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On the derivation and possibilities of the secant stiffness matrix for non linear finite element analysis

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Abstract In this paper the general non symmetric parametric form of the incremental secant stiffness matrix for non linear analysis of solids using the finite element metod is derived. A convenient symmetric expression for a particular value of the parameters is obtained. The geometrically non linear formulation is based on a Generalized Lagrangian approach. Detailed expressions of all the relevant matrices involved in the analysis of 3D solids are obtained. The possibilities of application of the secant stiffness matrix for non linear structural problems including stability, bifurcation and limit load analysis are also discussed. Examples of application are given for the non linear analysis of pin joined frames.

1 Introduction

The use of numerical solution strategies based on the secant stiffness matrix has received little attention in the context of non linear structural analysis by the finite element method. An exception to this is the use of quasi-Newton incremental iterative algorithms (i.e. *BFGS*, etc.) (Zienkiewicz and Taylor (1991); Bathe (1982); Geradin, Idelsohn and Hodge (1981)) where approximate pseudo-secant stiffness matrices are used as iteration matrices within each iteration loop.

The potential of using the "exact" form of the secant stiffness matrix for developing new solution algorithms for non linear structural problem has been recently recognized by different authors. Oñate, Oliver, Miquel and Suarez (1986) and Oñate (1991) proposed different secant iteration procedures for geometrically non linear problems. Kroplin and co-workers have successfully used the secant stiffness matrix, derived via mixed finite element methods, to propose new solution strategies based on energy perturbation techniques for predicting the degree of instability and estimating limit and bifurcation points for static and dynamic structural problems [see Kropin, Dinkler and Hillmann (1985); Kropin and Dinkler (1988); Duddeck, Kroplin, Dinkler, Hillmann and Wagenhuber (1989); Kropin, Wilhelm and Herrmann (1991); Kroplin (1992)]. The use of secant stiffness operators has also been exploited by Kroplin and Dinkler (1990) to reinterpret the concept of consistent tangent operators in plasticity.

One of the problems in using secant stiffness matrix based procedures is that the expression of this matrix is not unique and non symmetrical forms are typically found unless a careful derivation is performed.

The first attempt to obtain the expression of the secant stiffness matrix in the displacement finite element context was apparently due to Mallet and Marcal (1968) who presented a general approach for non linear finite element structural analysis based on a Total Lagrangian (TL) description. Starting from the general expression of the total potential energy Mallet and Marcal expressed the force-balance equilibrium equation in the form

$$[\mathbf{K}_0+\tfrac{1}{2}\mathbf{N}_1+\tfrac{1}{3}\mathbf{N}_2]\mathbf{a}-\mathbf{f}=\mathbf{K}_S\mathbf{a}-\mathbf{f}=\mathbf{0} \tag{1}$$

where $\mathbf{f}$ is the total external force vector, $\mathbf{K}_0$ is the linear symmetric stiffness matrix and $\mathbf{N}_1$ and $\mathbf{N}_2$ are non linear symmetric stiffness matrices that depend linearly and quadratically on the total

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nodal displacements a. The form of the secant stiffness matrix $K_s = K_0 + \frac{1}{2}N_1 + \frac{1}{3}N_2$ is therefore symmetric.

Later Rajasekaran and Murray (1973) recognized that the expressions used by Mallet and Marcal to evaluate N_1 and N_2 in (1) were not unique and several non symmetric forms of K_S were proposed. Following these ideas Felippa (1974) presented a general parametric expression of the secant stiffness matrix. More recently Felippa and Crivelli (1991) and Felippa, Crivelli and Haugen (1994) have re-formulated the expression of K_S derived in Felippa (1974) using a "core congruential" TL formulation based in the expression of the total potential energy in terms of the displacement gradients.

Other attempts to obtain "attractive" expressions of the secant stiffness matrix are due to Wood and Schrefler (1978) who obtained a symmetrical form of K_S equivalent to (1) using a B-notation. Badawi and Cusens (1992) have presented a simpler expression of the symmetric form derived by Wood and Schrefler. A symmetric secant stiffness matrix for analysis of composite beams and plates has been recently derived by Carrera (1992). Again a TL description was used in all these cases.

In this paper a general parametric form of the secant stiffness matrix for non linear finite element analysis of solids is proposed. The secant matrix is formulated in *an incremental* form using a Generalized Lagrangian (GL) description so as to provide a non linear relationship between *finite increments* of nodal displacements and forces as

$$K_S \Delta a - \Delta f = 0 \tag{2}$$

Eq. (2) is a more general secant expression than that given by (1) and it allows to derive different incremental iterative procedures. Moreover, it allows to directly obtain the tangent stiffness matrix as

$$K_T = \lim_{\Delta a \to 0} K_S \tag{3}$$

in a straightforward manner. Non linear material effects can be also taken into account in the derivation of (2) using an hyperelastic constitutive model. A convenient symmetric expression of K_S can be derived as a particular case of the general parametric form obtained.

The layout of the paper is the following: In the first part the basic geometric, kinematic and constitutive relationships using a *Generalized Lagrangian* description are presented. Then, starting from the virtual work expression the full incremental form of the finite element equations are derived and the corresponding secant stiffness matrix is obtained in a general non symmetric parametric form. A convenient symmetric expression is derived for a particular value of the parameters. The detailed form of the secant stiffness matrix for 3D solid elements is given. Different possibilities of application of the secant stiffness matrix for general non linear structural computations, stability and bifurcation analysis and the prediction of limit loads are then briefly discussed. In the last part of the paper the general formulation is particularized for pin joined bar structures and explicit expressions of all the relevant matrices are given as well as examples of application showing the possibilities of the secant stiffness approach for this kind of problems.

2
Basic kinematic equations

The incremental finite element equations for solving non linear problems can be derived in a variety of ways. The most popular approach is probably to proceed to a direct linearization of the non linear virtual work equation. This leads to the expression of the so called tangent stiffness matrix, traditionally used in the context of Newton-Raphson solution algorythms. This procedure can also be interpreted in several more "rigorous" or "intuitive" equivalent ways like: second variation of the total energy potential, total differential, linear incrementation, Taylor expansion limited to linear terms, direct consequence of the Newton-Raphson method for solution of non linear equation systems, etc. This approach, in one or other form, has been used by an extensive number of authors both in the context of total Lagrangian (TL) or Updated Lagrangian (UL) descriptions. For references see, for example, the list of references of chapter 8 of Vol. 2 of [Zienkiewicz and Taylor (1991)].

A second approach is to derive the incremental equations by substracting the virtual work equations written for two equilibrium configurations and then linearizing the resulting equation. This method was originally suggested in the finite element context by Yaghmai (1968) and then followed by others Horrigmoe (1970); Larsen (1971); Bathe, Ramm and Wilson (1975); Mondkar and Powell (1977). It has been shown by Frey (1978) and Frey and Cescotto (1978) that the final linearized equations are the same in both approaches if the displacement field is linearly interpolated in the displacement nodal unknowns in a standard finite element form.

In this paper we will exploit a less popular third alternative based in the development of the expression of the virtual work equation for an unknown incremental configuration at time $t+\Delta t$. This allows to obtain the *full incremental form* of the finite element equations from which the secant stiffness matrix (and subsequently the standard tangent matrix) can be derived.

We will consider a three dimensional body with initial volume ${}^{0}V$ in equilibrium at a known actual configuration ${}^{t}V$ corresponding to time t under body force ${}^{t}\mathbf{b}$, surface loads ${}^{t}\mathbf{t}$ and point loads ${}^{t}\mathbf{p}$. When these forces are incremented the body changes its configuration from ${}^{t}V$ to ${}^{t+\Delta t}V$. The coordinates of the body in each configuration are refered to the global Cartesian system x_1, x_2, x_3 (see Fig. 1). The displacements at the actual configuration ${}^{t}V$ are defined as

$$ {}^{t}u_i = {}^{t}x_i - {}^{0}x_i, \quad i = 1,2,3 \tag{4a} $$

and the displacement increments from ${}^{t}V$ to ${}^{t+\Delta t}V$ as

$$ \Delta u_i = {}^{t+\Delta t}x_i - {}^{t}x_i \tag{4b} $$

In (3) and (4) right indexes refer to the global coordinate axe along which the displacement is measured, while the left superindex denotes the configuration corresponding to each variable.

The displacement and displacement increment vectors are defined as

$$ {}^{t}\mathbf{u} = [{}^{t}u_1, {}^{t}u_2, {}^{t}u_3]^T, \quad \Delta\mathbf{u} = [\Delta u_1, \Delta u_2, \Delta u_3]^T \tag{4c} $$

Obviously, the total displacements at $t+\Delta t$ are

$$ {}^{t+\Delta t}\mathbf{u} = {}^{t}\mathbf{u} + \Delta\mathbf{u} \tag{4d} $$

A *Generalized Lagrangian* description will be used in which strains and stresses are referred to an intermediate reference configuration ${}^{r}V$ (see Fig. 1). Thus the strain tensor at time $t+\Delta t$ refered to the configuration ${}^{r}V$ can be written as (Malvern (1969))

$$ {}^{t+\Delta t}_{\ \ r}\varepsilon_{ij} = \tfrac{1}{2}\left({}^{t+\Delta t}_{\ \ r}u_{i,j} + {}^{t+\Delta t}_{\ \ r}u_{j,i} + {}^{t+\Delta t}_{\ \ r}u_{k,i}\,{}^{t+\Delta t}_{\ \ r}u_{k,j}\right) \tag{5} $$

where

$$ {}^{t+\Delta t}_{\ \ r}u_{i,j} = \frac{\partial\, {}^{t+\Delta t}u_i}{\partial\, {}^{r}x_j} \quad i,j = 1,2,3 \tag{6} $$

The left index in (5) and (6) denotes the configuration in which strains (and stresses) are refered. Note that for ${}^{r}V = {}^{0}V$ Eq. (5) yields precisely the well known expression of the Green-Lagrange strain tensor in the Total Lagrangian (TL) description. Also for ${}^{r}V = {}^{t}V$ the expression of the Almansi strain tensor, typical of the Updated Lagrangian (UL) formulation can be derived from (5).

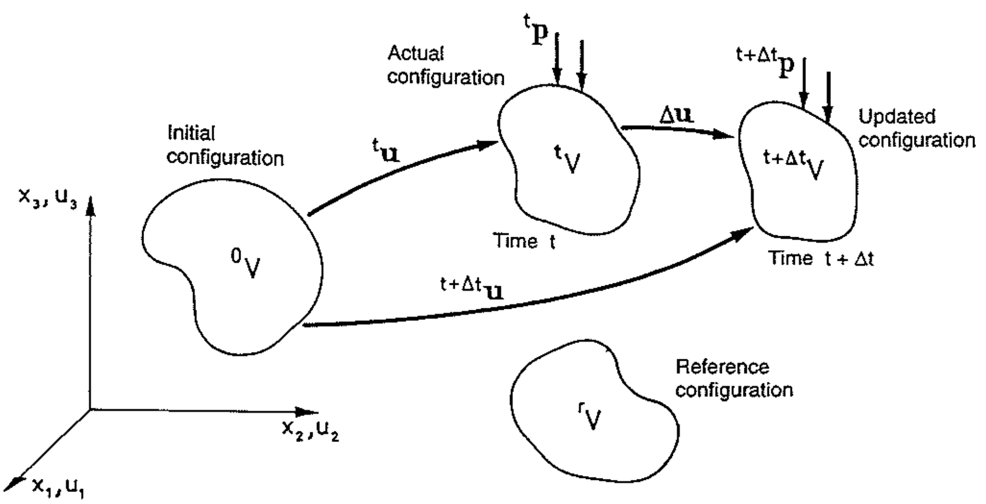


Fig. 1. Deformation of a body in a stationary Cartesian coordinate system

The strain increments are obtained as

$$ {}_r\Delta\varepsilon_{ij} = {}^{t+\Delta t}_{r}\varepsilon_{ij} - {}^{t}_{r}\varepsilon_{ij} = {}_r e_{ij} + {}_r\eta_{ij} \tag{7} $$

where ${}_r e_{ij}$ and ${}_r\eta_{ij}$ are the first and second order strain increments. From (4d) and (7) it can be obtained

$$ {}_r e_{ij} = \tfrac{1}{2}({}_r\Delta u_{i,j} + {}_r\Delta u_{j,i} + \underbrace{{}^{t}_{r}u_{k,i}\,{}_r\Delta u_{k,j} + {}_r\Delta u_{k,i}\,{}^{t}_{r}u_{k,j}}_{=0 \quad \text{for} \quad {}^rV = {}^tV}) \tag{8a} $$

$$ {}_r\eta_{ij} = \tfrac{1}{2}\,{}_r\Delta u_{k,i}\,{}_r\Delta u_{k,j} \tag{8b} $$

where

$$ {}_r\Delta u_{i,j} = \frac{\partial(\Delta u_i)}{\partial^r x_j} \qquad i,j = 1,2,3 \tag{9} $$

Equations (8a) and (8b) are easily particularized for the TL and UL formulations simply by making $r = 0$ and $r = t$, respectively. Note, that the underlined terms in (8a) are zero in the UL formulation (${}^rV = {}^tV$) (Malvern (1969); Bathe (1982)).

The virtual strains are defined as the first variation of the strains in the configuration ${}^{t+\Delta t}V$. On the other hand, the displacements ${}^t u_i$ can be considered as fixed during the deformation increment and thus $\delta\,{}^t u_i = 0$. Taking this into account we can write

$$ \delta\,{}^{t+\Delta t}_{r}\varepsilon_{ij} = \delta\,{}_r e_{ij} + \delta\,{}_r\eta_{ij} \tag{10} $$

where

$$ \delta_r e_{ij} = \tfrac{1}{2}(\delta_r\Delta u_{i,j} + \delta_r\Delta u_{j,i} + \underline{{}^{t}_{r}u_{k,i}\,\delta_r\Delta u_{k,j} + \delta_r\Delta u_{k,i}\,{}^{t}_{r}u_{k,j}}) \tag{11a} $$

$$ \delta_r\eta_{ij} = \tfrac{1}{2}(\delta_r\Delta u_{k,i}\,{}_r\Delta u_{k,j} + {}_r\Delta u_{k,i}\,\delta_r\Delta u_{k,j}) \tag{11b} $$

with

$$ \delta_r\Delta u_{i,j} = \frac{\partial(\delta\Delta u_i)}{\partial^r x_j}, \qquad i,j = 1,2,3 \tag{12} $$

where $\delta\Delta u_i$ are the virtual displacement increments. Again the underlined terms in (11a) are zero in an UL formulation.

Finally, the first and second order strain increments are defined in matrix form for 3D solids as

$$ {}_r\mathbf{e} = [{}_r e_{11}, {}_r e_{22}, {}_r e_{33}, 2\,{}_r e_{12}, 2\,{}_r e_{13}, 2\,{}_r e_{23}]^T \tag{13a} $$

$$ {}_r\boldsymbol{\eta} = [{}_r\eta_{11}, {}_r\eta_{22}, {}_r\eta_{33}, 2\,{}_r\eta_{12}, 2\,{}_r\eta_{13}, 2\,{}_r\eta_{23}]^T \tag{13b} $$

and their corresponding virtual expressions by

$$ \delta_r\mathbf{e} = [\delta_r e_{11}, \delta_r e_{22}, \delta_r e_{33}, 2\delta_r e_{12}, 2\delta_r e_{13}, 2\delta_r e_{23}]^T \tag{14a} $$

$$ \delta_r\boldsymbol{\eta} = [\delta_r\eta_{11}, \delta_r\eta_{22}, \delta_r\eta_{33}, 2\delta_r\eta_{12}, 2\delta_r\eta_{13}, 2\delta_r\eta_{23}]^T \tag{14b} $$

For convenience we will write the first and second order strain increment vectors as

$$ {}_r\mathbf{e} = [\mathbf{L}_0 + {}^{t}_{r}\mathbf{L}_1({}^{t}_{r}\mathbf{g})]\,{}_r g \tag{15a} $$

$$ {}_r\boldsymbol{\eta} = \tfrac{1}{2}\,{}_r\mathbf{L}_1\,{}_r g \tag{15b} $$

where the gradient vectors ${}^{t}_{r}g$ and ${}_r g$ are defined as

$$ {}^{t}_{r}g = \begin{Bmatrix} {}^{t}_{r}u_{,1} \\ {}^{t}_{r}u_{,2} \\ {}^{t}_{r}u_{,3} \end{Bmatrix} \quad \text{and} \quad {}_r g = \begin{Bmatrix} {}_r\Delta u_{,1} \\ {}_r\Delta u_{,2} \\ {}_r\Delta u_{,3} \end{Bmatrix} \tag{16a} $$

with

$$ {}^{t}_{r}\mathbf{u}_{,i} = \begin{Bmatrix} \dfrac{\partial^t u_1}{\partial^r x_i} \\ \dfrac{\partial^t u_2}{\partial^r x_i} \\ \dfrac{\partial^t u_3}{\partial^r x_i} \end{Bmatrix}; \quad {}_{r}\Delta\mathbf{u}_{,i} = \begin{Bmatrix} \dfrac{\partial \Delta u_1}{\partial^r x_i} \\ \dfrac{\partial \Delta u_2}{\partial^r x_i} \\ \dfrac{\partial \Delta u_3}{\partial^r x_i} \end{Bmatrix} \tag{16b} $$

In (15) $\mathbf{L}_0$ is a 6×9 rectangular matrix which contains ones and zeros (see Appendix I), ${}^{t}_{r}\mathbf{L}_1$ and ${}_{r}\mathbf{L}_1$ are displacement and increment dependent matrices defined as

$$ {}^{t}_{r}\mathbf{L}_1 = \begin{bmatrix} {}^{t}_{r}g^T \mathbf{H}_1 \\ {}^{t}_{r}g^T \mathbf{H}_2 \\ \vdots \\ {}^{t}_{r}g^T \mathbf{H}_6 \end{bmatrix}; \quad {}_{r}\mathbf{L}_1 = \begin{bmatrix} {}_{r}g^T \mathbf{H}_1 \\ {}_{r}g^T \mathbf{H}_2 \\ \vdots \\ {}_{r}g^T \mathbf{H}_6 \end{bmatrix} \tag{17} $$

where $\mathbf{H}_i$ are displacement independent matrices containing ones and zeros given in Appendix I. A more general case for non standard strain definitions where $\mathbf{H}_i$ can also be a function of the displacements is discussed in Felippa and Crivelli (1991) and Felippa, Crivelli and Haugen (1994).

From (15) it can be easily found (noting that $\delta^t u_i = 0$)

$$ \delta_r \mathbf{e} = (\mathbf{L}_0 + {}^{t}_{r}\mathbf{L}_1)\, \delta_r \mathbf{g} \tag{18a} $$

$$ \delta_r \eta = {}_{r}\mathbf{L}_1 \delta_r \mathbf{g} \tag{18b} $$

Equations (18) will be useful to derive the corresponding finite element expressions in a next section.

3
Stress field and constitutive equation

The second Piola-Kirchhoff stresses in the configurations ${}^{t}V$ and ${}^{t+\Delta t}V$ refered to the configuration ${}^{r}V$ are written as

$$ {}^{t}_{r}\sigma = [{}^{t}_{r}\sigma_{11}, {}^{t}_{r}\sigma_{22}, {}^{t}_{r}\sigma_{33}, {}^{t}_{r}\sigma_{12}, {}^{t}_{r}\sigma_{13}, {}^{t}_{r}\sigma_{23}]^T \tag{19a} $$

$$ {}^{t+\Delta t}_{r}\sigma = [{}^{t+\Delta t}_{r}\sigma_{11}, {}^{t+\Delta t}_{r}\sigma_{22}, {}^{t+\Delta t}_{r}\sigma_{33}, {}^{t+\Delta t}_{r}\sigma_{12}, {}^{t+\Delta t}_{r}\sigma_{13}, {}^{t+\Delta t}_{r}\sigma_{23}]^T \tag{19b} $$

Note that for ${}^{r}V = {}^{t}V$, ${}^{t}_{r}\sigma_{ij}$ coincide with the Cauchy stresses in the actual configuration (Malvern (1968), Bathe (1982)).

The stresses in the updated configuration ${}^{t+\Delta t}_{r}\sigma_{ij}$ are obtained in an incremental form as

$$ {}^{t+\Delta t}_{r}\sigma = {}^{t}_{r}\sigma + {}_{r}\Delta\sigma \tag{20} $$

where

$$ {}_{r}\Delta\sigma = [{}_{r}\Delta\sigma_{11}, {}_{r}\Delta\sigma_{22}, {}_{r}\Delta\sigma_{33}, {}_{r}\Delta\sigma_{12}, {}_{r}\Delta\sigma_{13}, {}_{r}\Delta\sigma_{23}]^T \tag{21} $$

is the vector of second Piola-Kirchhoff stress increments refered to ${}^{r}V$.

We will now assume a constitutive law relating the *finite* second Piola-Kirchhoff stress increments and the corresponding Green-Lagrange strain increments as

$$ {}_{r}\Delta\sigma = {}^{t}_{r}\mathbf{D}\, {}_{r}\Delta\varepsilon \tag{22} $$

where ${}^{t}_{r}\mathbf{D}$ is the incremental constitutive matrix in the configuration ${}^{t}V$ and refered to ${}^{r}V$. The form of this matrix can be obtained by adequate tensor transformation of the constitutive matrix refered to the initial configuration ${}^{\circ}V$ (Malvern (1969); Bathe (1982)).

Combining Eqs. (22) and (7) yields

$$ {}_{r}\Delta\sigma = {}^{t}_{r}\mathbf{D}\,({}_{r}\mathbf{e} + {}_{r}\eta) \tag{23} $$

Remark I. The incremental constitutive relationship (22) can be found for non linear elastic material behaviour using a hyperelastic model (Malvern (1968)).

4
Virtual work expression

Equilibrium at the updated configuration implies satisfaction of the Principle of Virtual Work (PVW) in ${}^{t+\Delta t}V$. This can be written in matrix form as

$$\int_{{}^{t}V} \delta^{t+\Delta t}_{\ r}\varepsilon^{T\ t+\Delta t}_{\ r}\sigma \, dV = \int_{{}^{t}V} \delta^{t+\Delta t}\mathbf{u}^{T\ t+\Delta t}\mathbf{b} \, dV \tag{24}$$

where

$${}^{t+\Delta t}\mathbf{b} = [{}^{t+\Delta t}b_1, {}^{t+\Delta t}b_2, {}^{t+\Delta t}b_3]^T \tag{25}$$

For simplicity only body forces b are assumed to act in (24).

From Eqs. (4d), (10), (20) and (23) and noting again that $\delta^t u_i = 0$, Eq. (24) can be rewritten as

$$\int_{{}^{t}V} [\delta_r \mathbf{e}^T{}_r\Delta\sigma + \delta_r \boldsymbol{\eta}^T{}_r\Delta\sigma + \delta_r \boldsymbol{\eta}^T{}^t_r\sigma] \, dV = \int_{{}^{t}V} \delta\Delta\mathbf{u}^{T\ t+\Delta t}\mathbf{b} \, dV - \int_{{}^{t}V} \delta_r \mathbf{e}^T{}^t_r\sigma \, dV \tag{26}$$

Substituting (23) in (26) we can finally write

$$\begin{aligned} &\int_{{}^{t}V} [\delta_r \mathbf{e}^T{}^t_r\mathbf{D}\,{}_r\mathbf{e} + \underline{(\delta_r \mathbf{e}^T{}^t_r\mathbf{D}\,{}_r\boldsymbol{\eta} + \delta_r \boldsymbol{\eta}^T{}^t_r\mathbf{D}\,{}_r\mathbf{e}) + \delta_r \boldsymbol{\eta}^T{}^t_r\mathbf{D}\,{}_r\boldsymbol{\eta}} + \delta_r \boldsymbol{\eta}^T{}^t_r\sigma] \, dV \\ &\quad = \int_{{}^{t}V} \delta\Delta\mathbf{u}^{T\ t+\Delta t}\mathbf{b} \, dV - \int_{{}^{t}V} \delta_r \mathbf{e}^T{}^t_r\sigma \, dV \end{aligned} \tag{27}$$

Equation (27) is the *full incremental form* of the PVW and it is also the basis for obtaining the incremental finite element equations. Note that the right hand side of (27) is independent of the displacement increments and it will lead to the expression of the out of balance or residual forces after simplification of the virtual displacement increments. On the the other hand, all the terms in the left hand side are a function of the displacement increments. In particular note that the underlined terms in (27) contain cuadratic and cubic expressions of the displacement increments. The consideration of these terms is the basis for the derivation of the secant stiffness matrix. A linearization of Eq. (27) will neglect these terms, yielding the standard tangent stiffness matrix. The derivation of these two matrices for elasticity problems is presented in next section.

Remark II. The form of the PVW in terms of the displacement increment gradients can be obtained simply by substituting eqs. (15) and (18) into (27). A compact form of this expression can be found in Felippa and Crivelli (1991) and Felippa, Crivelli and Haugen (1994).

5
Finite element interpolation. Derivation of the secant stiffness matrix for 3D elasticity problems

We will consider a discretization of a general solid in standard 3D isoparametric C° continuous finite elements with n nodes and nodal shape functions $N^k(\xi, \eta, \zeta)$ defined in the natural coordinate system ξ, η, ζ [Bathe (1982) and Zienkiewicz and Taylor (1989)].

The displacement and displacement increment fields within each element are defined by the standard interpolations

$${}^t\mathbf{u} = \mathbf{N}\,{}^t\mathbf{a} \quad \text{and} \quad \Delta\mathbf{u} = \mathbf{N}\,\Delta\mathbf{a} \tag{28}$$

where

$$\begin{gathered} \mathbf{N} = [\mathbf{N}^1, \mathbf{N}^2, \ldots, \mathbf{N}^n]; \quad \mathbf{N}^k = N^k \mathbf{I}_3 \\ {}^t\mathbf{a} = \begin{Bmatrix} {}^t\mathbf{a}^1 \\ \vdots \\ {}^t\mathbf{a}^n \end{Bmatrix}; \quad \Delta\mathbf{a} = \begin{Bmatrix} \Delta\mathbf{a}^1 \\ \vdots \\ \Delta\mathbf{a}^n \end{Bmatrix}; \quad \begin{aligned} {}^t\mathbf{a}^k &= [{}^tu_1^k, {}^tu_2^k, {}^tu_3^k]^T \\ \Delta\mathbf{a}^k &= [\Delta u_1^k, \Delta u_2^k, \Delta u_1^k]^T \end{aligned} \end{gathered} \tag{29}$$

are the shape function matrices and the displacement and displacement vectors of the element and of a node k, respectively and $\mathbf{I}_3$ is the 3×3 unit matrix.

Substitution of (28) into (16*a*) allows to express the vector of displacement increment gradients in terms of the nodal displacement increments as

$$ {}_r\mathbf{g} = {}_r\mathbf{G}\,\Delta\mathbf{a} \tag{30}$$

$$ {}_r\mathbf{g} = \begin{Bmatrix} {}_r\Delta\mathbf{u}_{,1} \\ {}_r\Delta\mathbf{u}_{,2} \\ {}_r\Delta\mathbf{u}_{,3} \end{Bmatrix}; \quad {}_r\Delta\mathbf{u}_{,i} = \begin{Bmatrix} \dfrac{\partial \Delta u_1}{\partial^r x_i} \\ \dfrac{\partial \Delta u_2}{\partial^r x_i} \\ \dfrac{\partial^r \Delta u_3}{\partial^r x_i} \end{Bmatrix} $$

$$ \boxed{{}_r\mathbf{g} = {}_r\mathbf{G}\,\Delta\mathbf{a}} $$

$$ {}_r\mathbf{G} = [{}_r\mathbf{G}^1, {}_r\mathbf{G}^2, \ldots, {}_r\mathbf{G}^n] $$

$$ {}_r\mathbf{G}^k = \begin{bmatrix} \dfrac{\partial N^k}{\partial^r x_1}\mathbf{I}_3 \\ \dfrac{\partial N^k}{\partial^r x_2}\mathbf{I}_3 \\ \dfrac{\partial N^k}{\partial^r x_3}\mathbf{I}_3 \end{bmatrix}; \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Box I. Expression of the gradient vector ${}_r\mathbf{g}$ and the gradient matrix ${}_r\mathbf{G}$ for 3D elasticity

The form of the gradient matrix ${}_r\mathbf{G}$ for 3D elasticity problems can be seen in Box I. Combining (30) and (15) gives

$$ {}_r\mathbf{e} = [\mathbf{L}_0 + {}^t_r\mathbf{L}_1]\,{}_r\mathbf{G}\,\Delta\mathbf{a} = {}^t_r\mathbf{B}_L\,\Delta\mathbf{a} \tag{31a}$$

$$ {}_r\eta = \tfrac{1}{2}\,{}_r\mathbf{L}_1\,{}_r\mathbf{G}\,\Delta\mathbf{a} = \tfrac{1}{2}\,{}_r\mathbf{B}_1\,\Delta\mathbf{a} \tag{31b}$$

where

$$ {}^t_r\mathbf{B}_L = [\mathbf{L}_0 + {}^t_r\mathbf{L}_1]\,{}_r\mathbf{G} = {}_r\mathbf{B}_{L_0} + {}^t_r\mathbf{B}_{L_1} \tag{32}$$

$$ {}_r\mathbf{B}_1 = {}_r\mathbf{L}_1\,{}_r\mathbf{G} \tag{33}$$

are the first and second order strain increment matrices for each element.

Remark III. Note that ${}_r\mathbf{B}_{L_0} = \mathbf{L}_0\,{}_r\mathbf{G}$ is the standard displacement-independent strain matrix as derived from infinitesimal theory [Bathe (1982), Zienkiewicz and Taylor (1989, 1991)] and ${}^t_r\mathbf{B}_{L_1} = {}^t_r\mathbf{L}_1\,{}_r\mathbf{G}$ is the *displacement-dependent* part of the first order strain increment matrix.

Remark IV. Matrices ${}^t_r\mathbf{B}_L$ and ${}_r\mathbf{B}_1$ can be computed node-wise in terms of the infinitesimal strain matrix in the following simple manner:

$$ {}^t_r\mathbf{B}^k_L({}^t\mathbf{a}) = {}_r\mathbf{B}^k_{L_0}[\mathbf{I}_3 + {}^t_r\mathbf{L}^T({}^t\mathbf{a})] = {}_r\mathbf{B}^k_{L_0} + {}^t_r\mathbf{B}^k_{L_1} \tag{34a}$$

$$ {}_r\mathbf{B}^k_1(\Delta\mathbf{a}) = {}_r\mathbf{B}^k_{L_0}\,{}_r\hat{\mathbf{L}}^T(\Delta\mathbf{a}) \tag{34b}$$

where ${}^t_r\mathbf{L}$ and ${}_r\hat{\mathbf{L}}$ are 3×3 matrices of the same form depending on the displacements ${}^t\mathbf{a}$ and the displacement increments $\Delta\mathbf{a}$, respectively. The form of all the matrices appearing in Eqs. (34) is shown in Box II for the case of 3D solids. Note that ${}^t_r\mathbf{L} = 0$ (and hence ${}^t_r\mathbf{B}^k_{L_1} = 0$) in an UL formulation.

Equations (18) and (31) allow to obtain the virtual strain increments as

$$ \delta_r\mathbf{e} = {}^t_r\mathbf{B}_L\,\delta(\Delta\mathbf{a}), \quad \delta_r\eta = {}_r\mathbf{B}_1\,\delta(\Delta\mathbf{a}) \tag{35}$$

5.1
Discretization of the incremental constitutive equation

Substituting (31) into (23) yields the constitutive equation in the form

$$ {}_r\Delta\sigma = {}^t_r\mathbf{D}\,[{}^t_r\mathbf{B}_L + \tfrac{1}{2}\,{}_r\mathbf{B}_1]\,\Delta\mathbf{a} \tag{36a} $$

Making use of Eq. (34*a*) and Box II allows to write (36*a*) as

$$ \begin{aligned} {}_r\Delta\sigma &= {}^t_r\mathbf{D}\sum_k {}_r\mathbf{B}^k_{L_0}[\mathbf{I}_3 + {}^t_r\mathbf{L}^T({}^t\mathbf{a}) + \tfrac{1}{2}\,{}_r\hat{\mathbf{L}}^T(\Delta\mathbf{a})]\,\Delta\mathbf{a}^k \\ &= {}_r\Delta\sigma^l + {}^t_r\mathbf{D}\sum_k {}_r\mathbf{B}^k_{L_0}[{}^t_r\mathbf{L}^T({}^t\mathbf{a}) + \tfrac{1}{2}\,{}_r\hat{\mathbf{L}}^T(\Delta\mathbf{a})]\,\Delta\mathbf{a}^k \end{aligned} \tag{36b} $$

where ${}_r\Delta\sigma^l$ is the stress increment obtained from infinitesimal elasticity theory. Equation (36*b*) can be useful for computational implementation.

5.2
Derivation of the secant stiffness matrix

By substituting the discretized finite element equations into the PVW (Eq. (27)) a expression relating the total applied forces with the corresponding nodal displacement increments can be obtained, thus yielding the form of the incremental secant stiffness matrix. However, the expression of this matrix is not unique and generally non symmetrix. A method for deriving a parametric expression of the incremental secant stiffness matrix is presented next.

$$ {}_r\mathbf{e} = {}^t_r\mathbf{B}_L\,\Delta\mathbf{a}; \quad {}_r\eta = \tfrac{1}{2}\,{}_r\mathbf{B}_1\,\Delta\mathbf{a} $$

$$ {}^t_r\mathbf{B}_L = [{}^t_r\mathbf{B}^1_L, {}^t_r\mathbf{B}^2_L, \dots, {}^t_r\mathbf{B}^n_1] $$

$$ {}_r\mathbf{B}_1 = [{}_r\mathbf{B}^1_1, {}_r\mathbf{B}^2_1, \dots, {}_r\mathbf{B}^n_1] $$

$$ {}^t_r\mathbf{B}^k_L = {}_r\mathbf{B}^k_{L_0} + {}^t_r\mathbf{B}^k_{L_1} $$

$$ {}_r\mathbf{B}^k_1 = {}_r\mathbf{B}^k_{L_0}\,{}_r\hat{\mathbf{L}}^T $$

$$ {}_r\mathbf{B}^k_{L_0} = \begin{bmatrix} {}_rN^k_{,1} & 0 & 0 \\ 0 & {}_rN^k_{,2} & 0 \\ 0 & 0 & {}_rN^k_{,3} \\ {}_rN^k_{,2} & {}_rN^k_{,1} & 0 \\ {}_rN^k_{,3} & 0 & {}_rN^k_{,1} \\ 0 & {}_rN^k_{,3} & {}_rN^k_{,2} \end{bmatrix}; \quad {}^t_r\mathbf{B}^k_{L_1} = {}_r\mathbf{B}^k_{L_0}\,{}^t_r\mathbf{L}^T $$

$$ {}^t_r\mathbf{L} = [{}^t_r\mathbf{u}_{,1}, {}^t_r\mathbf{u}_{,2}, {}^t_r\mathbf{u}_{,3}]; \quad {}_r\hat{\mathbf{L}} = [{}_r\Delta\mathbf{u}_{,1}, {}_r\Delta\mathbf{u}_{,2}, {}_r\Delta\mathbf{u}_{,3}] $$

with ${}_rN^k_{,i} = \dfrac{\partial N^k}{\partial^r x_i}$ and ${}^t_r\mathbf{u}_{,i}$, ${}_r\Delta\mathbf{u}_{,i}$ as given in Eq. (16*b*)

Box II. First and second order strain increment matrices for 3D solids

The terms involving displacement increments in the left hand side of the PVW expression, Eq. (27), can be written after some algebra (for details see Oñate (1994)), using Eqs. (31), (35) and (36*a*), as

$$ \delta_r\mathbf{e}^T\,{}^t_r\mathbf{D}\,{}_r\mathbf{e}_r = \delta(\Delta\mathbf{a})^T\,[{}^t_r\mathbf{B}^T_L\,{}^t_r\mathbf{D}\,{}^t_r\mathbf{B}_L]\,\Delta\mathbf{a} \tag{37a} $$

$$ \delta_r\eta^T\,{}^t_r\mathbf{D}\,{}_r\mathbf{e} + \delta_r\mathbf{e}^T\,{}^t_r\mathbf{D}\,{}_r\eta = \delta(\Delta\mathbf{a})^T[\tfrac{1}{2}\,{}^t_r\mathbf{B}^T_L\,{}^t_r\mathbf{D}\,{}_r\mathbf{B}_1 + \alpha\,{}_r\mathbf{B}^T_1\,{}^t_r\mathbf{D}\,{}^t_r\mathbf{B}_L + (1-\alpha)\,{}_r\mathbf{G}^T\,{}^t_r\mathbf{E}\,{}_r\mathbf{G}]\,\Delta\mathbf{a} \tag{37b} $$

$$ \delta_r\eta^T\,{}^t_r\mathbf{D}\,{}_r\eta = \delta(\Delta\mathbf{a})^T\left[\frac{1}{4}(2-\beta)\,{}_r\mathbf{B}^T_1\,{}^t_r\mathbf{D}\,{}_r\mathbf{B}_1 + \frac{\beta}{4}\,{}_r\mathbf{G}^T\,{}^t_r\mathbf{H}\,{}_r\mathbf{G}\right]\Delta\mathbf{a} \tag{37c} $$

$$ \delta_r\eta^T\,{}^t_r\sigma = \delta(\Delta\mathbf{a})^T\,[{}_r\mathbf{B}^T_{NL}\,{}^t_r\mathbf{S}\,{}_r\mathbf{B}_{NL}]\,\Delta\mathbf{a} \tag{37d} $$

In (37) all the matrices and vectors have been previously defined with the exception of ${}^t_r\mathbf{E}$, ${}_r\mathbf{B}_{NL}$, ${}^t_r\mathbf{H}$ and ${}^t_r\mathbf{S}$ which are shown in Box III for the case of 3D elastic solids.

$$
{}^t_r\mathbf{E} = \begin{bmatrix} a\mathbf{I}_3 & d\mathbf{I}_3 & e\mathbf{I}_3 \\ & b\mathbf{I}_3 & f\mathbf{I}_3 \\ \text{sym.} & & c\mathbf{I}_3 \end{bmatrix}; \quad [a, b, c, d, e, f\,]^T = {}^t_r\mathbf{D}\,{}_r\mathbf{e}
$$

$$
{}_r\mathbf{B}_{NL} = \begin{bmatrix} {}_r\bar{\mathbf{B}}_{NL} & \bar{0} & \bar{0} \\ \bar{0} & {}_r\bar{\mathbf{B}}_{NL} & \bar{0} \\ \bar{0} & \bar{0} & {}_r\bar{\mathbf{B}}_{NL} \end{bmatrix}; \quad {}_r\bar{\mathbf{B}}_{NL} = \begin{bmatrix} {}_rN^1_{,1} & 0 & 0 & {}_rN^2_{,1} & 0 & 0 & \cdots & {}_rN^n_{,1} \\ {}_rN^1_{,2} & 0 & 0 & {}_rN^2_{,2} & 0 & 0 & \cdots & {}_rN^n_{,2} \\ {}_rN^1_{,3} & 0 & 0 & {}_rN^2_{,3} & 0 & 0 & \cdots & {}_rN^n_{,3} \end{bmatrix}
$$

$$
{}^t_r\mathbf{S} = \begin{bmatrix} {}^t_r\bar{\mathbf{S}} & 0 & 0 \\ 0 & {}^t_r\bar{\mathbf{S}} & 0 \\ 0 & 0 & {}^t_r\bar{\mathbf{S}} \end{bmatrix}; \quad {}^t_r\bar{\mathbf{S}} = \begin{bmatrix} {}^t_r\sigma_{11} & {}^t_r\sigma_{12} & {}^t_r\sigma_{13} \\ {}^t_r\sigma_{12} & {}^t_r\sigma_{22} & {}^t_r\sigma_{23} \\ {}^t_r\sigma_{13} & {}^t_r\sigma_{23} & {}^t_r\sigma_{33} \end{bmatrix}
$$

$$
0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \bar{0} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}; \quad {}_rN^k_{,i} = \frac{\partial N^k}{\partial\, {}^r x_i}
$$

$$
{}^t_r\mathbf{H} = \sum_{i=1}^{6}\sum_{j=1}^{6} {}^t_r d_{ij}\, {}_r\eta_j \mathbf{H}_i; \quad {}_r\eta_j = {}_r\mathbf{g}^T \mathbf{H}_j\, {}_r\mathbf{g}
$$

${}^t_r d_{ij}$: element ij of constitutive matrix ${}^t_r\mathbf{D}$

Box III. Matrices ${}^t_r\mathbf{E}$, ${}_r\mathbf{B}_{NL}$, ${}^t_r\mathbf{S}$ and ${}^t_r\mathbf{H}$ for 3D elastic solids.

Substituting Eqs. (37) in (27) the following secant expression is obtained, after simplification of the arbitrary virtual displacement increments,

$$
{}^t_r\mathbf{K}_S(\Delta\mathbf{a})\,\Delta\mathbf{a} = -{}^t_r\mathbf{r} \tag{38}
$$

In Eq. (38) ${}^t_r\mathbf{r}$ is the standard residual force vector [Bathe (1982) and Zienkiewicz and Taylor (1991)] which can be written for each element with volume ${}^rV^{(e)}$ as

$$
{}^t_r\mathbf{r}^{(e)} = \int_{{}^rV^{(e)}} {}^t_r\mathbf{B}_L^T\, {}^t_r\boldsymbol{\sigma}\, dV - {}^{t+\Delta t}\mathbf{f}^{(e)} \tag{39a}
$$

with

$$
{}^{t+\Delta t}\mathbf{f}^{(e)} = \int_{{}^rV^{(e)}} \mathbf{N}^{T\,t+\Delta t}\mathbf{b}\, dV \tag{39b}
$$

being the equivalent nodal force vector for the element, and ${}^t_r\mathbf{K}_S$ is the *incremental secant stiffness matrix* which can be written as

$$
\boxed{{}^t_r\mathbf{K}_S(\Delta\mathbf{a}) = {}^t_r\mathbf{K}_L + {}^t_r\mathbf{K}_M(\Delta\mathbf{a}) + {}^t_r\mathbf{K}_N(\Delta\mathbf{a}^2) + {}^t_r\mathbf{K}_\sigma} \tag{40}
$$

where for each element

$$
{}^t_r\mathbf{K}_L = \int_{{}^rV^{(e)}} {}^t_r\mathbf{B}_L^T\, {}^t_r\mathbf{D}\, {}^t_r\mathbf{B}_L\, dV \tag{41a}
$$

$$
{}^t_r\mathbf{K}_M(\Delta\mathbf{a}) = \int_{{}^rV^{(e)}} \left[\tfrac{1}{2}\, {}^t_r\mathbf{B}_L^T\, {}^t_r\mathbf{D}\, {}_r\mathbf{B}_1 + \alpha\, {}_r\mathbf{B}_1^T\, {}^t_r\mathbf{D}\, {}^t_r\mathbf{B}_L + (1-\alpha)\, {}_r\mathbf{G}^T\, {}^t_r\mathbf{E}\, {}_r\mathbf{G} \right] dV \tag{41b}
$$

$$
{}^t_r\mathbf{K}_N(\Delta\mathbf{a}^2) = \int_{{}^rV^{(e)}} \left[\frac{1}{4}(2-\beta)\, {}_r\mathbf{B}_1^T\, {}^t_r\mathbf{D}\, {}_r\mathbf{B}_1 + \frac{\beta}{4}\, {}_r\mathbf{G}^T\, {}^t_r\mathbf{H}\, {}_r\mathbf{G} \right] dV \tag{41c}
$$

$$
{}^t_r\mathbf{K}_\sigma = \int_{{}^rV^{(e)}} {}_r\mathbf{B}_{NL}^T\, {}^t_r\mathbf{S}\, {}_r\mathbf{B}_{NL}\, dV \tag{41d}
$$

The global secant stiffness matrix and the residual force vector for the whole structure are assembled from the individual element contributions in the standard manner [Zienkiewicz and Taylor (1991)].

Remark V. A similar parametric form of the secant stiffness matrix relating total displacements and external forces was deduced by Felippa and Crivelli (1991) and Felippa, Crivelli and Haugen (1994) using a "core congruential formulation" and a TL description. The secant expression deduced here can be considered an extension of the work carried out in these references as it relates finite displacement *increments* and residual forces using a GL description.

Remark VI. Note that the expression of the incremental secant stiffness matrix is non symmetric for values of $\alpha \neq 1/2$. An infinite set of *symmetric forms* is obtained for $\alpha = 1/2$ depending on the values of the parameter β. The particular symmetric expression of the incremental secant stiffness matrix for $\alpha = 1/2$, $\beta = 0$ was derived by Oñate (1991) also using a GL description. Other symmetric forms of the secant stiffness matrix relating *total displacements* and applied forces using a TL description have been proposed in Mallet and Marcal (1968); Rajasekaran and Murray (1973); Wood and Schrefler (1978), Badawi and Cusens (1992); Carrera (1992). 581

Remark VII. Note that in (41) ${}^t_r\mathrm{K}_L$ and ${}^t_r\mathrm{K}_\sigma$ are the standard finite element initial displacement and initial stress matrices, respectively Bathe (1982); Zienkiewicz and Taylor (1991) The new matrices ${}^t_r\mathrm{K}_M$ and ${}^t_r\mathrm{K}_N$ are a linear and quadratic function of the displacement increments respectively and their contribution will be zero in a linearization of the PVW.

By substituting Eq. (34a) into (41a) the computation of the initial displacement matrix ${}^t_r\mathrm{K}_L$ can be splitted up in the usual manner [Zienkiewicz and Taylor (1991)]

$$ {}^t_r\mathrm{K}_L = {}^t_r\mathrm{K}_{L_0} + {}^t_r\mathrm{K}_{L_1} \tag{42} $$

where

$$ {}^t_r\mathrm{K}^{kj}_{L_0} = \int_{{}^rV^{(e)}} {}^t_r\mathrm{H}^{kj}_0 \, dV \tag{43} $$

with ${}^t_r\mathrm{H}^{kj}_0 = [{}_r\mathrm{B}^k_{L_0}]^T \, {}^t_r\mathrm{D} \, {}_r\mathrm{B}^j_{L_0}$, is the standard contribution to the stiffness matrix from the infinitesimal theory and

$$ {}^t_r\mathrm{K}^{kj}_{L_1} = \int_{{}^rV^{(e)}} [{}^t_r\mathrm{H}^{kj}_0 \, {}^t_r\mathrm{L}^T + {}^t_r\mathrm{L} \, [{}^t_r\mathrm{H}^{kj}_0]^T + {}^t_r\mathrm{L} \, {}^t_r\mathrm{H}^{kj}_0 \, {}^t_r\mathrm{L}^T] \, dV \tag{44} $$

where ${}^t_r\mathrm{L}\,({}^t\mathrm{a})$ was defined in Box II.

Remark VIII. Equations (41)-(44) can be particularized for the TL and UL formulations by making ${}^rV = {}^\circ V$ and ${}^rV = {}^tV$ respectively. Note that in the later case matrix ${}^t_t\mathrm{K}_{L_1} = 0$ due to the fact that ${}^t_t\mathrm{L}$ is then a null matrix as previously mentioned.

5.3
Derivation of the tangent stiffness matrix

The standard way of obtaining the expression of the tangent stiffness matrix is to linearize the PVW expression. This implies neglecting the underlined terms in Eq. (27) containing quadratic and cubic terms in the displacement increments. An alternative method, which should lead to identical results, is to derive the tangent matrix as the limit of the incremental secant matrix when the values of the displacement increments tend to zero. Thus starting from the expression of ${}^t_r\mathrm{K}_s$ of Eq. (40) we can write

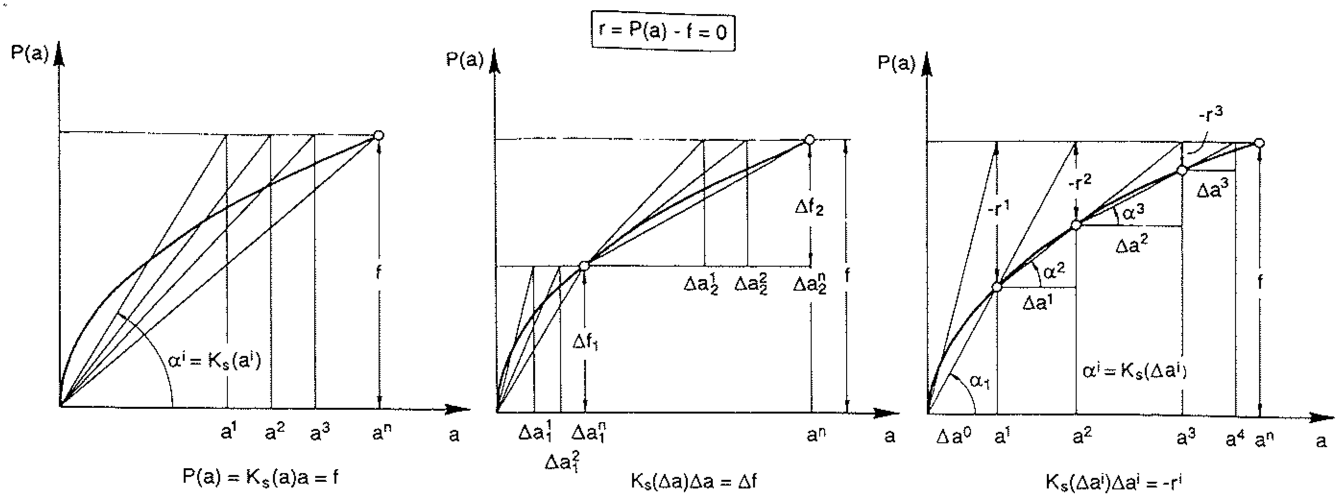
$$ {}^t_r\mathrm{K}_T = \lim_{\Delta\mathbf{a} \to 0} {}^t_r\mathrm{K}_S = {}^t_r\mathrm{K}_L + {}^t_r\mathrm{K}_\sigma \tag{45} $$

where ${}^t_r\mathrm{K}_T$ is the standard tangent stiffness matrix [Bathe (1982), Zienkiewicz and Taylor (1991)].

6
Possibilities of application of the secant matrix

Equation (38) can be used as a *total secant* equation to solve the displacements between the initial load-free configuration and any deformed one using a direct iteration scheme (see Fig. 2a) as

$$ {}^{t+\Delta t}\mathbf{a}^{i+1} = [{}^t_r\mathrm{K}_S({}^{t+\Delta t}\mathbf{a}^i)]^{-1} \, {}^{t+\Delta t}\mathbf{f} \tag{46} $$



a Total secant approach b Updated total secant approach c Incremental secant approach

Fig. 2. Total, secant and incremental secant approaches ($P(a) = K_s(a)a$)

where the superindex i denotes the iteration number and ${}^{t+\Delta t}\mathbf{f}$ is the nodal force vector due to the total load applied to the structure. In this case the option ${}^rV = {}^oV$ seems the most convenient one.

Solution of (46) follows until an adequate error norm in the displacement increments is satisfied. Obviously the efficiency of this approach will very much depend on the well known limitations of the direct iteration scheme to solve non linear problems.

Equation (38) is also the starting point for deriving an *updated total secant* approach. This implies dividing the total applied force in several increments and obtaining the displacement increments between two equilibrium configurations by using a total secant algorithm (see Fig. 2b) as

$$\Delta \mathbf{a}^{i+1} = [{}^t_r\mathbf{K}_S(\Delta \mathbf{a}^i)]^{-1}\Delta \mathbf{f} \tag{47}$$

A third alternative is to use an *incremental secant* approach. (Fig. 2c) This is based in the iterative solution of (38) by

$$\Delta \mathbf{a}^i = -[{}^t_r\mathbf{K}_S({}^{t+\Delta t}\mathbf{a}^i, \Delta \mathbf{a}^{i-1})]^{-1}\,{}^t_r\mathbf{r}^i \tag{48a}$$

$${}^{t+\Delta t}\mathbf{a}^{i+1} = {}^{t+\Delta t}\mathbf{a}^i + \Delta \mathbf{a}^i \tag{48b}$$

with ${}^{t+\Delta t}\mathbf{a}^0 = {}^t\mathbf{a}$ and $\Delta \mathbf{a}^{-1} = 0$. Convergence of this algorithm is controlled by means of satisfaction of any of the standard error norms in the displacement increments or the residual force vector [Bathe (1982), Zienkiwicz and Taylor (1991)].

Another interesting application of the secant stiffness matrix is the direct computation of the residual force vector from the known values of the displacement increments (see Eq. (38)). This circunvents the need for stress computations for evaluating the residual vector (i.e. for convergence checking) which might prove to be useful in some situations.

The structure of the secant stiffness matrix allows to combine the use of secant and tangent iteration algorithms in a simple unified manner. This should allow to take advantage of the best properties of both algorithm families to improve the solution of complex non linear problems.

6.1
Applications to predict the degree of structural stability

The secant stiffness matrix can be used to find the distance from an equilibrium point in the pre-buckling state to the nearest point in the neighbouring equilibrium post-buckling state, i.e. the distance between points A and B in Fig. 3. It is obvious that in this case the increment of external forces is zero and Eq. (2) reads

$${}^t_r\mathbf{K}_S({}^t\mathbf{a}, \Delta \mathbf{a})\,\Delta \mathbf{a} = 0 \tag{49}$$

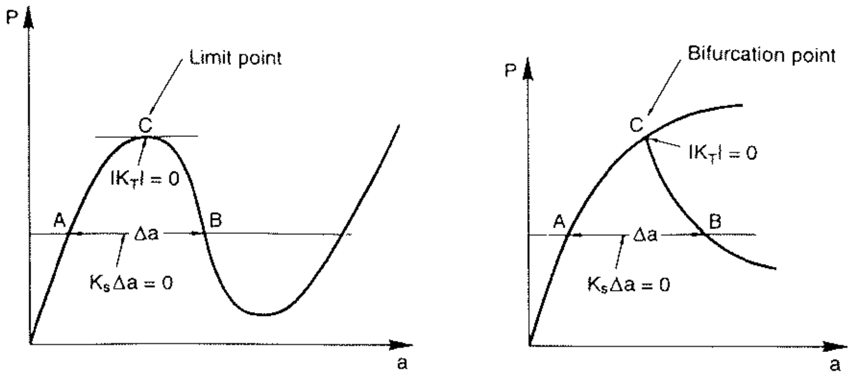


Fig. 3. Examples of limit and bifurcation points

Assuming now $\Delta a = \lambda\alpha$ where α is an estimate of the displacement pattern at the post-buckling state, Eq. (48) can be formulated as the following eigenvalue problem

$$[{}^t_r\mathrm{K}_L({}^t\mathrm{a}) + {}^t_r\mathrm{K}_\sigma({}^t\sigma) + \lambda\,{}^t_r\mathrm{K}_M(\alpha) + \lambda^2\,{}^t_r\mathrm{K}_N(\alpha)]\,\alpha = 0 \tag{50}$$

Equation (50) is a quadratic eigenvalue problem which can be solved using iterative techniques.

This procedure has been successfully exploited by Kroplin, Dinkler and Hillmann (1985) and Kroplin and Dinkler (1988) to define the degree of stability of a structure as

$$\gamma = \frac{U_{\text{crit}} - W_{\text{ext}}}{U_{\text{crit}}} \begin{matrix} > 0 & \text{stable} \\ = 0 & \text{indiferent} \\ < 0 & \text{unstable} \end{matrix} \tag{51}$$

where U_{crit} is the minimum strain energy required to push the structure to the nearest post-buckling state (i.e. from A to B in Fig. (3)) and W_{ext} is the energy of an external perturbation load (the actual loading can be considered a particular case).The values of U_{crit} and W_{ext} can be computed as

$$U_{\text{crit}} = \tfrac{1}{2}\Delta\mathrm{a}^T\,{}^t_r\mathrm{K}_U\,\Delta\mathrm{a}; \quad W_{\text{ext}} = \Delta\mathrm{a}_p^T\mathrm{p} \tag{52}$$

where p is the external perturbation force vector, $\Delta\mathrm{a}_p$ are the displacement increments induced by the perturbation forces (computed via an standard incremental analysis), $\Delta\mathrm{a} = \lambda\alpha$ where λ is the minimum eigenvalue obtained from (50) and K_U is the *energy stiffness matrix* given by

$${}^t_r\mathrm{K}_U = {}^t_r\mathrm{K}_L + {}^t_r\mathrm{K}_\sigma + {}^t_r\bar{\mathrm{K}}_M + {}^t_r\bar{\mathrm{K}}_N \tag{53}$$

with ${}^t_r\mathrm{K}_L$ and ${}^t_r\mathrm{K}_\sigma$ defined as in (41) and

$${}^t_r\bar{\mathrm{K}}_M = \int_{{}^rV^{(e)}} \left[\frac{\phi}{2}({}^t_r\mathrm{B}_L^T\,{}^t_r\mathrm{D}\,{}_r\mathrm{B}_1 + {}_r\mathrm{B}_1^T\,{}^t_r\mathrm{D}\,{}^t_r\mathrm{B}_L) + (1-\phi)\,{}_r\mathrm{G}\,{}^t_r\mathrm{E}\,{}_r\mathrm{G}\right] dV \tag{54}$$

$${}^t_r\bar{\mathrm{K}}_N = \int_{{}^rV^{(e)}} \left[\frac{\gamma}{4}\,{}_r\mathrm{B}_1^T\,{}^t_r\mathrm{D}\,{}_r\mathrm{B}_1 + \frac{1}{4}(1-\gamma)\,{}_r\mathrm{G}^T\,{}^t_r\mathrm{H}\,{}_r\mathrm{G}\right] dV \tag{55}$$

in (54) and (55) ϕ and γ are arbitrary parameters. Note, that ${}^t_r\mathrm{K}_U$ is always symmetric as expected.

The derivation of ${}^t_r\mathrm{K}_U$ follows identical arguments than those used previously to derive the expression of ${}^t_r\mathrm{K}_S$. A similar form of ${}^t_r\mathrm{K}_U$ using a TL formulation is given in Felippa and Crivelli (1991) and Felippa, Crivelli and Haugen (1994).

6.2
Estimates for limit and bifurcation points

The condition of limit and bifurcation points is the singularity of the tangent stiffness matrix ${}^t_r\mathrm{K}_T$. Having solved the non linear problem in (50) the point of vanishing tangent operator can be found at an intermediate position between points A and B (see Fig. 3). Taking for instance the middle point as an estimate, the critical load is found using (2) as

$$\Delta f_{crit} = \left[{}^t_r K_L({}^t a) + {}^t_r K_\sigma({}^t \sigma) + {}^t_r K_M\left(\frac{\Delta a}{2}\right) + {}^t_r K_N\left(\frac{\Delta a^2}{4}\right)\right]\frac{\Delta a}{2} \tag{56}$$

Remark IX. The value $\Delta a/2$ is found to be an exact estimate if a mixed formulation is used [Kroplin, Dinkler and Hillmann (1985) and Kroplin and Dinkler (1988)].

An alternative procedure which avoids the solution of the non linear eigenvalue problem (50) is estimating the *critical displacement values* instead of those of the forces as done in classical limit load theory. The displacement vector in the limit state is splitted now as

$${}^{t+\Delta t_c}a = {}^t a + \Delta a \tag{57}$$

where $\Delta a = \lambda\phi$ with ϕ being an estimate of the buckling pattern in the limit point ($\phi = {}^t a$, the first eigenmode in ${}^t a$ or the last converged value of the displacement increment vector can be taken) and λ is a multiplier. Introducing (54) in the expression of the tangent matrix (45) and requiring this to be singular at ${}^{t+\Delta t_c}a$ gives

$$|{}^{t+\Delta t_c}_r K_T| = {}^t_r K_T + \lambda[{}^t_r K_{L_2} + {}^t_r K_{\sigma_1}] + \lambda^2[{}^t_r K_{L_3} + {}^t_r K_{\sigma_2}] = 0 \tag{58}$$

where the form of matrices ${}^t_r K_{L_2}, {}^t_r K_{L_3}\, {}^t_r K_{\sigma_1}$ and ${}^t_r K_{\sigma_2}$ can be found in Appendix II.

Equation (58) can be solved for λ and the corresponding eigenvector ϕ. Obviously the solution of Eq. (58) can be simplified further neglecting the quadratic term in λ.

The critical load increment is now estimated using the secant relationship

$$\Delta f_{crit} = [{}^t_r K_T({}^t a) + {}^t_r K_M(\lambda\phi) + {}^t_r K_N(\lambda^2\phi^2)]\lambda\phi. \tag{59}$$

This procedure was first proposed by Kroplin (1992) in the context of a mixed formulation. Examples of the efficiency of this approach are given in next section.

7
Application to the analysis of 3D trusses

We will consider a 3D straight bar element defined in a cartesian frame $x_i (i = 1, 3)$ and subjected to axial forces only. A local coordinate system x'_1, x'_2, x'_3 is also defined where x'_1 is aligned wih the bar direction and x'_2 and x'_3 are any two orthogonal vectors contained in a plane orthogonal to x'_1 (see Fig. 4a).

For the sake of preciseness we will use here an Updated Lagrangian formulation (${}^r V = {}^t V$). The first and second order axial strain increments can be defined as

$${}_t e'_{11} = \frac{d(\Delta u'_1)}{d\,{}^t x'_1}; \quad {}_t \eta'_{11} = \frac{1}{2}\frac{d(\Delta u'_1)}{d\,{}^t x'_1}\frac{d(\Delta u'_1)}{d\,{}^t x'_1} \tag{60}$$

where $\Delta u'_1$ is the displacement increment along the local axis ${}^t u'_1$.

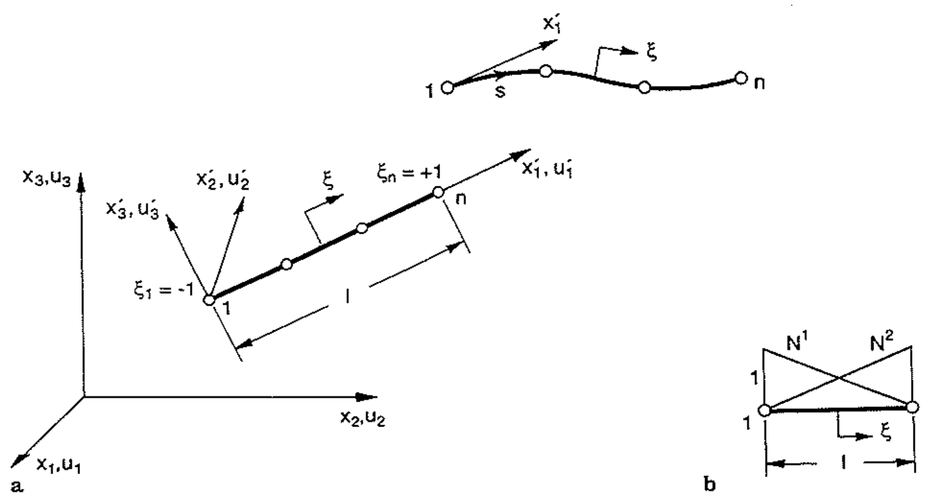


Fig. 4 a,b. a Bar element for truss analysis b Linear 2 node bar element

The constitutive equation is now written in the simpler form

$$ {}_t\Delta N = {}^t[EA][{}_te'_{11} + {}_t\eta'_{11}] \tag{61} $$

where ${}_t\Delta N$ is the axial force increment and ${}^tE$ and ${}^tA$ are respectively the Young modulus and the area of the bar cross section at the configuration t.

Local and global displacement increments are related by the standard transformation

$$ \Delta u'_j = \frac{\partial\, {}^tx_i}{\partial\, {}^tx_j} \Delta u_i; \quad i,j = 1,2,3 \tag{62} $$

The finite element interpolation is defined in the usual manner. Thus, if n noded one dimensional isoparametric elements are used we can write

$$ \Delta u_i = \sum_{k=1}^{n} N^k(\xi)\Delta u_i^k; \quad i = 1,2,3 \tag{63} $$

Substitution of (62) and (63) into (60) yields the expression of the strain matrices ${}^t_t\mathbf{B}_L$ and ${}_t\mathbf{B}_1$ shown in Box IV together with the rest of the relevant matrices necessary for computation of the secant stiffness matrix via Eqs. (40) and (41).

$$ {}_te'_{11} = {}^t_t\mathbf{B}_L \Delta\mathbf{a}; \quad \boxed{{}^t_t\mathbf{B}_L = \frac{1}{{}^tJ^2}\, {}^t\mathbf{x}^T \mathbf{N}^T_{,\xi}\mathbf{N}_{,\xi}} $$

$$ {}_t\eta'_{11} = {}_t\mathbf{B}_1 \Delta\mathbf{a}; \quad \boxed{{}_t\mathbf{B}_1 = \frac{1}{{}^tJ^2}\Delta\mathbf{a}^T \mathbf{N}^T_{,\xi}\mathbf{N}_{,\xi}} $$

$$ \boxed{{}^t_t\mathbf{B}_{NL} = \frac{1}{{}^tJ}\mathbf{N}_{,\xi}} $$

$$ \boxed{{}_t\mathbf{G} = \frac{1}{{}^tJ}\mathbf{N}_{,\xi}} $$

$$ \boxed{{}^t_t\mathbf{E} = \frac{d(\Delta u'_1)}{d\,{}^tx'_1}\, {}^t[EA]\mathbf{I}_3 = \frac{{}^t[EA]}{{}^tJ^2}\, {}^t\mathbf{x}^T \mathbf{N}^T_{,\xi}\mathbf{N}_{\xi}\Delta\mathbf{a}} $$

$$ \mathbf{N}_{,\xi} = \left[\frac{dN^1}{d\xi}\mathbf{I}_3, \ldots, \frac{dN^n}{d\xi}\mathbf{I}_3\right] $$

$$ {}^t\mathbf{x} = [{}^tx_1^1, {}^tx_2^1, {}^tx_3^1, \ldots, {}^tx_1^n, {}^tx_2^n, x_3^n]^T $$

$$ \Delta\mathbf{a} = [\Delta u_1^1, \Delta u_2^1, \Delta u_3^1, \ldots, \Delta u_1^n, \Delta u_2^n, \Delta u_3^n]^T $$

$$ {}^tJ = \frac{d\,{}^tx'_1}{d\xi} \quad \left(\text{usually } {}^tJ = \frac{{}^tl^{(e)}}{2}\right); \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Box IV. Relevant expressions involved in the computation of the secant stiffness matrix for 3D bar elements

7.1
Particularization for the linear bar element

The simplest and most widely used element for truss analysis is the 2 node one dimensional bar element with linear shape functions $N_i = \frac{1}{2}(1 + \xi\xi_i)$, $i = 1, 2$ (see Fig. 4b).

The simplicity of the element allows to obtain an explicit form of the different matrices involved in Eq. (40) for the computation of the secant stiffness matrix. A symmetric form with $\alpha = 1/2$ and $\beta = 0$ has been chosen here. The exact expression of these matrices for the case of constant cross section and homogeneous material is shown in Box V.

Example 1
2 bar truss under central point load

The secant stiffness formulation developed in previous section will be applied to the well known example of a simple two bar truss structure under a symmetrical vertical point load as shown in Fig. 5. Due to the symmetry of the problem only one degree of freedom, the vertical displacement of node 2, is involved.

The relationship between the external vertical force P and the angle rotated by the bar θ is given in "exact" form by Pignataro, Rizzi and Luongo (1991)

$$P = 2EAl\,[\sin\theta - \cos\theta_0 \operatorname{tang}\theta]$$

where l is the initial bar length and $\theta_0 = 15^\circ$ in this case. The values of the critical angles and the critical load are simply obtained from the condition $dP/d\theta = 0$ as

$$\boxed{\theta_{\text{crit}} = \operatorname{arcos}(\cos\theta_0)^{1/3}} = \pm 8.6937^\circ \tag{64}$$

$$\boxed{P_{\text{crit}} = 2EA\,(\cos\theta_0)\,(\operatorname{tang}\theta_0)^{1/3}} = \pm 69.066 \tag{65}$$

The "exact", critical vertical displacement of node 2 is thus

$$(v_2)_{\text{crit}} = \pm l\cos\theta_0 \operatorname{tang}\theta_{\text{crit}} - l\operatorname{sen}\theta_0 = \begin{cases} -1.112 \\ -4.0651 \end{cases} \tag{66}$$

$$ {}^t_t\mathbf{K}_{L_{ij}} = {}^t\left[\frac{EA}{l^3}\right](-1)^{i+j}\begin{bmatrix} {}^t(x_{12})^2 & {}^tx_{12}\,{}^ty_{12} & {}^tx_{12}\,{}^tz_{12} \\ & {}^t(y_{12})^2 & {}^ty_{12}\,{}^tz_{12} \\ \text{sym.} & & {}^t(z_{12})^2 \end{bmatrix}$$

$$ {}^t_t\mathbf{K}_{M_{ij}} = {}^t\left[\frac{EA}{2l^3}\right](-1)^{i+j}\begin{bmatrix} 2\,{}^tx_{12}u_{12} & ({}^tx_{12}v_{12} + {}^ty_{12}u_{12}) & ({}^tx_{12}w_{12} + {}^tz_{12}u_{12}) \\ & 2\,{}^ty_{12}v_{12} & ({}^ty_{12}w_{12} + {}^tz_{12}v_{12}) \\ \text{sym.} & & 2\,{}^tz_{12}w_{12} \end{bmatrix}$$

$$ + {}^t\left[\frac{EA}{2l^3}\right](-1)^{i+j}\,[{}^tx_{12}u_{12} + {}^ty_{12}v_{12} + {}^tz_{12}w_{12}]\,\mathbf{I}_3$$

$$ {}^t_t\mathbf{K}_{N_{ij}} = {}^t\left[\frac{EA}{2l^3}\right](-1)^{i+j}\begin{bmatrix} (u_{12})^2 & u_{12}v_{12} & u_{12}w_{12} \\ & (v_{12})^2 & v_{12}w_{12} \\ \text{sym.} & & (w_{12})^2 \end{bmatrix}$$

$$ {}^t_t\mathbf{K}_{\sigma_{ij}} = {}^t\left[\frac{N}{l}\right](-1)^{i+j}\,\mathbf{I}_3$$

$$ {}^tx_{12} = {}^tx_1 - {}^tx_2, \quad u_{12} = \Delta u_1 - \Delta u_2 \quad \text{etc.} \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Box V. Expressions of the different matrices involved in the computation of the secant stiffness matrix for the two node 3D bar element ($\alpha = 1/2$, $\beta = 0$)

Finite element solution From the expressions of Box V the secant equilibrium Eq. (38) can be written in the form

$$\left[{}^tk\,{}^tS^2 + \frac{3}{2}\,{}^t\!\left(\frac{k}{l}\right){}^tS\,\Delta v_2 + \frac{{}^tk}{2\,{}^tl^2}(\Delta v_2)^2 + \frac{{}^tN}{{}^tl}\right]\Delta v_2 = -\frac{{}^{t+\Delta t}P}{2} + {}^tN\,{}^tS \tag{67}$$

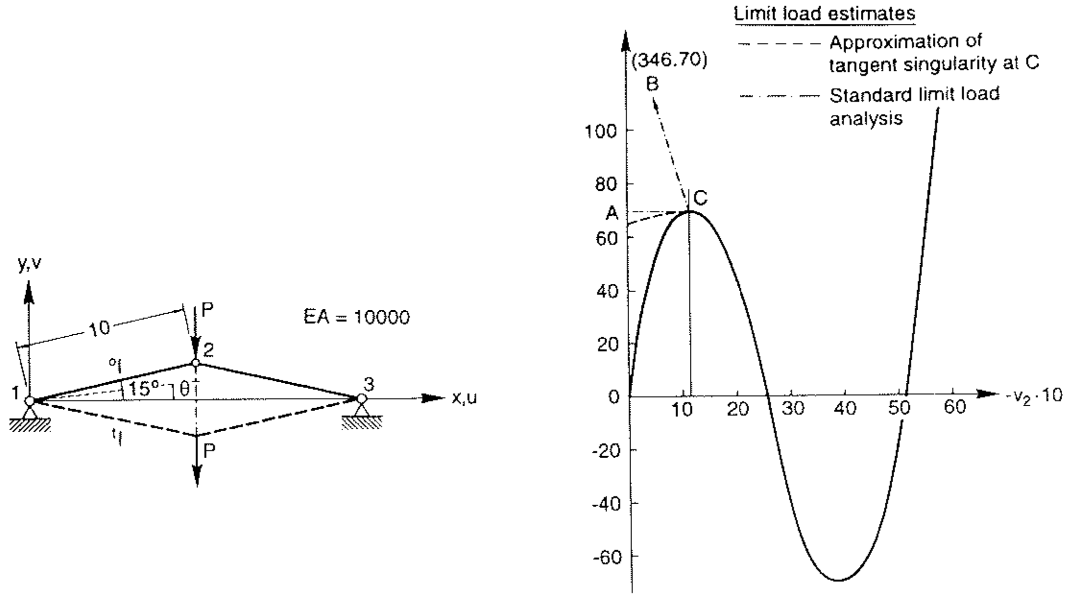


Fig. 5. Two bar truss under point load. Geometry, load-displacement curve and limit-load estimates

where Δv_2 is the increment of vertical displacement of node 2, ${}^{t+\Delta t}P$ the total vertical load applied, ${}^tN$ the corresponding axial force in the bar $1-2$. Also in (67)

$$ {}^tk = {}^t\left(\frac{EA}{l}\right) \quad \text{and} \quad {}^tS = \sin{}^t\theta = -{}^t\left(\frac{y_{21}}{l}\right) \tag{68}$$

Equation (67) is a cubic polinomial in Δv_2 relating the total vertical load applied and the corresponding displacement increment. If a *total secant algorithm* is used with the unloaded configuration taken as the initial configuration ${}^tk = {}^\circ k$, ${}^tS = {}^\circ S$, $\Delta v_2 = {}^{t+\Delta t}v_2 = v_2$, ${}^tN = 0$ and ${}^{t+\Delta t}P = P$. Then

$$\left[{}^\circ k({}^\circ S)^2 + \frac{3}{2}{}^\circ\left(\frac{k}{l}\right){}^\circ S\, v_2 + \frac{{}^\circ k}{2({}^\circ l)^2}(v_2)^2\right] v_2 = -\frac{P}{2} \tag{69}$$

Equation (69) can be solved by direct iteration for values of v_2 for each load level P. The load-displacement curve obtained is displayed in bold line in Fig. 5. Clearly the numerical computation of the descending branch of the load-displacement curve requires the use of arch length or similar displacement control techniques (Bathe (1982)).

In particular, the solution of (69) for $P = 0$ yields

$$\boxed{\begin{array}{l} v_2 = 0 \\ v_2 = -{}^\circ l\,{}^\circ S \\ v_2 = -2\,{}^\circ l\,{}^\circ S \end{array}} \begin{array}{l} \\ = -2.5882 \\ = -5.176 \end{array} \tag{70}$$

which corresponds exactly with the three intersecting points of the load–displacement curve with the horizontal axis, $P = 0$.

Solution of Eq. (69) for any value of P yields all the possible vertical displacement solutions ($\leqq 3$) obtained by intersecting the load–displacement curve of Fig. 5 with a horizontal line $P = \bar{P}$.

Axial force-displacement relationship The axial force-displacement relationship can be obtained from Eq. (27) and the expressions of Box IV (taking into account that $(\partial N_i/\partial\xi) = (-1)^i/2$). Thus using a *total secant approach* gives

$$N = {}^\circ\left(\frac{EA}{l}\right)\left[{}^\circ S + \frac{v_2}{2\,{}^\circ l}\right] v_2 \tag{71}$$

where $N = {}^{t+\Delta t}N$.

From (71) it is found that the axial force in the bar reaches a maximum compression value for $v_2 = -{}^{\circ}l\,{}^{\circ}S$ and

$$\boxed{N_{\max} = -\tfrac{1}{2}\,{}^{\circ}(EA)\,({}^{\circ}S)^2} \; = -334.934 \text{ (exact)} \tag{72}$$

Also from (71) we see that for $v_2 = -2\,{}^{\circ}l\,{}^{\circ}S$ the axial force is zero and for $v_2 < -2\,{}^{\circ}l\,{}^{\circ}S$ the axial force has an increasingly positive (traction) value as shown in Fig. 6. A plot of the axial force in the bar versus the applied external load is also shown in Fig. 6.

Computation of critical points The critical points are obtained from the condition $dP/dv_2 = 0$. Thus, using (69) gives

$$({}^{\circ}S)^2 + 3\frac{{}^{\circ}S}{{}^{\circ}l}v_2 + \frac{3}{2}\frac{{}^{\circ}k}{{}^{\circ}l^2}(v_2)^2 = 0 \tag{73}$$

which yields

$$(v_2)_{\text{crit}} = {}^{\circ}S\,{}^{\circ}l\left[-1 \pm \frac{\sqrt{3}}{3}\right] = \begin{cases} -1.0938 \text{ (1.56\% error)} \\ -4.0824 \text{ (0.42\% error)} \end{cases} \tag{74}$$

Substituting (74) into (69) gives the two values of the critical load as

$$P_{\text{crit}} = \frac{\pm 2\,{}^{\circ}k\,({}^{\circ}S)^3\,{}^{\circ}l}{9}\sqrt{3} = \pm 66.732 \text{ (3.38\% error)} \tag{75}$$

Approximate computation of critical points The estimation of the first critical point following the mid -point procedure described in a previous section gives (using (70)).

$$(v_2)_{\text{crit}} = -\frac{{}^{\circ}l\,{}^{\circ}S}{2} = -1.2941 \text{ (14.13\% error)} \tag{76}$$

Substituting (76) into (69) gives the approximate critical load as

$$P_{\text{crit}} = \tfrac{3}{8}\,{}^{\circ}k\,({}^{\circ}S)^3\,{}^{\circ}l = 65.01 \text{ (5.87\% error)} \tag{78}$$

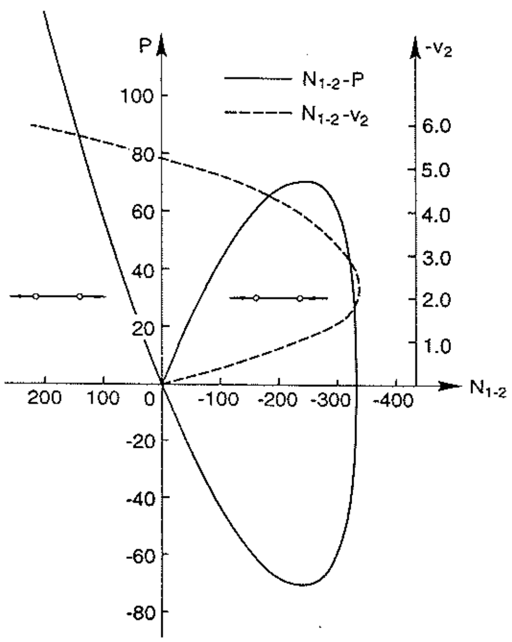


Fig. 6. Two bar truss under point load. Axial force versus displacement and load curves

The second estimation procedure is based on the approximation of the tangent stiffness singularity condition at the critical point corresponding to time t_c. Thus, we can write from (67)

$$ {}^{t_c}_{t}K_T = \left[{}^{t_c}k \, ({}^{t_c}S)^2 + {}^{t_c}\left(\frac{N}{l}\right) \right] = 0 \tag{78} $$

From Fig. 5 we note

$$ {}^{t_c}S = \frac{{}^0S \, {}^0l + v_2}{{}^0l} \tag{79} $$

Substituting this expression and (71) into (78) and assuming ${}^{t_c}l = {}^0l$ gives the singularity condition for the tangent stiffness in a form *identical* to (73). Neglecting the quadratic term in this expression yields

$$ {}^{t_c}_{t}K_T \simeq ({}^0S)^2 + \frac{3\,{}^0S}{{}^0l} v_2 = 0 \tag{80} $$

and the approximate critical displacement is now

$$ (v_2)_{\text{crit}} = \frac{-\,{}^0l \, {}^0S}{3} = -\,0.863 \text{ (22.34\% relative error)} \tag{81} $$

The approximate critical load is obtained substituting this displacement value into (69) giving

$$ P_{\text{crit}} = \tfrac{10}{27} \, {}^0k \, ({}^0S)^3 \, {}^0l = 64.20 \text{ (7.04\% relative error)} \tag{82} $$

Note that the error in the estimation of both the critical displacement and the critical load values is slightly bigger in this case. However, it is important to point out that the first procedure requires the knowledge of two points in the equilibrium curve (i.e. the solution of a non linear system is mandatory), whereas in the second procedure only a simple linear equation is solved.

Also note that the critical load values have been estimated from the *initial unloaded configuration.* A higher precission will be obtained if the same procedure would be attempted starting from an intermediate equilibrium point. The evolution of the prediction of the critical load using this technique has been plotted in curve AC of Fig. 5. It is interesting to point out that a lower bound of the critical load is obtained in all cases.

Remark X. Note the accuracy of above approximation with respect the crude estimate provided by the solution of the standard "initial" stability problem giving

$$ ({}^0S)^2 + \frac{{}^0N_{\text{crit}}}{{}^0l} = 0, \quad \textit{i.e.} \quad {}^0N_{\text{crit}} = -\,{}^0l \, ({}^0S)^2 \tag{83} $$

and

$$ P_{\text{crit}} = 2 \, {}^0k \, {}^0l \, ({}^0S)^3 = 346.70 \text{ (432\% relative error)} \tag{84} $$

The evolution of the critical load predicted using standard limit load analysis is plotted in curve BC of Fig. 5.

Example 2
3D pin-joined star structure

The geometry, loading and material properties are shown in Fig. 7a.

Figure 7b shows the evolution of the load with the displacement of the central point, obtained using an *incremental secant approach.* The cost of the numerical solution in this case was identical to that of the standard Newton-Raphson solution.

Figure 7b displays also the approximation of the limit load values obtained using the classical incremental limit load analysis (curve BC) and the method based on the approximation of the tangent matrix singularity condition, as proposed in this paper (curve AC). Note the accuracy of this second

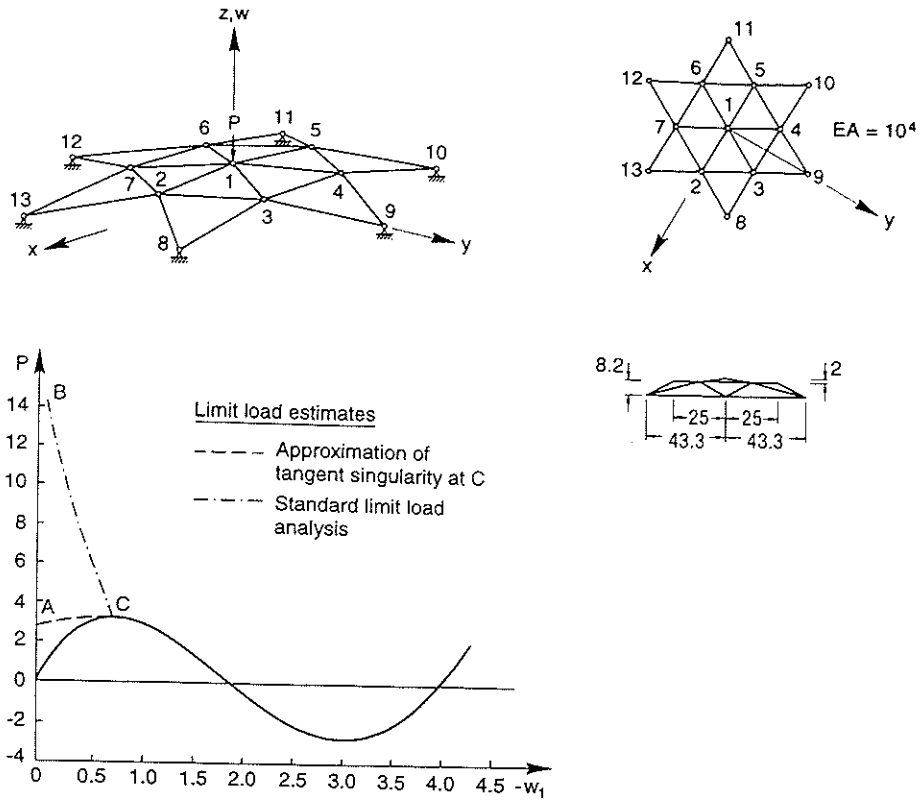


Fig. 7a,b. 3D pin-joined truss. a Geometry and load. b Load-displacement curve and limit load estimates

procedure, even for predictions based on initial deformed configurations far from the critical value sought. Also note that curve AC provides a lower bound of the critical load values.

Further details on this example and other similar analyses of two and three dimensional pin-joined bar structures can be found in Oñate and Matias (1995).

8
Conclusions

The formulation proposed in this paper provides a simple procedure to derive the general expression of the incremental secant stiffness matrix for non linear finite element analysis of solids and pin-joined trusses.

The symmetric form obtained emerges in a natural manner from the general non symmetric expression and it can be simply particularized for both TL and UL descriptions. The formulation proposed opens many possibilities for deriving new iterative solution strategies and some of them have been suggested in the paper. The possible merits of each one and also those of using particular non-symmetric forms of the secant stiffness matrix should be investigated in more detail.

One of the more interesting applications of the secant matrix approach could be its potential for predicting instability points in a simple manner. The methodology proposed in the paper has proved to be extremely efficient to predict very accurate lower bounds for the limit load in the two examples presented.

Results for a wide range of examples analyzed by the author's group show the same promising trend (Oñate and Matias (1995)).

Appendix I

The first and second order strain increments can be written as

$$ {}_r e = [L_0 + {}^t_r L_1({}^t_r g)]\,{}_r g $$

$$ {}_r \eta = \tfrac{1}{2}\,{}_r L_{1\,r} g \qquad \text{(I.2)} $$

For 3D solids:

$$\mathbf{L}_0 = \begin{bmatrix} 100 & 000 & 000 \\ 000 & 010 & 000 \\ 000 & 000 & 001 \\ 010 & 100 & 000 \\ 001 & 000 & 100 \\ 000 & 001 & 001 \end{bmatrix} \tag{I.3}$$

$${}^t_r\mathbf{L}_1 = \begin{Bmatrix} {}^t_r\mathbf{g}\mathbf{H}_1 \\ {}^t_r\mathbf{g}\mathbf{H}_2 \\ \vdots \\ {}^t_r\mathbf{g}\mathbf{H}_6 \end{Bmatrix}; \quad {}_r\mathbf{L}_1 = \begin{Bmatrix} {}_r\mathbf{g}\mathbf{H}_1 \\ {}_r\mathbf{g}\mathbf{H}_2 \\ \vdots \\ {}_r\mathbf{g}\mathbf{H}_6 \end{Bmatrix} \tag{I.4}$$

where

$${}^t_r\mathbf{g} = \begin{Bmatrix} {}^t_r\mathbf{g}_1 \\ {}^t_r\mathbf{g}_2 \\ {}^t_r\mathbf{g}_3 \end{Bmatrix}; \quad {}_r\mathbf{g} = \begin{Bmatrix} {}_r\mathbf{g}_1 \\ {}_r\mathbf{g}_2 \\ {}_r\mathbf{g}_3 \end{Bmatrix} \tag{I.5}$$

with

$${}^t_r\mathbf{g}_i = \frac{\partial\, {}^t\mathbf{u}}{\partial\, {}^r x_i} \quad \text{and} \quad {}_r\mathbf{g}_i = \frac{\partial(\Delta\mathbf{u})}{\partial\, {}^r x_i} \tag{I.6}$$

In (I.4)

$$\mathbf{H}_1 = \begin{bmatrix} \mathbf{I}_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{H}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{I}_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{H}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_3 \end{bmatrix}; \quad \mathbf{H}_4 = \begin{bmatrix} 0 & \mathbf{I}_3 & 0 \\ \mathbf{I}_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{H}_5 = \begin{bmatrix} 0 & 0 & \mathbf{I}_3 \\ 0 & 0 & 0 \\ \mathbf{I}_3 & 0 & 0 \end{bmatrix}; \quad \mathbf{H}_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_3 \\ 0 & \mathbf{I}_3 & 0 \end{bmatrix}$$

with

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{I.7}$$

Appendix II

Approximate expression of the tangent stiffness matrix at the limit point

Let us assume a known equilibrium solution at time t with displacements ${}^t\mathbf{a}$ and stresses ${}^t\boldsymbol{\sigma}$. The limit point will occur at time $t_c = t + \Delta t_c$ for which

$${}^{t_c}\mathbf{a} = {}^t\mathbf{a} + \Delta\mathbf{a}$$

$${}^{t_c}\boldsymbol{\sigma} = {}^t\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma} \tag{II.1}$$

The displacement increments Δa will be assumed to be of the form $\Delta a = \lambda\phi$ with ϕ being an estimate of the buckling pattern in the limit point. The simplest choices are $\phi = {}^t a$, ϕ equal to the first eigenmode of ${}^t_r K_T$ or ϕ equal to the last converged displacement increment vector. For simplicity ${}^{t_c}_r D = {}^t_r D$ will also be assumed.

With these assumptions the stress field at the limit point can be written as (using Eqs. (20) and (36))

$$ {}^{t_c}\sigma = {}^t\sigma + {}^t_r D\,[{}^t_r B_L({}^t a) + \frac{\lambda}{2}\,{}_r\bar{B}_1(\phi)]\,\lambda\phi = {}^t\sigma + \lambda\sigma_1 + \lambda^2\sigma_2 \tag{II.2} $$

where

$$ \sigma_1 = {}^t_r D\, {}^t_r B_L({}^t a)\,\phi \tag{II.3} $$

$$ \sigma_2 = \tfrac{1}{2}\,{}^t_r D\, {}_r\bar{B}_1(\phi)\,\phi \tag{II.4} $$

The first order strain matrix at the critical point can be written as

$$ {}^{t_c}_r B_L = {}^t_r B_L({}^t a) + \lambda\, {}_r\bar{B}_1(\phi) \tag{II.5} $$

In above ${}_r\bar{B}_1(\phi)$ is obtained from the expression of ${}_r B_1$ given in Box II simply substituting the displacement increment vector by the estimated increment ϕ. Note that when $\phi = {}^t a$ then ${}_r B_1 = {}^t_r B_{L_1}$ (see Eq. (32) and Box II).

Substituting (II.2) and (II.5) into (45) the tangent stiffness matrix at the critical point can be found to have the following expression

$$ {}^{t_c}_r K_T = {}^t_r K_T + \lambda\,[{}^t_r K_{L_2} + {}^t_r K_{\sigma_1}] + \lambda^2\,[{}^t_r K_{L_3} + {}^t_r K_{\sigma_2}] \tag{II.6} $$

where ${}^t_r K_T$ is the standard tangent stiffness matrix at time t and

$$ {}^t_r K_{L_2} = \int_{{}^rV^{(e)}} [{}^t_r B_L^T\, {}^t_r D\, {}_r\bar{B}_1 + {}_r\bar{B}_1^T\, {}^t_r D\, {}^t_r B_L] \tag{II.7} $$

$$ {}^t_r K_{L_3} = \int_{{}^rV^{(e)}} {}_r\bar{B}_1^T\, {}^t_r D\, {}_r\bar{B}_1\, dV \tag{II.8} $$

$$ {}^t_r K_{\sigma_1} = \int_{{}^rV^{(e)}} {}_r B_{NL}^T\, {}^t_r S_1\, {}_r B_{NL}\, dV \tag{II.9} $$

$$ {}^t_r K_{\sigma_2} = \int_{{}^rV^{(e)}} {}_r B_{NL}^T\, {}^t_r S_2\, {}_r B_{NL}\, dV \tag{II.10} $$

where ${}^t_r S_1$ and ${}^t_r S_2$ are obtained by substituting the "stresses" σ_1 and σ_2 given by (II.3) and (II.4) into the expression of matrix ${}^t_r S$ of Box III, respectively.

The condition $|{}^{t_c}_r K_T| = 0$ yields a quadratic eigenvalue problem which can be solved iteratively for λ and an approximation of the limit point position is then obtained as ${}^{t_c}a = {}^t a + \lambda\phi$. Obviously Eq. (I.6) can be further simplified simply neglecting the quadratic terms in λ.

The value of the critical load can be subsequently computed using the incremental secant relationship (2) as

$$ \Delta f_{crit} = {}^t_r K_S(\lambda\phi)\,\lambda\phi \tag{II.11} $$

and

$$ f_{crit} = {}^t f + \Delta f_{crit} \tag{II.12} $$

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