106×10^{-8}	1.852×10^{-8}	87.9%

Table 1. *Example 1. Summary of the relevant error quantities along the adaptive processes. Reference and estimated values and effectivity index (estimated/reference)*

iteration is larger than the error of the mesh obtained with criterion 1. Nevertheless, the mesh obtained with criterion 2 is finer than the mesh obtained with criterion 1. Thus, the remeshing process is stopped at the first iteration. In only one iteration, the remeshing criteria give meshes where the elements are concentrated around the point where the output is measured and also, in the vicinity of the supports (sources of pollution errors).

Table 2 displays the values of the relevant error quantities for all the meshes. The remarks of example 1 are also valid here.

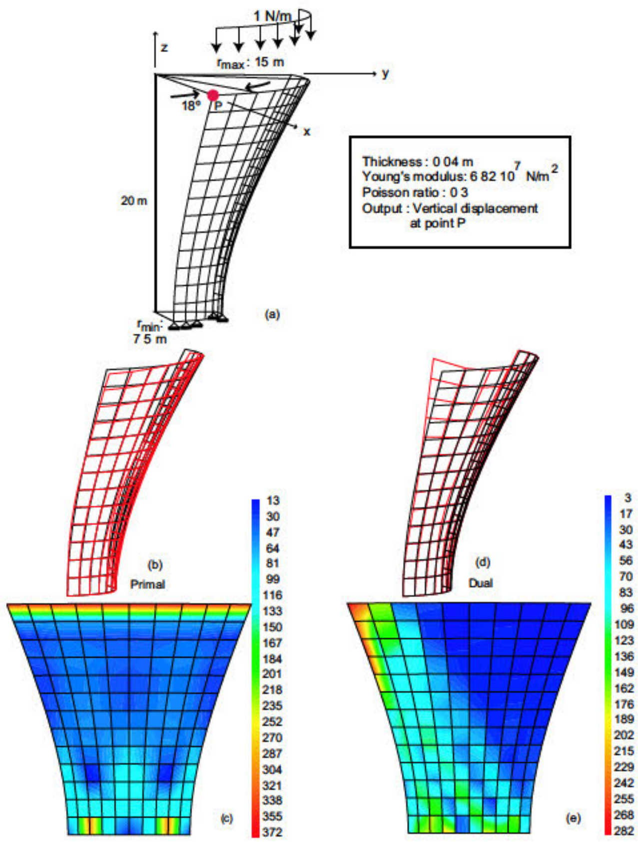

Figure 11. *Problem statement for example 2 (**a**). Deformed shape, (**b**) and (**d**), and Von Mises stress distribution, (**c**) and (**e**), for the primal problem (left) and the dual problem (right)*

6. Conclusions

A new method to assess and control the error in quantities of interest is presented in the context of Reissner-Mindlin shell models with assumed-strains. The Assumed-strain model for shells bring an additional difficulty in the representation of the error that differs from the usual representation. The Galerkin orthogonality does not stand in this framework and, consequently, two additional terms appear in the expression of the error in the output of interest. This new representation of the error in the quantity of interest is an original contribution introduced in this paper. These additional terms are difficult to evaluate accurately using the standard estimates.

In order to perform adaptive computations, a remeshing criterion is introduced for the general case of goal-oriented adaptivity. The remeshing criterion is formulated in

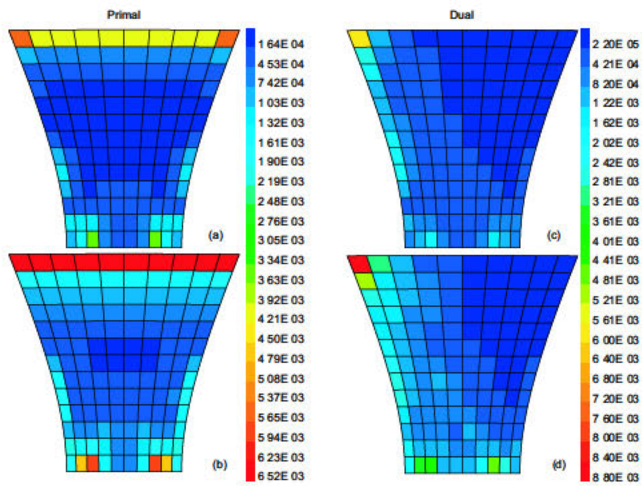

Figure 12. *Error distribution for the primal (left) and dual (right) problems. The magnitudes represented are the local contributions to the squared energy norms: $\|e^h\|^2_{\Omega_k}$ in **b**, $\|\varepsilon^h\|^2_{\Omega_k}$ in **d**, and their estimated counterparts (plots **a** and **c**, respectively). The estimated error (top, plots **a** and **c**) is in good agreement with the reference error (bottom, plots **b** and **d**)*

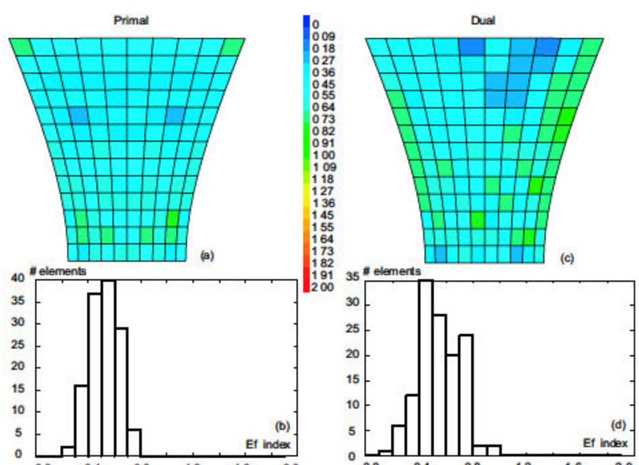

Figure 13. *Distribution of the local effectivity index for the primal (**a**) and dual (**c**) problems. Histogram showing the number of elements with a given local effectivity index for the primal (**b**) and dual (**d**) problems. The spatial distribution is quite uniform and, consequently, the histogram is narrow*

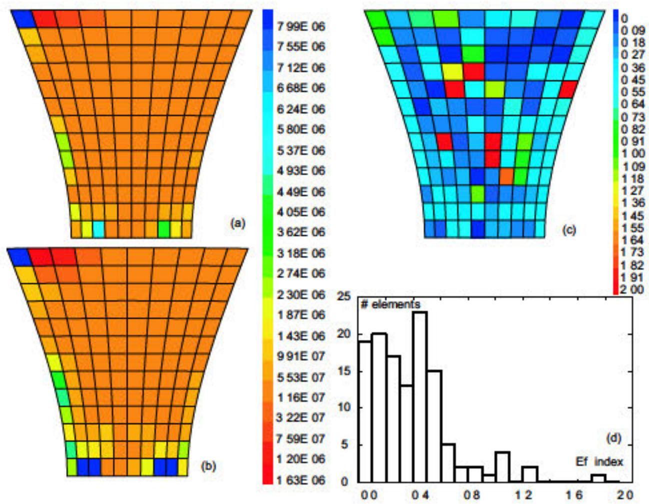

Figure 14. *Distribution of the local quantity* $a_{\Omega_k}\left(e^h, \varepsilon^h\right)$*. The estimated values* ***(a)*** *are fair approximations of the reference values* ***(b)****. The effectivity index associated with this quantity: spatial distribution* ***(c)*** *and histogram* ***(d)***

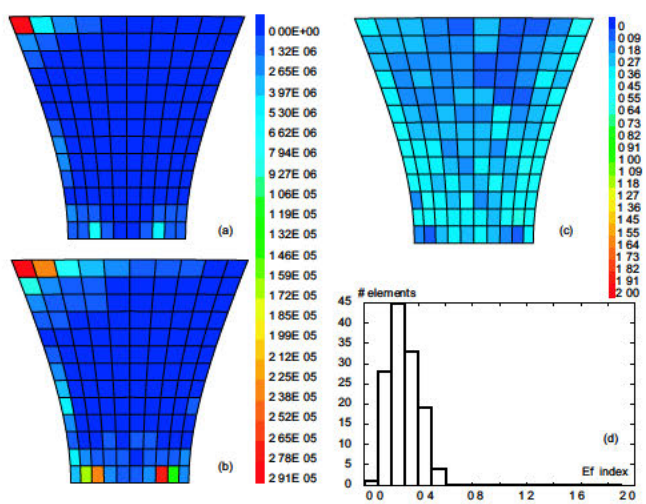

Figure 15. *Distribution of the local quantity* $\|e^h\|_{\Omega_k}\|\varepsilon^h\|_{\Omega_k}$*. The estimated values* ***(a)*** *are fair approximations of the reference values* ***(b)****. The effectivity index associated with this quantity: spatial distribution* ***(c)*** *and histogram* ***(d)***

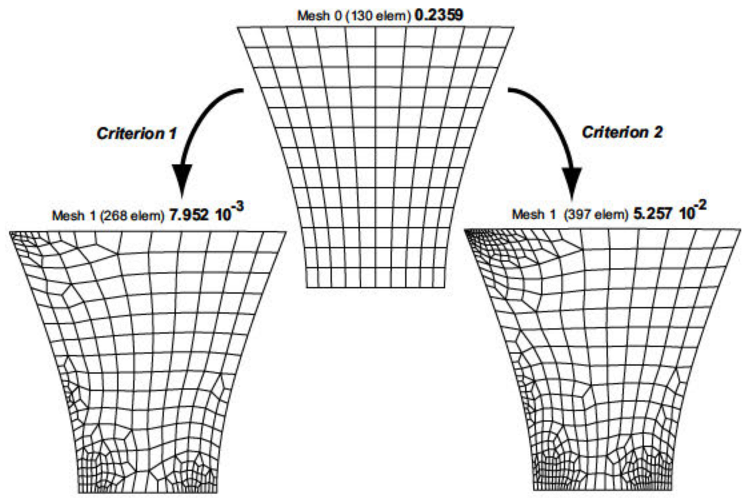

Figure 16. *Meshes obtained in the adaptive process for the two criteria. The prescribed relative error in the output of interest is* 5×10^{-2}

	Reference	Estimated	Effectivity
	mesh0: $J(e^h) = -1.004 \times 10^{-4} = 23.6\%$ of $J(u^h)$		
$\Vert e^h\Vert/\Vert u^h\Vert$	5.039×10^{-1}	2.976×10^{-1}	59.1%
$\Vert \varepsilon^h\Vert/\Vert \varphi^h\Vert$	4.899×10^{-1}	2.604×10^{-1}	53.1%
$a_h(e^h, \varepsilon^h)$	-1.136×10^{-4}	-4.499×10^{-5}	39.6%
$a_h(e^h, \varphi^H)$	3.626×10^{-6}	2.192×10^{-5}	604.5%
$a_h(u^H, \varepsilon^h)$	9.472×10^{-6}	2.960×10^{-5}	312.4%
	mesh1 (crit 1): $J(e^h) = -3.638 \times 10^{-6} = 0.079\%$ of $J(u^h)$		
$\Vert e^h\Vert/\Vert u^h\Vert$	6.412×10^{-1}	2.847×10^{-1}	44.4%
$\Vert \varepsilon^h\Vert/\Vert \varphi^h\Vert$	4.655×10^{-1}	2.737×10^{-1}	58.8%
$a_h(e^h, \varepsilon^h)$	-9.544×10^{-5}	-3.162×10^{-5}	33.1%
$a_h(e^h, \varphi^H)$	9.703×10^{-4}	5.705×10^{-4}	58.8%
$a_h(u^H, \varepsilon^h)$	2.735×10^{-5}	3.461×10^{-5}	126.5%
	mesh1 (crit 2): $J(e^h) = -2.414 \times 10^{-5} = 5.257\%$ of $J(u^h)$		
$\Vert e^h\Vert/\Vert u^h\Vert$	6.269×10^{-1}	2.633×10^{-1}	42.0%
$\Vert \varepsilon^h\Vert/\Vert \varphi^h\Vert$	3.955×10^{-1}	2.634×10^{-1}	66.8%
$a_h(e^h, \varepsilon^h)$	-7.120×10^{-5}	-2.587×10^{-5}	36.3%
$a_h(e^h, \varphi^H)$	3.136×10^{-5}	2.607×10^{-5}	83.2%
$a_h(u^H, \varepsilon^h)$	1.586×10^{-5}	2.694×10^{-5}	169.9%

Table 2. *Example 2. Summary of the relevant error quantities along the adaptive processes. Reference and estimated values and effectivity index (estimated/reference)*

oder to control the standard representation of the error. Numerical tests demonstrate that the adaptive procedure reduces the error in the output of interest such that the prescribed accuracy is reached.

7. References

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