INTERNATIONAL CENTER FOR NUMERICAL METHODS IN ENGINEERING

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On some pressure segregation methods of fractional-step type for the finite element approximation of incompressible flow problems

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Abstract

In this paper we treat several aspects related to time integration methods for the incompressible Navier-Stokes equations that allow to uncouple the calculation of the velocities and the pressure. The first family of schemes consists of classical fractional step methods, of which we discuss several possibilities for the pressure extrapolation and the time integration of first and second order. The second family consists of schemes based on an explicit treatment of the pressure in the momentum equation followed by a Poisson equation for the pressure. It turns out that this "staggered" treatment of the velocity and the pressure is stable. Finally, we present predictor-corrector methods based on the above schemes that aim to converge to the solution of the monolithic time integration method. Apart from presenting these schemes and check its numerical performance, we also present a complete set of stability results for the fractional step methods that are independent of the space stability of the velocity-pressure interpolation, that is, of the classical inf-sup condition.

1 Introduction

The uncoupling of the pressure from the velocity in the numerical approximation of the incompressible Navier-Stokes equations has been traditionally attempted using different approaches. Perhaps the most well known is the use of fractional step methods, although there are other techniques based on the solution of a pressure Poisson equation (see [9] for a review of this type of methods) and predictor-multicorrector algorithms (either using a single stage or a multistage time integration). The interest for this type of methods is their computational efficiency, since only scalar equations need to be solved (see e.g. [12, 17]).

The objective of this paper is to study several aspects of some pressure segregation methods for the *transient* incompressible Navier-Stokes equations using a finite element approximation for the space discretization. Our reference will be the solution of the *monolithic* problem, that is, the coupled calculation of the velocity and the pressure. Obviously, the fully discrete and linearized monolithic scheme leads to an algebraic system the structure of which can be exploited so as to solve independently for the velocity and pressure degrees of freedom. However, we consider this a particular algebraic treatment of the final linear system that we will not study in this paper (even though it might have a counterpart at the space-continuous level).

It is *not* the purpose of this paper to discuss how to deal with the pressure interpolation. All our discussion will be based on a Galerkin finite element interpolation for the velocity and the pressure. If the velocity-pressure pairs satisfy the classical inf-sup condition, that will yield a stable pressure approximation. Otherwise, if for example equal velocity-pressure interpolation is used, the Galerkin formulation can be modified to a stabilized finite element method for which all the discussion that follows could be easily adapted. Likewise, no stabilization methods for the convective term when it dominates the viscous one will be taken into account, although these could also be easily incorporated to what follows. An example of this is the work presented in [4], of which part of the present work is a continuation.

Referring to the time integration, we will concentrate on first and second order implicit finite difference schemes. The backward Euler method will be used for the former, whereas for second order methods we will consider both the Crank-Nicolson and backward differencing (or Gear) schemes. We will refer to the first and second order backward differencing schemes as BDF1 (which coincides with backward Euler) and BDF2, respectively.

After describing the problem and its time and space discretization in Section 2, the methods we wish to consider are presented in Section 3. The first family is the classical fractional step methods. Our approach here is to present the splitting at the pure algebraic level, as in [13, 14], rather than at the space continuous level as it was done in the original works of Chorin [3] and Temam [16] and which is still the most common approach. The algebraic viewpoint obviates the discussion on the pressure boundary conditions (which nevertheless it is obviously there).

The second family of methods is the one based on a particular pressure Poisson equation presented in [11]. The momentum equation can be solved treating the pressure explicitly, and then updating the pressure. Contrary to classical fractional step methods, in this case there is no intermediate velocity to deal with. However, the velocity obtained with this scheme is not (weakly) divergence free for the discrete problem. Our contribution here is to compare methods based on this approach with the fractional step methods discussed previously. We explicitly show the equivalence between this *momentum-pressure Poisson equation* approach and classical splitting methods.

The third and last family of methods is of predictor-multicorrector type in the spirit of [1] (see also references therein). Starting from fractional step methods, we propose an iterative scheme the goal of which is to converge to the solution of the monolithic problem. The robustness of this scheme relies on the presence of a term that is motivated precisely by the starting splitting method.

An important part of this paper is devoted to the *stability* analysis of the fractional step methods presented. This is done in Section 4. Four methods are considered. In the first two, a first order time integration is used, both with a first and a second order splitting error. For the third and fourth methods the splitting error is of second order, the same as the time integration error. In one case the time integration is performed using the Crank-Nicolson method, whereas the BDF2 scheme is used in the last case. Two of these four stability results were already presented in [4], whereas the other two are new. The important issue is that we do not rely on the pressure interpolation, which leads to poor stability estimates for the pressure. They could be improved either by making use of the inf-sup condition or by resorting to stabilized finite element methods. Nevertheless, we consider important to point out that *some* pressure stability is obtained even if none of these possibilities is used.

In the last two sections of the paper (Section 5 and 6) we present some numerical examples and draw some conclusions.

2 Problem statement

2.1 Continuous problem

Let Ω be the domain of $\mathbb{R}^{n_{\rm sd}}$ occupied by the fluid, where $n_{\rm sd} = 2$ or 3 is the number of space dimensions, $\Gamma = \partial\Omega$ its boundary and $[0,T]$ the time interval of analysis. The Navier-Stokes problem consists in finding a velocity $\boldsymbol{u}$ and a pressure p such that

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\nabla^2\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega,\ t\in(0,T), \tag{1}$$
$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega,\ t\in(0,T), \tag{2}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma,\ t\in(0,T), \tag{3}$$
$$\boldsymbol{u} = \boldsymbol{u}^0 \quad \text{in } \Omega,\ t=0, \tag{4}$$

where ν is the kinematic viscosity, $\boldsymbol{f}$ is the force vector and $\boldsymbol{u}^0$ is the velocity initial condition. We have considered the homogeneous Dirichlet boundary condition (3) for simplicity.

To write the weak form of problem (1)-(4) we need to introduce some notation. We denote by $H^1(\Omega)$ the Sobolev space of functions whose first derivatives belong to $L^2(\Omega)$, and by $H^1_0(\Omega)$ the subspace of $H^1(\Omega)$ of functions with zero trace on Γ. A bold character is used for the vector counterpart of these spaces. The L^2 scalar product in Ω is denoted by $(\cdot,\cdot)$, and the L^2 norm by $\|\cdot\|$. To pose the problem, we also need the functional spaces $\boldsymbol{V}_{\rm st} = \boldsymbol{H}^1_0(\Omega)^{n_{\rm sd}}$, and $Q_{\rm st} = \left\{ q\in L^2(\Omega) \mid \int_\Omega q = 0\right\}$, as well as $\boldsymbol{V} = \boldsymbol{L}^2(0,T;\boldsymbol{V}_{\rm st})$ and $Q = L^2(0,T;Q_{\rm st})$ for the transient problem.

Assuming for simplicity the force vector to be square integrable, the weak form of problem (1)-(4) consists in finding $(\boldsymbol{u},p)\in\boldsymbol{V}\times Q$ such that

$$(\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}) + \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) - (p,\nabla\cdot\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}), \tag{5}$$
$$(q,\nabla\cdot\boldsymbol{u}) = 0, \tag{6}$$

for all $(\boldsymbol{v},q)\in\boldsymbol{V}_{\rm st}\times Q_{\rm st}$, and satisfying the initial condition in a weak sense. For the stability analysis of Section 4 it will be convenient to write the convective term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ in its skew-symmetric form, that is, $\boldsymbol{u}\cdot\nabla\boldsymbol{u} + (1/2)\boldsymbol{u}\nabla\cdot\boldsymbol{u}$, although this is irrelevant for the space continuous case, where $\nabla\cdot\boldsymbol{u} = 0$.

2.2 Monolithic time discretization

Let us introduce some notation that we will use throughout the paper. Consider a uniform partition of the time interval of size δt, and let us denote by f^n the approximation of a time dependent function f at time level $t^n = n\delta t$. For a parameter $\theta \in [0,1]$, we will denote

$$f^{n+\theta} = \theta f^{n+1} + (1-\theta) f^n,$$
$$\delta f^{n+1} \equiv \delta^{(1)} f^{n+1} = f^{n+1} - f^n,$$
$$\delta^{(i+1)} f^{n+1} = \delta^{(i)} f^{n+1} - \delta^{(i)} f^n,\ i = 1, 2, 3, ...$$

The discrete operators $\delta^{(i+1)}$ are centered. We will also use the backward difference operators

$$D_1 f^{n+1} = \delta f^{n+1} = f^{n+1} - f^n,$$
$$D_2 f^{n+1} = \frac{1}{2}(3 f^{n+1} - 4 f^n + f^{n-1}),$$

as well as the backward extrapolation operators

$$\begin{aligned} \widehat{f}_i^{n+1} &= f^{n+1} - \delta^{(i)} f^{n+1} = f^{n+1} + \mathcal{O}(\delta t^i), \\ \widehat{f}_1^{n+1} &= f^n, \\ \widehat{f}_2^{n+1} &= 2 f^n - f^{n-1}. \end{aligned} \tag{7}$$

For the time integration of problem (5)-(6) we consider two types of finite difference approximation. The first is the generalized trapezoidal rule, which consists of solving the following problem: from known $\boldsymbol{u}^n$, find $\boldsymbol{u}^{n+1} \in \boldsymbol{V}_{\mathrm{st}}$ and $p^{n+1} \in Q_{\mathrm{st}}$ such that

$$(\frac{1}{\delta t} D_1 \boldsymbol{u}^{n+1} + \boldsymbol{u}^{n+\theta} \cdot \nabla \boldsymbol{u}^{n+\theta}, \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}^{n+\theta}, \nabla \boldsymbol{v}) - (p^{n+\theta}, \nabla \cdot \boldsymbol{v}) = (\bar{\boldsymbol{f}}^{n+\theta}, \boldsymbol{v}), \tag{8}$$

$$(q, \nabla \cdot \boldsymbol{u}^{n+\theta}) = 0, \tag{9}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_{\mathrm{st}} \times Q_{\mathrm{st}}$. The force term $\bar{\boldsymbol{f}}^{n+\theta}$ in (8) and below has to be understood as the time average of the force in the interval $[t^n, t^{n+1}]$, even though we use a superscript $n+\theta$ to characterize it. The pressure value computed here has been identified as the pressure evaluated at $t^{n+\theta}$, although this is irrelevant for the velocity approximation. The values of interest of θ are $\theta = 1/2$, corresponding to the second order Crank-Nicolson scheme, and $\theta = 1$, which corresponds to the backward Euler method.

The second scheme is BDF2. In this case, $\boldsymbol{u}^1$ can be computed from the backward Euler method, whereas for $n \geq 1$ the unknowns $\boldsymbol{u}^{n+1} \in \boldsymbol{V}_{\mathrm{st}}$ and $p^{n+1} \in Q_{\mathrm{st}}$ are found by solving the problem

$$(\frac{1}{\delta t} D_2 \boldsymbol{u}^{n+1} + \boldsymbol{u}^{n+1} \cdot \nabla \boldsymbol{u}^{n+1}, \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}^{n+1}, \nabla \boldsymbol{v}) - (p^{n+1}, \nabla \cdot \boldsymbol{v}) = (\bar{\boldsymbol{f}}^{n+1}, \boldsymbol{v}), \tag{10}$$

$$(q, \nabla \cdot \boldsymbol{u}^{n+1}) = 0, \tag{11}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_{\mathrm{st}} \times Q_{\mathrm{st}}$.

2.3 Finite element discretization

Let $\mathcal{T}_h$ denote a finite element partition of the domain Ω of diameter h, from which we construct the finite element spaces Q_h and $\boldsymbol{V}_{h,0}$, approximations to Q_{st} and $\boldsymbol{V}_{\mathrm{st}}$, respectively. The former is made up with continuous functions of degree $k_{\rm q}$ and the other with continuous vector functions of degree $k_{\rm v}$ verifying the homogeneous Dirichlet boundary conditions. In the following, finite element functions will be identified with a subscript h.

The discrete problem is obtained by approximating $\boldsymbol{u}$ and p. We assume that $\boldsymbol{u}_h^n$ and p_h^n are constructed using the standard finite element interpolation from the nodal values. From problem (8)-(9), these are solution of the nonlinear algebraic system

$$\mathsf{M}\frac{1}{\delta t}D_1\mathsf{U}^{n+1} + \mathsf{K}(\mathsf{U}^{n+\theta})\mathsf{U}^{n+\theta} + \mathsf{G}\mathsf{P}^{n+\theta} = \mathsf{F}^{n+\theta}, \tag{12}$$

$$\mathsf{D}\mathsf{U}^{n+\theta} = \mathsf{0}, \tag{13}$$

where U and P are the arrays of nodal unknowns for $\boldsymbol{u}$ and p, respectively. If we denote the node indexes with superscripts a, b, the space indexes with subscripts i, j, and the standard shape function of node a by N^a, the components of the arrays involved in these equations are:

$$\mathsf{M}_{ij}^{ab} = (N^a, N^b)\delta_{ij} \quad (\delta_{ij} \text{ is the Kronecker } \delta),$$
$$\mathsf{K}(\mathsf{U}^{n+\theta})_{ij}^{ab} = (N^a, \boldsymbol{u}_h^{n+\theta}\cdot\nabla N^b)\delta_{ij} + \frac{1}{2}\left(N^a, (\nabla\cdot\boldsymbol{u}_h^{n+\theta})N^b\right)\delta_{ij} + \nu(\nabla N^a, \nabla N^b)\delta_{ij},$$
$$\mathsf{G}_i^{ab} = (N^a, \partial_i N^b),$$
$$\mathsf{D}_j^{ab} = (N^a, \partial_j N^b),$$
$$\mathsf{F}_i^a = (N^a, f_i).$$

It is understood that all the arrays are matrices (except F, which is a vector) whose components are obtained by grouping together the left indexes in the previous expressions (a and possibly i) and the right indexes (b and possibly j). Likewise, (12) and (13) need to be modified to account for the Dirichlet boundary conditions (matrix G can be replaced by $-\mathsf{D}^{\rm t}$ when this is done). Observe also that we have used the skew-symmetric form of the convective term, which yields the convective contribution to matrix $\mathsf{K}(\mathsf{U}^{n+\theta})$ skew-symmetric.

For problem (10)-(11) the resulting algebraic system is analogous to (12)-(13), simply replacing $D_1\mathsf{U}$ by $D_2\mathsf{U}$ and evaluating the rest of the terms at $n+1$ instead of $n+\theta$.

3 Some pressure segregation methods

3.1 Fractional step methods

The first family of schemes we consider is classical fractional step methods. They can be introduced at this point, applied to the fully discrete problem (12)-(13). This is exactly equivalent to

$$\mathsf{M}\frac{1}{\delta t}(\tilde{\mathsf{U}}^{n+1} - \mathsf{U}^n) + \mathsf{K}(\mathsf{U}^{n+\theta})\mathsf{U}^{n+\theta} + \gamma\mathsf{G}\mathsf{P}^n = \mathsf{F}^{n+\theta}, \tag{14}$$

$$\mathsf{M}\frac{1}{\delta t}(\mathsf{U}^{n+1} - \tilde{\mathsf{U}}^{n+1}) + \mathsf{G}(\mathsf{P}^{n+1} - \gamma\mathsf{P}^n) = \mathsf{0}, \tag{15}$$

$$\mathsf{D}\mathsf{U}^{n+1} = \mathsf{0}, \tag{16}$$

where $\tilde{\mathsf{U}}^{n+1}$ is an auxiliary variable and γ is a numerical parameter, whose values of interest are 0 and 1. At this point we can make the essential approximation

$$\mathsf{K}(\mathsf{U}^{n+\theta})\mathsf{U}^{n+\theta} \approx \mathsf{K}(\tilde{\mathsf{U}}^{n+\theta})\tilde{\mathsf{U}}^{n+\theta}, \tag{17}$$

where $\tilde{\mathsf{U}}^{n+\theta} := \theta\tilde{\mathsf{U}}^{n+1} + (1-\theta)\mathsf{U}^n$. Expressing U^{n+1} in terms of $\tilde{\mathsf{U}}^{n+1}$ using (15) and inserting the result in (16), the set of equations to be solved is

$$\mathsf{M}\frac{1}{\delta t}(\tilde{\mathsf{U}}^{n+1} - \mathsf{U}^n) + \mathsf{K}(\tilde{\mathsf{U}}^{n+\theta})\tilde{\mathsf{U}}^{n+\theta} + \gamma\mathsf{G}\mathsf{P}^n = \mathsf{F}^{n+\theta}, \tag{18}$$

$$\delta t\mathsf{D}\mathsf{M}^{-1}\mathsf{G}(\mathsf{P}^{n+1} - \gamma\mathsf{P}^n) = \mathsf{D}\tilde{\mathsf{U}}^{n+1}, \tag{19}$$

$$\mathsf{M}\frac{1}{\delta t}(\mathsf{U}^{n+1} - \tilde{\mathsf{U}}^{n+1}) + \mathsf{G}(\mathsf{P}^{n+1} - \gamma\mathsf{P}^n) = \mathsf{0}, \tag{20}$$

which have been ordered according to the sequence of solution, for $\tilde{\mathsf{U}}^{n+1}$, P^{n+1} and U^{n+1}. This uncoupling of variables has been made possible by (17).

Even though problem (18)-(20) can be implemented as such, it is very convenient to make a further approximation. Observe that $\mathsf{D}\mathsf{M}^{-1}\mathsf{G}$ represents an approximation to the Laplacian operator. In order to avoid dealing with this matrix (which is computationally feasible only if M is approximated by a diagonal matrix), we can approximate

$$\mathsf{D}\mathsf{M}^{-1}\mathsf{G} \approx \mathsf{L}, \quad \text{with components } \mathsf{L}^{ab} = -(\nabla N^a, \nabla N^b). \tag{21}$$

Matrix L is the standard approximation to the Laplacian operator. Clearly, this approximation is only possible when continuous pressure interpolations are employed. Likewise, it introduces implicitly the same wrong pressure boundary condition as when the splitting is performed at the continuous level (see [9] for a discussion on boundary conditions for the pressure Poisson equation). In [10], the use of approximation (21) is referred to as "approximate projection".

After using (17) and (21) the problem to be solved is:

$$\mathsf{M}\frac{1}{\delta t}(\tilde{\mathsf{U}}^{n+1} - \mathsf{U}^n) + \mathsf{K}(\tilde{\mathsf{U}}^{n+\theta})\tilde{\mathsf{U}}^{n+\theta} + \gamma\mathsf{G}\mathsf{P}^n = \mathsf{F}^{n+\theta}, \tag{22}$$

$$\delta t\mathsf{L}(\mathsf{P}^{n+1} - \gamma\mathsf{P}^n) = \mathsf{D}\tilde{\mathsf{U}}^{n+1}, \tag{23}$$

$$\mathsf{M}\frac{1}{\delta t}(\mathsf{U}^{n+1} - \tilde{\mathsf{U}}^{n+1}) + \mathsf{G}(\mathsf{P}^{n+1} - \gamma\mathsf{P}^n) = \mathsf{0}. \tag{24}$$

In Section 4 we will consider three possibilities depending on the choice of θ and γ. Formally, it is easy to see that the splitting error, introduced by approximation (17), is of order $\mathcal{O}(\delta t)$ when $\gamma = 0$ and of order $\mathcal{O}(\delta t^2)$ when $\gamma = 1$ (observe from (15) that $\mathcal{O}(\|\mathsf{U}^{n+1} - \tilde{\mathsf{U}}^{n+1}\|) = \delta t\mathcal{O}(\|\mathsf{P}^{n+1} - \gamma\mathsf{P}^n\|)$ in any norm $\|\cdot\|$). Thus, we will refer to the case $\gamma = 0$ as the case with splitting error of order 1, called SE1 in the following, and the case $\gamma = 1$ as the case with splitting error of order 2, called SE2. The three possibilities mentioned are:

- $\theta = 1$, $\gamma = 0$. Method BDF1-SE1.
- $\theta = 1$, $\gamma = 1$. Method BDF1-SE2.
- $\theta = 1/2$, $\gamma = 1$. Method CN-SE2.

Method BDF1-SE2 will obviously be first order, and thus the second order splitting error unnecessary. However, this method has some interesting properties that will be discussed below.

So far, we have considered the trapezoidal rule for the time integration. If, instead, we use BDF2 with a second order splitting error, the final algebraic system will be

$$\mathsf{M}\frac{1}{2\delta t}(3\tilde{\mathsf{U}}^{n+1} - 4\mathsf{U}^n + \mathsf{U}^{n-1}) + \mathsf{K}(\tilde{\mathsf{U}}^{n+1})\tilde{\mathsf{U}}^{n+1} + \mathsf{G}\mathsf{P}^n = \mathsf{F}^{n+1}, \tag{25}$$

$$\frac{2}{3}\delta t \mathsf{L}(\delta \mathsf{P}^{n+1}) = \mathsf{D}\tilde{\mathsf{U}}^{n+1}, \tag{26}$$

$$\mathsf{M}\frac{1}{2\delta t}(3\mathsf{U}^{n+1} - 3\tilde{\mathsf{U}}^{n+1}) + \mathsf{G}(\delta \mathsf{P}^{n+1}) = \mathsf{0}. \tag{27}$$

We will call this method BDF2-SE2.

3.2 Momentum-pressure Poisson equation methods

In this section we discuss a family of methods that allow us to segregate the pressure calculation recently proposed in [11]. The idea is to start with a formulation of the continuous problem equivalent to (1)-(2) obtained by replacing the continuity equation by a pressure Poisson equation. The system of equations to be solved is thus

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \nabla^2 \boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{28}$$

$$\nabla^2 p = \nabla \cdot (\boldsymbol{f} + \nu \nabla^2 \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{u}). \tag{29}$$

The pressure boundary condition is obtained by imposing that the normal component of the pressure gradient be equal to the normal component of the term within parenthesis in the right-hand-side of (29). The viscous term in this equation could be deleted by assuming that the divergence and the Laplacian operator commute, but this would lead to a non-physical pressure boundary condition.

The key point is the way the pressure appearing in (28) and (29) is treated in the time discretization. In principle, to guarantee that the incompressibility condition holds, both pressures should be the same. However, another possibility is to *use an explicit treatment of the pressure* in (28). This implies that the incompressibility constraint will be relaxed, but allows to uncouple the velocity and pressure calculation. Using a BDF time integration method of order k, $k = 1, 2$, the equations to be solved are

$$\frac{1}{\delta t} D_k \boldsymbol{u}^{n+1} + \boldsymbol{u}^{n+1} \cdot \nabla \boldsymbol{u}^{n+1} - \nu \nabla^2 \boldsymbol{u}^{n+1} + \nabla p_k^{*,n+1} = \boldsymbol{f}^{n+1}, \tag{30}$$

$$\nabla^2 p^{n+1} = \nabla \cdot (\boldsymbol{f}^{n+1} - \boldsymbol{u}^{n+1} \cdot \nabla \boldsymbol{u}^{n+1} + \nu \nabla^2 \boldsymbol{u}^{n+1}), \tag{31}$$

where $p_k^{*,n+1}$ is an explicit approximation to p^{n+1} of order k. The right-hand-side of (31) is cumbersome to evaluate numerically. Making use of (30) in (31) we can alternatively solve

$$\frac{1}{\delta t} D_k \boldsymbol{u}^{n+1} + \boldsymbol{u}^{n+1} \cdot \nabla \boldsymbol{u}^{n+1} - \nu \nabla^2 \boldsymbol{u}^{n+1} + \nabla p_k^{*,n+1} = \boldsymbol{f}^{n+1}, \tag{32}$$

$$\delta t \nabla^2 (p^{n+1} - p_k^{*,n+1}) = \nabla \cdot (D_k \boldsymbol{u}^{n+1}). \tag{33}$$

Let us compare now these *momentum-pressure Poisson equation* methods with the fractional step methods of the previous subsection when the space discretization is carried out. For $k = 1$, problem (32)-(33) leads to

$$\mathsf{M}\frac{1}{\delta t}(D_1\mathsf{U}^{n+1}) + \mathsf{K}(\mathsf{U}^{n+1})\mathsf{U}^{n+1} + \mathsf{G}\mathsf{P}_1^{*,n+1} = \mathsf{F}^{n+1}, \tag{34}$$

$$\delta t\mathsf{L}(\mathsf{P}^{n+1} - \mathsf{P}_1^{*,n+1}) = \mathsf{D}(D_1\mathsf{U}^{n+1}), \tag{35}$$

whereas the algebraic equations of the BDF1-SE1 of the previous subsection can be rearranged to yield

$$\mathsf{M}\frac{1}{\delta t}(D_1\tilde{\mathsf{U}}^{n+1}) + \mathsf{K}(\tilde{\mathsf{U}}^{n+1})\tilde{\mathsf{U}}^{n+1} + \mathsf{G}\mathsf{P}^{n} = \mathsf{F}^{n+1}, \tag{36}$$

$$\delta t\mathsf{L}(\mathsf{P}^{n+1} - \mathsf{P}^{n}) = \mathsf{D}(D_1\tilde{\mathsf{U}}^{n+1}). \tag{37}$$

It is observed that *problems (34)-(35) and (36)-(37) are identical*, provided the intermediate velocity of the fractional step method is identified with the velocity to be computed at each time step and the following two conditions hold: an initial pressure (which is unnecessary in fractional step methods) is obtained from the equation $\delta t\mathsf{L}\mathsf{P}^0 = \mathsf{D}\mathsf{U}^0$ and the explicit first order approximation to the pressure is taken as $\mathsf{P}_1^{*,n+1} = \mathsf{P}^n$.

In the case $k = 2$, problem (32)-(33) leads to

$$\mathsf{M}\frac{1}{\delta t}(D_2\mathsf{U}^{n+1}) + \mathsf{K}(\mathsf{U}^{n+1})\mathsf{U}^{n+1} + \mathsf{G}(\mathsf{P}_2^{*,n+1}) = \mathsf{F}^{n+1}, \tag{38}$$

$$\mathsf{L}(\mathsf{P}^{n+1} - \mathsf{P}_2^{*,n+1}) = \mathsf{D}(\frac{1}{\delta t}D_2\mathsf{U}^{n+1}), \tag{39}$$

whereas the algebraic equations of the BDF2-SE2 of the previous subsection, with an appropriate choice for the initial conditions, can be written as

$$\mathsf{M}\frac{1}{\delta t}(D_2\tilde{\mathsf{U}}^{n+1}) + \mathsf{K}(\tilde{\mathsf{U}}^{n+1})\tilde{\mathsf{U}}^{n+1} + \mathsf{G}(\widehat{\mathsf{P}}_2^{n+1} - \frac{1}{3}\delta^2\mathsf{P}^n) = \mathsf{F}^{n+1}, \tag{40}$$

$$\mathsf{L}(\mathsf{P}^{n+1} - (\widehat{\mathsf{P}}_2^{n+1} - \frac{1}{3}\delta^2\mathsf{P}^n)) = \mathsf{D}(\frac{1}{\delta t}D_2\tilde{\mathsf{U}}^{n+1}). \tag{41}$$

Problems (38)-(39) and (40)-(41) are again *identical*, also identifying the intermediate velocity of the fractional step method with the velocity to be computed at each time step and taking $\mathsf{P}_2^{*,n+1} = \widehat{\mathsf{P}}_2^{n+1} - \frac{1}{3}\delta^2\mathsf{P}^n$ as explicit second order approximation to the pressure at t^{n+1}. Therefore, scheme BDF2-SE2 can be consideread a particular case of (38)-(39).

3.3 Predictor corrector schemes

Starting from the fractional step method (22)-(24), a predictor multicorrector scheme is proposed in [7] whose goal is to converge to the monolithic time discretized problem. We will omit the details of the motivation. Denoting by a superscript i the ith iteration of the scheme, the resulting linearized system is

$$\mathsf{M}\frac{1}{\delta t}(\mathsf{U}^{n+1,i+1} - \mathsf{U}^{n}) + \mathsf{K}(\mathsf{U}^{n+\theta,i})\mathsf{U}^{n+\theta,i+1} + \mathsf{G}\mathsf{P}^{n+\theta,i} = \mathsf{F}^{n+\theta}, \tag{42}$$

$$\delta t\mathsf{L}(\mathsf{P}^{n+\theta,i+1} - \mathsf{P}^{n+\theta,i}) = \mathsf{D}\mathsf{U}^{n+\theta,i+1}. \tag{43}$$

Apparently, this is a straightforward iteration procedure for solving the original *monolithic* problem (12)-(13) freezing the pressure gradient in the momentum equation. However, there is a term whose presence would be hardly motivated by looking only at this system, namely, the term $\delta t \mathsf{L}(\mathsf{P}^{n+\theta,i+1} - \mathsf{P}^{n+\theta,i})$. The motivation to introduce it comes from the inspection of what happens in the fractional step scheme.

If instead of starting from the generalized trapezoidal rule the second order BDF scheme is employed, the iterative scheme we propose is

$$\mathsf{M}\frac{1}{2\delta t}(3\mathsf{U}^{n+1,i+1} - 4\mathsf{U}^n + \mathsf{U}^{n-1}) + \mathsf{K}(\mathsf{U}^{n+1,i})\mathsf{U}^{n+1,i+1} + \mathsf{G}\mathsf{P}^{n+1,i} = \mathsf{F}^{n+1}, \tag{44}$$

$$\frac{2}{3}\delta t \mathsf{L}(\mathsf{P}^{n+1,i+1} - \mathsf{P}^{n+1,i}) = \mathsf{D}\mathsf{U}^{n+1,i+1} \tag{45}$$

with

$$\mathsf{P}^{n+1,0} = 2\mathsf{P}^n - \mathsf{P}^{n-1}, \tag{46}$$

$$\mathsf{U}^{n+1,0} = 2\mathsf{U}^n - \mathsf{U}^{n-1}. \tag{47}$$

Both (42)-(43) and (44)-(45) are iterative schemes in which the pressure calculation is uncoupled from the velocity. This is why we have included them in this section about pressure segregation methods. Their numerical performance will be discussed in Section 5.

4 Stability of fractional step methods

The goal of this section is to present stability estimates for the fractional step methods introduced previously. Let us first introduce some additional notation. If X, Y are arrays, $\{\mathsf{X}^n\}_{n=0,1,...,N}$ is a sequence of arrays of $N+1$ terms and A a symmetric positive semi-definite matrix, we define

$$\begin{aligned}
(\mathsf{X},\mathsf{Y})_{\mathsf{A}} &:= \mathsf{X}\cdot\mathsf{A}\mathsf{Y},\\
\|\mathsf{X}\|_{\mathsf{A}} &:= (\mathsf{X}\cdot\mathsf{A}\mathsf{X})^{1/2},\\
\|\mathsf{Y}\|_{-\mathsf{A}} &:= \sup_{\mathsf{X}\neq 0}\frac{\mathsf{Y}\cdot\mathsf{X}}{\|\mathsf{X}\|_{\mathsf{A}}} \quad \text{(here } \mathsf{A} \text{ is assumed to be positive definite)},\\
\{\mathsf{X}^n\}\in\ell^\infty(\mathsf{A}) &\iff \|\mathsf{X}^n\|_{\mathsf{A}} \le C < \infty \quad \forall\, n = 0,1,...,N,\\
\{\mathsf{X}^n\}\in\ell^p(\mathsf{A}) &\iff \sum_{n=0}^{N}\delta t\|\mathsf{X}^n\|_{\mathsf{A}}^p \le C < \infty, \quad 1\le p<\infty.
\end{aligned}$$

Here and in the following, C denotes a positive constant, not necessarily the same at different appearances.

A remark is needed when $\mathsf{A} = \mathsf{K}$. This matrix is not symmetric, but it has the contribution from the convective term, which is skew-symmetric, and the contribution from the viscous term, $\mathsf{K}_{\rm visc}$, which is symmetric and positive definite. We will simply write $\mathsf{U}\cdot\mathsf{K}(\mathsf{U})\mathsf{U} = \mathsf{U}\cdot\mathsf{K}_{\rm visc}\mathsf{U} \equiv \|\mathsf{U}\|_{\mathsf{K}}^2$.

We will make use also of $\mathsf{L}^+ := -\mathsf{L}$, which is the positive semi-definite matrix corresponding to the discretization of $-\nabla^2$.

These definitions will allow us to express our stability results in a compact manner. The basic assumption in all the cases will be that

$$\sum_{n=0}^{N} \delta t \|\mathsf{F}^n\|_{-\mathsf{K}}^2 \leq C < \infty, \tag{48}$$

which is the matrix version of the classical condition required for the problem to be well posed. Apart from this, *no other regularity assumptions will be required*. Thus, the following estimates hold for the minimum velocity-pressure regularity.

The first stability result we present was proved already in [4]. For method BDF1-SE1, we have:

Stability of BDF1-SE1:

$$\{\mathsf{U}^n\} \in \ell^\infty(\mathsf{M}), \ \ \{\tilde{\mathsf{U}}^n\} \in \ell^\infty(\mathsf{M}) \cap \ell^2(\mathsf{K}), \ \ \{\sqrt{\delta t}\,\mathsf{P}^n\} \in \ell^2(\mathsf{L}^+)$$

The stability estimate for the pressure shows that the pressure gradient *multiplied by* δt is ℓ^2 bounded. When δt is of order $\mathcal{O}(h^2)$ this is optimal [8, 2, 15]. For the velocity, the stability estimates are optimal.

Method BDF1-SE1 is first order because of the order of both the time integration and the splitting error. However, if we consider method BDF1-SE2, with a second order splitting error, we obtain the same estimates for the velocity but much weaker estimates for the pressure. The result is:

Stability of BDF1-SE2:

$$\{\mathsf{U}^n\} \in \ell^\infty(\mathsf{M}), \ \ \{\tilde{\mathsf{U}}^n\} \in \ell^\infty(\mathsf{M}) \cap \ell^2(\mathsf{K}), \ \ \{\delta t \mathsf{P}^n\} \in \ell^\infty(\mathsf{L}^+)$$

The stability estimate for the pressure is now multiplied by δt instead of $\sqrt{\delta t}$ as in the previous case, which makes it weaker (even though the temporal norm is stronger). The way to improve it is by making use of the inf-sup condition, if it holds for the velocity-pressure interpolation employed, or by using stabilization techniques.

Let us prove this result. First, let us write the scheme as

$$\mathsf{M}\frac{1}{\delta t}(\tilde{\mathsf{U}}^{n+1} - \mathsf{U}^n) + \mathsf{K}(\tilde{\mathsf{U}}^{n+1})\tilde{\mathsf{U}}^{n+1} + \mathsf{G}\mathsf{P}^n = \mathsf{F}^{n+1}, \tag{49}$$

$$\delta t \mathsf{L}(\delta \mathsf{P}^{n+1}) = \mathsf{D}\tilde{\mathsf{U}}^{n+1}, \tag{50}$$

$$\mathsf{M}\frac{1}{\delta t}(\mathsf{U}^{n+1} - \tilde{\mathsf{U}}^{n+1}) + \mathsf{G}(\delta \mathsf{P}^{n+1}) = \mathsf{0}. \tag{51}$$

Taking the inner product of (49) with $2\delta t\tilde{\mathsf{U}}^{n+1}$ and using the identity

$$(2a, a-b) := a^2 - b^2 + (a-b)^2,$$

we get

$$\begin{aligned} &\|\tilde{\mathsf{U}}^{n+1}\|_{\mathsf{M}}^2 - \|\mathsf{U}^n\|_{\mathsf{M}}^2 + \|\tilde{\mathsf{U}}^{n+1} - \mathsf{U}^n\|_{\mathsf{M}}^2 + 2\delta t\|\tilde{\mathsf{U}}^{n+1}\|_{\mathsf{K}}^2 + 2\delta t\tilde{\mathsf{U}}^{n+1} \cdot \mathsf{G}\mathsf{P}^n \\ &\quad = 2\delta t\tilde{\mathsf{U}}^{n+1} \cdot \mathsf{F}^{n+1} \leq \delta t\|\mathsf{F}^{n+1}\|_{-\mathsf{K}}^2 + \delta t\|\tilde{\mathsf{U}}^{n+1}\|_{\mathsf{K}}^2. \end{aligned} \tag{52}$$

Multiplying (51) by $2\delta t\mathsf{U}^{n+1}$ we obtain

$$\|\mathsf{U}^{n+1}\|_{\mathsf{M}}^2 - \|\tilde{\mathsf{U}}^{n+1}\|_{\mathsf{M}}^2 + \|\mathsf{U}^{n+1} - \tilde{\mathsf{U}}^{n+1}\|_{\mathsf{M}}^2 + 2\delta t\mathsf{U}^{n+1}\cdot\mathsf{G}\delta\mathsf{P}^{n+1} = 0. \tag{53}$$

Adding up (52) and (53) it is found that

$$\begin{aligned}\|\mathsf{U}^{n+1}\|_{\mathsf{M}}^2 - \|\mathsf{U}^{n}\|_{\mathsf{M}}^2 + \|\mathsf{U}^{n+1} - \tilde{\mathsf{U}}^{n+1}\|_{\mathsf{M}}^2 + \|\tilde{\mathsf{U}}^{n+1} - \mathsf{U}^{n}\|_{\mathsf{M}}^2 + \delta t\|\tilde{\mathsf{U}}^{n+1}\|_{\mathsf{K}}^2 \\ + 2\delta t\mathsf{U}^{n+1}\cdot\mathsf{G}\delta\mathsf{P}^{n+1} + 2\delta t\tilde{\mathsf{U}}^{n+1}\cdot\mathsf{G}\mathsf{P}^{n} \leq \delta t\|\mathsf{F}^{n+1}\|_{-\mathsf{K}}^2.\end{aligned} \tag{54}$$

Since (51) implies

$$\mathsf{U}^{n+1} - \tilde{\mathsf{U}}^{n+1} = -\delta t\mathsf{M}^{-1}\mathsf{G}(\delta\mathsf{P}^{n+1}),$$

and $\mathsf{G}^{\mathrm{t}} = -\mathsf{D}$ with the boundary conditions considered, we have that

$$\begin{aligned}\|\mathsf{U}^{n+1} - \tilde{\mathsf{U}}^{n+1}\|_{\mathsf{M}}^2 &= \delta t^2\mathsf{M}^{-1}\mathsf{G}(\delta\mathsf{P}^{n+1})\cdot\mathsf{G}(\delta\mathsf{P}^{n+1}) \\ &= -\delta t^2(\delta\mathsf{P}^{n+1})\cdot\mathsf{D}\mathsf{M}^{-1}\mathsf{G}(\delta\mathsf{P}^{n+1}) = \delta t^2\|\delta\mathsf{P}^{n+1}\|_{\mathsf{L}_D^+}^2,\end{aligned} \tag{55}$$

being $\mathsf{L}_D^+ := -\mathsf{D}\mathsf{M}^{-1}\mathsf{G}$ positive semi-definite.

From (50) we now obtain

$$\begin{aligned}2\delta t\tilde{\mathsf{U}}^{n+1}\cdot\mathsf{G}\mathsf{P}^{n} &= -2\delta t\mathsf{P}^{n}\cdot\mathsf{D}\tilde{\mathsf{U}}^{n+1} = 2\delta t^2\mathsf{P}^{n}\cdot\mathsf{L}^{+}\delta\mathsf{P}^{n+1} \\ &= \delta t^2(\|\mathsf{P}^{n+1}\|_{\mathsf{L}^+}^2 - \|\mathsf{P}^{n}\|_{\mathsf{L}^+}^2 - \|\delta\mathsf{P}^{n+1}\|_{\mathsf{L}^+}^2).\end{aligned} \tag{56}$$

On the other hand, using again (51) and (50) we get

$$\begin{aligned}2\delta t\mathsf{U}^{n+1}\cdot\mathsf{G}(\delta\mathsf{P}^{n+1}) &= -2\delta t(\delta\mathsf{P}^{n+1})\cdot\mathsf{D}(\tilde{\mathsf{U}}^{n+1} - \delta t\mathsf{M}^{-1}\mathsf{G}\delta\mathsf{P}^{n+1}) \\ &= 2\delta t^2(\delta\mathsf{P}^{n+1})\cdot\mathsf{L}^{+}(\delta\mathsf{P}^{n+1}) + 2\delta t^2(\delta\mathsf{P}^{n+1})\cdot\mathsf{D}\mathsf{M}^{-1}\mathsf{G}(\delta\mathsf{P}^{n+1}) \\ &= \delta t^2\|\delta\mathsf{P}^{n+1}\|_{\mathsf{L}^+}^2 + \delta t^2\|\delta\mathsf{P}^{n+1}\|_{\mathsf{B}}^2 - \delta t^2\|\delta\mathsf{P}^{n+1}\|_{\mathsf{L}_D^+}^2,\end{aligned} \tag{57}$$

being $\mathsf{B} := \mathsf{D}\mathsf{M}^{-1}\mathsf{G} - \mathsf{L} = \mathsf{L}^{+} - \mathsf{L}_D^{+}$ positive semi-definite (see [4]).

Using (55), (56) and (57) in (54), we find that

$$\|\mathsf{U}^{n+1}\|_{\mathsf{M}}^2 - \|\mathsf{U}^{n}\|_{\mathsf{M}}^2 + \delta t\|\tilde{\mathsf{U}}^{n+1}\|_{\mathsf{K}}^2 + \delta t^2\|\mathsf{P}^{n+1}\|_{\mathsf{L}^+}^2 - \delta t^2\|\mathsf{P}^{n}\|_{\mathsf{L}^+}^2 \leq \delta t\|\mathsf{F}^{n+1}\|_{-\mathsf{K}}^2,$$

and summing from $n = 1$ to $n = N$, an arbitrary time level, we obtain

$$\|\mathsf{U}^{N}\|_{\mathsf{M}}^2 + \sum_{n=1}^{N}\delta t\|\tilde{\mathsf{U}}^{n}\|_{\mathsf{K}}^2 + \delta t^2\|\mathsf{P}^{N}\|_{\mathsf{L}^+}^2 \leq C, \tag{58}$$

where C includes the norm of the force vector and the initial condition. This inequality (58) proves the desired stability estimate for method BDF1-SE2, except for the $\ell^\infty(\mathsf{M})$ estimate for $\{\tilde{\mathsf{U}}^n\}$, which is easily obtained from (51), the $\ell^\infty(\mathsf{M})$ estimate for $\{\mathsf{U}^n\}$ and noting that the right-hand-side of (57) is non-negative.

For the CN-SE2 the stability estimate was already obtained in [4]. The result is the following:

Stability of CN-SE2:
$$\{\mathsf{U}^n\} \in \ell^\infty(\mathsf{M}),\ \ \{\tilde{\mathsf{U}}^n\} \in \ell^\infty(\mathsf{M}),\ \ \{\tilde{\mathsf{U}}^{n+1/2}\} \in \ell^2(\mathsf{K}),$$
$$\{\delta t \mathsf{P}^n\} \in \ell^\infty(\mathsf{L}^+),\ \ \{\sqrt{\delta t}\,\delta \mathsf{P}^n\} \in \ell^2(\mathsf{L}^+)$$

The same remarks as those made concerning the stability of method BDF1-SE2 apply now. We therefore conclude that the pressure stability depends on how the splitting is done rather than on the time integration scheme. This is also ratified by the stability estimate for method BDF2-SE2, which we present now:

Stability of BDF2-SE2:
$$\{\mathsf{U}^n\} \in \ell^\infty(\mathsf{M}),\ \ \{\tilde{\mathsf{U}}^n\} \in \ell^2(\mathsf{K}),$$
$$\{\delta t \mathsf{P}^n\} \in \ell^\infty(\mathsf{L}^+),\ \ \{\sqrt{\delta t}\,\delta \mathsf{P}^n\} \in \ell^2(\mathsf{L}^+)$$

We conclude this section by proving this result. Let us start by the method obtained *without using approximation (21)*. This method reads

$$\mathsf{M}\frac{1}{2\delta t}(3\tilde{\mathsf{U}}^{n+1} - 4\mathsf{U}^n + \mathsf{U}^{n-1}) + \mathsf{K}(\tilde{\mathsf{U}}^{n+1})\tilde{\mathsf{U}}^{n+1} + \mathsf{G}\mathsf{P}^n = \mathsf{F}^{n+1}, \tag{59}$$
$$\mathsf{D}\mathsf{U}^{n+1} = \mathsf{0}, \tag{60}$$
$$\mathsf{M}\frac{1}{2\delta t}(3\mathsf{U}^{n+1} - 3\tilde{\mathsf{U}}^{n+1}) + \mathsf{G}(\delta \mathsf{P}^{n+1}) = \mathsf{0}. \tag{61}$$

In this case, the velocity at the end of the step is divergence free (in the discrete weak sense). Alternatively, (60) could be replaced by

$$\frac{2}{3}\delta t \mathsf{D}\mathsf{M}^{-1}\mathsf{G}(\delta \mathsf{P}^{n+1}) = \mathsf{D}\tilde{\mathsf{U}}^{n+1}, \tag{62}$$

in contrast with the pressure Poisson equation

$$\frac{2}{3}\delta t \mathsf{L}(\delta \mathsf{P}^{n+1}) = \mathsf{D}\tilde{\mathsf{U}}^{n+1}, \tag{63}$$

that would be obtained making use of approximation (21).

To obtain the stability of problem (59)-(61), let us start by multiplying (59) by $4\delta t\tilde{\mathsf{U}}^{n+1}$, getting

$$\begin{aligned}(2\tilde{\mathsf{U}}^{n+1}, 3\tilde{\mathsf{U}}^{n+1} - 4\mathsf{U}^n + \mathsf{U}^{n-1})_{\mathsf{M}} + 4\delta t\|\tilde{\mathsf{U}}^{n+1}\|^2_{\mathsf{K}} + 4\delta t\tilde{\mathsf{U}}^{n+1}\cdot\mathsf{G}\mathsf{P}^n \\ = 4\delta t\tilde{\mathsf{U}}^{n+1}\cdot\mathsf{F}^{n+1} \leq 2\delta t\|\mathsf{F}^{n+1}\|^2_{-\mathsf{K}} + 2\delta t\|\tilde{\mathsf{U}}^{n+1}\|^2_{\mathsf{K}}\end{aligned} \tag{64}$$

Expanding the first term of (64) we get

$$\begin{aligned}&(2\tilde{\mathsf{U}}^{n+1}, 3\tilde{\mathsf{U}}^{n+1} - 4\mathsf{U}^n + \mathsf{U}^{n-1})_{\mathsf{M}} \\ &= (2\mathsf{U}^{n+1} + 2\tilde{\mathsf{U}}^{n+1} - 2\mathsf{U}^{n+1}, 3\mathsf{U}^{n+1} - 4\mathsf{U}^n + \mathsf{U}^{n-1} + 3\tilde{\mathsf{U}}^{n+1} - 3\mathsf{U}^{n+1})_{\mathsf{M}} \\ &= (2\mathsf{U}^{n+1}, 3\mathsf{U}^{n+1} - 4\mathsf{U}^n + \mathsf{U}^{n-1})_{\mathsf{M}} + (2\tilde{\mathsf{U}}^{n+1} - 2\mathsf{U}^{n+1}, 3\mathsf{U}^{n+1} - 4\mathsf{U}^n + \mathsf{U}^{n-1})_{\mathsf{M}} \\ &+ (2\tilde{\mathsf{U}}^{n+1}, 3\tilde{\mathsf{U}}^{n+1} - 3\mathsf{U}^{n+1})_{\mathsf{M}}.\end{aligned} \tag{65}$$

Using the identity

$$(2a, 3a - 4b + c) := a^2 - b^2 + (2a - b)^2 - (2b - c)^2 + (a - 2b + c)^2,$$

we can manipulate the first term in the right-hand-side of (65) as follows:

$$\begin{aligned}&(2\mathsf{U}^{n+1}, 3\mathsf{U}^{n+1} - 4\mathsf{U}^n + \mathsf{U}^{n-1})_{\mathsf{M}}\\ &\quad = \|\mathsf{U}^{n+1}\|^2_{\mathsf{M}} - \|\mathsf{U}^n\|^2_{\mathsf{M}} + \|2\mathsf{U}^{n+1} - \mathsf{U}^n\|^2_{\mathsf{M}} - \|2\mathsf{U}^n - \mathsf{U}^{n-1}\|^2_{\mathsf{M}} + \|\delta^2\mathsf{U}^{n+1}\|^2_{\mathsf{M}}.\end{aligned} \tag{66}$$

From (61) it follows that

$$\tilde{\mathsf{U}}^{n+1} - \mathsf{U}^{n+1} = \frac{2}{3}\delta t \mathsf{M}^{-1}\mathsf{G}(\delta \mathsf{P}^{n+1}),$$

which can be used to express the second term in the right-hand-side of (65) as

$$(2\tilde{\mathsf{U}}^{n+1} - 2\mathsf{U}^{n+1}, 3\mathsf{U}^{n+1} - 4\mathsf{U}^n + \mathsf{U}^{n-1})_{\mathsf{M}} = -\frac{4}{3}\delta t(\delta\mathsf{P}^{n+1})\cdot\mathsf{D}(3\mathsf{U}^{n+1} - 4\mathsf{U}^n + \mathsf{U}^{n-1}) = 0, \tag{67}$$

which is zero because of (60).

For the last term in (65) we have

$$3(2\tilde{\mathsf{U}}^{n+1}, \tilde{\mathsf{U}}^{n+1} - \mathsf{U}^{n+1})_{\mathsf{M}} = 3\|\tilde{\mathsf{U}}^{n+1}\|^2_{\mathsf{M}} - 3\|\mathsf{U}^{n+1}\|^2_{\mathsf{M}} + 3\|\tilde{\mathsf{U}}^{n+1} - \mathsf{U}^{n+1}\|^2_{\mathsf{M}}. \tag{68}$$

Using (66), (67) and (68) in (65), and applying the result in (64) we obtain

$$\begin{aligned}&3\|\tilde{\mathsf{U}}^{n+1}\|^2_{\mathsf{M}} - 3\|\mathsf{U}^{n+1}\|^2_{\mathsf{M}} + 3\|\tilde{\mathsf{U}}^{n+1} - \mathsf{U}^{n+1}\|^2_{\mathsf{M}} + \|\mathsf{U}^{n+1}\|^2_{\mathsf{M}} - \|\mathsf{U}^n\|^2_{\mathsf{M}}\\ &\quad + \|2\mathsf{U}^{n+1} - \mathsf{U}^n\|^2_{\mathsf{M}} - \|2\mathsf{U}^n - \mathsf{U}^{n-1}\|^2_{\mathsf{M}} + \|\delta^2\mathsf{U}^{n+1}\|^2_{\mathsf{M}} + 2\delta t\|\tilde{\mathsf{U}}^{n+1}\|^2_{\mathsf{K}}\\ &\quad + 4\delta t\tilde{\mathsf{U}}^{n+1}\cdot\mathsf{G}\mathsf{P}^n \le 2\delta t\|\mathsf{F}^{n+1}\|^2_{-\mathsf{K}}.\end{aligned} \tag{69}$$

On the other hand, (61) can be reordered to get

$$\frac{3}{2\delta t}\mathsf{U}^{n+1} + \mathsf{M}^{-1}\mathsf{G}\mathsf{P}^{n+1} = \frac{3}{2\delta t}\tilde{\mathsf{U}}^{n+1} + \mathsf{M}^{-1}\mathsf{G}\mathsf{P}^n.$$

Squaring both terms of this equation with the inner product $(\cdot,\cdot)_{\mathsf{M}}$ we obtain

$$\begin{aligned}&(\frac{3}{2\delta t}\mathsf{U}^{n+1} + \mathsf{M}^{-1}\mathsf{G}\mathsf{P}^{n+1})\cdot(\mathsf{M}\frac{3}{2\delta t}\mathsf{U}^{n+1} + \mathsf{G}\mathsf{P}^{n+1})\\ &\qquad = (\frac{3}{2\delta t}\tilde{\mathsf{U}}^{n+1} + \mathsf{M}^{-1}\mathsf{G}\mathsf{P}^n)\cdot(\mathsf{M}\frac{3}{2\delta t}\tilde{\mathsf{U}}^{n+1} + \mathsf{G}\mathsf{P}^n).\end{aligned}$$

After expanding the terms of this equality and using the fact that the velocity at the end of step is divergence free, it is found

$$\frac{9}{4\delta t^2}\|\mathsf{U}^{n+1}\|^2_{\mathsf{M}} + \|\mathsf{P}^{n+1}\|^2_{\mathsf{L}^+_D} = \frac{9}{4\delta t^2}\|\tilde{\mathsf{U}}^{n+1}\|^2_{\mathsf{M}} + \frac{6}{2\delta t}\tilde{\mathsf{U}}^{n+1}\cdot\mathsf{G}\mathsf{P}^n + \|\mathsf{P}^n\|^2_{\mathsf{L}^+_D}. \tag{70}$$

Recall that $\mathsf{L}^+_D := -\mathsf{D}\mathsf{M}^{-1}\mathsf{G}$ is positive semi-definite. Multiplying this equation by $\frac{4}{3}\delta t^2$ and adding it to (69) we obtain

$$\begin{aligned}&\|\mathsf{U}^{n+1}\|^2_{\mathsf{M}} - \|\mathsf{U}^n\|^2_{\mathsf{M}} + 3\|\tilde{\mathsf{U}}^{n+1} - \mathsf{U}^{n+1}\|^2_{\mathsf{M}} + \|2\mathsf{U}^{n+1} - \mathsf{U}^n\|^2_{\mathsf{M}} - \|2\mathsf{U}^n - \mathsf{U}^{n-1}\|^2_{\mathsf{M}}\\ &\quad + \|\delta^2\mathsf{U}^{n+1}\|^2_{\mathsf{M}} + \frac{4}{3}\delta t^2\|\mathsf{P}^{n+1}\|^2_{\mathsf{L}^+_D} - \frac{4}{3}\delta t^2\|\mathsf{P}^n\|^2_{\mathsf{L}^+_D} + 2\delta t\|\tilde{\mathsf{U}}^{n+1}\|^2_{\mathsf{K}} + \le 2\delta t\|\mathsf{F}^{n+1}\|^2_{-\mathsf{K}}.\end{aligned} \tag{71}$$

Using (61) we can express $\|\tilde{\mathsf{U}}^{n+1} - \mathsf{U}^{n+1}\|_{\mathsf{M}}$ as

$$3\|\tilde{\mathsf{U}}^{n+1} - \mathsf{U}^{n+1}\|_{\mathsf{M}}^2 = \frac{4}{3}\delta t^2 \mathsf{M}^{-1}\mathsf{G}(\mathsf{P}^{n+1} - \mathsf{P}^n) \cdot \mathsf{M}\mathsf{M}^{-1}\mathsf{G}(\mathsf{P}^{n+1} - \mathsf{P}^n)$$
$$= -\frac{4}{3}\delta t^2 (\mathsf{P}^{n+1} - \mathsf{P}^n) \cdot \mathsf{D}\mathsf{M}^{-1}\mathsf{G}(\mathsf{P}^{n+1} - \mathsf{P}^n) = \frac{4}{3}\delta t \|\sqrt{\delta t}(\mathsf{P}^{n+1} - \mathsf{P}^n)\|_{\mathsf{L}_D^+}^2.$$

Using this in (71), adding the result up from $n = 1$ to an arbitrary time level N and neglecting some positive terms we get

$$\|\mathsf{U}^N\|_{\mathsf{M}}^2 + \frac{4}{3}\sum_{n=1}^{N} \delta t \|\sqrt{\delta t}\delta \mathsf{P}^n\|_{\mathsf{L}_D^+}^2 + \frac{4}{3}\|\delta t \mathsf{P}^N\|_{\mathsf{L}_D^+}^2 + 2\sum_{n=1}^{N} \delta t \|\tilde{\mathsf{U}}^n\|_{\mathsf{K}}^2 \leq C, \tag{72}$$

where C involves the norm of the force vector and the initial condition. Therefore, the stability results obtained are:

$$\{\mathsf{U}^n\} \in \ell^\infty(\mathsf{M}),\ \{\tilde{\mathsf{U}}^n\} \in \ell^2(\mathsf{K}),\ \{\sqrt{\delta t}\delta \mathsf{P}^n\} \in \ell^2(\mathsf{L}_D^+),\ \{\delta t \mathsf{P}^n\} \in \ell^\infty(\mathsf{L}_D^+).$$

Let us consider now scheme BDF2-SE2 using approximation $\mathsf{D}\mathsf{M}^{-1}\mathsf{G} \cong \mathsf{L}$. In this case, the velocity at the end of the step is not divergence free. Making use of (27) in (26) the scheme can be written as follows:

$$\mathsf{M}\frac{1}{2\delta t}(3\tilde{\mathsf{U}}^{n+1} - 4\mathsf{U}^n + \mathsf{U}^{n-1}) + \mathsf{K}(\tilde{\mathsf{U}}^{n+1})\tilde{\mathsf{U}}^{n+1} + \mathsf{G}\mathsf{P}^n = \mathsf{F}^{n+1}, \tag{73}$$

$$\mathsf{D}\mathsf{U}^{n+1} + \frac{2}{3}\delta t \mathsf{B}(\delta \mathsf{P}^{n+1}) = 0, \tag{74}$$

$$\mathsf{M}\frac{1}{2\delta t}(3\mathsf{U}^{n+1} - 3\tilde{\mathsf{U}}^{n+1}) + \mathsf{G}(\delta \mathsf{P}^{n+1}) = \mathsf{0}. \tag{75}$$

Recall that $\mathsf{B} := \mathsf{D}\mathsf{M}^{-1}\mathsf{G} - \mathsf{L}$ is positive semi-definite.

The places of the previous stability analysis where there will be differences are those where the divergence free property of the end-of-step velocity has been used. This assumption has just been made in equations (67) and (70). Using (74), expression (67) in this case is:

$$-\frac{4}{3}\delta t(\delta \mathsf{P}^{n+1}) \cdot \mathsf{D}(3\mathsf{U}^{n+1} - 4\mathsf{U}^n + \mathsf{U}^{n-1}) = \frac{4}{9}\delta t^2 (2\delta \mathsf{P}^{n+1}) \cdot (3\mathsf{B}\delta \mathsf{P}^{n+1} - 4\mathsf{B}\delta \mathsf{P}^n + \mathsf{B}\delta \mathsf{P}^{n-1})$$
$$= \frac{4}{9}\delta t^2 \cdot (\|\delta \mathsf{P}^{n+1}\|_{\mathsf{B}}^2 - \|\delta \mathsf{P}^n\|_{\mathsf{B}}^2 + \|2\delta \mathsf{P}^{n+1} - \delta \mathsf{P}^n\|_{\mathsf{B}}^2 - \|2\delta \mathsf{P}^n - \delta \mathsf{P}^{n-1}\|_{\mathsf{B}}^2 + \|\delta^3 \mathsf{P}^n\|_{\mathsf{B}}). \tag{76}$$

On the other hand, the following term has to be added to the left-hand-side of (70):

$$\frac{6}{2\delta t^2}\mathsf{U}^{n+1} \cdot \mathsf{G}\mathsf{P}^{n+1} = -\frac{6}{2\delta t^2}\mathsf{P}^{n+1} \cdot \mathsf{D}\mathsf{U}^{n+1}$$
$$= 2\mathsf{P}^{n+1} \cdot \mathsf{B}\delta \mathsf{P}^{n+1} = \|\mathsf{P}^{n+1}\|_{\mathsf{B}}^2 - \|\mathsf{P}^n\|_{\mathsf{B}}^2 + \|\delta \mathsf{P}^{n+1}\|_{\mathsf{B}}^2. \tag{77}$$

With the change (76) in (67) and (77) in (70), inequality (71) has to be replaced by

$$\begin{aligned}
&\|\mathsf{U}^{n+1}\|_{\mathsf{M}}^2 - \|\mathsf{U}^n\|_{\mathsf{M}}^2 + \frac{4}{3}\delta t\|\sqrt{\delta t}\delta \mathsf{P}^{n+1}\|_{\mathsf{L}_D^+}^2 + \frac{4}{3}\|\delta t\mathsf{P}^{n+1}\|_{\mathsf{L}_D^+}^2 - \frac{4}{3}\|\delta t\mathsf{P}^n\|_{\mathsf{L}_D^+}^2 \\
&\quad + \frac{4}{3}\|\delta t\mathsf{P}^{n+1}\|_{\mathsf{B}}^2 - \frac{4}{3}\|\delta t\mathsf{P}^n\|_{\mathsf{B}}^2 + \frac{4}{3}\delta t\|\sqrt{\delta t}\delta \mathsf{P}^{n+1}\|_{\mathsf{B}}^2 \\
&\quad + 2\delta t\|\tilde{\mathsf{U}}^{n+1}\|_{\mathsf{K}}^2 + \|2\mathsf{U}^{n+1} - \mathsf{U}^n\|_{\mathsf{M}}^2 - \|2\mathsf{U}^n - \mathsf{U}^{n-1}\|_{\mathsf{M}}^2 + \|\delta^2\mathsf{U}^{n+1}\|_{\mathsf{M}}^2 \\
&\quad + \frac{4}{9}\delta t^2 \cdot (\|\delta\mathsf{P}^{n+1}\|_{\mathsf{B}}^2 - \|\delta\mathsf{P}^n\|_{\mathsf{B}}^2 + \|2\delta\mathsf{P}^{n+1} - \delta\mathsf{P}^n\|_{\mathsf{B}}^2 - \|2\delta\mathsf{P}^n - \delta\mathsf{P}^{n-1}\|_{\mathsf{B}}^2 + \|\delta^3\mathsf{P}^n\|_{\mathsf{B}}) \\
&\leq 2\delta t\|\mathsf{F}^{n+1}\|_{-\mathsf{K}}^2.
\end{aligned}$$

Adding up from $n = 1$ to an arbitrary time level N and neglecting some positive terms we find that

$$\begin{aligned}
\|\mathsf{U}^N\|_{\mathsf{M}}^2 + \frac{4}{3}\sum_{n=1}^{N}\delta t\|\sqrt{\delta t}\delta\mathsf{P}^n\|_{\mathsf{L}_D^+}^2 + \frac{4}{3}\|\delta t\mathsf{P}^N\|_{\mathsf{L}_D^+}^2 + 2\sum_{n=1}^{N}\delta t\|\tilde{\mathsf{U}}^n\|_{\mathsf{K}}^2 \\
+ \frac{4}{3}\sum_{n=1}^{N}\delta t\|\sqrt{\delta t}\delta\mathsf{P}^n\|_{\mathsf{B}}^2 + \frac{4}{3}\|\delta t\mathsf{P}^N\|_{\mathsf{B}}^2 \leq C. \qquad (78)
\end{aligned}$$

which, noting that $\mathsf{B} + \mathsf{L}_D^+ = \mathsf{L}^+$, yields the stability result we wished to prove.

5 Numerical tests

In this section we present some numerical results to test the time integration schemes described in this paper. Even though all the exposition has been based on the Galerkin method for the spatial discretization, in the following examples we have used equal velocity-pressure interpolation and the pressure stabilization technique presented in [5]. In particular, we have taken $k_{\rm q} = k_{\rm v} = 1$, with the notation of Section 2.

5.1 Convergence test

The first example we consider is a simple convergence test whose goal is to check numerically the rate of convergence in time for some of the numerical methods described.

The computational domain is the unit square, discretized using a uniform triangular mesh of 11×11 nodal points (200 triangles). The boundary and initial conditions and the force term are prescribed so that the analytic solution is $\boldsymbol{u} = (y, -x)\sin(\pi t/10)\exp(t/25),\ p = 0$. Note that the exact solution belongs to the finite element space, and thus the only source of numerical error is the time approximation. The nonlinear term of the Navier-Stokes equations is neglected (it is zero for the exact solution), that is, we consider the transient Stokes problem.

Results are shown in Figure 1. The error E is measured in the ℓ^2 norm of the sequence $\{\boldsymbol{u}^n - \boldsymbol{u}(t^n)\}$. It is seen that all the methods show the expected rate of convergence. This is particularly relevant for the predictor-corrector schemes, whose error is affected by the convergence tolerance adopted in the iterative loop of each time step.

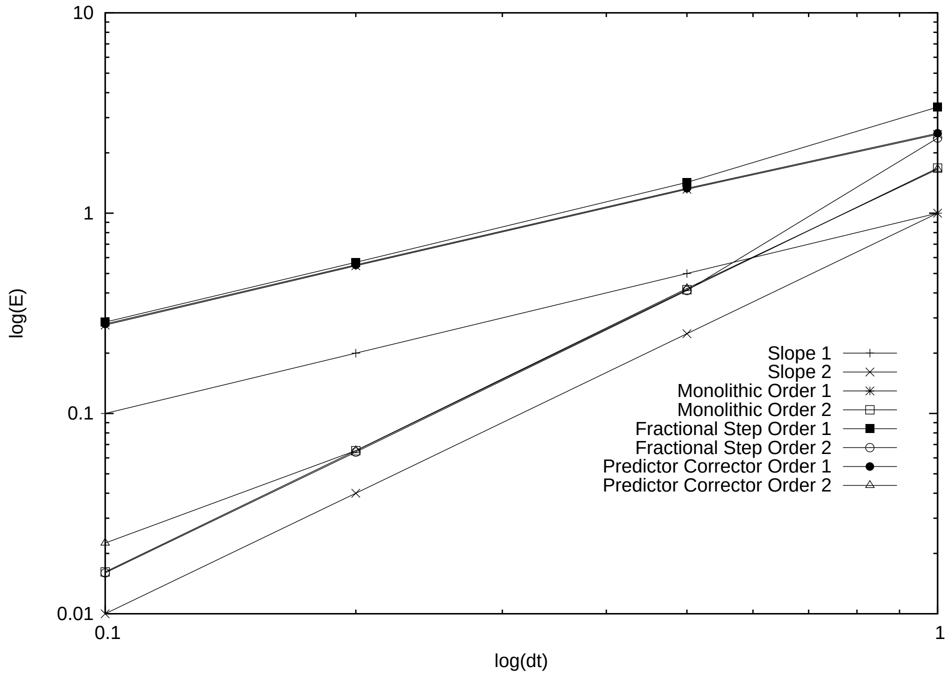

Figure 1: Convergence test

5.2 Flow in a cavity

In this second example we solve the classical cavity flow problem at a Reynolds number Re $=$ 100. The computational domain is the unit square, discretized using a mesh of 21$\times$21 nodal points (400 triangles). The velocity is fixed to zero everywhere except on the top boundary, where it is prescribed to (1,0).

Even though the solution in this simple example is stationary, we obtain it by stepping in time. The goal of this test is precisely to check the properties of the schemes proposed for the long-term time integration of stationary solutions (very often difficult to obtain in a stationary calculation) and, particularly, their numerical dissipation. The time step employed is $\delta t = 1$.

Figure 2 shows the evolution towards the steady state obtained. It is observed that the monolithic scheme is more dissipative than the fractional step method, particularly for the second order scheme (BDF2-SE2). In this particular example, BDF2 seems to be slightly more dissipative than BDF1 for the monolithic case.

When the predictor-corrector scheme is employed, the evolution towards the steady-state depends on the final error of the iterative scheme within each time step. For loose convergence requirements, it is expected that the predictor-corrector method will behave in a way similar to the fractional step method, whereas the behavior will approach that of the monolithic scheme as the iterative error per time step decreases. This is what is observed in Figure 3. With a small tolerance (10^{-6}) the error is directly given by the number of iterations allowed per time step. It is observed that for 20 iteration the behavior in time is similar to that of the monolithic scheme, whereas for 5 iterations it is much less dissipative.

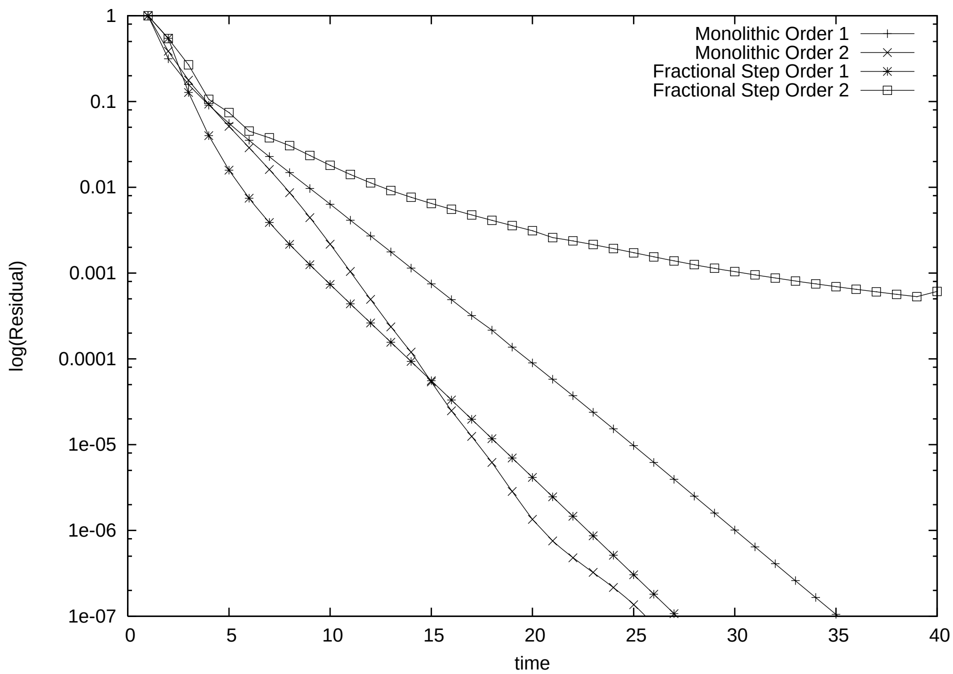

Figure 2: Evolution towards the steady-state for the cavity flow problem using different schemes

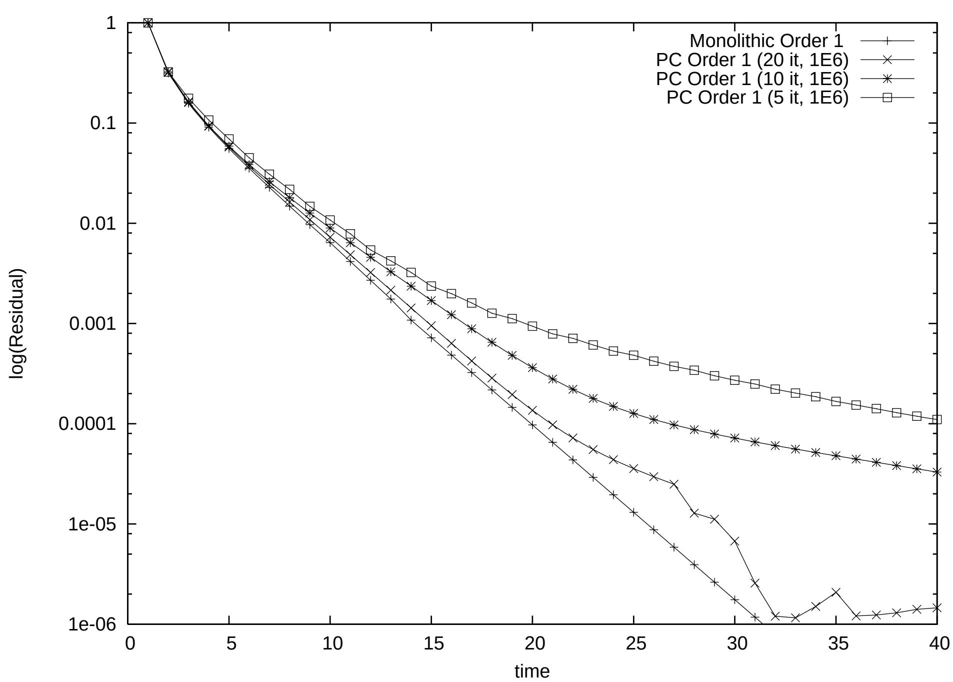

Figure 3: Time dissipation of the predictor-corrector scheme in terms of the time step iterations

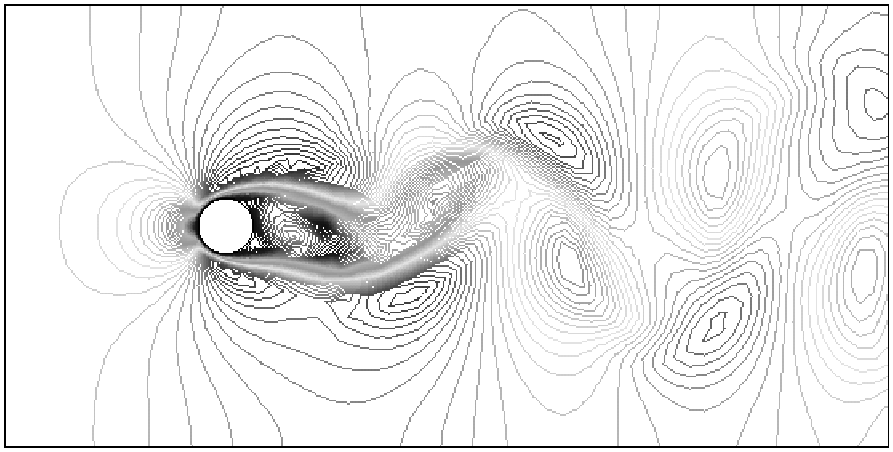

Figure 4: Contours of velocity norm for the flow over a cylinder

5.3 Flow over a cylinder

The last example is also a classical benchmark, namely, the flow over a cylinder. The computational domain is $\bar{\Omega} = [0, 16] \times [0, 8] \setminus D$, with the cylinder D of diameter 1 and centered at $(4, 4)$. The velocity at $x = 0$ is prescribed to $(1, 0)$, whereas at $y = 0$ and $y = 8$ the y-velocity component is prescribed to 0 and the x-component is left free. The outflow (where both the x- and y-components are free) is $x = 16$. The Reynolds number is 100, based on the cylinder diameter and the prescribed inflow velocity. The finite element mesh employed consists of 3604 linear triangles, with 1902 nodal points. A snapshot of the contours of the velocity norm and pressure isolines is shown in Figures 4 and 5. The purpose of this example is not to compare the quality of these results with those presented in the literature, but rather to discuss the time behavior of the schemes proposed in this paper.

The evolution of the y-velocity component at the control point located at (6,4) is shown in Figure 6. The time step size used in all the cases is $\delta t = 0.05$. The tolerance of the iterative procedure of each time step has been set to 0.0001 (0.01%), although a maximum of only 5 iterations has been permitted. It is observed that the least dissipative scheme (with higher frequency and amplitude) is the second order monolithic method (using BDF2 for the time integration), and the most dissipative one (with the smaller frequency and smaller amplitude) is BDF1-SE1. This example serves to show that even though fractional step schemes are attractive for their low computational cost, they are usually *less* accurate for a given time step size than monolithic methods. Likewise, for this particular example predictor-corrector methods lie in between fractional-step and monolithic schemes (recall that only 5 iterations have been

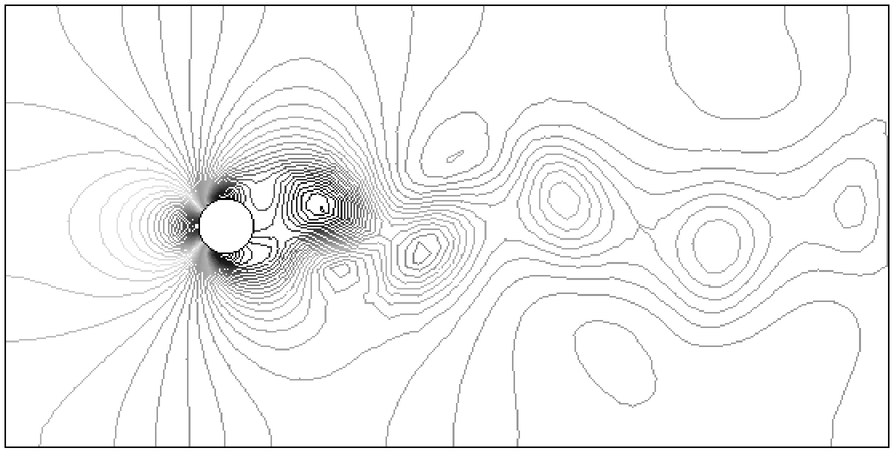

Figure 5: Contours of pressure for the flow over a cylinder

allowed per time step). The improvement with respect to fractional step methods is particularly significant for the first order method.

Finally, concerning the number of iterations performed (for a tolerance of 0.01%) it is interesting to study the effect of the initial guess. Independently of the time integration employed, it is always possible to use a second order extrapolation for the unknowns as given by (46)-(47) to start the iterative procedure. The following table shows the influence of how the initial guess is taken. The numbers listed are the total number of iterations from $t = 0$ to $t = 10$ using two time steps sizes, predictor-corrector and monolithic schemes of second order and first or second order extrapolations to take the initial guess (X refers here to both velocity and pressure degrees of freedom). It is observed that the monolithic schemes need fewer iterations, which can be easily explained since the iterations for the predictor-corrector scheme need to deal not only with the nonlinearity of the problem but also with the velocity-pressure coupling to converge to the monolithic solution. Nevertheless, what is interesting is that an important gain in the total number of iterations is obtained using the second order extrapolation. For first order schemes, it implies that the unknowns at two previous time steps need to be stored (which is unnecessary) but it certainly pays off in view of the significant reduction in the calculation time.

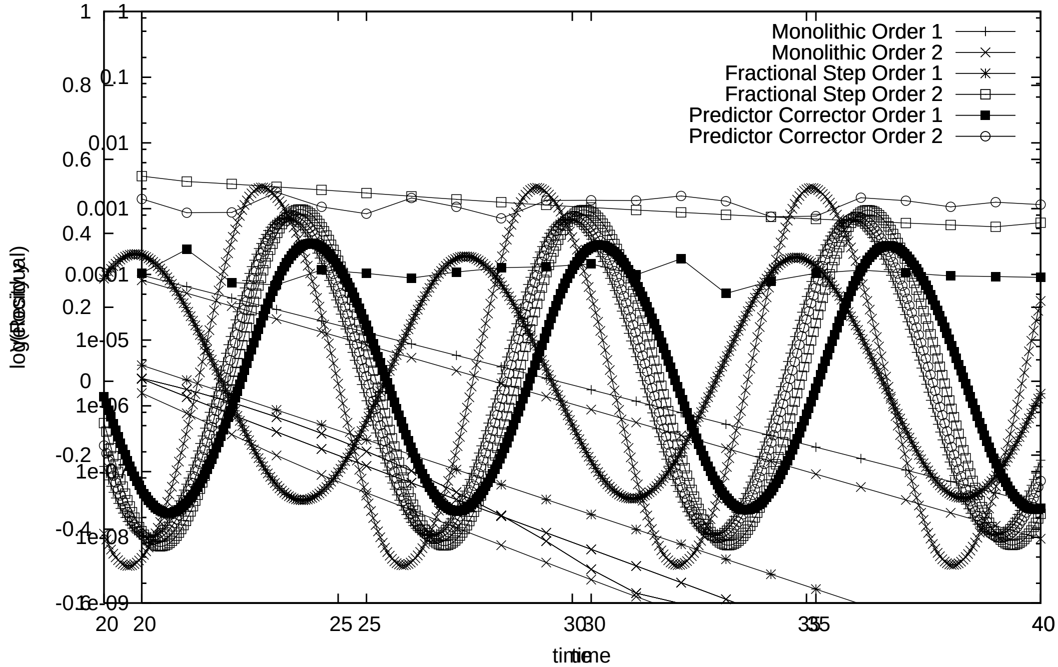

Figure 6: Temporal evolution of the y-velocity component at the control point

	PC using BDF2		Monolithic using BDF2	
	$\delta t = 0.10$	$\delta t = 0.01$	$\delta t = 0.10$	$\delta t = 0.01$
$\mathsf{X}^{n+1,0} = \mathsf{X}^n$	414	2025	300	2000
$\mathsf{X}^{n+1,0} = 2\mathsf{X}^n - \mathsf{X}^{n-1}$	325	1046	204	1006

6 Conclusions

In this paper we have discussed several ways to implement first and second order time integration schemes for the incompressible Navier-Stokes equations whose objective is to uncouple the calculation of the velocity and the pressure.

Taking as a reference the monolithic approach, the advantage of all these methods is that they are more effective from the computational point of view, although other aspects need to be taken into account.

Fractional steps methods based on a pressure Poisson equation offer some pressure stability independent of the space approximation. We have presented here stability results for four of these methods that show this. However, numerical evidence also shows that they are less dissipative in the evolution towards a steady solution, and thus less effective when a transient calculation is used to reach a steady state. Only one example showing this effect has been presented here, but we have observed this in many cases. Likewise, for a given time step they tend to be less accurate than fractional step methods, in spite of the fact that the rate of convergence be optimal. Again, an example of this behavior has been presented in this paper.

Another family of methods discussed is what we have called momentum-pressure Poisson equation with an explicit treatment of the pressure gradient in the momentum equation. Even though the derivation of these methods is different, we have shown that the bottom line is a method that can be considered a fractional step one.

Finally, predictor-corrector schemes can be considered as a way to avoid the shortcomings of fractional step methods while allowing to uncouple the velocity and pressure calculations. As it has been shown in the last numerical example presented, the price is a higher number of iterations to be done per time step. We believe that in some cases this can be worthy (see also [6]). Also in this last example and as a by-product, we have observed that using as initial guess for the iterations within each time step a second order extrapolation of the unknowns leads to significantly fewer iterations. This also can be worthy in many cases and for different methods.

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