14th World Congress on Computational Mechanics (WCCM)
ECCOMAS Congress 2020)
Virtual Congress: 11–-15 January 2021
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A DECOMPOSITION OF THE RAVIART-THOMAS FINITE ELEMENT INTO A SCALAR AND AN ORIENTATION-PRESERVING PART

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Key words: mixed finite element methods, Raviart-Thomas element

Abstract. This contribution considers the conforming finite element discretizations the vector-valued function space $H(div,\Omega)$ in 2 and 3 dimensions. A new set of basis functions on simplices is introduced, using a decomposition into an orientation setting part with the edgewise constant normal flux as a degree of freedom and an orientation preserving higher-order part. As a simple combination of lowest-order Raviart-Thomas elements and higher order Lagrange-elements, the basis is suited for fast assembling strategies.

1 Introduction

Accurate flux and stress approximations are of crucial interest in many applications, although the standard Galerkin approximation usually minimize an energy depending on a primal variable. The divergence of those approximation does usually not belong to the Sobolev space $H(div,\Omega)$, consisting of vector fields for which the components and the weak divergence are square-integrable. A lot of attention has therefore been devoted to the reconstruction of the flux from a primal formulation. The reconstruction procedures for fluxes are also of particular importance for a posteriori error estimation and have a long history with ideas dating back at least as far as [17] and [18]. A unified framework for Stokes is presented in [16], polynomial-degree robustness is shown in [13] and extensions to three space dimensions in [14].

An alternative approach uses flux-based variational formulations involving the flux as an independent variable approximated in a suitable $H(\text{div})$-conforming finite element spaces. Such approaches may either lead to a saddle-point problem or a symmetric positive definite system. The first one, have been intensively studied and we refer to [8] for an overview. The second type includes in particular the Least-Squares Method (see [7] for a comprehensive overview) and the discontinuous Petrov-Galerkin method, introduced in a series of papers [9, 10, 11].

The use of $H(div,\Omega)$-conforming finite elements is therefore crucial in challenging applications. Throughout this paper, a regular triangulation $\mathcal{T}$ of a domain $\Omega \subset \mathbf{R}^d$ is considered, and $P_k(\mathcal{T})$ denotes the space of discontinuous polynomials of degree k with respect to $\mathcal{T}$. Recall that for a finite element family to be $H(div,\Omega)$-conforming, the normal component must be continuous. For a given polynomial degree $k \geq 0$

[1]Acknowledgments: The author gratefully acknowledge support by the Deutsche Forschungsgemeinschaft in the Priority Program SPP 1748 *Reliable simulation techniques in solid mechanics, Development of non standard discretization methods, mechanical and mathematical analysis* under the project number BE 6511/1-1.

the smallest polynomial space from which the divergence maps onto $P_0(\mathcal{T})$ is the so-called the so-called Raviart-Thomas element, denoted by $RT_k(\mathcal{T})$ and introduced in [?].

The construction of the basis function and the choice of the degrees of freedom for the Raviart-Thomas element are an important issue in view of fast assembling and computations. They are less evident than for the standard element, as not all degrees of freedom are pointvalues. Computational bases for $H(div,\Omega)$ with Lagrangian property are derived in [15], althought the autor restrict himself to the two-dimensional case. This basis is defined on the reference triangle and the continuity of the normal component of the approximation across the edges in the triangulation is satisfied by the use of the Piola transformation and the Lagrangian property of the basis functions. However, using an affine transformation to map a reference triangle on any triangle in the triangulation is less appropriate regarding an adaptive mesh refinement such that an alternative approach forming local bases on any triangle in the triangulation is proposed in [2]. A construction of basis functions which implies sparse element matrices has been developped in [6], where the authors are concerned with fast assembling and fast matrix-vector multiplications.

The efficient assembly of $H(div)$ is an important issue, discussed e.g. in [19], where the authors propose an implementing strategy relying on a decomposition of the element tensor into a precomputable reference tensor and a mesh-dependent geometry tensor. An alternative effective and fexible way to assemble finite element stiffness and mass matrices with edge elements is proposed in [1].

The aim of this paper is to propose a decomposition of the Raviart-Thomas space into a scalar and an orientation-preserving part, as this will increase the speed of the assembling strategies. In fact, the orientation of the basis function can be defined once and for all for the lowest-order Raviart-Thomas space, and be conserved in the higher-order case. Such a decomposition is also usefull for the analysis of the effect of approximated flux boundary conditions (see [5]). For the implementation of the parametric Raviart-Thomas elements (see [4]) this decomposition can be used to reduce the computational cost of the parametric mapping, since it is then sufficient to apply the $H(div)$ conforming mapping on the lower-order orientation-preserving part and to use the standard isoparametric mapping for the scalar part. An other application of this decomposition is the derivation of hierarchical error estimator, simplifying the analysis and dumbing computational cost down.

The main result of this paper is that given a basis of $RT_0(\mathcal{T})$, it is possible to find a set of basis functions for $RT_k(T)$, that are given as a product of the $RT_0(\mathcal{T})$ basis functions and a scalar function. This means that the lowest order basis functions can be considered as the orientation part of all the cell-based higher-order basis functions. Since these are easily computable and since the scalar part can be seen as a Lagrange basis function this implies a reduction of the computational cost. This paper is organized as follows: section 2 recalls some properties of the Raviart-Thomas element while in section 3 the orientation-preserving basis function are provided. In section 4, a basis retaining the orientation is constructed for the quadratic case. Finally, the results are extended to the higher-order case in section 5.

2 The Raviart-Thomas element

In this section, we review the different definitions of the degrees of freedom. Note that even the indexing of the Raviart-thomas space is not consistent in the literature. We choose to work with the following definition of the Raviart-Thomas element:

$$RT_k(T) = (\mathcal{P}_k(T))^d + \mathbf{x}\mathcal{P}_k(T) \tag{1}$$

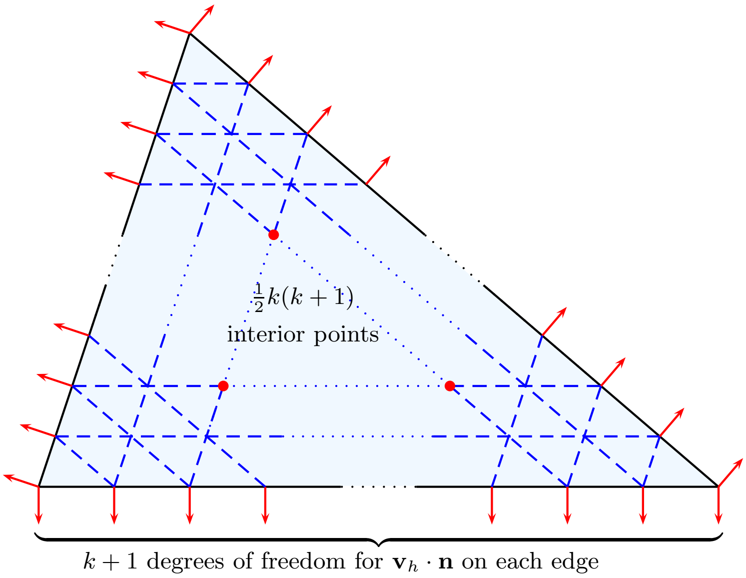


Figure 1: Degrees of freedom for the Raviart-Thomas element

forming the Raviart-Thomas space

$$RT_k(\mathcal{T},\Omega) = \{\mathbf{v} \in H(div,\Omega) \; : \; \mathbf{v}|_T \in RT_k(T) \text{ for all } T \in \mathcal{T}\} \; . \tag{2}$$

We start by recalling from [**?**] (see also [8]) that $k \geq 0$ and for any $\mathbf{v}_h \in \mathcal{RT}_k(T)$, it holds

$$\begin{aligned} \text{div } \mathbf{v}_h &\in \mathcal{P}_k(T) \; , \\ \mathbf{v}_h \cdot \mathbf{n} &\in R_k(\partial T) \; , \end{aligned} \tag{3}$$

where ∂T denote the boundary of an element $T \in \mathcal{T}$. Further let $\mathcal{F}(T)$ denote the set of its facets, $\mathbf{n}$ the outward oriented normal and

$$R_k(\partial T) = \{\phi \in L^2(\partial T) : \phi|_e \in \mathcal{P}_k(e) \quad \forall \, e \in \mathcal{F}(T)\} \tag{4}$$

the polynomial space on the facets. Recall that if

$$\int_{\partial T} (\mathbf{n} \cdot \mathbf{v}_h) p_k \; ds, \quad \forall \; p_k \in R_k(\partial T) \tag{5a}$$

$$\int_{T} \mathbf{v}_h \cdot \mathbf{p}_{k-1} \; d\mathbf{x}, \quad \forall \; \mathbf{p}_{k-1} \in (P_{k-1}(T))^d \tag{5b}$$

holds for $\mathbf{v}_h \in RT_k(T)$, then $\mathbf{v}_h = \mathbf{0}$. As represented in figure 2 and 3 for $k=0$ and $k=1$ (see [12] as well), this motivate the fact that the degrees of freedom $\mathbf{\Sigma}_k(\mathbf{v}_h) = \mathbf{\Sigma}^{\mathbf{n}}_{\mathbf{R},k}(\mathbf{v}_h) \cup \mathbf{\Sigma}^{int}_{\mathbf{P},k}(\mathbf{v}_h)$ are typically given by

$$\Sigma^{\mathbf{n}}_{\mathbf{R},k}(\mathbf{v}_h) = \{ \int_{\partial T} (\mathbf{n}\cdot\mathbf{v}_h) p_k \, ds \ : \ p_k \in \mathbf{R} \} \tag{6a}$$

$$\Sigma^{int}_{\mathbf{P},k}(\mathbf{v}_h) = \{ \int_{T} \mathbf{v}_h \cdot \mathbf{p}_{k-1} \, d\mathbf{x} \ : \ \mathbf{p}_{k-1} \in \mathbf{P} \} \tag{6b}$$

where $\mathbf{R}$ and $\mathbf{P}$ are basis for $R_k(\partial T)$ and $(P_{k-1}(T))^d$ respectively. Note that the degrees of freedom depend on this basis. Denoting the standard conforming Lagrangian basis function on a element E with $\mathcal{L}^k(E)$, one possible choice are

$$\mathbf{R} = \{ q \in L^2(\partial T) : q|_e \in \mathcal{L}^k(e) \quad \forall\, e \in \mathcal{F}(T) \} \tag{7}$$

and $\mathbf{P} = \mathcal{L}^{k-1}(T)$. Constructing the set of functions

$$\{ \mathbf{v}_i \in RT_k(T) : \{ \Sigma_k(\mathbf{v}_i) \}_j = \delta_{ij} \}_{i=1}^{\dim RT_k(T)} \tag{8}$$

leads to a dual basis of $RT_k(T)$. Throughout this paper, $RT^{\mathbf{n}}_k(\mathcal{T})$ denotes the linear combination of the basis functions assiociated with (6a) such that

$$RT_k(\mathcal{T},\Omega) = RT^{\mathbf{n}}_k(\mathcal{T}) \bigoplus RT^{int}_k(\mathcal{T}) \tag{9}$$

holds. Note that for $k = 0$, $RT^{int}_k(\mathcal{T}) = \emptyset$. Further, recall that

$$\dim RT_k(T) = (k+d+1)\binom{k+d-1}{d-1} = \begin{cases} (k+3)(k+1) & d = 2 \\ \frac{1}{2}(k+1)(k+2)(k+4) & d = 3 \end{cases}. \tag{10}$$

Due to $\mathcal{P}^{k-1}(K) \subset RT_k(T)$ and $\mathbf{v}_h \cdot \mathbf{n} \in R_k(\partial T)$ for $\mathbf{v}_h \in RT_k(T)$, an alternative way to define the degrees of freedom are point evaluations, i.e. $\tilde{\Sigma}_k(\mathbf{v}_h) = \tilde{\Sigma}^{\mathbf{n}}_k(\mathbf{v}_h) \cup \tilde{\Sigma}^{int}_k(\mathbf{v}_h)$ with

$$\tilde{\Sigma}^{\mathbf{n}}_{k,\Xi}(\mathbf{v}_h) = \{ (\mathbf{v}_h(\boldsymbol{\xi}_i)\cdot\mathbf{n})|_e \ : \ \boldsymbol{\xi}_i \in \Xi,\ e \in \mathcal{F}(T) \} \tag{11a}$$

$$\tilde{\Sigma}^{int}_{k,\mathbf{H}}(\mathbf{v}_h) = \{ \mathbf{v}_{h,j}(\boldsymbol{\eta}_i) \ : \ \boldsymbol{\eta}_i \in \mathbf{H} \}_{j=1}^{d} \tag{11b}$$

where $\mathbf{H}$ is a point set on an element T with

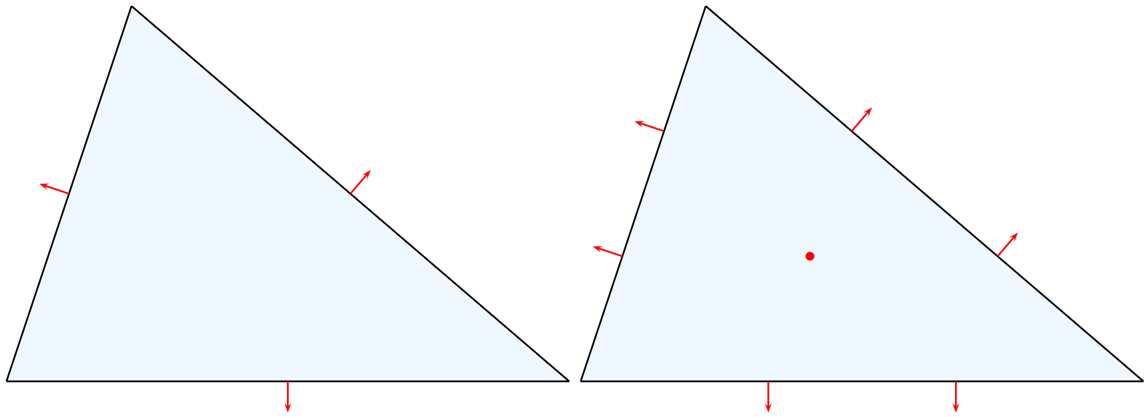
$$\dim(\mathbf{H}) = \binom{k-1+d}{d} \tag{12}$$

and Ξ is a point set on a facet, with

$$\dim(\Xi) = \binom{k-1+d}{k}\ . \tag{13}$$

Now, constructing the set of functions

$$\{ \mathbf{v}_i \in RT_k(T) : \{ \tilde{\Sigma}_k(\mathbf{v}_i) \}_j = \delta_{ij} \}_{i=1}^{\dim RT_k(T)} \tag{14}$$



(a) $k=0$ (b) $k=1$

Figure 2: Degrees of freedom for the Raviart-Thomas element for $k=0$ and $k=1$

leads to basis of $RT_k(T)$. For $k \geq 1$, the points for (6b) can be chosen as the interior points of the Lagrange element of type $k+d$, each point represents d degrees of freedom (see figure 1). Note that in fact, there is no need to enforce the evaluations of the different components at the same point, such that $\bar{\boldsymbol{\Sigma}}_k(\mathbf{v}_h) = \tilde{\boldsymbol{\Sigma}}_k^{\mathbf{n}}(\mathbf{v}_h) \cup \bar{\boldsymbol{\Sigma}}_k^{int}(\mathbf{v}_h)$ with

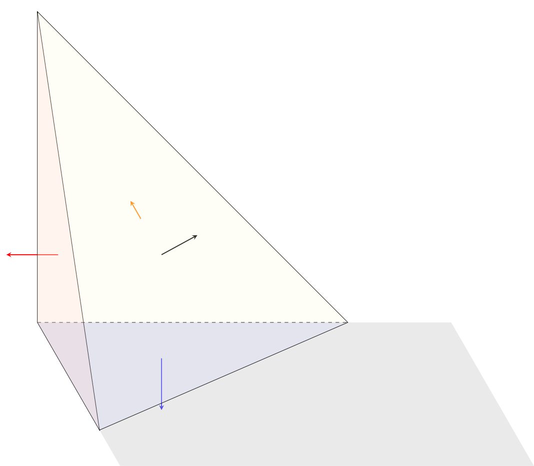
$$\bar{\boldsymbol{\Sigma}}^{int}_{k,\mathbf{H}_1,\ldots,\mathbf{H}_d}(\mathbf{v}_h) = \{\mathbf{v}_{h,j}(\boldsymbol{\eta}_i) \; : \; \boldsymbol{\eta}_i \in \mathbf{H}_j\}_{j=1}^{d} \tag{15}$$

where $\mathbf{H}_1, \ldots, \mathbf{H}_d$ are sets of points on an element T with $\dim(\mathbf{H}_j) = \begin{pmatrix} k-1+d \\ d \end{pmatrix}$, $j = 1..d$. In particular, this definition of the degrees of freedom allows to reduce the computational cost could be reduced using the coordinate system provided from the RT_0 functions.

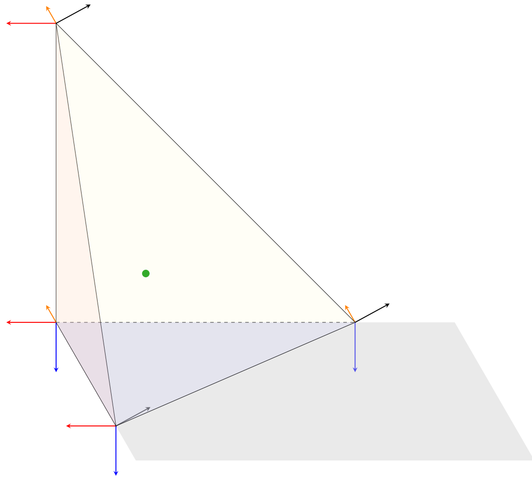
3 Orientation preserving Basis Functions

[3] provides three short Matlab implementations of the lowest-order Raviart-Thomas mixed finite elements for the numerical solution of a Laplace equation with mixed Dirichlet and Neumann boundary conditions. In particular, a section is devoted to the edge-basis functions for the lowest-order Raviart-Thomas finite elements. Although these results are written down for the 2D case, the representation of a Raviart-Thomas function directly carries over to three dimensions. In the lowest-order case, the Raviart-Thomas Function space $RT_0(\mathcal{T})$ is given by facet-functions, i.e. there is a basis $\hat{\Psi} = \{\hat{\phi}_F\}_{F\in\mathcal{F}}$, where ϕ_F lives on the two adjacent elements of F. This together with the $H(div)$ conformity implies that the facets will be oriented. Therefore, for a given facet $F \in \mathcal{F}$ a left and a right triangle denoted by T_+ and T_- are defined, corresponding to left and right d-th points denoted by P_+ and P_-. This defines one for all the orientation of the given edge, and the purpose of the next section is to retain the orientation in the corresponding basis function. The basis functions are given by

$$\hat{\psi}_F(\mathbf{x}) = \begin{cases} \frac{|F|}{2|T_+|}(\mathbf{x} - P_+) & \text{on } |T_+| \\ -\frac{|F|}{2|T_-|}(\mathbf{x} - P_-) & \text{on } |T_-| \\ 0 & \text{elsewhere,} \end{cases} \tag{16}$$



(a) $k = 0$



(b) $k = 1$

Figure 3: Degrees of freedom for the Raviart-Thomas element ($d = 3$)

where $|F|$ denotes the measure of the facet F and $|T|$ denotes the measure of the element T. These basis functions correspond to the nodal set of degrees of freedom $\tilde{\mathbf{\Sigma}}^{\mathbf{n}}_{0,\cdot} = \mathbf{\Sigma}^{\mathbf{n}}_{\tilde{\mathbf{R}}^0,k}$, with

$$\tilde{\mathbf{R}}^0 = \{q : e \mapsto \mathbf{R} \,:\, q(\mathbf{x}) = |F|^{-1}\}_{e\in\mathcal{F}(T)} \,. \tag{17}$$

Since the normal component of the basis function is constant on an edge, the evaluation point does not have to be specified. In the two-dimensional case, denoting the barycentric coordinates by λ associated with the points of an edge E by λ_E^+ and λ_E^-, the basis function $\hat{\psi}_E$ for $E \in \mathcal{F}$ can be written as

$$\hat{\psi}_E(\lambda)_{|T} = \lambda_E^-(\nabla\times\lambda_E^+) - \lambda_E^+(\nabla\times\lambda_E^-) \,. \tag{18}$$

In the three-dimensional case, the basis function $\hat{\psi}_F$ for $F \in \mathcal{F}$ can be written as

$$\hat{\psi}_F(\lambda)_{|T} = \sum_{i=1}^{3} \lambda_{F,i}^-(\nabla\lambda_{F,i}^+ \times \nabla\lambda_{F,i}^-) \,. \tag{19}$$

where $\{\lambda_{F,i}\}_{i=1}^3$ denotes the barycentric coordinates associated with the $i-th$ vertice of the face F, and $\nabla\lambda_{F,i}^+$ and $\nabla\lambda_{F,i}^-$ its adjacent points on the face.

The alternative definition of the degrees of freedom with the moments leads to

$$\hat{\phi}_F(\mathbf{x}) = \begin{cases} \frac{1}{2|T_+|}(\mathbf{x}-P_+) & \text{on } |T_+| \\ -\frac{1}{2|T_-|}(\mathbf{x}-P_-) & \text{on } |T_-| \\ 0 & \text{elsewhere,} \end{cases} \tag{20}$$

corresponding to $\mathbf{\Sigma}^{\mathbf{n}}_{\mathbf{R}^0,k}$, with

$$\mathbf{R}^0 = \{q : e \mapsto \mathbf{R} \,:\, q(\mathbf{x}) = 1\}_{e\in\mathcal{F}(T)} \,. \tag{21}$$

Throughout the further sections, the choice of the set of basis functions for RT_0 does not matter, the decomposition into a orientation-part and the scalar part will conserve the scaling of the RT_0 basis functions into the higher-order case either way.

4 Quadratic Raviart-Thomas Element

In this section, a basis for the facet-based shape functions $RT_2^{\mathbf{n}}(T)$ is proposed. Recall that we want to conserve the orientation defined by the lowest-order facet-based functions. Therefore, define the linear scalar-valued polynomials $p_{F,P,T} : T \mapsto \mathbf{R}$ such that

$$\begin{aligned} p_{F,P,T}(P) &= 1 \\ p_{F,P,T}(P_m(T)) &= 0 \text{ and} \\ p_{F,P,T}(P_i) &= 0 \text{ for } P_i \in Q(F) \backslash P \ . \end{aligned} \tag{22}$$

where $P_m(T)$ denote the center of gravity of the element T and $Q(F)$ the set of the vertices of the facet F. Note that these functions are unique and correspond to a Lagrange Basis on a uniform refined mesh. This leads to the following lemma.

Lemma 1. *A basis for $RT_2^{\mathbf{n}}(T)$ is given by $\{\psi_{P_i,F}\}_{F\in\mathcal{F},i=1,\dots,d}$, with*

$$\psi_{P_i,F}(\mathbf{x}) = \begin{cases} \frac{|F|}{2|T_+|}(\mathbf{x}-P_+)p_{F,P_i,T_+}(\mathbf{x}) & on\ |T_+| \\ -\frac{|F|}{2|T_-|}(\mathbf{x}-P_-)p_{F,P_i,T_-}(\mathbf{x}) & on\ |T_-| \\ 0 & elsewhere \end{cases} . \tag{23}$$

Using the lowest-order basis function from (16)*, it holds*

$$\psi_{P_i,F} = \hat{\psi}_F \cdot p_{F,i},\ i = 1,\dots,d \tag{24}$$

with the scalar-valued, piecewise linear polynomial $p_{F,i} : \mathcal{T}_h \to \mathbf{R}$ such that

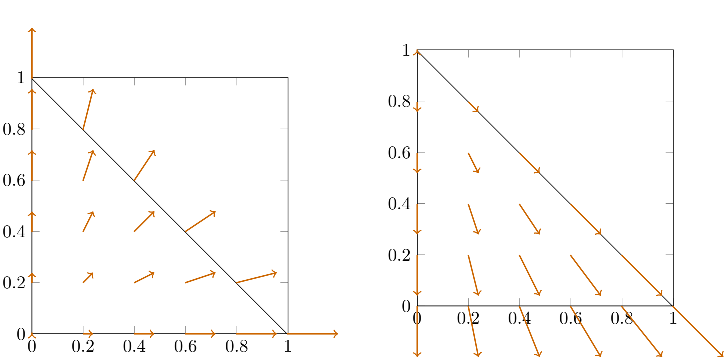
$$p_{F,i} = \begin{cases} p_{F,P_i,T_+} & on\ |T_+| \\ p_{F,P_i,T_-} & on\ |T_-| \quad i = 1,\dots,d\ . \\ 0 & elsewhere \end{cases} \tag{25}$$

Proof. Due to construction, $\psi_{P_i,F} \in RT_2^{\mathbf{n}}(T)$. Further for $\mathbf{x} \notin F$, since $\hat{\psi}_F(\mathbf{x}) \cdot \mathbf{n} = 0$ it follows from (24) that $\psi_{P_i,F}(\mathbf{x}) \cdot \mathbf{n} = 0$. It remains to proove that for a given facet F, the d functions $\psi_{P_i,F}(\mathbf{x}) \cdot \mathbf{n} = 0$ for $P_i \in Q(F)$ are linearly independent. Due to (24), this is equivalent to the fact that the d functions $p_{F,i}, i = 1,\dots,d$ are linearly independent for a given facet F, and this is ensure by the conditions (22), and the fact that the triangulation is regular, since these are $d+1$ conditions for a linear polynomial in $\mathbf{R}^d$. □

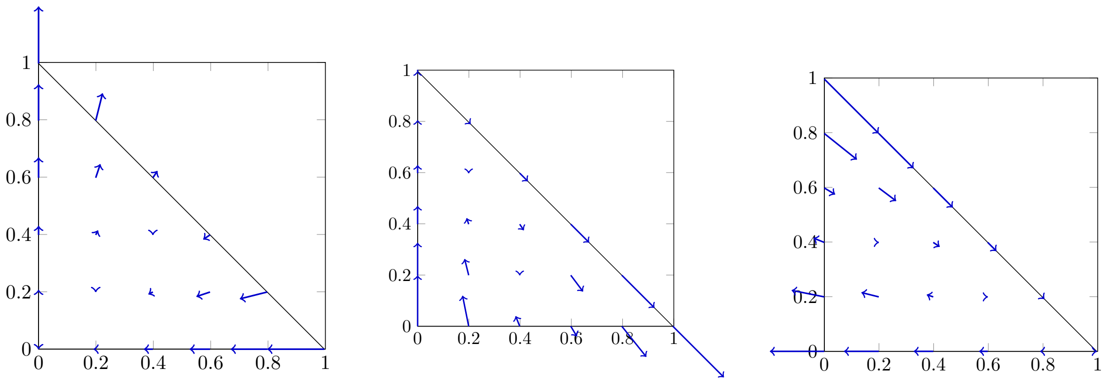
Note that in the two-dimensional case, due to symmetric properties, it holds

$$\begin{aligned} p_{F,P,T}(P) &= 1 \\ p_{F,P,T}(P_m(T)) &= 0 \text{ and} \\ p_{F,P,T}(P_+(F)) &= -1\ . \end{aligned} \tag{26}$$

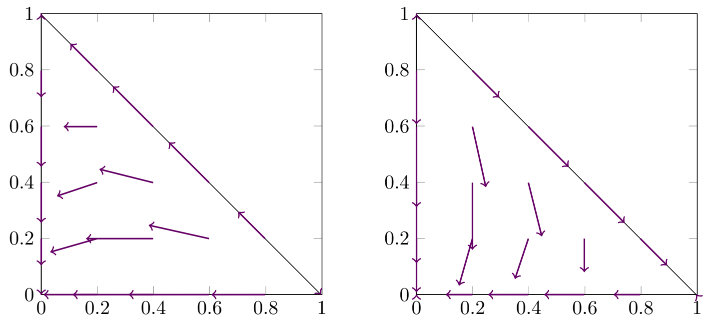
To obtain an orientation preserving basis, it remains to construct a basis for $RT_2^{int}(\mathcal{T})$, using the lowest-order basis functions. Therefore for a facet F of an element $T \in \mathcal{T}_h$ consider the scalar-valued linear



(a) $\boldsymbol{\psi}_1 = (x,y)^\top$ (b) $\boldsymbol{\psi}_2 = (x-1,y)^\top$ (c) $\boldsymbol{\psi}_3 = (x,y-1)^\top$



(d) $\boldsymbol{\psi}_4 = (x,y)^\top(y-x)$ (e) $\boldsymbol{\psi}_5 = (x-1,y)^\top(y-1+2x)$ (f) $\boldsymbol{\psi}_6 = (x,y-1)^\top(1-x-2y)$



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(g) $\boldsymbol{\psi}_7 = (3x(-2+2x+y),3y(-1+2x+y))^\top$ (h) $\boldsymbol{\psi}_8 = (3x(x+2y-1),3y(-2+x+2y))^\top$

Figure 4: Example for ansatzfunctions on reference triangle

polynomial $p_{F,T} : T \to \mathbf{R}$ such that

$$\begin{aligned} p_{F,T}(P_i(F)) &= 0, \ i \in \{1,...,d\} \\ p_{F,T}(P_m(T)) &= 1 \end{aligned} \tag{27}$$

and let $i_{min}(T)$ be the index such that $P_{i_{min}(T)}$ is the vertice of T with minimal angle. Note that $p_{F,T}$

althought correspond to a Lagrange Basis on a uniform refined mesh, such that the following lemma ensures a simple computation of the basis functions for $\mathcal{T}$.

Lemma 2. *A basis for $RT_2^{int}(\mathcal{T})$ is given by $\{\psi_{i,T}\}_{T\in\mathcal{T}, i\in\{1,2,3,4\}\setminus\{i_{min}(T)\}}$ with*

$$\psi_{i,T} = \hat{\psi}_{F(P_i,T)} \cdot p_{F,T} \tag{28}$$

where $F(P_i,T)$ denotes the facet of T which intersection with P_i is empty.

Proof. Since $p_{F(P_i),T} = 0$ for each vertices of the facet F and $\hat{\psi}_{F(P_i,T)}$ vanishes for the remaining verctice of the element T, $\psi_{i,T}$ vanishes at all vertices of T. This implies that the normal component of $\psi_{i,T}$ vanishes on all facet, such that it remains to check that the d functions $\psi_{i,T}$ are linearly independent for a given T, and that holds due to the fact that $\hat{\psi}_{F(P_i,T)}$, $i \in \{1,2,3,4\}\setminus\{i_{min}(T)\}$ are linearly independent. □

5 Higher order case

The results from the previous section can be extended to higher order elements. Therefore recall that a Lagrangian element of order $k+d$ has

$$N_I(k) = \binom{k-1+d}{d} \tag{29}$$

degrees of Freedom located in the interior of an element $T \in \mathcal{T}_h$. Let $\mathcal{S}_k(T) = \{S_{T,k,i}\}_{i=1}^{\mathcal{N}_I}$ be the points corresponding to these degrees of Freedom. Further $\mathcal{R}_k(F) = \{R_{F,k,i}\}_{i=1}^{\mathcal{N}_F}$ with $\mathcal{N}_F = \dim \mathcal{P}_k(F) = \binom{d-1+k}{k}$ denotes the Lagrange points on a facet. Then, the following lemma states that one can define a unique scalar-valued polynomial of order k on a element $T \in \mathcal{T}$ using the nodes values at $S_k(T)$ and $\{R_k(F)\}_{F\in\mathcal{F}(T)}$.

Lemma 3. *Consider $j \in \{1,...,\mathcal{N}_F\}$. There exist a unique scalar-valued polynomial $q_{F,j,T,k}$ of order k such that*

$$\begin{aligned} &q_{F,j,T,k}(R_{F,k,j}) = 1 \\ &q_{F,j,T,k}(R_{F,k,i}) = 0 \textit{ for } i = 1,...,\mathcal{N}_F,\ i \neq j \textit{ and} \\ &q_{F,j,T,k}(S_{T,k,i}) = 0 \textit{ for } i = 1,...,\mathcal{N}_I\ . \end{aligned} \tag{30}$$

Proof. Since all points are distinct, the equations (30) corresponds to $\mathcal{N}_F + \mathcal{N}_I$ equations and it is sufficient to prove that $\dim \mathcal{P}_k(T) = \mathcal{N}_F + \mathcal{N}_I$. In fact, it holds

$$\begin{aligned} \mathcal{N}_F + \mathcal{N}_I - \dim \mathcal{P}_k(T) &= \binom{d-1+k}{k} + \binom{k-1+d}{d} - \binom{d+k}{k} \\ &= \frac{(d-1+k)!}{(d-1)!(k-1)!}\left(\frac{1}{k} + \frac{1}{d} - \frac{d+k}{kd}\right) \\ &= 0\ . \end{aligned}$$

□

Note that these functions again correspond to a Lagrange Basis on a uniform refined mesh, such that the following theorem implies a possible fast implementation of the facet-based Raviart-Thomas basis functions.

Theorem 1. *A basis for $RT_k^{\mathbf{n}}(T)$ is given by $\{\psi_{i,F,k}\}_{F\in\mathcal{F},i=1..d}$, with*

$$\psi_{i,F,k}(\mathbf{x}) = \begin{cases} \frac{|F|}{2|T_+|}(\mathbf{x}-P_+)q_{F,i,T_+,k}(\mathbf{x}) & on\ |T_+| \\ -\frac{|F|}{2|T_-|}(\mathbf{x}-P_-)q_{F,i,T_-,k}(\mathbf{x}) & on\ |T_-| \quad i=1,...,d\ . \\ 0 & elsewhere \end{cases} \tag{31}$$

Using the lowest-order basis function, it holds

$$\psi_{i,F,k} = \hat{\psi}_F \cdot q_{F,i,k},\ i=1,...,d \tag{32}$$

with

$$q_{F,i,k} = \begin{cases} q_{F,i,T_+,k} & on\ |T_+| \\ q_{F,i,T_-,k} & on\ |T_-| \quad i=1,...,d\ . \\ 0 & elsewhere \end{cases} \tag{33}$$

Proof. Similarly to the quadratic case, the construction of the basis function implies $\psi_{i,F,k}\in RT_k^{\mathbf{n}}(T)$. Further for $\mathbf{x}\notin F$, since $\hat{\psi}_{i,F,k}(\mathbf{x})\cdot\mathbf{n}=0$ it follows from (32) that $\psi_{i,F,k}(\mathbf{x})\cdot\mathbf{n}=0$. It remains to prove that for a given facet F, the d functions $\{\psi_{i,F,k}(\mathbf{x})\cdot\mathbf{n}=0\}_{i=1,...,d}$ are linearly independent. Due to (32), this is equivalent to the fact that the d functions $q_{F,i,k}, i=1,...,d$ are linearly independent for a given facet F, and this is the statement of lemma 3. □

It remains to construct a basis for $RT_k^{int}(T)$. Therefore, for a point $S\in\mathcal{S}_k(T)$ consider the scalar-valued polynomial $q_{F,T,S}: T\to\mathbf{R}$ of order k such that

$$\begin{aligned} &q_{F,T,S,k}(S)=1 \\ &q_{F,T,S,k}(P)=0,\ P\in\mathcal{R}_k(F) \\ &q_{F,T,S,k}(P)=0,\ P\in\mathcal{S}_k(T)\backslash S\ . \end{aligned} \tag{34}$$

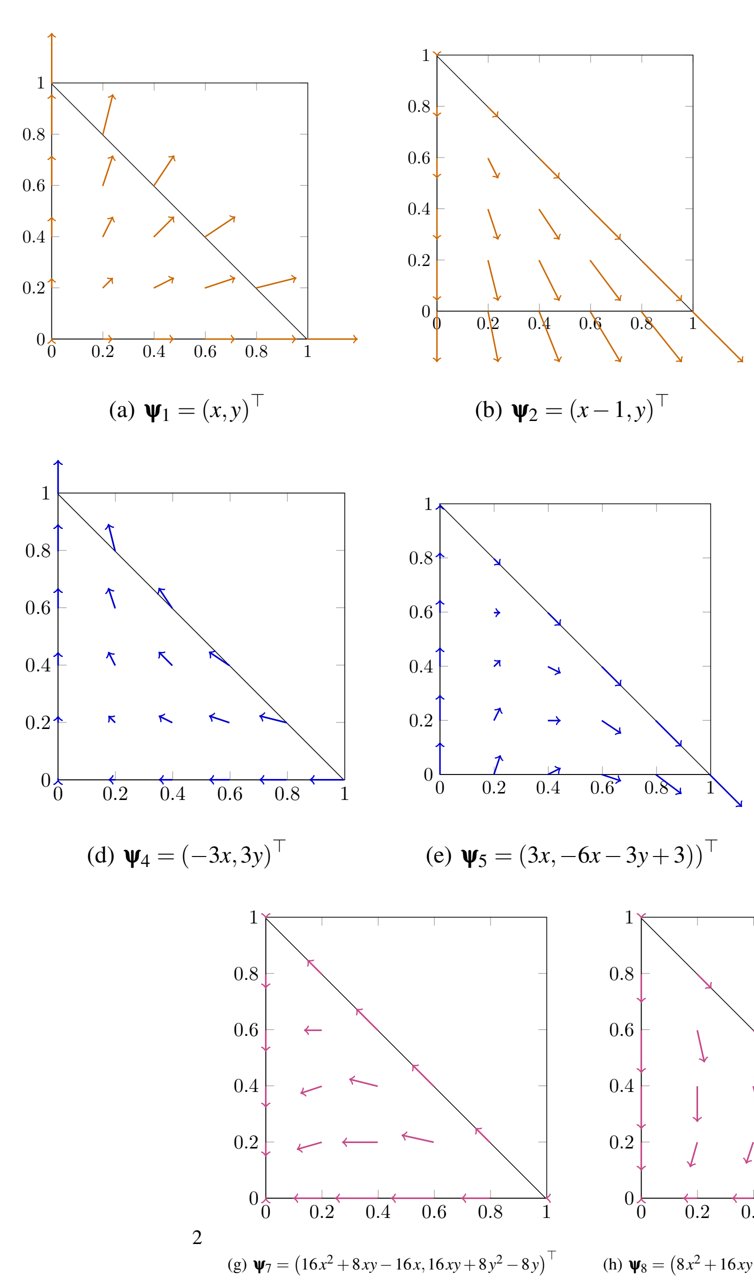
These polynomial functions leads to a basis of $RT_k^{int}(T)$, as the following theorem states.

Theorem 2. $\{\psi_{i,j,T}\}_{T\in\mathcal{T},i\in\{1,2,3,4\}\backslash\{i_{min}\},j=1,...,\mathcal{N}_i}$ *with*

$$\psi_{i,j,T} = \hat{\psi}_{F(P_i)}|_T \cdot q_{F(P_i),T,S_{T,k,i},k} \tag{35}$$

define a basis for $RT_k^{int}(T)$.

Proof. Recall that the normal component of $\hat{\psi}_{F(P_i}$ vanishes on all facet of T but F. Thus, $q_{F,T,S,k}=0$ on F implies that the normal component of $\psi_{i,j,T}$ vanishes on all facet, such that it remains to check that the $d\cdot\mathcal{N}_i$ functions $\psi_{i,j,T}$ are linearly independent for a given T. This is ensured by the combiation of the linearly independency of $\hat{\psi}_{F(P_i,T)}$, $i\in\{1,2,3,4\}\backslash\{i_{min}(T)\}$ and the N_i nodes values. □



(a) $\boldsymbol{\psi}_1 = (x, y)^\top$ (b) $\boldsymbol{\psi}_2 = (x-1, y)^\top$ (c) $\boldsymbol{\psi}_3 = (x, y-1)^\top$

(d) $\boldsymbol{\psi}_4 = (-3x, 3y)^\top$ (e) $\boldsymbol{\psi}_5 = (3x, -6x-3y+3))^\top$ (f) $\boldsymbol{\psi}_6 = (3x+6y-3, -3y))^\top$

(g) $\boldsymbol{\psi}_7 = (16x^2+8xy-16x, 16xy+8y^2-8y)^\top$ (h) $\boldsymbol{\psi}_8 = (8x^2+16xy-8x, 8xy+16y^2-16y)^\top$

Figure 5: An other set of Ansatzfunctions on reference triangle

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