

An Innovative Distribution that Incorporates Actuarial Parameters and Applications, Real Data in a Variety of Domains

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INFORMATION

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ABSTRACT

The complexity of data necessitates the development of novel distributions. The introduction of new models allows us to enhance this data and remain contemporary. In this paper, we proposed and explored a new flexible distribution, referred to as the extended inverse Weibull model. This study presents a novel probability model, the extended inverse Weibull (E-IW) distribution. The recently introduced distribution combines the E-X approach with the inverse Weibull distribution. The novel distribution facilitates the evaluation of real-world data due to its analytical feasibility and applicability. The suggested distribution may accommodate several types of datasets. Numerous statistical characteristics of the proposed model were acquired. The features include quantile functions, ordinary moments, order statistics, and moment-generating functions. The estimated parameters of the new model are determined using several estimation techniques, including maximum likelihood, least squares, weighted least squares, maximum product spacing, and the Bayesian method under the square error loss function. The simulation analysis performed on the suggested model validated the dependability and consistency of its parameters. Furthermore, actuarial measures were computed, along with simulation study for these actuarial metrics was executed. Five real data sets were taken from several sectors to illustrate the importance and usefulness of the proposed model. Our empirical findings underscore the significance of the recommended model as a flexible and reliable tool for statistical modeling, with implications for improving data-driven analyses and integrating parametric modeling into modern applications.

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1 Introduction

Asymmetrical statistical distributions are significant because they can provide a useful understanding of natural phenomena. Their value encompasses several fields, notably healthcare, finance, engineering, and social sciences. Numerous classical probability distributions have shown little adaptability in recent decades when successfully representing the complexity of real-world data sets. This has encouraged researchers to add more parameters to traditional models to create more flexible families of distributions. For example, Meraou et al. [1] introduced a new extension of exponential

distribution using the compound Poisson strategy, Khalid et al. [2] proposed the log-logistic exponential Poisson model, Cordeiro and Brito [3], Mahdavi and Kundu [4] added one parameter to the classical distribution for generating a novel model, and Elbatal et al. [5] established a novel family of distributions by employing the Topp-Leone Weibull generator. Another well known generated family of distributions are Yildirim et al. [6] which examined new three parameter exponentiated Benini model, Almetwally [7] introduced the Marshall-Olkin alpha power extended Weibull distribution and studied several methods of estimation based on type I and type II censoring schemes, Sapkota et al. [8] used the sine transformation to construct new family of distributions, Marshall and Olkin [9] proposed a novel tool for adding one parameter to a family of distributions, Eugene et al. [10] studied several statistical properties of the beta-normal distribution. Also, various notable transformations illustrate this extension, including Moakof et al. [11], which provide the type II exponentiated half-logistic Topp-Leone Marshall-Olkin-G family of distributions, El-Alosey et al. [12] constructed a new two-parameter mixture family of generalized distributions, and they established various distributional and mathematical properties, Meraou and Raqab [13], Meraou et al. [14,15] investigated several models using the combining transformation and applied the resulting distributions to several data sets from many fields. Recent studies to extended this line of research to other flexible models. For instance, Almetwally and Meraou [16] defined a novel approach to Nadarajah-Haghighi Distribution, Teamah et al. [17] proposed an alternative version of half-Logistic distribution and they estimated the model parameters by various estimation procedures, Hassan et al. [18] investigated a novel asymmetric form of the power half-logistic distribution, Alzaatreh et al. [19] provided a new technique for generating class of distributions, AL-Dayian et al. [20] studied the Bayesian and maximum likelihood estimation for mixture models of the new Topp-Leone-G family. At the end, one may refer to the works of Tahir et al. [21], Ahsan-ul-Haq et al. [22], and Hassan et al. [23], who developed a new extension model using several methods to create a novel family of distributions.

The latest study suggests that standard models often need improvement. Because of this limitation, more adaptable types of these models are needed, leading to significant advancements in the generalization of established distributions. These extended distributions have been successfully applied in various domains, including radiation and finance. A collection of distributional generalities in the financial and radiation domains is the extended-X (E-X) technique, one of the most powerful methods for obtaining new extension models. The E-X technique is based on adding an extra parameter to the classical model. It has been suggested by Alshanbari et al. [24]. The E-X technique has numerous uses, particularly fitting financial and insurance claims. The cumulative and probability densities (PDF) and (CDF) of the E-X transformation can be formulated as

$$\Lambda(t) = \frac{e^\theta - e^{\theta(1-(F(t))^2)}}{e^\theta - 1} \quad t \in \mathbb{R}, \quad \theta > 0, \quad (1)$$

and

$$\lambda(t) = \frac{2\theta f(t)F(t)}{e^\theta - 1} e^{\theta[1-(F(t))^2]}, \quad (2)$$

where θ and $f(t)$, $F(t)$ are denoted the the PDF and CDF of the parent distribution.

The suggested transformation is adaptable and capable of representing different kinds of data sets. Its density function can be either unimodal or decreasing, and its cdf and pdf have closed-form expressions. Additionally, this family of distributions has been successful in applications across these domains and can overcome the limitations of fitting complex data.

As far as we know, the inverse Weibull (IW) is a more adaptable model for various applications, notably in actuarial measurements, economics, lifetime analysis, wind speed, rainfall, and cancer patient studies. We aim to highlight some major scientific studies to help readers better understand the application of the IW distribution. Some examples are Tashkandy et al. [25] who studied various statistical inference for the extended IW distribution under progressive type-II censored sample, Rasjid et al. [26] proposed a new lifetime distribution using the alpha power transformation and taking the IW as classical distribution, Muhammed and Almetwally [27] studied several estimation methods for the IW versions of bivariate IW distribution based on progressive type II censoring, Alshanbari et al. [28] proposed a new version of the IW distribution and applied it to varied data sets, Elbiely and Yousof [29], and Tushar et al. [30] proposed a alternative version of the IW distribution using the generalized odd log-logistic-G and the cubic transmuted transformations, respectively. The CDF and PDF of the IW model are given below.

$$\Phi(t) = e^{-\alpha t^{-\beta}}, \sim t > 0, \sim \alpha, \beta > 0, \quad (3)$$

and

$$\phi(t) = \alpha \beta t^{-(\beta+1)} e^{-\alpha t^{-\beta}}. \quad (4)$$

In this study, we will focus on proposing a more efficient IW distribution by combining the standard classical IW distribution into a more adapted class of distributions utilizing the E-X tool. The distribution that was constructed is the E-IW model. The primary gaps in the available research serve as the driving force behind this investigation:

1. The E-IW distribution can be assumed to be a suitable model for modeling heavy right-tail data.
2. Complex models can hinder practical implementation; our formulation strikes a balance between parsimony and adaptability.
3. The E-IW model applies to various data sets and employs both classical and Bayesian approaches for parameter estimation, yielding a significantly high fit compared to baseline models.
4. Comparative analyses reveal that the proposed distribution provides superior fits to empirical data in scenarios where classical models fail.

Moreover, classical and Bayesian methods are used to estimate the unknown parameters of our recommended E-IW model. including notably the maximum likelihood estimator (MLE), the least squares estimator (LSE), the weighted least squares estimator (WLSE), and the maximum product spacing (MPS) estimators. Additionally, the Bayesian tool can be implemented by applying the square error (SE) function via the Metropolis-Hastings (MH) algorithm. This research focuses on evaluating four risk metrics for the proposed E-IW model: the value at risk (VaR), tail value at risk (TVaR), variance premium (VP), and tail variance premium (TVP); some may cited the studies of Atchadé et al. [31] studied the actuarial metrics for the shifted lomax Weibull distributions, Alzeley et al. [32] provided Bayesian and non-Bayesian estimation techniques for the new exponential-X Frechet model, While Hussam et al. [33] used the Metropolis–Hastings strategy to estimate the unknown parameters for the stress strengths reliability of the IW distribution, Abdelall et al. [34], Abonongo et al. [35] provided several estimation procedures to obtain the final estimate of the model parameters.

The structure of this research study is organized as follows: Section 2 establishes this. The E-IW is constructed in Section 2. In Section 3, we investigated different statistical properties. In Section 4, we present classical and Bayesian methods for estimating the model parameters, and in Section 5, we explore various actuarial metrics of the E-IW distribution. In Section 6, a numerical simulation of parameter estimation is performed. Finally, in Section 7, four examples of real financial and radiation data sets are presented to demonstrate their applications and applicability. Our illustration demonstrates that the recommended E-IW model fits the data sets better than its competitors. In the final Section 8, a conclusion is provided.

2 The New Extended Inverse Weibull Distribution

Motivated by the advantages of the inverse Weibull distribution, we present a new distribution, namely, the E-IW model. Its CDF is defined as follows

$$K(t) = \frac{e^\theta - e^{\theta(1-e^{-2\alpha t^{-\beta}})}}{e^\theta - 1}, \quad t > 0, \quad \alpha, \beta, \theta > 0. \quad (5)$$

The corresponding PDF is

$$k(t) = \frac{2\theta\alpha\beta t^{-(\beta+1)} e^{-2\alpha t^{-\beta}}}{e^\theta - 1} e^{\theta(1-e^{-2\alpha t^{-\beta}})}. \quad (6)$$

We plot the CDF and PDF curves for different values of parameters presented in Fig. 1a,b, respectively. The E-IW exhibits unimodal, right-skewed, and decreasing shapes.

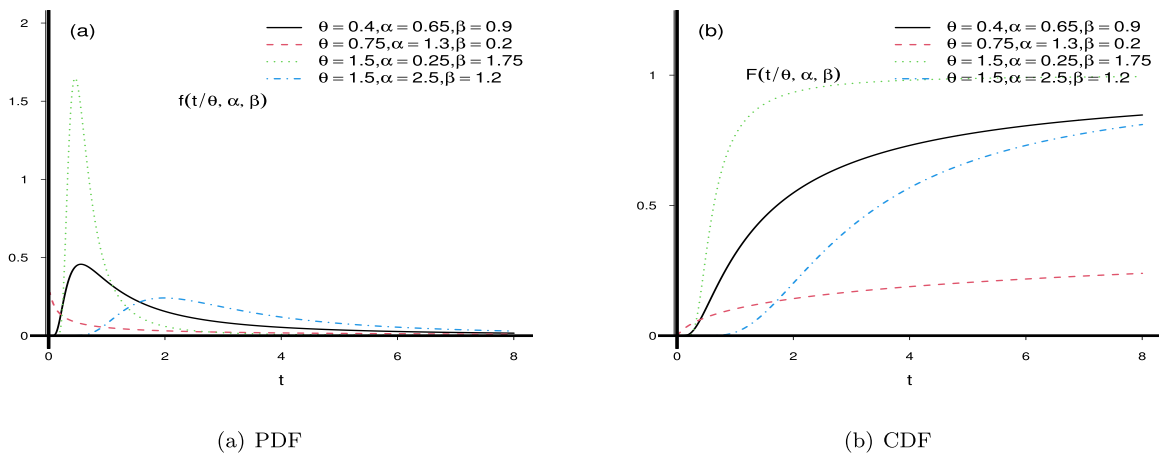


Figure 1: (Continued)

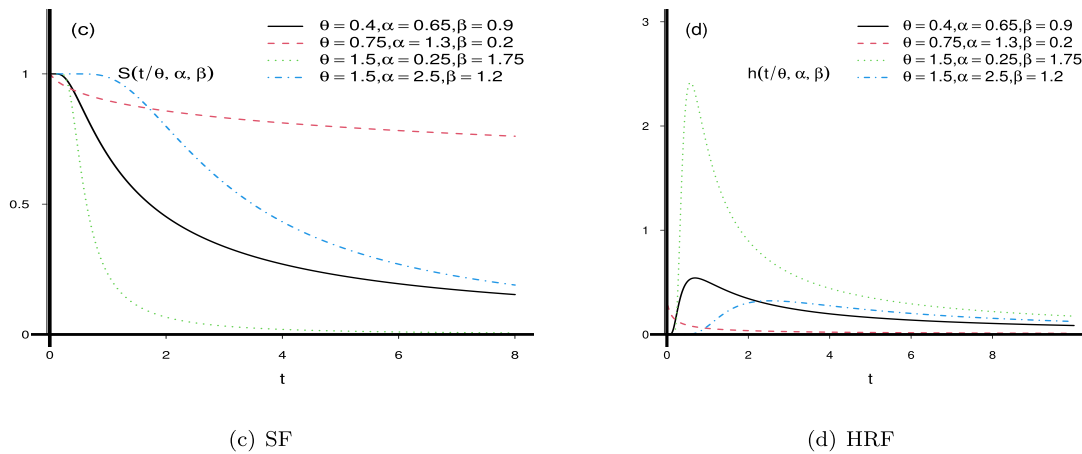


Figure 1: Various shapes of the of PDF, CDF, SF, and HRF of the E-IW

Subsequently, several reliability functions of the suggested E-IW are derived, including the survival function (SF) and the hazard rate function (HRF). They are provided by

$$S(t) = \frac{e^{\theta(1-e^{-2\alpha t^{-\beta}})} - 1}{e^{\theta} - 1},$$

and

$$h(t) = \frac{2\theta\alpha\beta t^{-(\beta+1)} e^{-2\alpha t^{-\beta}} e^{\theta(1-e^{-2\alpha t^{-\beta}})}}{e^{\theta(1-e^{-2\alpha t^{-\beta}})} - 1}.$$

The HRF of the E-IW model is shown for the parameter values in Fig. 1c,d. It is decreasing and has unimodal shapes, making it suitable for a wide range of real-world applications.

3 Some Statistical Characterizations

3.1 Quantile Function

In simulation, the quantile function serves as a bridge between the theoretical distribution of a variable and its empirical representation in the simulated data. One key advantage of using the quantile function Q_p in simulation is its ability to accurately generate random variables that follow specific distributions. Also, the Q_p is defined as the function that gives the value below which a given proportion of data falls. The Q_p of order p of the E-IW distribution can be derived by inverting Eq. (5), then

$$Q_p = 2^{1/\beta} \left(-\frac{\alpha}{\log \left(1 - \frac{\log(e^{\theta} - e^{\theta} p + p)}{\theta} \right)} \right)^{1/\beta}, \quad 0 < p < 1. \quad (7)$$

Proof. By definition, we have

$$\frac{e^\theta - e^{\theta(1-e^{-2\alpha t^{-\beta}})}}{e^\theta - 1} = p$$

$$e^\theta - e^{\theta(1-e^{-2\alpha t^{-\beta}})} = p(e^\theta - 1)$$

$$e^{\theta(1-e^{-2\alpha t^{-\beta}})} = e^\theta - p(e^\theta - 1)$$

$$\theta(1 - e^{-2\alpha t^{-\beta}}) = \log(e^\theta - p(e^\theta - 1))$$

$$1 - e^{-2\alpha t^{-\beta}} = \frac{1}{\theta} \log(e^\theta - p(e^\theta - 1))$$

$$e^{-2\alpha t^{-\beta}} = 1 - \frac{1}{\theta} \log(e^\theta - p(e^\theta - 1))$$

$$t = 2^{1/\beta} \left(-\frac{\alpha}{\log\left(1 - \frac{\log(e^\theta - e^\theta p + p)}{\theta}\right)} \right)^{1/\beta}. \quad \square$$

Now, if we put $p = \frac{1}{2}$, the value of the median is obtained, and it can be formulated as below

$$Q_{0.5} = 2^{1/\beta} \left(-\frac{\alpha}{\log\left(1 - \frac{\log(e^\theta - 0.5e^\theta + 0.5)}{\theta}\right)} \right)^{1/\beta}.$$

3.2 Linear Representation of the E-IW Model

Using the following expansion

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!},$$

the PDF of the proposed E-IW model is rewritten as

$$k(t) = \frac{2\alpha\beta t^{-(\beta+1)} e^{-2\alpha t^{-\beta}}}{e^\theta - 1} \sum_{j=0}^{\infty} \frac{\theta^{j+1}}{j!} (1 - e^{-2\alpha t^{-\beta}})^j.$$

Using the binomial expansion, Alanazi et al. [36]

$$(1 - z)^j = \sum_{l=0}^{\infty} (-1)^l \binom{j}{l} z^l, \quad |z| < 1.$$

The Eq. (6) is obtained to be

$$k(t|\theta, \alpha, \beta) = \frac{2\alpha\beta}{e^\theta - 1} t^{-(\beta+1)} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} (-1)^l \binom{j}{l} \frac{\theta^{j+1}}{j!} e^{-2\alpha(l+1)t^{-\beta}} \quad (8)$$

$$= \frac{2\alpha\beta t^{-(\beta+1)}}{e^\theta - 1} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \eta_{l,j} e^{-2\alpha(l+1)t^{-\beta}} \quad (9)$$

where, $\eta_{l,j} = (-1)^j \binom{j}{l} \frac{\theta^{j+1}}{j!}$.

3.3 Moments

Utilizing moments allows for the calculation of several statistical measures, including the mean, variance, coefficient of variation, and index of dispersion, which can be used to describe the shape and specific features of a distribution. Moments serve as effective tools for data summarization, comparison of probability distributions, and risk management.

Theorem 1. By using the linear representation, the r th moment of the E-IW is obtained to be

$$u'_r = \frac{2\alpha}{e^\theta - 1} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \eta_{l,j} \frac{\Gamma\left(1 - \frac{r}{\beta}\right)}{[2\alpha(l+1)]^{1-\frac{r}{\beta}}}. \quad (10)$$

Proof. By definition,

$$\begin{aligned} u'_r &= \int_0^\infty t^r k(t|\theta, \alpha, \beta); dt \\ &= \frac{2\alpha\beta}{e^\theta - 1} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \eta_{l,j} \int_0^\infty t^{r-\beta-1} e^{-2\alpha(l+1)t^{-\beta}} dt \end{aligned}$$

Let $y = t^{-\beta}$, then

$$\begin{aligned} u'_r &= \frac{2\alpha}{e^\theta - 1} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \eta_{l,j} \int_0^\infty y^{\frac{-r}{\beta}} e^{-2\alpha(l+1)y} dy \\ &= \frac{2\alpha}{e^\theta - 1} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \eta_{l,j} \frac{\Gamma\left(1 - \frac{r}{\beta}\right)}{[2\alpha(l+1)]^{1-\frac{r}{\beta}}} \quad \square \end{aligned}$$

Under Eq. (10), the mean, second moment, variance, and their corresponding index of dispersion of the E-IW distribution can be calculated in the following forms

$$\begin{aligned} \mu'_1 &= \frac{2\alpha}{e^\theta - 1} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \eta_{l,j} \frac{\Gamma\left(1 - \frac{1}{\beta}\right)}{[2\alpha(l+1)]^{1-\frac{1}{\beta}}}, \\ \mu'_2 &= \frac{2\alpha}{e^\theta - 1} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \eta_{l,j} \frac{\Gamma\left(1 - \frac{2}{\beta}\right)}{[2\alpha(l+1)]^{1-\frac{2}{\beta}}}, \end{aligned}$$

$$Var = \mu'_2 - (\mu'_1)^2,$$

and

$$ID = \frac{Var}{u'_1}.$$

The coefficient of Bowley's skewness (S_T) and Moor's kurtosis (K_T) metrics can be calculated as

$$S_T = \frac{u'_{1/4} + u'_{3/4} - 2u'_{1/2}}{u'_{3/4} - u'_{1/4}},$$

and

$$K_T = \frac{u'_{7/8} - u'_{5/8} + u'_{3/8} - u'_{1/8}}{u'_{6/8} - u'_{2/8}}.$$

Additionally, the moment generating function (MGF) of the E-IW, say $M(u)$, is obtained to be

$$\begin{aligned} M(u) &= \int_0^\infty e^{ut} k(t|\theta, \alpha, \beta) dt \\ &= \int_0^\infty \sum_{r=0}^\infty \frac{u^r}{r!} u'_r \\ &= \frac{2\alpha}{e^\theta - 1} \sum_{j=0}^\infty \sum_{l=0}^\infty \sum_{m=0}^\infty \frac{u^r}{r!} \eta_{l,j} \frac{\Gamma\left(1 - \frac{r}{\beta}\right)}{[2\alpha(l+1)]^{1-\frac{r}{\beta}}}. \end{aligned} \quad (11)$$

In Tables 1 and 2, we report various mathematical characterizations of our E-IW model. Figs. 2 and 3 present the 3-dimensional curves of these statistical properties. These several non-monotonic forms indicate that the suggested E-IW distribution can accommodate modeling various types of data sets.

Table 1: Numerical values for μ'_1 , Var , ID , S_T , and K_T when $\theta = 0.5$

	β	μ'_1	Var	ID	S_T	K_T
$\alpha = 0.3$	2	1.23733	7.13561	5.76694	0.30612	1.57855
	4	1.02909	0.17839	0.17335	0.21851	1.40585
	6	1.00577	0.05902	0.05868	0.18866	1.36241
	8	0.99968	0.02973	0.02974	0.17452	1.34345
$\alpha = 0.6$	2	1.74813	9.37685	5.36393	0.30598	1.57824
	4	1.22380	0.25228	0.20615	0.21805	1.40624
	6	1.12894	0.07436	0.06587	0.18894	1.36109
	8	1.09016	0.03535	0.03243	0.17199	1.34286

(Continued)

Table 1 (continued)

	β	μ'_1	Var	ID	S_T	K_T
$\alpha = 0.9$	2	2.14050	13.50493	6.30923	0.30589	1.57864
	4	1.35436	0.30898	0.22814	0.21887	1.40593
	6	1.20787	0.08512	0.07047	0.18852	1.36326
	8	1.14684	0.03912	0.03411	0.17297	1.34060
$\alpha = 1.2$	2	2.47115	17.47687	7.07238	0.30579	1.57837
	4	1.45535	0.35678	0.24515	0.21781	1.40619
	6	1.26719	0.09369	0.07393	0.18930	1.36307
	8	1.18883	0.04204	0.03536	0.17402	1.34174

Table 2: Numerical values for μ'_1 , Var , ID, S_T , and K_T when $\theta = 1.75$

	β	μ'_1	Var	ID	S_T	K_T
$\alpha = 0.3$	2	0.96241	2.52224	2.62075	0.28040	1.56673
	4	0.92312	0.11069	0.11991	0.20822	1.42315
	6	0.93837	0.03942	0.04201	0.18290	1.38710
	8	0.95003	0.02056	0.02164	0.17249	1.37066
$\alpha = 0.6$	2	1.36079	4.73903	3.48255	0.28063	1.56762
	4	1.09778	0.15655	0.14260	0.20740	1.42506
	6	1.05329	0.04967	0.04716	0.18359	1.38812
	8	1.03602	0.02445	0.02360	0.17054	1.37288
$\alpha = 0.9$	2	1.66638	6.84072	4.10514	0.28038	1.56661
	4	1.21490	0.19173	0.15781	0.20720	1.42497
	6	1.12692	0.05686	0.05045	0.18296	1.38610
	8	1.08988	0.02705	0.02482	0.17044	1.36883
$\alpha = 1.2$	2	1.92394	8.86775	4.60917	0.28038	1.56623
	4	1.30549	0.22139	0.16958	0.20767	1.42545
	6	1.18227	0.06258	0.05293	0.18350	1.38725
	8	1.12979	0.02907	0.02573	0.17178	1.36773

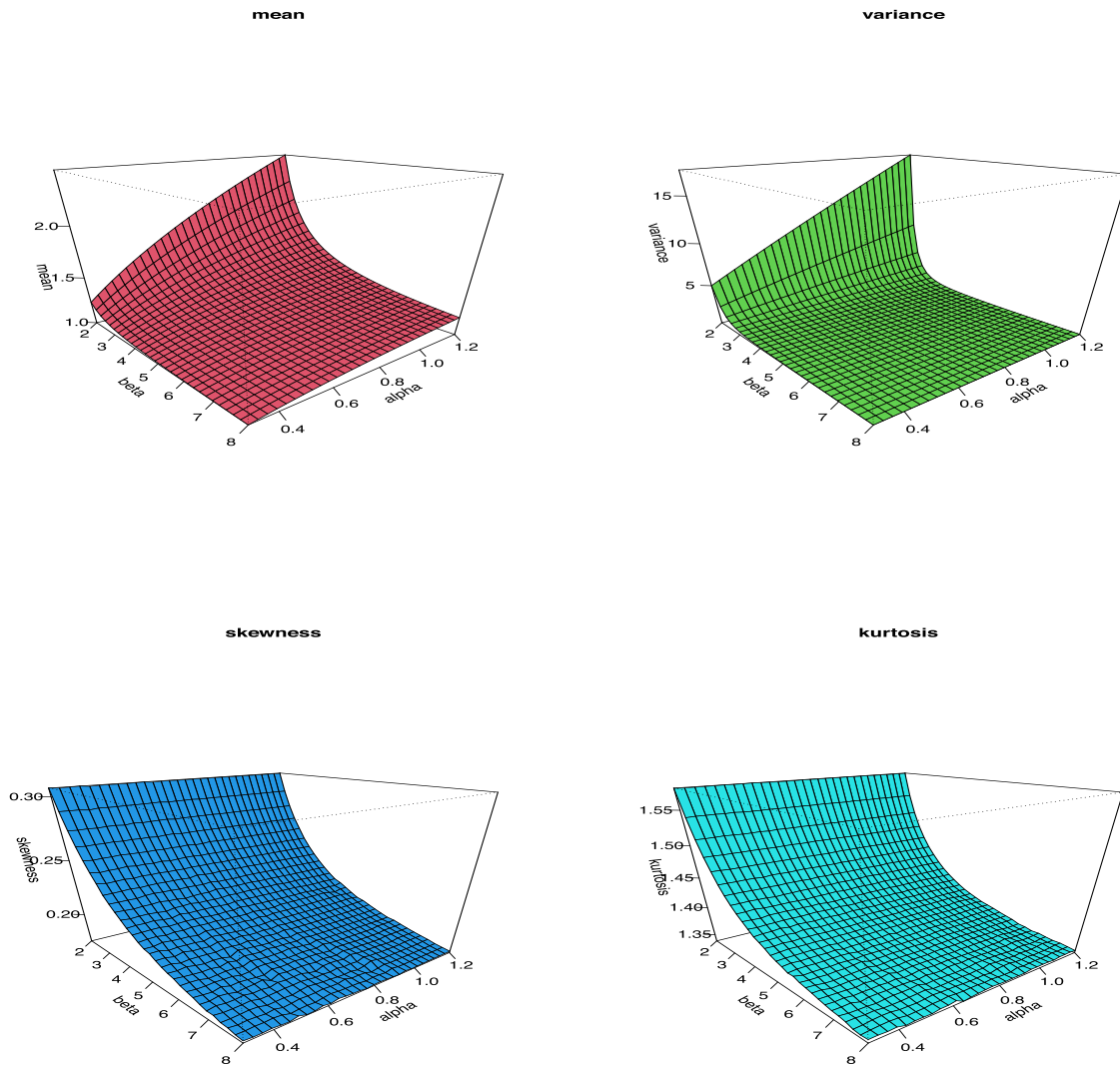


Figure 2: Graphs of μ'_1 , Var , S_T , and K_T of the E-IW model for $\theta = 0.5$

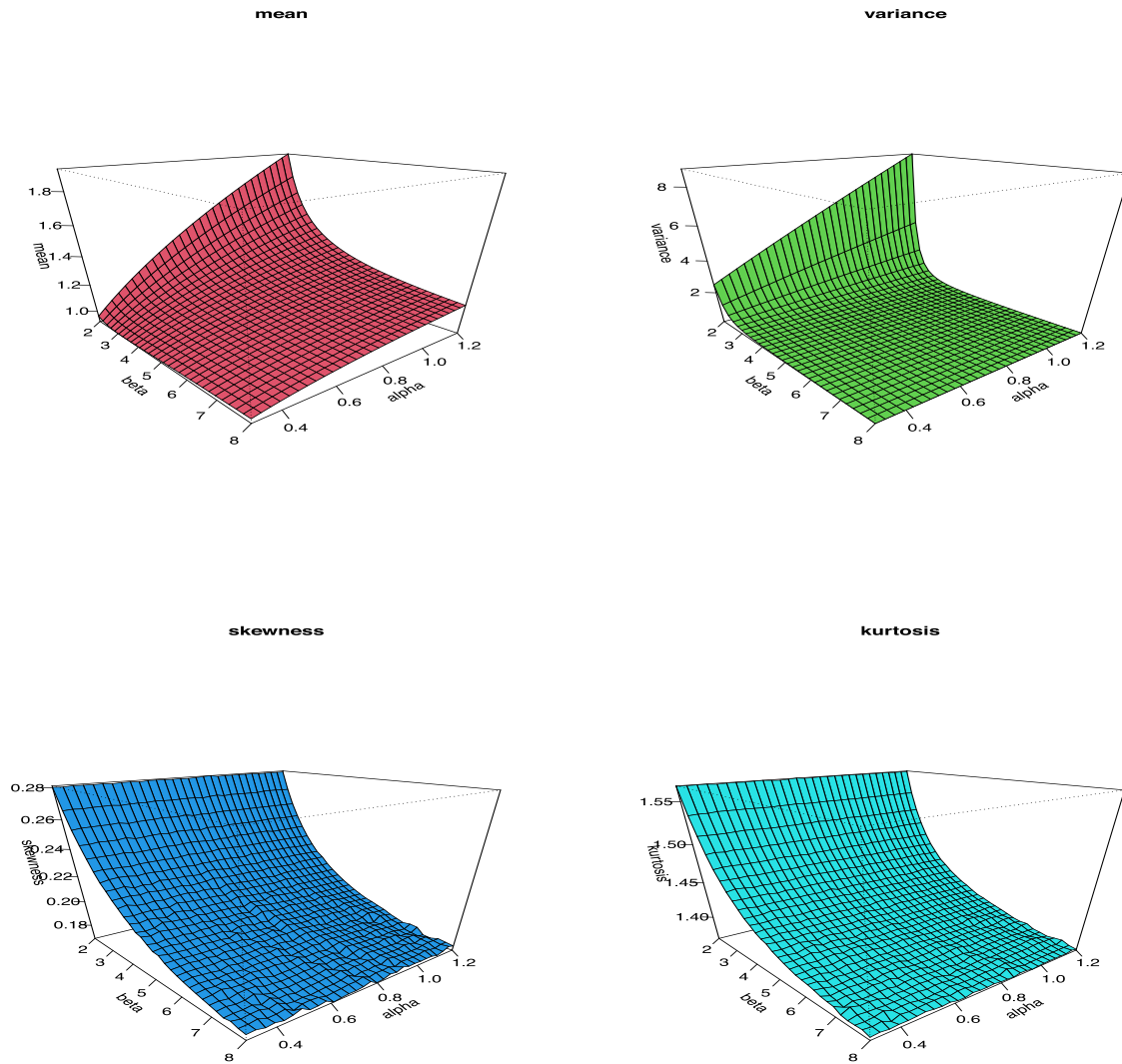


Figure 3: Graphs of μ'_1 , Var , S_T , and K_T of the E-IW model for $\theta = 1.75$

3.4 Conditional Moments

For lifetime models, it is also of interest to find the conditional moments. For the E-IW model, clearly the conditional moments, $E(T^s | T > y)$, is provided by

$$E(T^s | T > y) = \frac{1}{S(y)} \kappa_s(y),$$

where

$$\begin{aligned} \kappa_s(y) &= \int_y^\infty t^s k(t) dt \\ &= \frac{2\theta\alpha\beta}{(e^\theta - 1)} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \eta_{l,j} \psi(t; \theta, \alpha, \beta), \end{aligned} \quad (12)$$

where $\psi(t; \theta, \alpha, \beta) = \int_y^\infty t^{-\beta-1} e^{\theta(1-e^{-2\alpha t^{-\beta}})} dt$.

3.5 Mean Residual Life (MRL) Function

It is an essential element of maintenance and replacement programs, offering a detailed overview of the residual life function. The MRL function for the NE-LD is derived by

$$\begin{aligned} \delta(y) &= E((T - y) | T > y) = \left\{ \frac{1}{S(y)} \int_y^\infty t k(t) dt \right\} - y \\ &= \frac{2\theta\alpha\beta}{(e^\theta - 1)S(y)} \int_y^\infty t^{-\beta} e^{-2\alpha t^{-\beta}} e^{\theta(1-e^{-2\alpha t^{-\beta}})} - y \\ &= \frac{2\theta\alpha\beta}{(e^\theta - 1)S(y)} \sum_{j=0}^\infty \sum_{l=0}^\infty \eta_{l,j} \Psi(t; \theta, \alpha, \beta) - y, \end{aligned} \quad (13)$$

where, $\Psi(t; \theta, \alpha, \beta) = \int_y^\infty t^{-\beta} e^{\theta(1-e^{-2\alpha t^{-\beta}})} dt$.

3.6 Mean Inactivity Time (MIT) Function

It is commonly called the mean waiting time, and denotes the mean duration that has transpired since the hazard of Y_i , assuming that $Y_i \leq y$. In this case the random variable of interest is $(y - Y_i | Y_i \leq y)$, $i = 1, 2, \dots, n$. It shows the time elapsed since the hazard of Y_i given that it failed at or before t . It can be defined by

$$\begin{aligned} \Delta(y) &= E(y - T | T \leq y) = y - \frac{1}{K(y)} \int_0^y tk(t) dt \\ &= y - \frac{2\theta\alpha\beta}{(e^\theta - 1)K(y)} \sum_{j=0}^\infty \sum_{l=0}^\infty \eta_{l,j} \varphi(t; \theta, \alpha, \beta) \end{aligned} \quad (14)$$

where, $\varphi(t; \theta, \alpha, \beta) = \int_0^y t^{-\beta} e^{\theta(1-e^{-2\alpha t^{-\beta}})} dt$.

3.7 Order Statistics

The order statistics are a significant component of distribution theory that is utilized to identify the model of sample peaks and various statistics derived from the results of the sample. These concepts are utilized in reliability testing.

Let $\{t_1, t_2, \dots, t_n\}$ be a random sample of size n drawn from the E-IW. The PDF of i th order statistics is computed as below:

$$\begin{aligned} \pi_{(i:n)}(t) &= C_{i:n} k(t) [K(t)]^{i-1} [1 - k(t)]^{n-i} \\ &= \frac{2\theta\alpha\beta}{(e^\theta - 1)} C_{i:n} t^{-(\beta+1)} e^{\theta(1-e^{-2\alpha t^{-\beta}})} e^{-2\alpha t^{-\beta}} \\ &\quad \times \left(\frac{e^\theta - e^{\theta(1-e^{-2\alpha t^{-\beta}})}}{e^\theta - 1} \right)^{i-1} \left(\frac{e^{\theta(1-e^{-2\alpha t^{-\beta}})} - 1}{e^\theta - 1} \right)^{n-i}, \end{aligned} \quad (15)$$

where, $C_{in} = \frac{n!}{(i-1)!(n-i)!}$. Next, the CDF of i th order statistic is

$$\begin{aligned} \Pi_{(i:n)}(t) &= \sum_{s=i}^n K^s(t) [1 - K(t)]^{n-s} \\ &= \sum_{s=i}^n \left(\frac{e^\theta - e^{\theta(1-e^{-2\alpha t^{-\beta}})}}{e^\theta - 1} \right)^s \left(\frac{e^{\theta(1-e^{-2\alpha t^{-\beta}})} - 1}{e^\theta - 1} \right)^{n-s}. \end{aligned} \quad (16)$$

In particular, we can obtain the PDF of the first and last random variables. They are formulated, respectively, as follows:

$$\begin{aligned} \pi_{(n:n)}(t) &= \frac{2n\theta\alpha\beta}{e^\theta - 1} t^{-(\beta+1)} e^{\theta(1-e^{-2\alpha t^{-\beta}})-2\alpha t^{-\beta}} \\ &\quad \times \left(\frac{e^\theta - e^{\theta(1-e^{-2\alpha t^{-\beta}})}}{e^\theta - 1} \right)^{n-1}, \end{aligned} \quad (17)$$

and

$$\begin{aligned} \pi_{(1:n)}(t) &= \frac{2n\theta\alpha\beta}{e^\theta - 1} t^{-(\beta+1)} e^{\theta(1-e^{-2\alpha t^{-\beta}})-2\alpha t^{-\beta}} \\ &\quad \times \left(\frac{e^{\theta(1-e^{-2\alpha t^{-\beta}})} - 1}{e^\theta - 1} \right)^{n-1}. \end{aligned} \quad (18)$$

3.8 Double E-IW Model

This section will define a double E-IW model, as a double distribution is typically used for analyzing images and signals. The PDF of the double inverse Weibull model is given by

$$k(t) = \frac{\theta\alpha\beta|t|^{-(\beta+1)}e^{-\alpha|t|^{-\beta}}}{e^\theta - 1} e^{\theta(1-e^{-2\alpha|t|^{-\beta}})}. \quad \alpha, \beta, \theta > 0, \quad t \in \mathbb{R}. \quad (19)$$

The corresponding CDF of the double inverse Weibull model is given by

$$K(t|\theta, \alpha, \beta) = \begin{cases} \int_{-\infty}^t \frac{\theta\alpha\beta|u|^{-(\beta+1)}e^{-\alpha u^{-\beta}}}{e^\theta - 1} e^{\theta(1-e^{-2\alpha u^{-\beta}})} du, & \text{if } u < 0 \\ \frac{1}{2} + \int_{-\infty}^t \frac{\theta\alpha\beta|u|^{-(\beta+1)}e^{-\alpha u^{-\beta}}}{e^\theta - 1} e^{\theta(1-e^{-2\alpha u^{-\beta}})} du, & \text{if } u \geq 0 \end{cases} \quad (20)$$

Fig. 4a shows the PDF plot. As the parameter β increases, the PDF has two modes.

The numerical method will be used to compute the CDF. To provide a simple explanation for this method, the following algorithm offers a clear understanding of how to compute complex integration.

The CDF is obtained using the Algorithm 1. **Fig. 4b** shows the CDF plot. As the parameter β increases, the CDF reaches one equally.

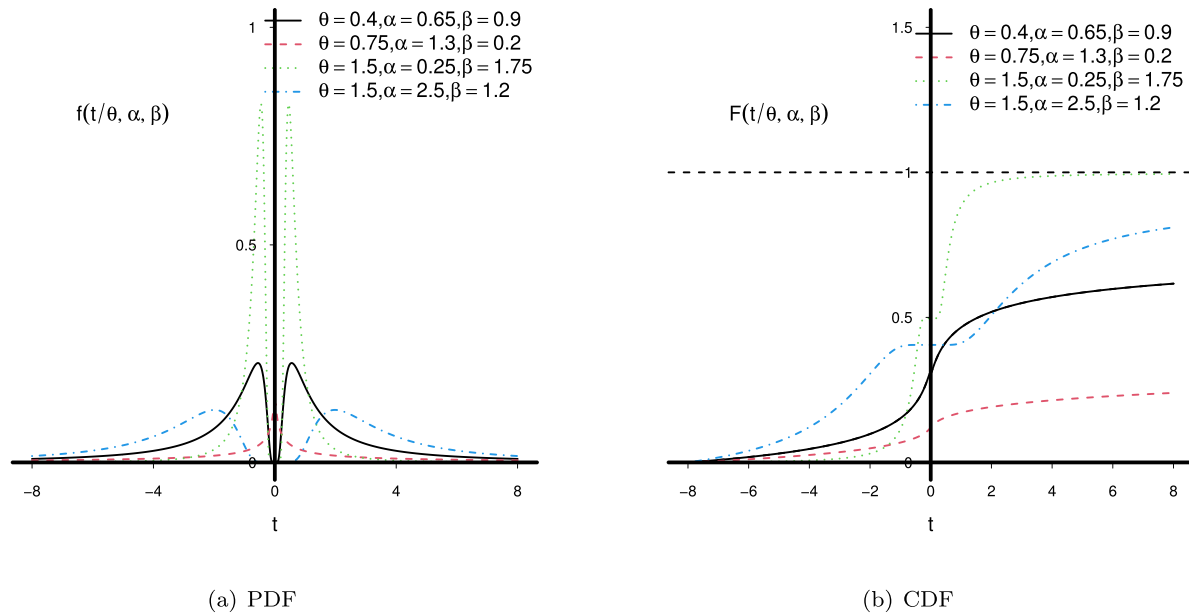


Figure 4: Various shapes of the PDF of the double E-IW model

Algorithm 1 : Computing CDF

Result: CDF.

```

1 Starting with value  $\theta^*$ ,  $\alpha^*$ , and  $\beta^*$ . Let  $t \in \mathbb{R}$ 
2 for r = 3 to R do
3   Let  $fx = 0$ 
4   for s = 2 to r do
5      $fx = fx + f(t) * \Delta(t)$ 
6   end
7   print( $fx$ )
8 end
```

4 Statistical Inference

4.1 MLE Method

Let a random sample $\{t_1, t_2, \dots, t_n\}$ be a random sample from the the E-IW model and $\Gamma = (\theta, \alpha, \beta)$ its parameter vector. The log-likelihood function of Eq. (6) without constant terms is given by

$$\begin{aligned}
 \mathcal{LL}(t; \Gamma) &= \sum_{i=1}^n \log k(t) = \sum_{i=1}^n \log \left\{ \frac{2\theta\alpha\beta t_i^{-(\beta+1)} e^{-2\alpha t_i^{-\beta}}}{e^\theta - 1} e^{\theta \left(1 - e^{-2\alpha t_i^{-\beta}}\right)} \right\} \\
 &\propto n \log \theta + n \log \alpha + n \log \beta - n \log(e^\theta - 1) - (\beta + 1) \sum_{i=1}^n \log t_i - 2\alpha \sum_{i=1}^n t_i^{-\beta} \\
 &\quad + \theta \sum_{i=1}^n \left(1 - e^{-2\alpha t_i^{-\beta}}\right).
 \end{aligned} \tag{21}$$

To obtain the MLEs of parameters, we differentiate (21) partially with respect to the unknown parameters. The nonlinear equations are given in detail as

$$\frac{\partial \mathcal{L}(t; \Gamma)}{\partial \theta} = \frac{n}{\theta} - \frac{ne^\theta}{e^\theta - 1} + \sum_{i=1}^n \left(1 - e^{-2\alpha t_i^{-\beta}}\right), \quad (22)$$

$$\frac{\partial \mathcal{L}(t; \Gamma)}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^n t_i^{-\beta} + 2\theta \sum_{i=1}^n t_i^{-\beta} e^{-2\alpha t_i^{-\beta}}, \quad (23)$$

and

$$\frac{\partial \mathcal{L}(t; \Gamma)}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^n \log t_i + 2\alpha\beta \sum_{i=1}^n \log t_i t_i^{-\beta} + 2\alpha\theta \sum_{i=1}^n \log t_i e^{-2\alpha t_i^{-\beta}}. \quad (24)$$

The MLEs of the parameters $\Gamma = (\theta, \alpha, \beta)$ are determined by concurrently resolving the three relations derived above. Since no closed-form solutions exist for these nonlinear equations, we used the Newton-Raphson or fixed-point optimization technique to get the final estimates of $\Gamma = (\theta, \alpha, \beta)$. We can apply numerically from the `optim()` function in R.

4.2 LSE Method

Let $K(t)$ denote the CDF of the E-IW model and let $t_1 < t_2 < \dots < t_n$ be the n th ordered random sample. The LSEs, say $\mathcal{D}_2$ are obtained upon minimizing

$$\mathcal{D}_2 = \sum_{i=1}^n [K(t_{(in)}; \Gamma) - \mathcal{C}_1]^2,$$

where, $\mathcal{C}_1 = \frac{i}{n+1}$. The final estimates of Γ can be derived by minimizing

$$\sum_{i=1}^n \left[K(t_{(in)}; \Gamma) - \frac{i}{n+1} \right] \varphi_1(t_{(in)}; \Gamma) = 0,$$

$$\sum_{i=1}^n \left[K(t_{(in)}; \Gamma) - \frac{i}{n+1} \right] \varphi_2(t_{(in)}; \Gamma) = 0,$$

and

$$\sum_{i=1}^n \left[K(t_{(in)}; \Gamma) - \frac{i}{n+1} \right] \varphi_3(t_{(in)}; \Gamma) = 0,$$

with

$$\varphi_1(t_{(in)}; \Gamma) = \frac{\partial}{\partial \theta} K(t_{(in)}; \Gamma), \quad (25)$$

$$\varphi_2(t_{(in)}; \Gamma) = \frac{\partial}{\partial \alpha} K(t_{(in)}; \Gamma), \quad (26)$$

and

$$\varphi_3(t_{(in)}; \Gamma) = \frac{\partial}{\partial \beta} K(t_{(in)}; \Gamma), \quad (27)$$

as these equations lack a closed-form solution, the least squares estimate of Γ will be computed numerically using the `optim()` function in R.

4.3 WLSE Method

The WLSEs, say $\mathcal{D}_3$ are calculated by minimizing the following expression

$$\mathcal{D}_3 = \sum_{i=1}^n \mathcal{C}_2 \left[K(t_{(i:n)}; \Gamma) - \mathcal{C}_1 \right]^2,$$

with $\mathcal{C}_2 = \frac{(n+1)^2(n+2)}{i(n-i+1)}$. The WLSE estimates of Γ is determined by minimizing

$$\sum_{i=1}^n \mathcal{C}_2 \left[K(t_{(i:n)}; \Gamma) - \mathcal{C}_1 \right] \varphi_1(t_{(i:n)}; \Gamma) = 0,$$

$$\sum_{i=1}^n \mathcal{C}_2 \left[K(t_{(i:n)}; \Gamma) - \mathcal{C}_1 \right] \varphi_2(t_{(i:n)}; \Gamma) = 0,$$

and

$$\sum_{i=1}^n \mathcal{C}_2 \left[K(t_{(i:n)}; \Gamma) - \mathcal{C}_1 \right] \varphi_3(t_{(i:n)}; \Gamma) = 0.$$

As these equations lack a closed-form solution, the WLSE of Γ will be computed numerically using the `optim()` function in R.

4.4 MPS Method

This method is an alternative to the MLE method. It depends on the geometric mean of the spacing, which is

$$\mathcal{B}_i = K(t_{(i:n)}; \Gamma) - K(t_{(i-1:n)}; \Gamma); \quad i = 1, \dots, n+1,$$

where

$$K(t_{(0:n)}; \Gamma) = 0, \quad \text{and} \quad K(t_{(n+1:n)}; \Gamma) = 1.$$

The MPS estimators of $\Gamma = (\theta, \alpha, \beta)$ can be determined by maximizing the geometric mean of the spacing

$$\mathcal{D}_4 = \frac{1}{n+1} \sum_{i=1}^{n+1} \log \mathcal{B}_i.$$

It can be given respectively by solving

$$\frac{\partial \mathcal{D}_4}{\partial \theta} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{\mathcal{B}_i} \{ \varphi_1(t_{(i:n)}; \Gamma) - \varphi_1(t_{(i-1:n)}; \Gamma) \} = 0,$$

$$\frac{\partial \mathcal{D}_4}{\partial \alpha} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{\mathcal{B}_i} \{ \varphi_2(t_{(i:n)}; \Gamma) - \varphi_2(t_{(i-1:n)}; \Gamma) \} = 0,$$

and

$$\frac{\partial \mathcal{D}_4}{\partial \beta} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{\mathcal{B}_i} \{ \varphi_3(t_{(i:n)}; \Gamma) - \varphi_3(t_{(i-1:n)}; \Gamma) \} = 0,$$

as these equations lack a closed-form solution, the MPSE of Γ will be computed numerically using the `optim()` function in R.

4.5 Bayesian Estimation

Let $\Gamma = (\theta, \alpha, \beta)$ represented random variables that have the gamma distribution with parameters a_1, b_1, a_2, b_2, a_3 and b_3 . Consequently, the joint density function is

$$\pi(\Gamma) \propto \theta^{a_1-1} \alpha^{a_2-1} \beta^{a_3-1} e^{-b_1\theta-b_2\alpha-b_3\beta}.$$

Based on Bayes theorem, the joint posterior density of Γ is formulated as

$$\begin{aligned} \pi^*(\Gamma | t) &= \mathcal{L}(t; \Gamma) \pi(\Gamma | t) \\ &= \frac{\theta^{n+a_1-1} \alpha^{a_2-\alpha n-1} \beta^{a_3-1}}{(e^\theta - 1)^n} e^{-b_1\theta-b_2\alpha-b_3\beta} \prod_{i=1}^n e^{-2\alpha t_i^{-\beta}} e^{\theta \left(1 - e^{-2\alpha t_i^{-\beta}}\right)}. \end{aligned}$$

Hence, the Bayes based on SE

$$\mathcal{E}_1 = (\Gamma - \hat{\Gamma})^2$$

has been obtained to be

$$\hat{\mathcal{E}}_{SE} = \int_{\Gamma} \mathcal{E}_1 \pi^*(\Gamma | t) d\Gamma. \quad (28)$$

Also, for the LI, the Bayes estimation is written as

$$\mathcal{E}_2 = e^{p(\Gamma - \hat{\Gamma})} - p(\Gamma - \hat{\Gamma}), \quad p \neq 0.$$

is obtained to be

$$\hat{\mathcal{E}}_{LI} = -\frac{1}{p} \log \left(\int_{\Gamma} e^{-p\mathcal{E}_2} \sim \pi^*(\Gamma | t) d\Gamma \right). \quad (29)$$

At the end, based on GE, the Bayesian estimation, which is presented by

$$\mathcal{E}_3 = \left(\frac{\hat{\Gamma}}{\Gamma} \right)^p - p \log \left(\frac{\hat{\Gamma}}{\Gamma} \right) - 1, \quad p \neq 0.$$

has the following form

$$\hat{\mathcal{E}}_{GE} = \left(\int_{\Gamma} \mathcal{E}_3^{-p} \sim \pi^*(\Gamma | t) d\Gamma \right)^{-1/p}. \quad (30)$$

The posterior expectations for these estimators are assessed using Monte Carlo methods, as closed-form solutions are not attainable. We use the Metropolis-Hastings algorithm to sample from the posterior distribution.

Metropolis-Hastings algorithm strategy

1. Start with initial values $\alpha^{(0)} = \hat{\alpha}$, $\beta^{(0)} = \hat{\beta}$, and $\theta^{(0)} = \hat{\theta}$.
2. Set $m = 1$.
3. obtain α^* , β^* and θ^* , respectively, from $N(\hat{\alpha}, V(\hat{\alpha}))$, $N(\hat{\beta}, V(\hat{\beta}))$, and $N(\hat{\theta}, V(\hat{\theta}))$.
4. Compute $H_\alpha = \min \left[1, \frac{\pi^*(\alpha^*|\beta^{m-1}, \theta^{m-1})}{\pi^*(\alpha^{m-1}|\beta^{m-1}, \theta^{m-1})} \right]$, $H_\beta = \min \left[1, \frac{\pi^*(\beta^*|\alpha^{m-1}, \theta^{m-1})}{\pi^*(\beta^{m-1}|\alpha^{m-1}, \theta^{m-1})} \right]$ and $H_\theta = \min \left[1, \frac{\pi^*(\theta^*|\alpha^{m-1}, \beta^{m-1})}{\pi^*(\theta^{m-1}|\alpha^{m-1}, \beta^{m-1})} \right]$.
5. We derive a random samples Q_j that follow $U(0, 1)$ for $1 \leq j \leq 3$.
6. If $Q_1 \leq H_\alpha$, $Q_2 \leq H_\beta$, $Q_3 \leq H_\theta$, then $\alpha^{(m)} = \alpha^*$, $\beta^{(m)} = \beta^*$, $\theta^{(m)} = \theta^*$, else $\alpha^{(m)} = \alpha^{m-1}$, $\beta^{(m)} = \beta^{m-1}$, $\theta^{(m)} = \theta^{m-1}$.
7. $m = m + 1$
8. For M repeat of process, the steps (3) to (7) are repeated, and finally, we obtain $\alpha^{(m)}$, $\beta^{(m)}$, $\theta^{(m)}$ for $m = 1, \dots, M$.

5 Simulation Investigation

This section assesses the suggested estimation technique's efficacy in detail, including MLE, LSE, WLSE, MPS, and Bayes estimators under several loss functions related to the three parameters of the E-IW. This exploration highlights the use of this innovative distribution as a specialized tool designed specifically for simulation purposes. The results are tested concerning several records of the model parameters $\Gamma = (\theta, \alpha, \beta)$ (case 1: $\Gamma = (1.2, 0.5, 0.25)$, case 2: $\Gamma = (1.5, 0.7, 0.65)$) and under numerous sample sizes, $n = \{50, 100, 300, 600, 1000\}$.

Under $M = 1000$ replications of the process, random samples of the original model were used using Eq. (7) are obtained. The simulation findings are assessed using the mean estimates ($\mathcal{A}_1$), mean biases ($\mathcal{A}_2$), and mean square error ($\mathcal{A}_3$). The following formulae formally delineate the suggested metrics:

- Mean estimate

$$\mathcal{A}_1(\Gamma) = \frac{1}{M} \sum_{j=1}^M \hat{\Gamma}_j$$

- Mean bias

$$\mathcal{A}_2(\Gamma) = \frac{1}{M} \sum_{i=1}^M |\Gamma_j - \Gamma|.$$

and

- Mean square error

$$\mathcal{A}_3(\Gamma) = \frac{1}{M} \sum_{i=1}^M (\Gamma_j - \Gamma)^2.$$

The estimation based on Bayes technique is computed applying the MCMC methodology under the following parameters: $p = 1.5$, $D = 10,000$, a burn-in period of $N = 5000$, and prior distribution values of $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 1$. Various diagnostic representations of the θ , α , and β are sketched in Figs. 5–8 to assess MCMC convergence. Note that the simulation experiments are conducted using the R language software with the rootSolve package.

All the simulation findings are presented in Tables 3–6. From these results, we can observe that the AEs approximate the initial values for all recommended estimators. Additionally, the MSEs decrease in size as n increases. This declining trend ensures that the estimated parameter value becomes consistent and stable as the sample size increases. Further, it can be seen that the MSE of Bayes based on the SE loss function is minimal in all cases among all other methods. So, we can conclude that the Bayes under the SE procedure is superior for estimating parameters of the E-IW distribution.

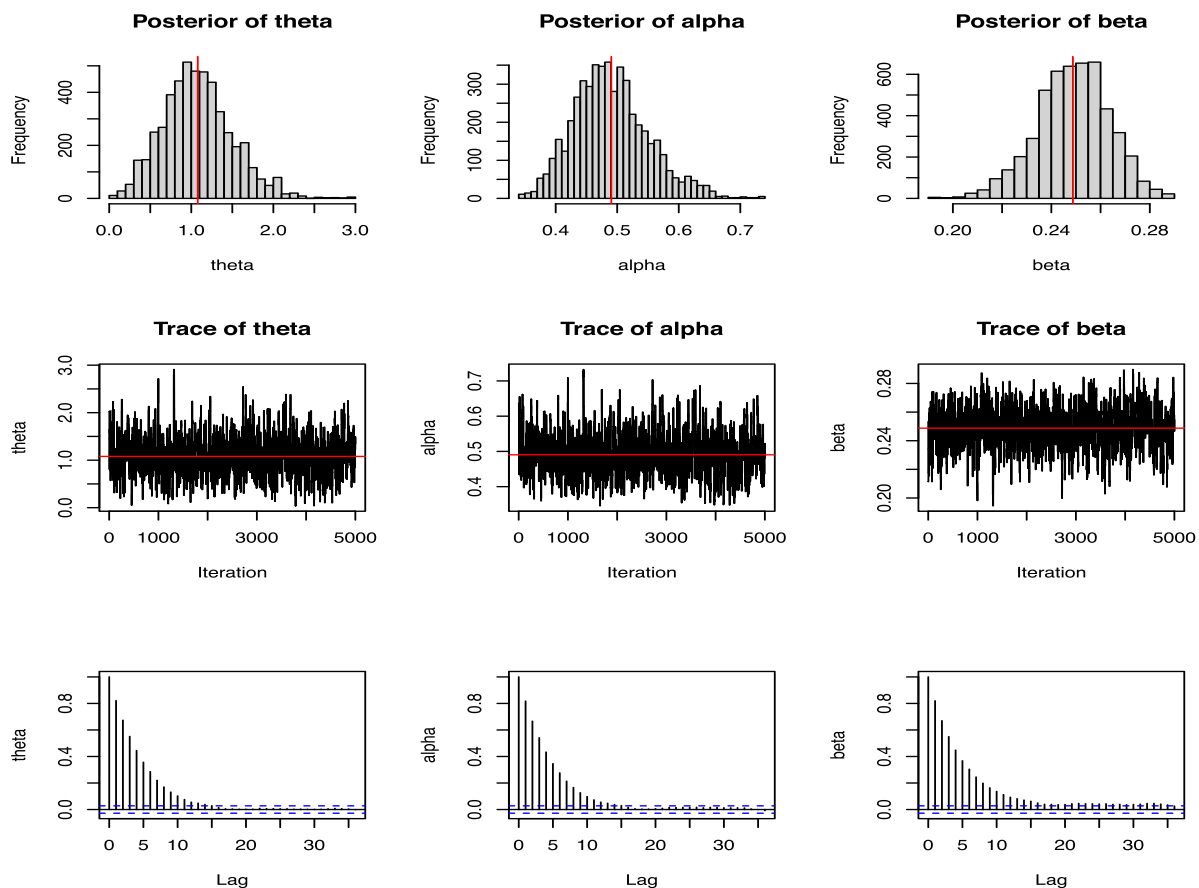


Figure 5: Bayesian diagnostic curves of E-IW model for Table 3

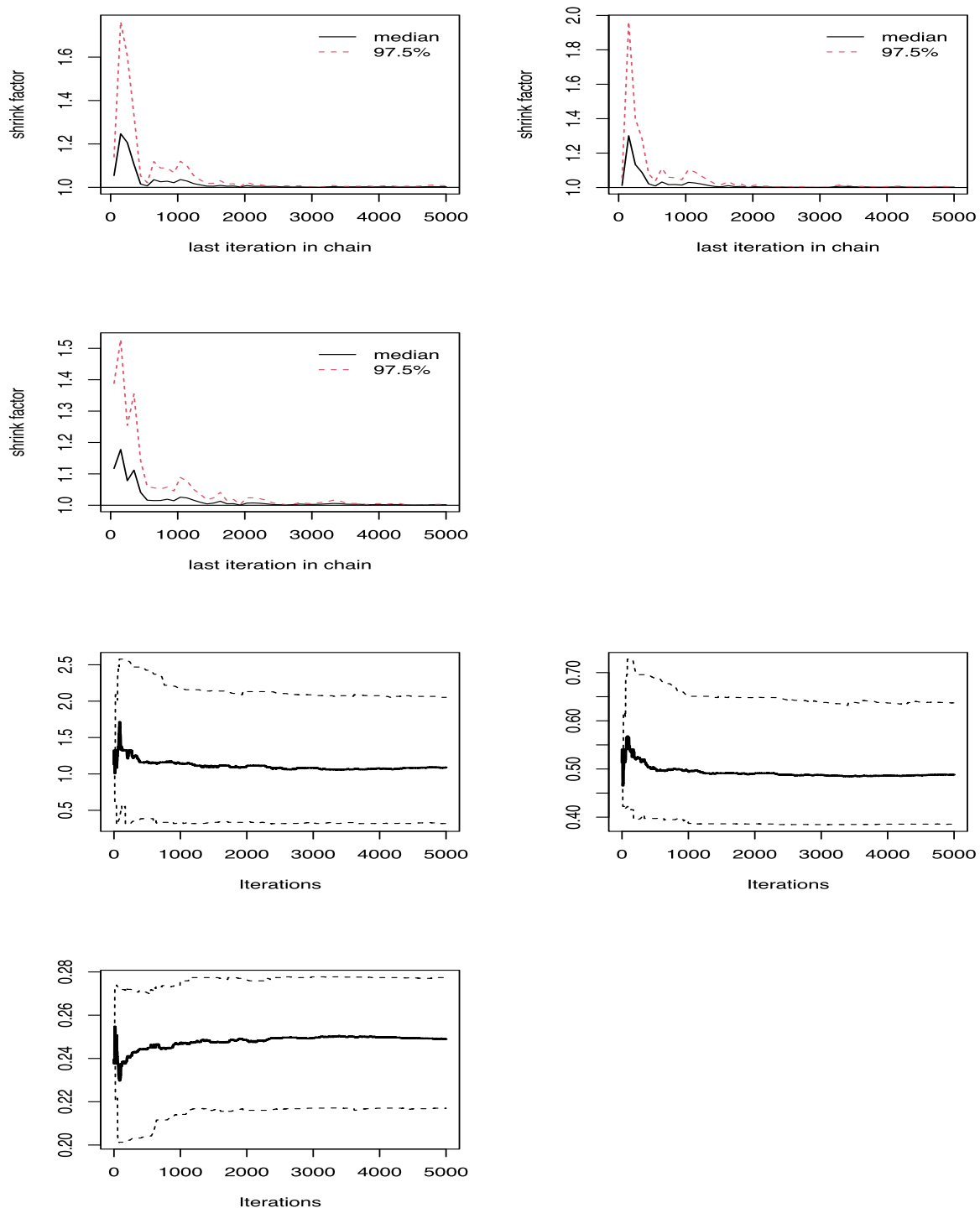


Figure 6: Bayesian diagnostic curves of E-IW model for [Table 3](#)

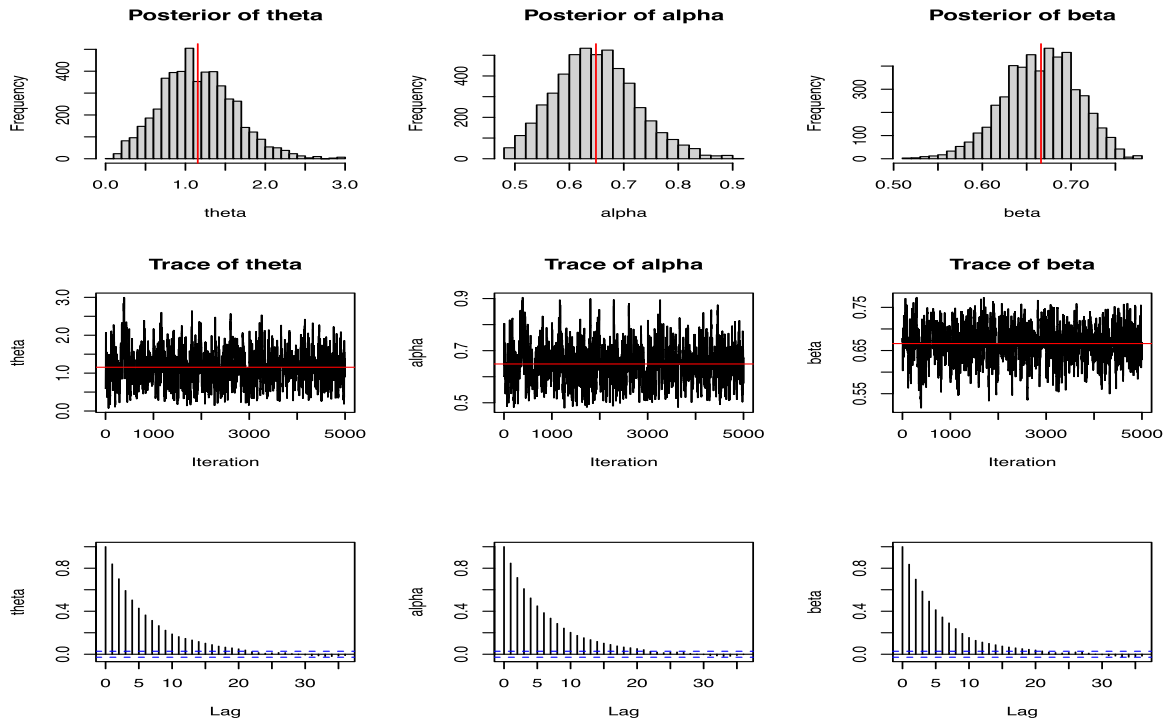


Figure 7: Bayesian diagnostic curves of E-IW model for [Table 4](#)

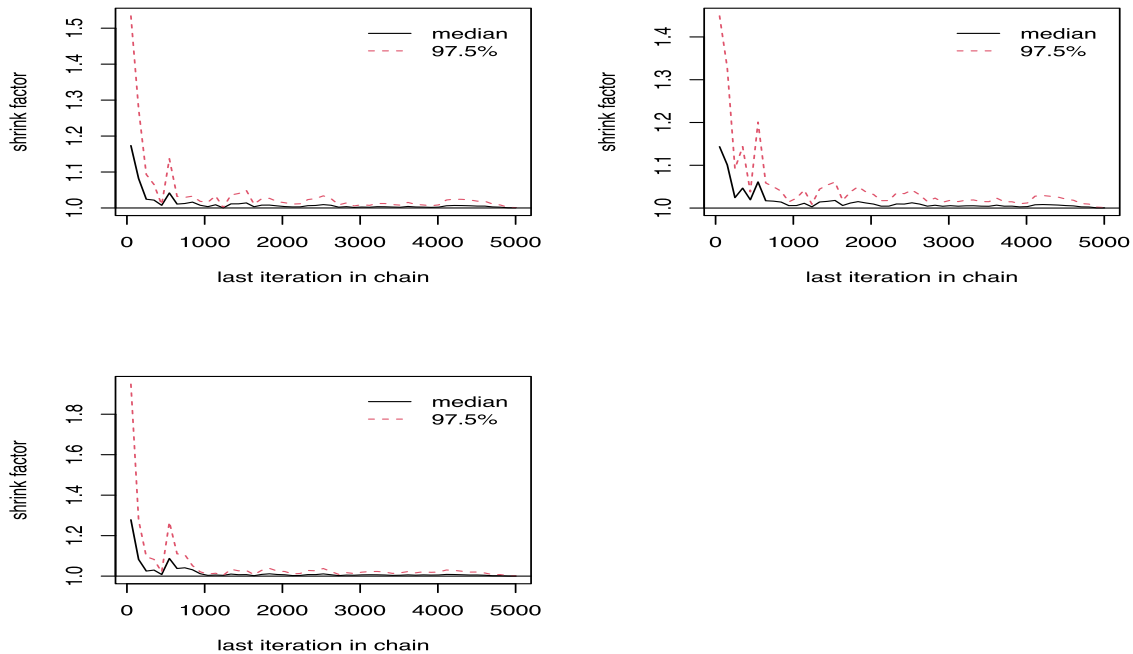


Figure 8: (Continued)

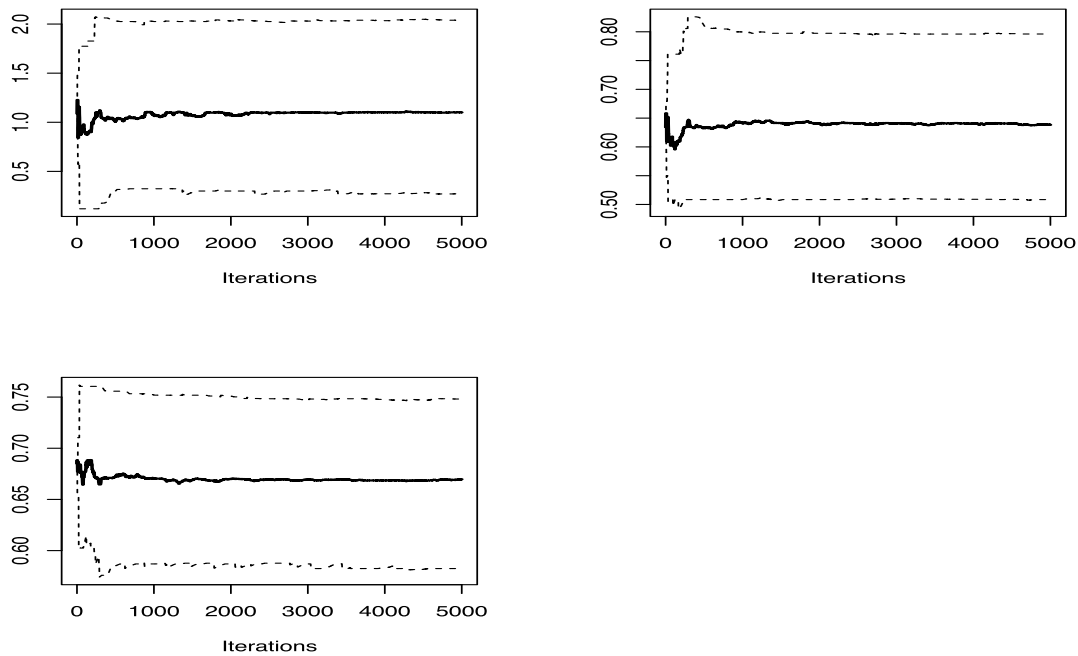


Figure 8: Bayesian diagnostic curves of E-IW model for [Table 4](#)

Table 3: Numerical outcomes for the simulation experiments under case 1

n	Method	$\hat{\theta}$			$\hat{\alpha}$			$\hat{\beta}$		
		$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$
50	$\mathcal{D}_1$	0.6770	0.5230	3.8191	0.4541	0.0459	0.0667	0.2806	0.0306	0.0049
	$\mathcal{D}_2$	1.4782	0.2782	4.4546	0.6139	0.1139	0.12083	0.2279	0.02202	0.0048
	$\mathcal{D}_3$	1.3347	0.1347	5.1448	0.6455	0.1455	0.1405	0.2437	0.0062	0.0051
	$\mathcal{D}_4$	2.2881	1.0881	6.1567	0.6490	0.1490	0.1439	0.2123	0.0376	0.0054
100	$\mathcal{D}_1$	1.2603	0.0603	2.7568	0.5209	0.0209	0.0597	0.2534	0.0034	0.0031
	$\mathcal{D}_2$	1.1290	0.0709	3.4665	0.5692	0.0692	0.0863	0.2379	0.0120	0.0037
	$\mathcal{D}_3$	1.1347	0.0652	3.9395	0.5550	0.0550	0.0949	0.2433	0.0066	0.0039
	$\mathcal{D}_4$	2.1263	0.9263	5.5575	0.6357	0.1357	0.0962	0.2208	0.0291	0.0041
300	$\mathcal{D}_1$	1.1511	0.0489	1.6997	0.5010	0.0010	0.0309	0.2550	0.0050	0.0017
	$\mathcal{D}_2$	1.4730	0.2730	2.6212	0.5459	0.0459	0.0447	0.2411	0.0088	0.0019
	$\mathcal{D}_3$	1.2377	0.0377	2.8109	0.5161	0.0161	0.0517	0.2475	0.0024	0.0022
	$\mathcal{D}_4$	1.8352	0.6352	2.9401	0.6051	0.1051	0.0588	0.2275	0.02243	0.0024
600	$\mathcal{D}_1$	1.2144	0.0144	1.3375	0.5070	0.0070	0.0240	0.2517	0.0017	0.0014
	$\mathcal{D}_2$	1.3503	0.1503	1.5189	0.5268	0.0268	0.0276	0.24438	0.0056	0.0015

(Continued)

Table 3 (continued)

n	Method	$\hat{\theta}$			$\hat{\alpha}$			$\hat{\beta}$		
		$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$
	$\mathcal{D}_3$	1.2230	0.02305	1.6337	0.5057	0.0057	0.0305	0.2505	0.0005	0.0017
	$\mathcal{D}_4$	1.710	0.5103	1.7119	0.5761	0.0761	0.0356	0.2346	0.0153	0.0019
1000	$\mathcal{D}_1$	1.2041	0.0041	0.7452	0.4857	0.0143	0.0113	0.2559	0.0059	0.0007
	$\mathcal{D}_2$	1.1928	0.0071	1.1244	0.5043	0.0043	0.0205	0.2511	0.0011	0.0011
	$\mathcal{D}_3$	1.2055	0.0055	1.1851	0.5028	0.0028	0.0250	0.2498	0.0001	0.0013
	$\mathcal{D}_4$	1.6096	0.4096	1.2666	0.56167	0.0616	0.0256	0.2359	0.0140	0.0016

Table 4: Numerical outcomes for the simulation experiments under case 2

n	Method	$\hat{\theta}$			$\hat{\alpha}$			$\hat{\beta}$		
		$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$
50	$\mathcal{D}_1$	1.7036	0.2036	4.6541	0.7173	0.0173	0.0855	0.6231	0.0231	0.0256
	$\mathcal{D}_2$	2.7418	1.2418	10.047	0.8180	0.1180	0.1085	0.5725	0.0774	0.0432
	$\mathcal{D}_3$	2.1150	0.6150	12.557	0.8711	0.1711	0.1393	0.5734	0.0765	0.0511
	$\mathcal{D}_4$	2.8613	1.3613	14.658	0.8588	0.1588	0.1467	0.5582	0.0917	0.0564
100	$\mathcal{D}_1$	1.8255	0.3255	3.3602	0.7354	0.0354	0.0686	0.5973	0.0027	0.0154
	$\mathcal{D}_2$	2.1456	0.6456	4.2981	0.7883	0.0883	0.0827	0.6115	0.0384	0.0205
	$\mathcal{D}_3$	1.8958	0.3958	4.6251	0.7522	0.0522	0.0701	0.6277	0.0222	0.0266
	$\mathcal{D}_4$	2.6708	1.1708	5.9953	0.8507	0.1507	0.1058	0.5699	0.0800	0.0272
300	$\mathcal{D}_1$	1.3347	0.1653	1.7439	0.6845	0.0155	0.0394	0.6185	0.0185	0.0108
	$\mathcal{D}_2$	2.0300	0.5300	2.1579	0.7765	0.0765	0.0475	0.6173	0.03263	0.0138
	$\mathcal{D}_3$	1.8482	0.3482	2.8852	0.7514	0.0514	0.0509	0.6274	0.0225	0.0218
	$\mathcal{D}_4$	2.0939	0.5939	2.9070	0.7853	0.0853	0.0544	0.6033	0.0466	0.0242
600	$\mathcal{D}_1$	1.3184	0.1816	1.3252	0.6745	0.0255	0.0293	0.6213	0.0213	0.0081
	$\mathcal{D}_2$	1.8096	0.3096	1.4815	0.7429	0.0429	0.0343	0.6266	0.0233	0.0104
	$\mathcal{D}_3$	1.4844	0.0155	1.5317	0.6974	0.0025	0.0341	0.6556	0.0056	0.0172
	$\mathcal{D}_4$	1.8225	0.3225	1.6120	0.7544	0.0544	0.0477	0.6205	0.0294	0.0179
1000	$\mathcal{D}_1$	1.4597	0.0403	0.5807	0.6945	0.0055	0.0147	0.6076	0.0076	0.0040
	$\mathcal{D}_2$	1.5583	0.0583	0.8267	0.7062	0.0062	0.0200	0.6443	0.0056	0.0058
	$\mathcal{D}_3$	1.4956	0.0043	0.8964	0.6951	0.0048	0.0226	0.6544	0.0044	0.0067
	$\mathcal{D}_4$	1.8580	0.3580	0.9579	0.7524	0.0524	0.0234	0.6219	0.0280	0.0075

Table 5: Numerical outcomes for the simulation experiments using Bayesian estimation under case 1

n	Method	$\hat{\theta}$			$\hat{\alpha}$			$\hat{\beta}$		
		$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$
50	Bayes _{SE}	1.3927	0.1927	0.0566	0.4828	0.0172	0.0020	0.2909	0.0409	0.0022
	Bayes _{LI}	1.2985	0.1002	0.0571	0.4975	0.0022	0.0023	0.2661	0.0163	0.0024
	Bayes _{GE}	1.298	0.1002	0.0574	0.4968	0.0022	0.0026	0.2651	0.0163	0.0027
100	Bayes _{SE}	1.1806	0.0194	0.0338	0.5033	0.0033	0.0017	0.2387	0.0113	0.0019
	Bayes _{LI}	1.1629	0.0365	0.0342	0.5155	0.0156	0.0019	0.2827	0.0328	0.0022
	Bayes _{GE}	1.1626	0.0365	0.0344	0.5151	0.0156	0.0022	0.2818	0.0328	0.0026
300	Bayes _{SE}	1.1554	0.0446	0.0082	0.4525	0.0475	0.0015	0.2644	0.0144	0.0016
	Bayes _{LI}	1.1704	0.0291	0.0085	0.4988	0.001	0.0018	0.2501	0.0006	0.0019
	Bayes _{GE}	1.1703	0.0291	0.0087	0.4982	0.001	0.0021	0.2497	0.0004	0.0022
600	Bayes _{SE}	1.1777	0.0223	0.0018	0.4638	0.0362	0.0013	0.2560	0.0060	0.0014
	Bayes _{LI}	1.1853	0.0146	0.0021	0.4972	0.0026	0.0016	0.2626	0.0126	0.0018
	Bayes _{GE}	1.1852	0.0146	0.0023	0.4967	0.0026	0.0018	0.2624	0.0126	0.0020
1000	Bayes _{SE}	1.2254	0.0254	0.0009	0.5179	0.0179	0.0010	0.2461	0.0039	0.0008
	Bayes _{LI}	1.2226	0.0229	0.0012	0.5124	0.0126	0.0012	0.2482	0.0017	0.0011
	Bayes _{GE}	1.2225	0.0229	0.0014	0.5119	0.0126	0.0015	0.2481	0.0017	0.0012

Table 6: Numerical outcomes for the simulation experiments using Bayesian estimation under case 2

n	Method	$\hat{\theta}$			$\hat{\alpha}$			$\hat{\beta}$		
		$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$
50	Bayes _{SE}	1.5772	0.0772	0.0221	0.7755	0.0755	0.0143	0.6767	0.0267	0.0048
	Bayes _{LI}	1.5556	0.0566	0.0224	0.7365	0.0368	0.0146	0.6512	0.0021	0.0051
	Bayes _{GE}	1.5555	0.0566	0.0226	0.7361	0.0368	0.0148	0.6498	0.0021	0.0054
100	Bayes _{SE}	1.3296	0.1704	0.0211	0.5975	0.1025	0.0125	0.6988	0.0488	0.0047
	Bayes _{LI}	1.379	0.1185	0.0214	0.6466	0.0525	0.0128	0.6481	0.0011	0.0049
	Bayes _{GE}	1.3785	0.1185	0.0216	0.6452	0.0525	0.0130	0.6467	0.0011	0.0052
300	Bayes _{SE}	1.591	0.0918	0.0171	0.7742	0.0742	0.0075	0.5878	0.0622	0.0044

(Continued)

Table 6 (continued)

n	Method	$\hat{\theta}$			$\hat{\alpha}$			$\hat{\beta}$		
		$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$
600	Bayes _{LI}	1.5463	0.0466	0.0173	0.7357	0.0363	0.0079	0.6155	0.0341	0.0046
	Bayes _{GE}	1.5462	0.0466	0.0175	0.735	0.0363	0.0081	0.6149	0.0341	0.0048
	Bayes _{SE}	1.4822	0.0178	0.0039	0.7015	0.0015	0.0052	0.6810	0.0310	0.0015
	Bayes _{LI}	1.487	0.0128	0.0041	0.733	0.0333	0.0055	0.6698	0.0206	0.0017
	Bayes _{GE}	1.4872	0.0128	0.0044	0.7326	0.0333	0.0057	0.6695	0.020	0.0020
	Bayes _{SE}	1.4843	0.0157	0.0021	0.7131	0.0131	0.0043	0.6496	0.0004	0.0009
1000	Bayes _{LI}	1.4599	0.0399	0.0023	0.7037	0.0039	0.0045	0.6309	0.0189	0.0012
	Bayes _{GE}	1.4596	0.0399	0.0025	0.7034	0.0039	0.0047	0.6306	0.0189	0.0014

6 Actuarial Measures

6.1 VaR Indicator

A method of statistical analysis that determines the risk of losses on a portfolio of funds across time and with an its level of trust is the value at risk ($\mathcal{X}_1$) indicator. The $\mathcal{X}_1$ of the E-IW distribution was obtained by finding the inverse of Eq. (5). Consequently, the $\mathcal{X}_1$ is established by

$$\mathcal{X}_1 = 2^{1/\beta} \left(-\frac{\alpha}{\log \left(1 - \frac{\log(e^\beta - e^\beta q + q)}{\theta} \right)} \right)^{1/\beta}, \quad 0 < q < 1.$$

6.2 TVaR Indicator

The TVaR, represented by ($\mathcal{X}_2$), is a risk indicator that calculates the potential loss across a given confidence level in the event of a catastrophic event. For the E-IW distribution, the TVaR metric, denoted as $\mathcal{X}_2$, is obtained to be

$$\begin{aligned} \mathcal{X}_2 &= \frac{1}{(1-q)} \int_{\mathcal{X}_1}^{\infty} t k(t) dt \\ &= \frac{2\alpha\beta}{(1-q)(e^\theta - 1)} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \eta_{l,j} \int_{\mathcal{X}_1}^{\infty} t^{-\beta} e^{-2(l+1)t^{-\beta}} dt \\ &= \frac{2\alpha\beta}{(1-q)(e^\theta - 1)} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \eta_{l,j} \Omega_l(t), \end{aligned}$$

$$\text{where } \Omega_l(t) = \int_{\mathcal{X}_1}^{\infty} t^{-\beta} e^{-2(l+1)t^{-\beta}} dt.$$

6.3 TV Indicator

The TV metric, denoted as $(\mathcal{X}_3)$, shows the variance of the loss model above the critical threshold. Also, it is most famous significant risk metrics that considers the tail variance outside of the $\mathcal{X}_1$. It is given as follows:

$$\begin{aligned}\mathcal{X}_3 &= \frac{1}{(1-p)} \int_{\mathcal{X}_1}^{\infty} t^2 k(t) dt - (\mathcal{X}_2)^2 \\ &= \frac{2\alpha\beta}{(1-q)(e^\theta - 1)} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \eta_{l,j} \int_{\mathcal{X}_1}^{\infty} t^{-\beta+1} e^{-2(l+1)t^{-\beta}} dt - (\mathcal{X}_2)^2.\end{aligned}$$

6.4 TVP Indicator

An indicator of risk that evaluates the range of losses in an estimated tail. This is the TVP metric, represented by $(\mathcal{X}_4)$. It can be defined below

$$\mathcal{X}_4 = \mathcal{X}_2 + q\mathcal{X}_3.$$

6.5 Simulation Experiments of the Actuarial Indicators

In this part, we provide several results for the these measures by applying our recommended E-IW and IW distributions with two scenarios of unknown parameters θ , α , and β . The outcomes are tabulated in [Tables 7](#) and [8](#). Henceforth, the comparisons between the two proposed distributions are plotted graphically in [Figs. 9](#) and [10](#). From these figures, the E-IW model can be considered heavier among the IW distribution since it has higher values of $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$, and $\mathcal{X}_4$ than the IW. Thus, the E-IW is better suitable for modeling heavy-tailed datasets.

Table 7: Numerical computations of $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$, and $\mathcal{X}_4$ for the E-IW and IW distributions

Distribution	Par	q	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$
E-IW	$\theta = 0.2, \alpha = 1.2, \beta = 1.5$	0.65	2.9776	9.2504	464.05	310.88
		0.75	3.8815	11.594	630.40	484.40
		0.85	5.6574	16.226	996.88	863.58
		0.95	12.157	32.773	2573.7	2477.7
		0.99	35.981	87.608	8963.6	8961.6
IW	$\alpha = 1.2, \beta = 1.5$	0.65	1.9797	6.2853	268.256	180.65
		0.75	2.5912	7.8960	366.466	282.74
		0.85	3.7918	11.087	585.241	508.54
		0.95	8.1798	22.578	1554.8	1499.6
		0.99	24.2475	61.535	5810.4	5813.9

Table 8: Numerical computations of $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$, and $\mathcal{X}_4$ for the E-IW and IW distributions

Distribution	Par	q	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$
E-IW	$\theta = 0.8, \alpha = 1.75, \beta = 2.25$	0.65	2.2047	4.0900	20.202	17.221
		0.75	2.6032	4.7703	26.657	24.763
		0.85	3.3095	6.0073	40.577	40.498
		0.95	5.4459	9.8124	99.324	104.17
		0.99	11.170	20.042	356.952	373.42
IW	$\alpha = 1.75, \beta = 2.25$	0.65	1.8645	3.5690	16.279	14.150
		0.75	2.2310	4.1829	21.467	20.283
		0.85	2.8755	5.2942	32.669	33.063
		0.95	4.8009	8.6940	80.110	84.799
		0.99	9.9068	17.801	289.814	304.71

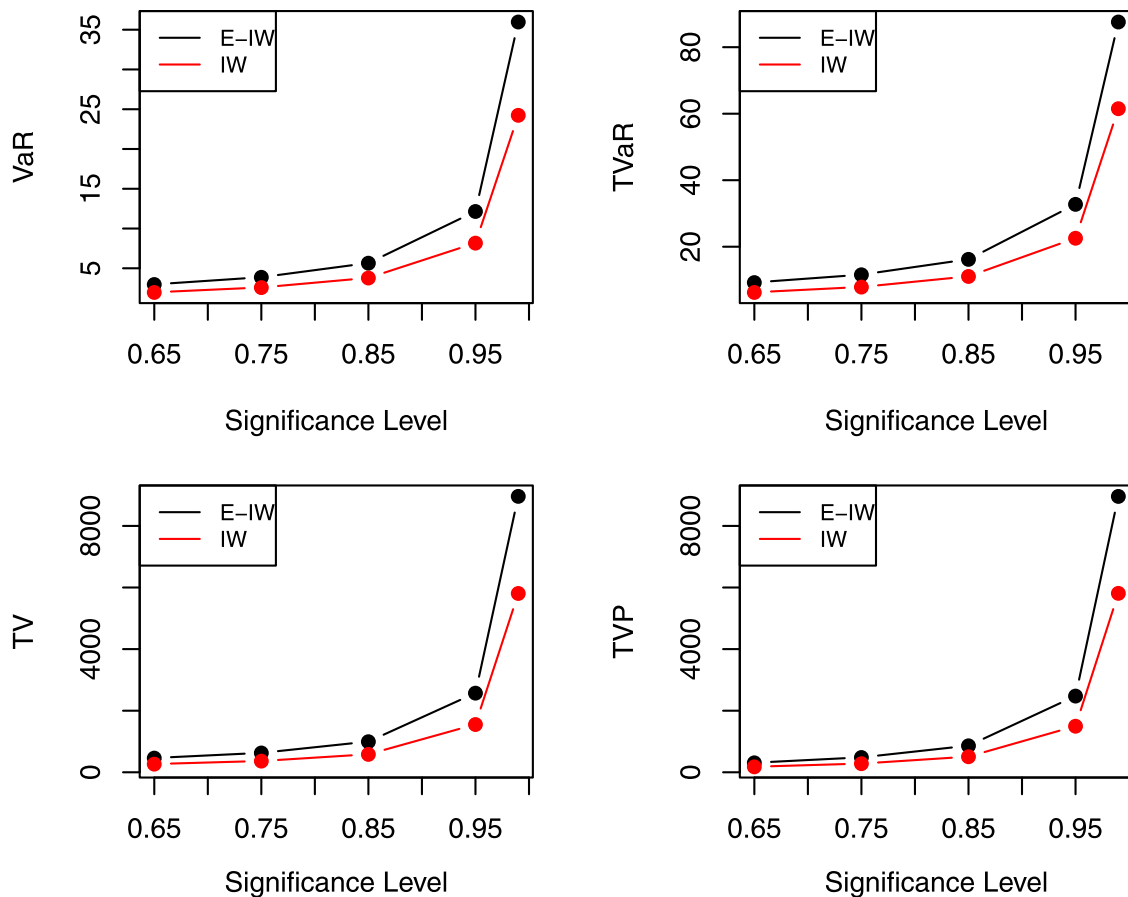


Figure 9: Graphical comparison between VAR, TVaR, TV, and TVP presented in [Table 7](#)

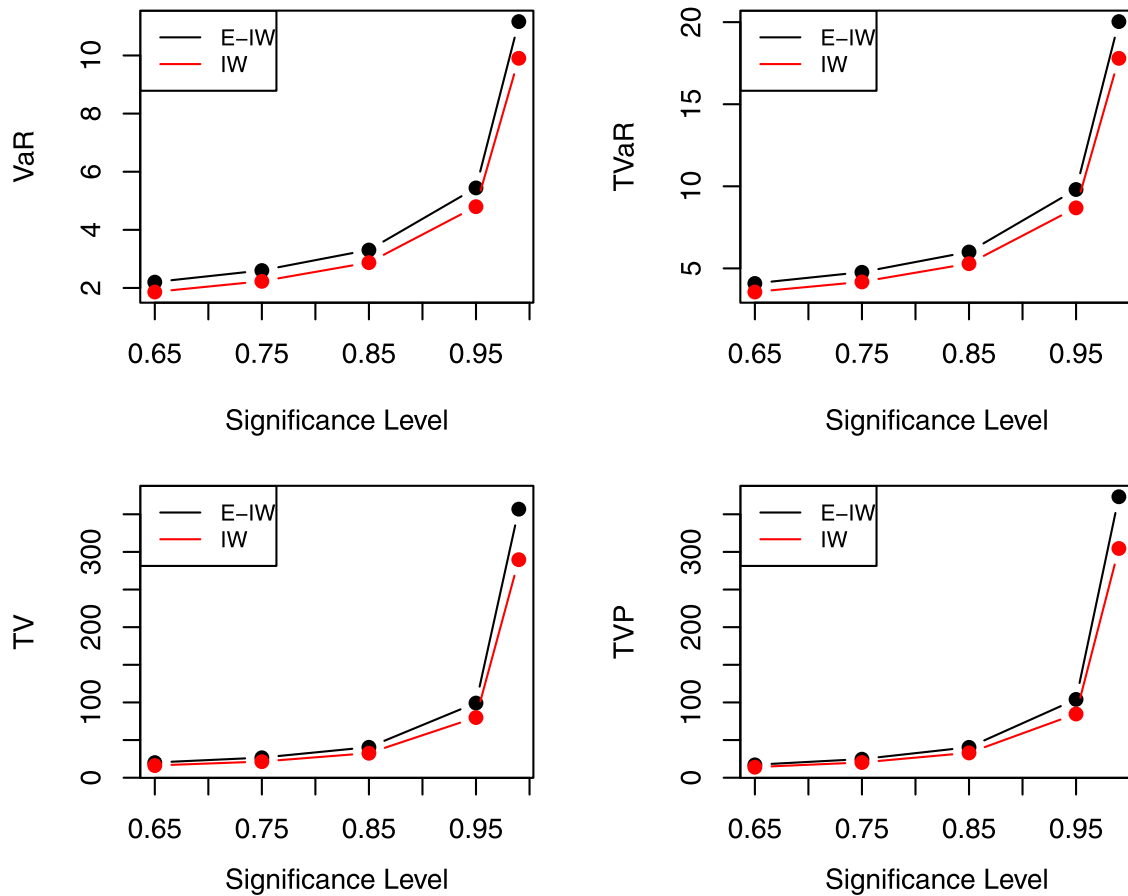


Figure 10: Graphical comparison between VAR, TVaR, TV, and TVP presented in [Table 8](#)

7 Data Analysis

This section shows how the E-IW can be applied to real-world data. Four data sets from the radiotherapy and financial sectors are used to test its performance.

7.1 Data I: Radiotherapy Data

A range of cancers of the mouth, throat, and other parts of the head and neck are collectively referred to as head and neck cancer (HNC). Most cancer cases are preventable, and many can be cured if detected early. This study covers the survival periods (measured in days) of 42 HNC patients who underwent radiation and chemotherapy at the same time (RT). The data were first applied by Efron [37]. The considered values are tabulated in [Table 9](#).

7.2 Data II: Leukemia Patients Data

Here, the considered data set represents the survival times (in months) of 20 patients with leukemia caused by acute myeloid leukemia. This data was studied by Afify et al. [38], and its values are reported in [Table 10](#).

Table 9: Head and neck cancer data

0.1220	0.2356	0.2374	0.2587	0.3198	0.3700	0.4135	0.4738	0.5546	0.5836
0.6347	0.6846	0.7447	0.7826	0.8143	0.8400	0.9200	0.9400	1.1000	1.1200
1.1900	1.2700	1.3000	1.3300	1.4000	1.4600	1.5500	1.5900	1.7300	1.7900
1.9400	1.9500	2.0900	2.4900	2.8100	3.1900	3.3900	4.3200	4.6900	5.1900
6.3300	7.2500								

Table 10: Values of leukemia data

2.226	2.113	3.631	2.473	2.720	2.050	2.061
3.915	0.871	1.548	2.746	1.972	2.265	1.200
2.967	2.808	1.079	2.353	0.726	1.958	

7.3 Data III: Total Economic Loss of the Claimant Data

This dataset represents the total economic loss of the claimant (in thousands) as scrutinized by the Insurance Research Council (IRC). This data, collected in 2002, contains demographic information about the claimant, the attorney's involvement, and the economic loss. The suggested data was accessible on <http://instruction.bus.wisc.edu/jfrees/jfreesbooks/Regression%20Modeling/BookWebDec2010/data.html>, and it was considered by Frees [39]. Table 11 summarizes the data considered.

Table 11: Total economic loss of the claimant data

34.94	10.892	0.330	11.037	0.138	0.309	3.538	4.882	0.874	1.351
6.290	29.620	2.678	11.673	0.243	1.053	0.150	0.758	0.100	4.754
3.200	0.230	26.262	1.619	20.040	0.787	0.250	1.405	0.415	0.595
3.994	0.337	0.603	2.673	4.752	3.538	3.285	13.711	0.386	2.970
1.914	8.090	8.190	23.201	0.460	3.982	2.740	0.100	2.103	3.200
3.777	1.975	0.497	0.335	2.669	33.633	0.305	6.694	0.239	0.987
1.000	1.530	5.563	0.270	2.523	0.730	3.638	1.081	1.320	0.300
44.061	2.600	1.895	1.442	0.445	3.510	1.919	0.100	2.875	3.889
2.765	1.527	9.730	2.720	10.834	2.132	3.370	0.250	4.227	4.440
0.800	4.506	1.860	13.028	2.467	0.198	3.682	34.572	8.698	0.460

7.4 Data IV: Medical Insurance Group Data

Here, the considered data set was obtained from a medical insurance group and represented the total loss amount in euros (EUR) for all claims in 1991 that exceeded 25,000 USD. It was accessible at <http://www.soa.org> and was provided by Meraou et al. [40]. The values of the data are reported in Table 12.

Table 12: Values of the fourth data

44.73127	41.48841	25.66236	38.12284	28.63050	55.83410	29.15282
48.73259	49.67760	25.77334	30.16309	34.91208	31.23646	33.40271
43.28791	88.28655	66.27829	30.08967	60.61784	28.43689	28.96690
71.28330	66.52654	28.79237	37.46642	47.70202	27.23568	150.03092
85.38754	61.78040	43.03892	38.24468	36.19602	51.66454	66.97332
38.37818	26.50199	118.90862	42.51939	46.21111	58.78668	26.32199
93.25783	33.33290	45.37340	51.43795	64.60545	26.95302	29.90154
57.80053	46.43980	65.26666	30.57411	51.02350	163.52556	62.49751
52.12305	29.32303	31.89054	151.06354			

7.5 Data V: Precipitation Data

The highest annual flood rivers of North Saskatchewan throughout a 45-year duration are used in this subsection. It contains has 45 records; and Table 13 summarizes the results of the suggested data set examined by Van Montfort [41].

Table 13: Values of data set V

20.940	21.820	23.700	24.888	25.460	25.760	26.720	27.500	28.100
28.600	30.200	30.380	31.500	32.600	32.680	34.400	35.347	35.700
38.100	39.020	39.200	40.000	40.400	40.400	42.250	44.020	44.730
44.900	46.300	50.330	51.442	57.220	58.700	58.800	61.200	61.740
65.440	65.597	66.000	74.100	75.800	84.100	106.600	109.700	121.970

Based on the proposed data sets, several mathematical statistics and curves are presented in Table 14 and Figs. 11–15, including kernel density, scaled total time on the test (TTT), quantile-quantile (Q-Q), probability-probability (P-P), Histogram, and box plots.

Table 14: Basic summary statistics for the proposed data sets

Data Set	Min	Q_1	Q_2	Mean	Q_3	Max	S_T	K_T	SD
I	0.122	0.647	1.230	1.723	1.947	7.250	1.6834	2.2690	1.6667
II	0.726	1.855	2.170	2.184	2.727	3.915	0.1139	−0.5683	0.8396
III	0.100	0.601	2.495	5.147	4.457	44.061	2.7485	7.3445	8.4351
IV	25.66	30.47	44.01	52.00	60.91	163.53	2.0757	4.3033	30.431
V	20.94	30.38	40.40	47.65	58.80	121.97	1.3797	1.5138	23.676

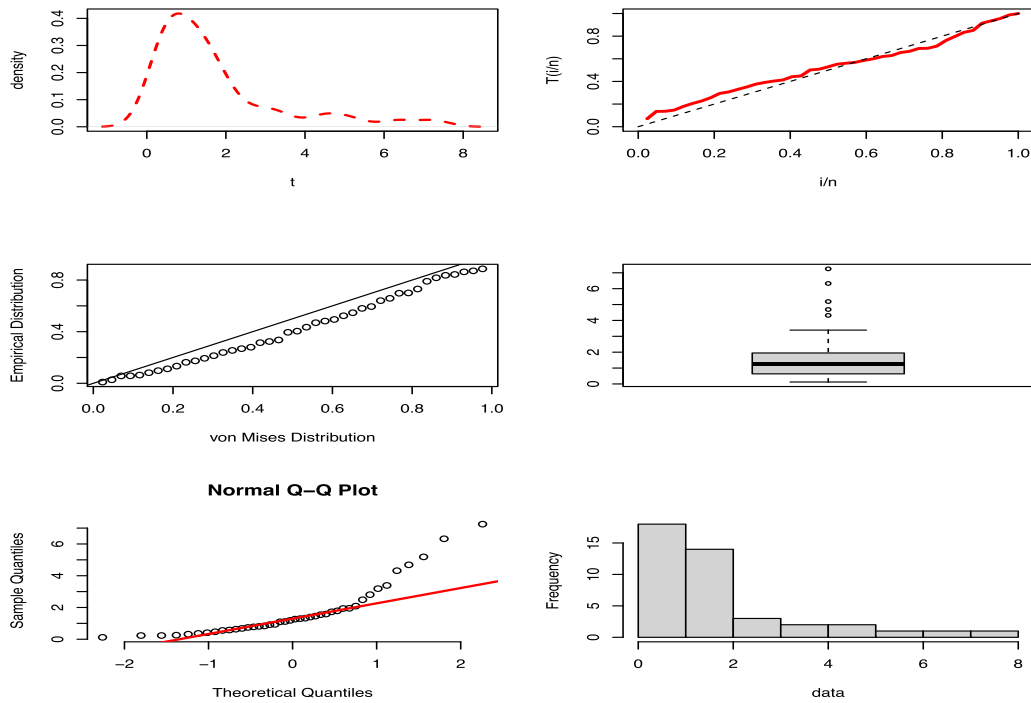


Figure 11: Some non-parametric plots for data I

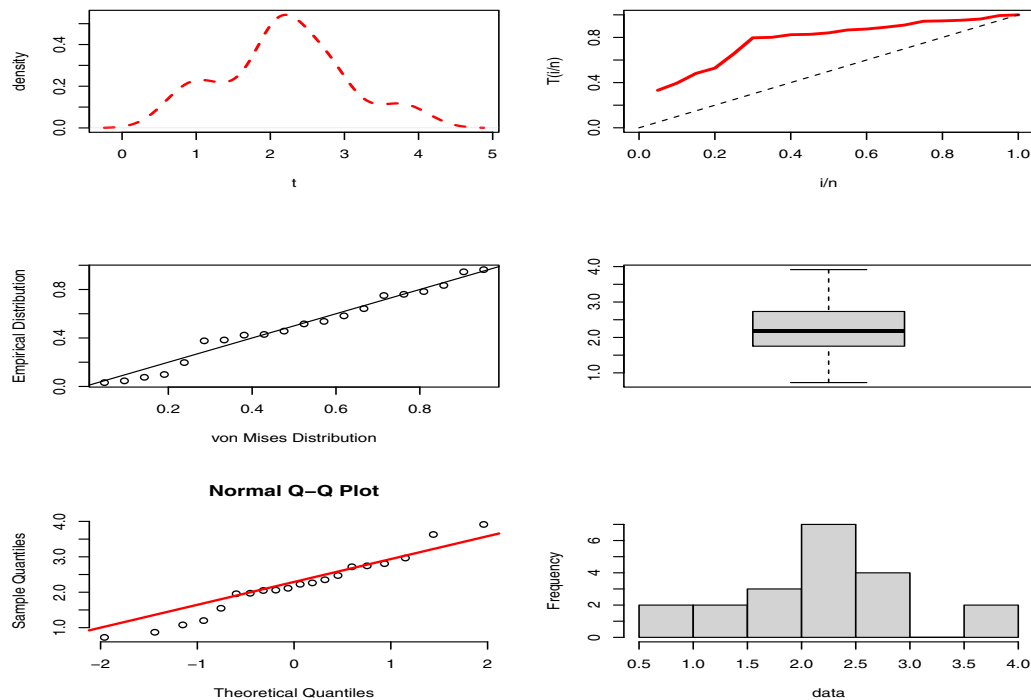


Figure 12: Some non-parametric plots for data II

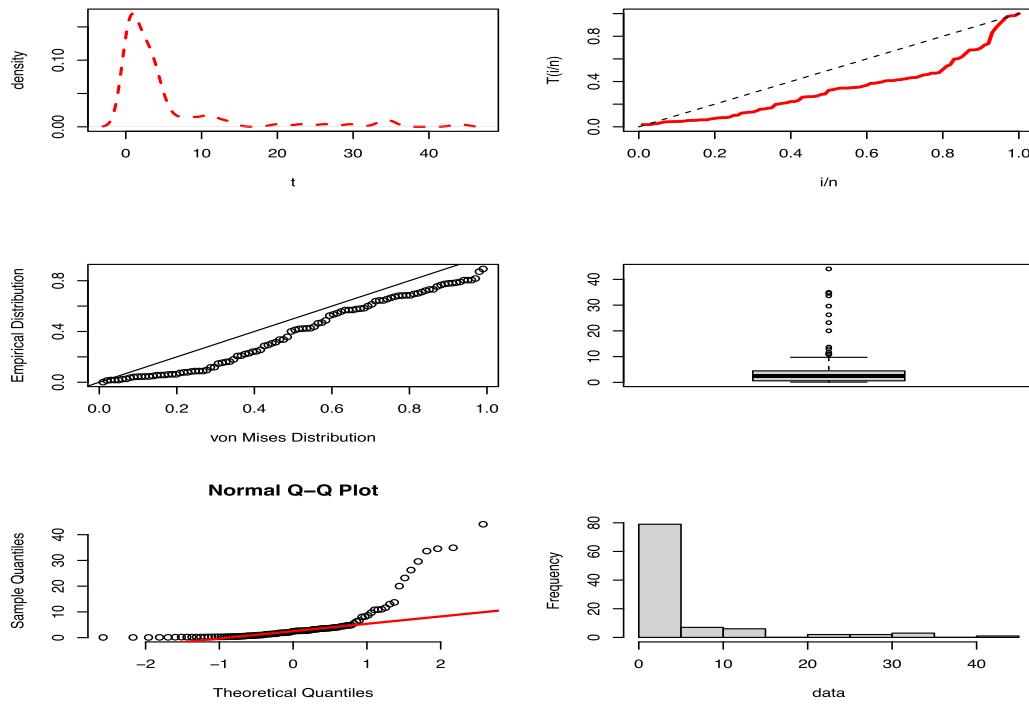


Figure 13: Some non-parametric plots for data III

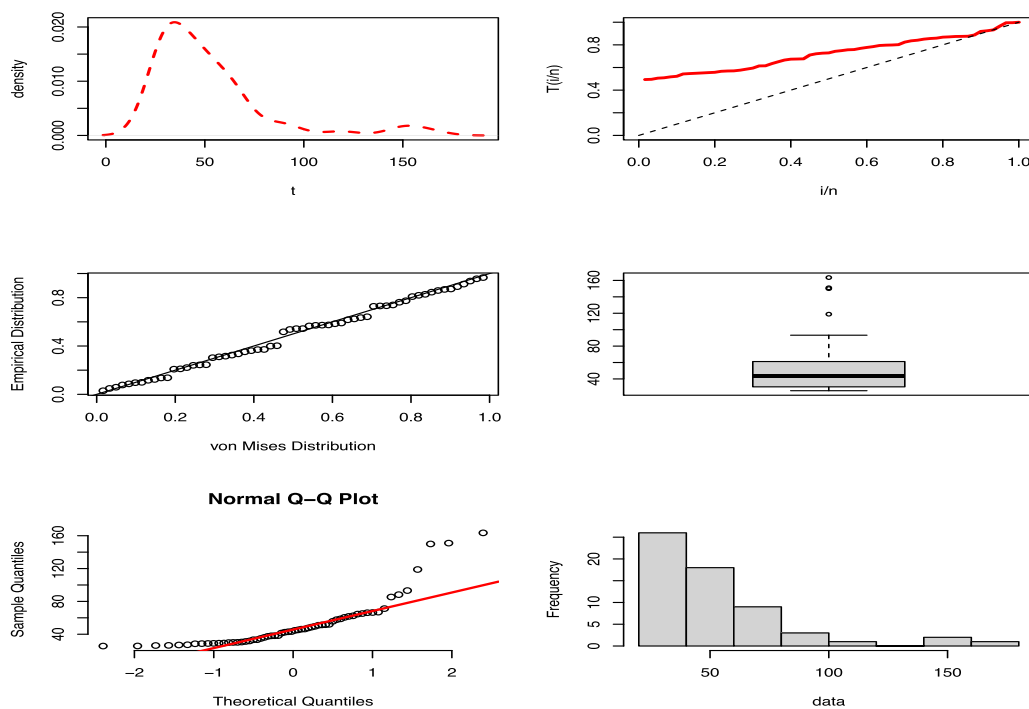


Figure 14: Some non-parametric plots for data IV

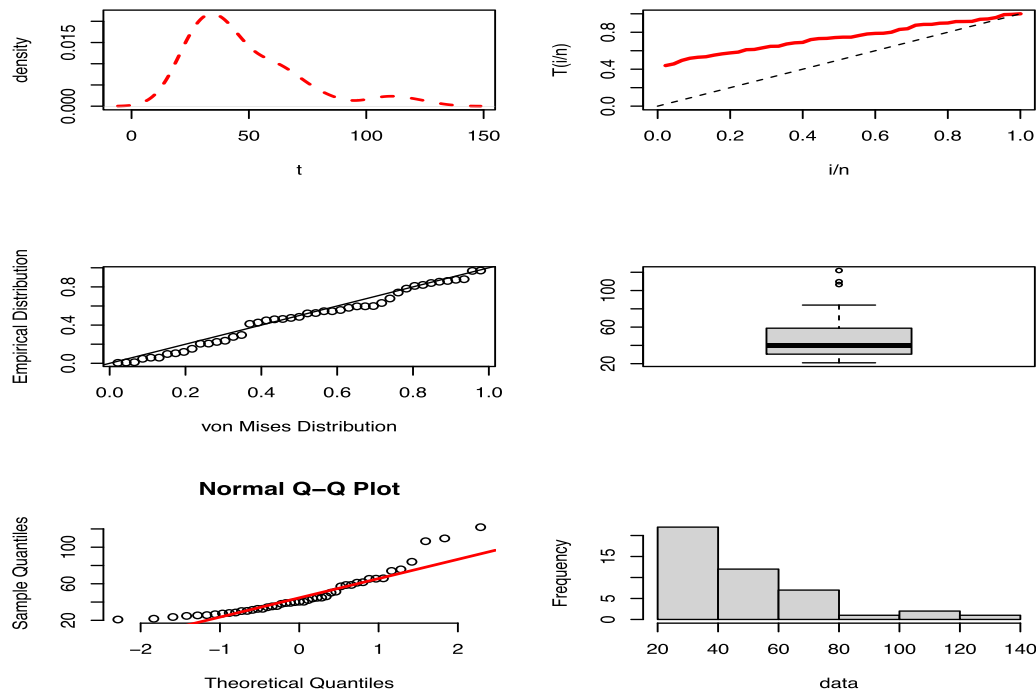


Figure 15: Some non-parametric plots for data V

We fitted numerous distributions, namely alpha power transformed inverse Weibull (APT-IW, see Basheer [42]), zero-truncated Poisson inverse Weibull (ZTP-IW, see Chakrabarty and Chowdhury [43]), Lomax (LD, see Lomax [44]), Inverse Weibull (IW), generalized exponential (GE, see Gupta and Kundu [45]), Rayleigh (RD), inverse Rayleigh (IRD, see Smadi and Alrefaei [46]), and generalized exponentiated moment exponential (EGME, see Mohie El-Din et al. [47]) models for selecting about the more suitable distribution for fitting all recommended data sets.

We first find the MLEs of the distributions together as four metrics: Kolmogorov-Smirnov ($\mathcal{KS}$) statistics, their associated p -values, the Akaike Information Criterion ($\mathcal{A}$), and the Bayesian Information Criterion ($\mathcal{B}$). These measurements are computed using the formula below:

$$\mathcal{A} = -2 \log(\mathcal{LL}) + 2d,$$

$$\mathcal{B} = -2 \log(\mathcal{LL}) + m \log(m).$$

and

$$\mathcal{KS} = \sup \left[\max_{1 \leq j \leq m} \left| \frac{j}{m} - K_0(t_{(j)}) \right|, \max_{1 \leq j \leq m} \left| K_0(t_{(j)}) - \frac{j-1}{m} \right| \right].$$

The amount of samples is denoted by m , $\mathcal{LL}$ is the log-likelihood function, and the total number of associated variables in the distribution by d . All of these computations are carried out by the software for statistics using the “BFGS” methodology and the R package (AdquacyModel). It should be noted that the best-fitted probability distribution is determined by taking the minimum values of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{KS}$, as well as the maximum p -values. The so obtained results are tabulated in Tables 15 and 16. Based on the numerical values of Tables 15 and 16, it is inferred that the E-IW distribution fits the proposed datasets better than the other distributions considered for comparison. Furthermore,

we have presented graphs comparing the fitted PDF, CDF, P-P, and Q-Q plots in Figs. 16–30 for the various fitting models under consideration, which yield the same conclusions.

Table 15: Estimated results and analytical measures of the fitted distributions employing the suggested data sets

Data	Model	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\mathcal{KS}$	p -Value	$\mathcal{A}$	$\mathcal{B}$
I	E-IW	45.054	2.1952	0.2971	0.0540	0.9991	130.872	136.085
	APT-IW	1.2321	2.0251	4.2595	0.0971	0.7872	135.957	141.170
	ZTP-IW	1.2939	4.0798	4.8859	0.0886	0.8675	134.836	140.049
	LD		47.697	28.644	0.1039	0.7159	133.906	137.382
	IW		0.6959	1.0697	0.1048	0.7058	136.222	139.697
	GE		0.7277	1.4365	0.0901	0.8539	132.046	136.522
	RD			1.6856	0.3075	0.0005	162.408	164.146
	IRD			0.4931	0.3893	<0.00001	180.515	182.253
	EGME	0.0800	1.0374	0.1130	0.0889	0.8650	132.279	137.492
II	E-IW	709.254	4.7122	0.4151	0.1831	0.4598	54.027	57.014
	APT-IW	2.3321	0.7301	3.0039	0.2683	0.0922	62.776	65.763
	ZTP-IW	2.5951	0.8916	3.2344	0.2371	0.1791	61.392	64.379
	LD		54.816	25.217	0.3373	0.0156	75.923	77.914
	IW		2.6041	2.0651	0.2719	0.0849	62.173	64.164
	GE		1.1403	6.6363	0.2210	0.24371	55.047	57.039
	RD			2.5645	0.4121	0.0013	66.003	66.999
	IRD			1.6021	0.2619	0.1064	60.213	61.209
	EGME	2.5919	2.9227	1.3415	0.2079	0.3083	56.028	59.015
III	E-IW	29.753	2.1727	0.2103	0.0687	0.7323	497.994	505.810
	APT-IW	0.8470	2.5062	10.699	0.1060	0.2108	504.090	511.905
	ZTP-IW	0.8463	2.4787	2.3382	0.1059	0.2116	504.091	511.906
	LD		10.803	3.3076	0.1017	0.2516	504.558	509.769
	IW		0.9431	0.7164	0.1262	0.0824	507.017	512.227
	GE		0.1373	0.6183	0.1479	0.0251	515.133	520.344
	RD			6.9646	0.5783	<0.00001	847.346	849.951
	IRD			0.396	0.5638	<0.00001	826.318	828.923
	EGME	0.0401	0.4490	0.3259	0.1365	0.0480	508.924	516.739
IV	E-IW	2.8469	962.61	1.9150	0.0906	0.6736	532.783	539.066
	APT-IW	1.9086	0.0184	0.0365	0.1081	0.4521	533.293	539.576
	ZTP-IW	3.0343	0.0325	1.9530	0.0978	0.5794	533.717	539.572
	LD		136.481	3.1552	0.4193	<0.00001	611.631	615.820
	IW		169.72	1.4288	0.2080	0.0093	563.607	567.796
	GE		0.0541	8.2970	0.1071	0.4647	543.315	547.503
	RD			42.503	0.1666	0.0637	562.185	564.279

(Continued)

Table 15 (continued)

Data	Model	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\mathcal{KS}$	p -Value	$\mathcal{A}$	$\mathcal{B}$
	IRD			39.285	0.1577	0.0901	537.076	539.170
	EGME	0.0281	7.3017	0.5191	0.1068	0.4679	5545.233	551.516

Table 16: Estimated results and analytical measures of the fitted distributions employing the suggested data set

Data	Model	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\mathcal{KS}$	p -Value	$\mathcal{A}$	$\mathcal{B}$
V	E-IW	5.8105	149.295	1.3289	0.0821	0.9219	396.554	401.974
	APT-IW	2.0806	0.0593	210.5	0.1295	0.4371	404.242	409.662
	ZTP-IW	2.4282	0.0366	2.0071	0.0968	0.79210	399.449	404.869
	LD		990.55	21.407	0.3609	<0.00001	443.359	446.972
	IW		1505.1	2.0492	0.1116	0.6286	398.567	402.180
	GE		0.0438	4.1835	0.1187	0.5496	402.333	405.946
	RD			37.539	0.1440	0.3076	405.967	407.774
	IRD			36.645	0.1319	0.4135	397.304	399.111
	EGME	0.0384	7.8345	0.6324	0.098	0.7764	398.700	404.120

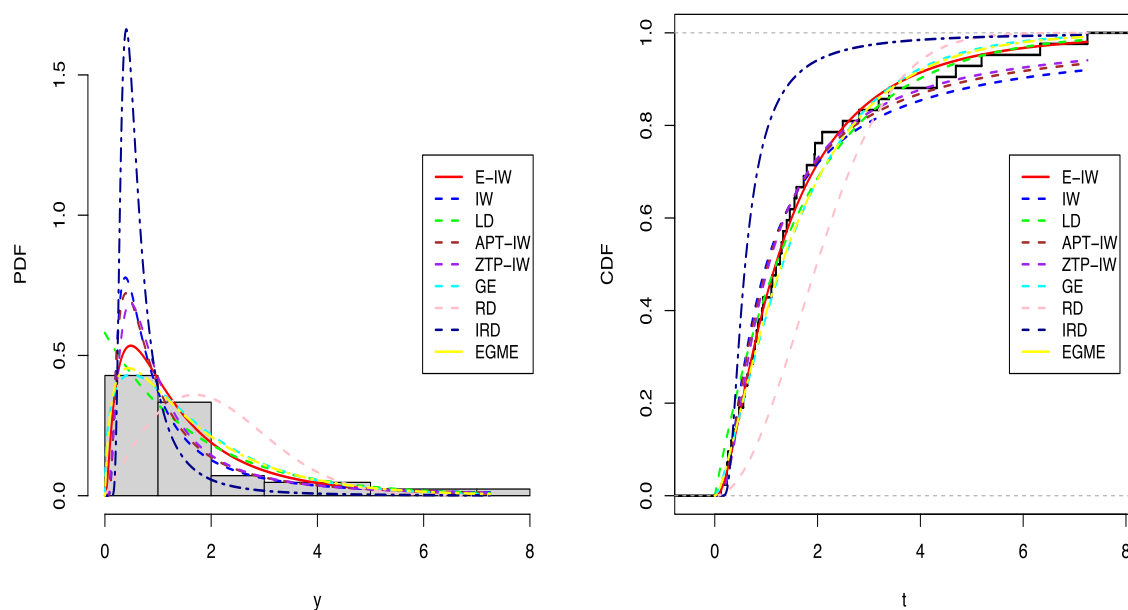


Figure 16: Empirical and fitted PDF and CDF of all models (data I)

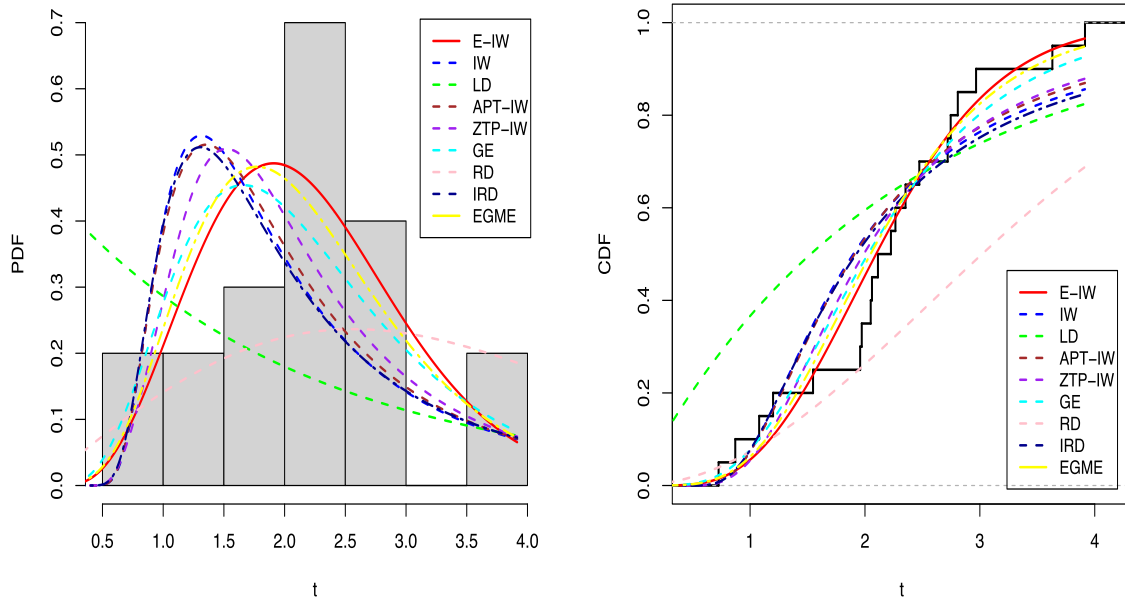


Figure 17: Empirical and fitted PDF and CDF of all models (data II)

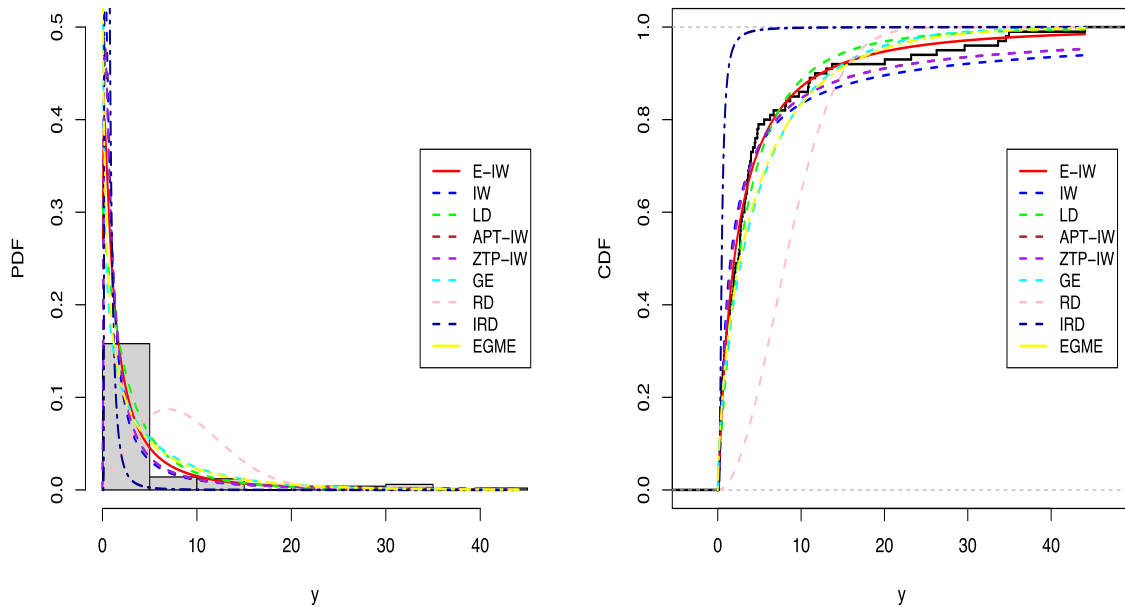


Figure 18: Empirical and fitted PDF and CDF of all models (data III)

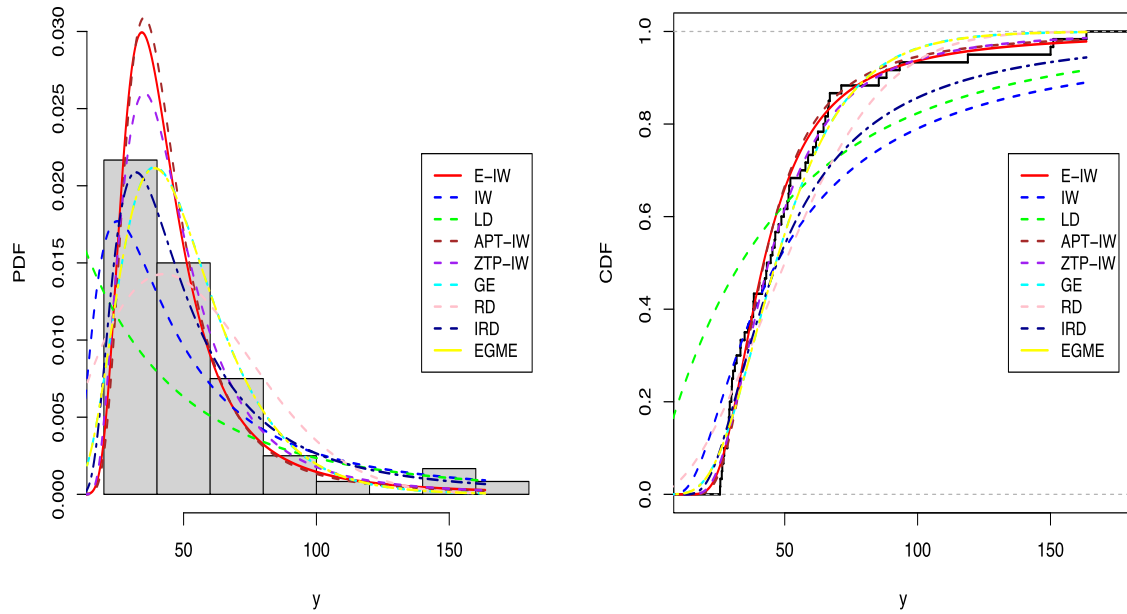


Figure 19: Empirical and fitted PDF and CDF of all models (data IV)

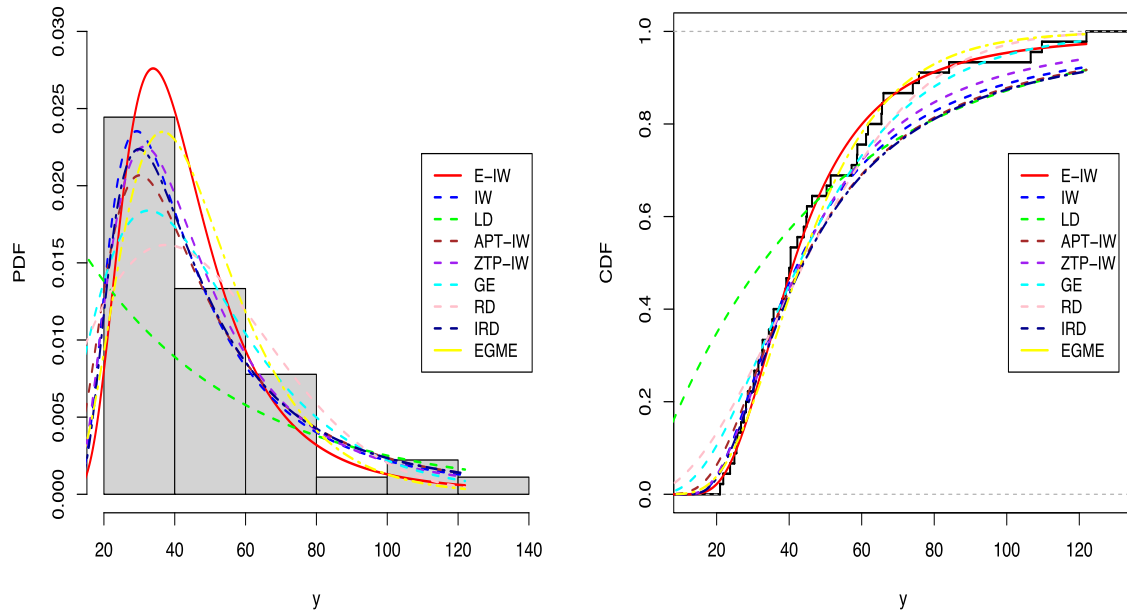


Figure 20: Empirical and fitted PDF and CDF of all models (data V)

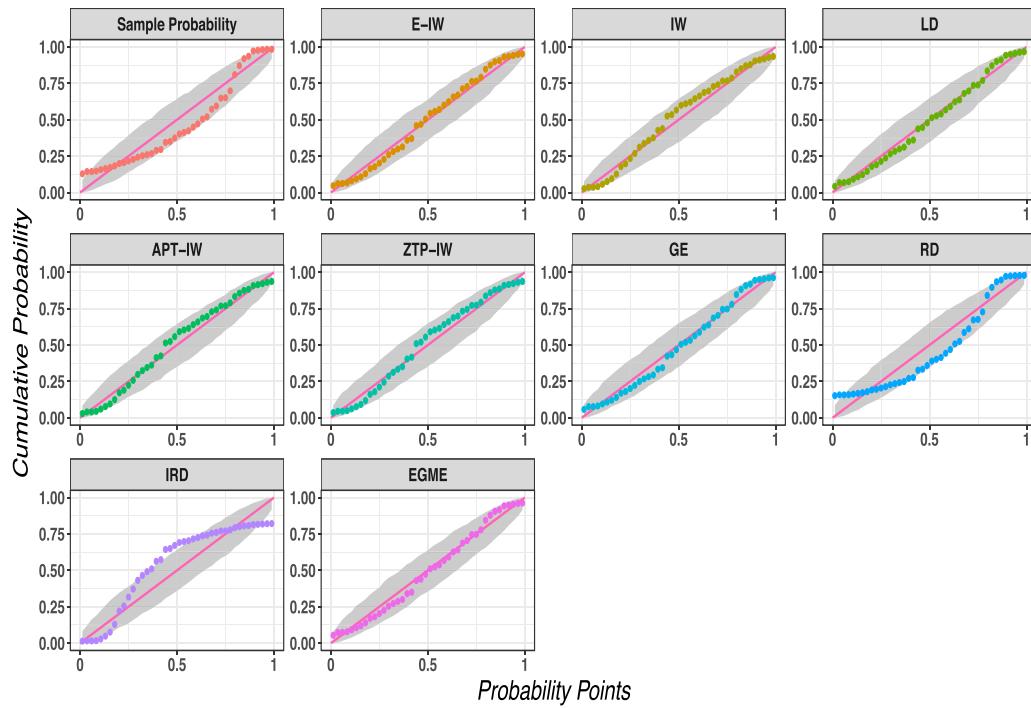


Figure 21: Fitted Probability-Probability curves for data I

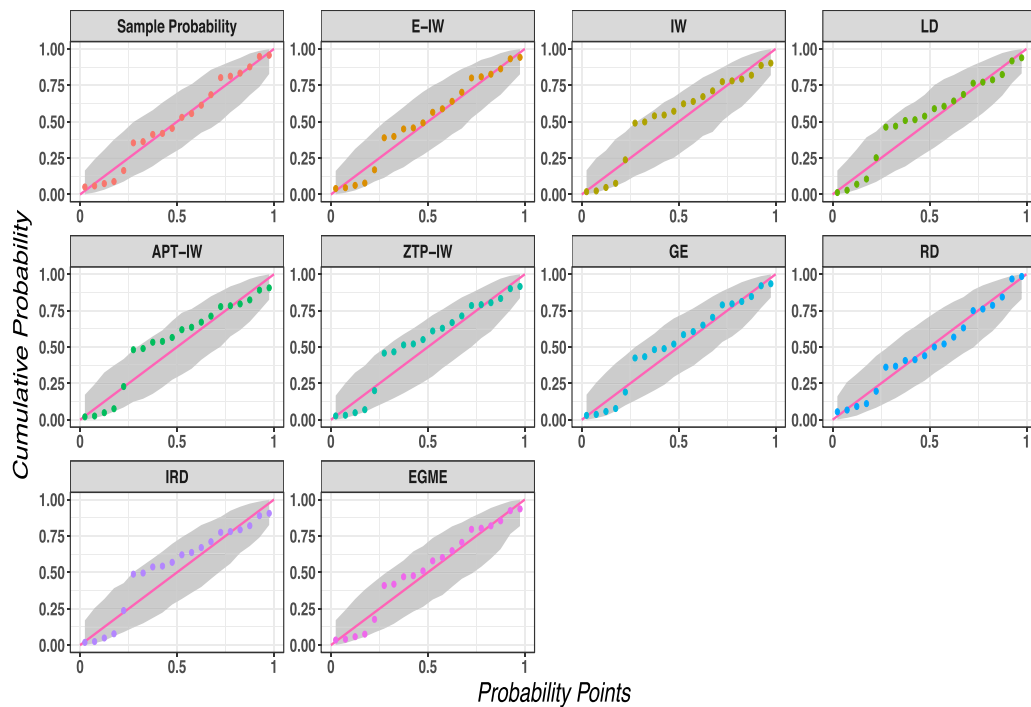


Figure 22: Fitted Probability-Probability curves for data II

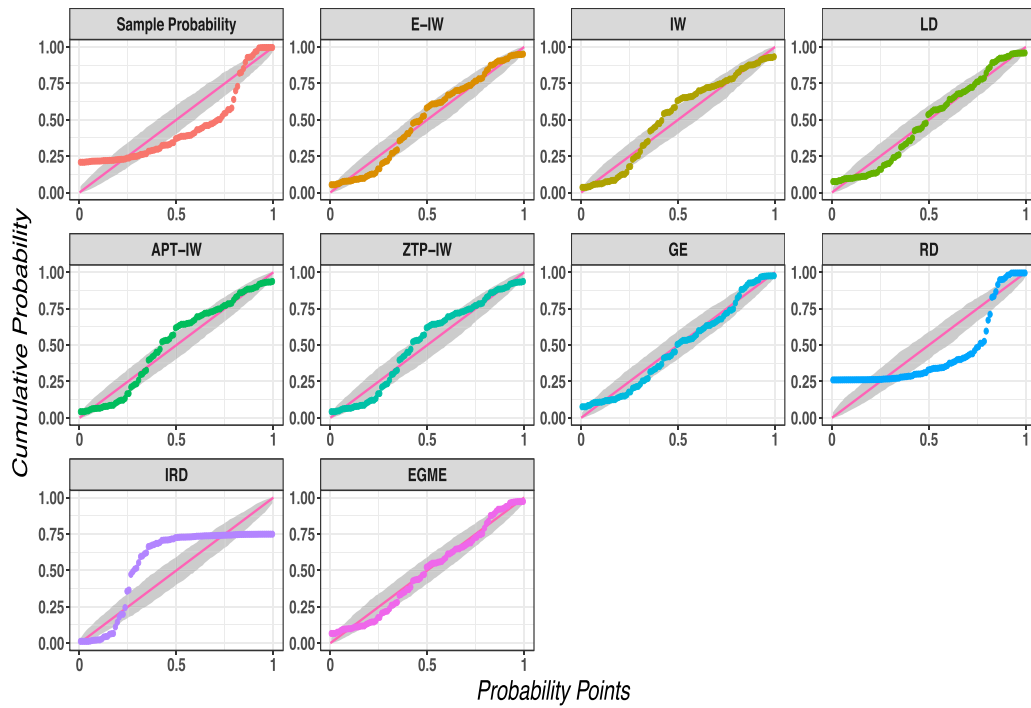


Figure 23: Fitted Probability-Probability curves for data III

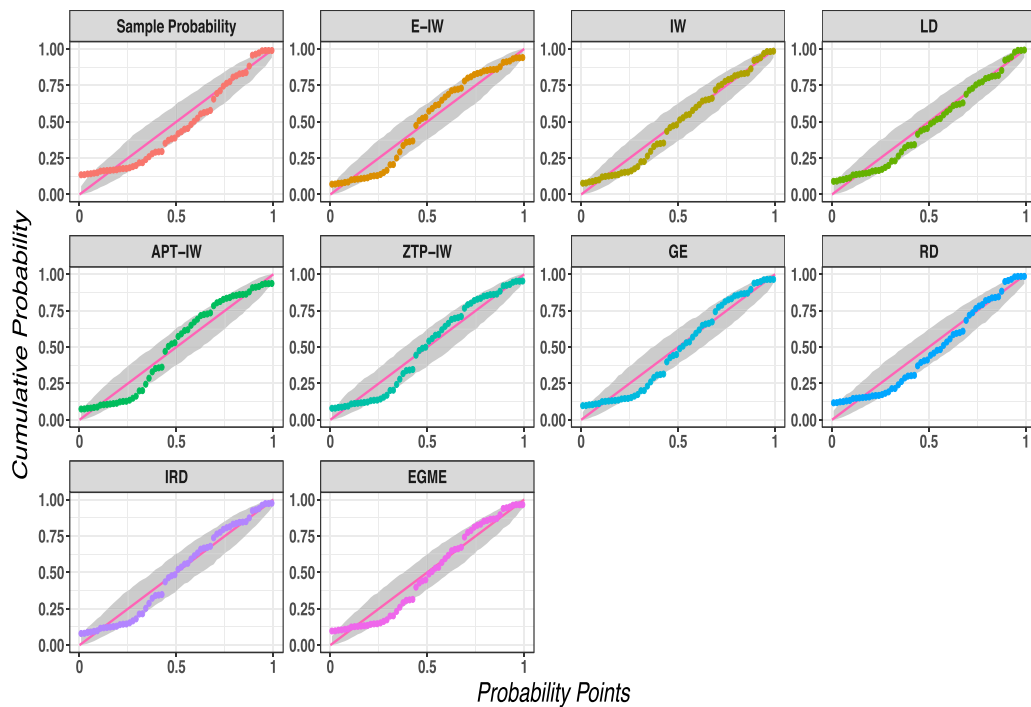


Figure 24: Fitted Probability-Probability curves for data IV

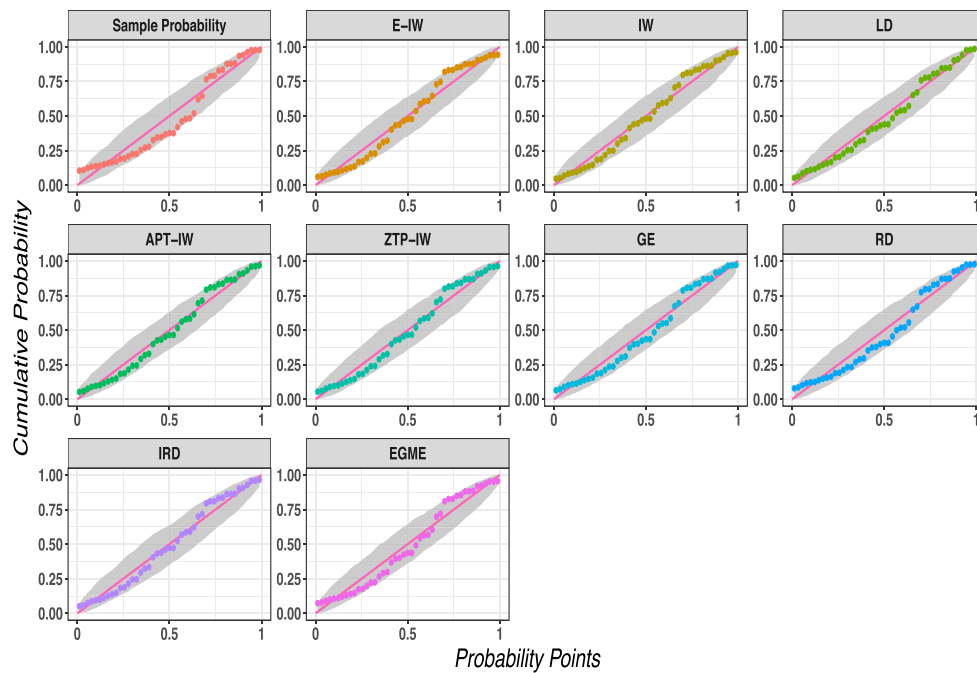


Figure 25: Fitted Probability-Probability curves for data V

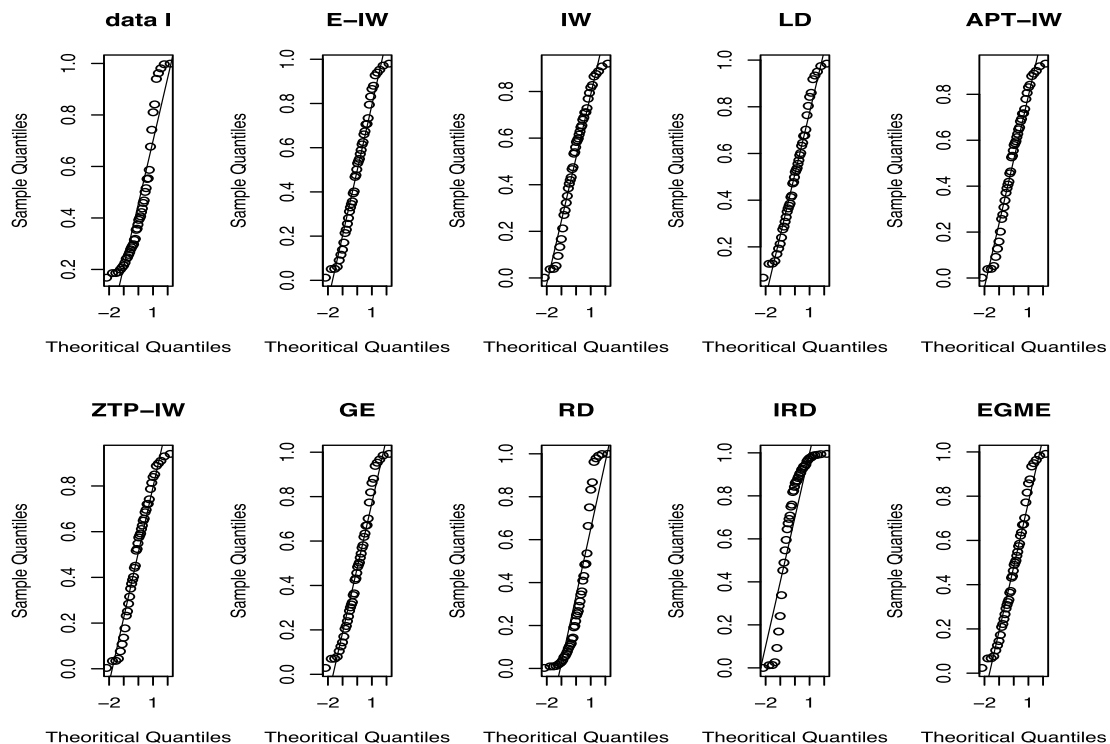


Figure 26: Fitted Quantile-Quantile curves for data I

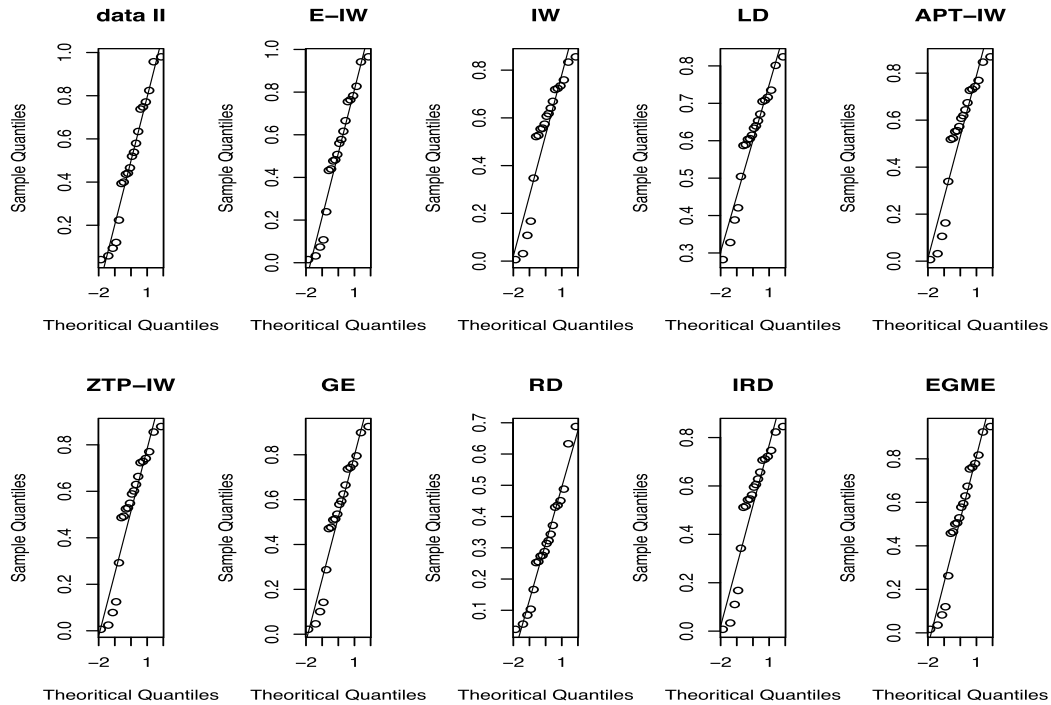


Figure 27: Fitted Quantile-Quantile curves for data II

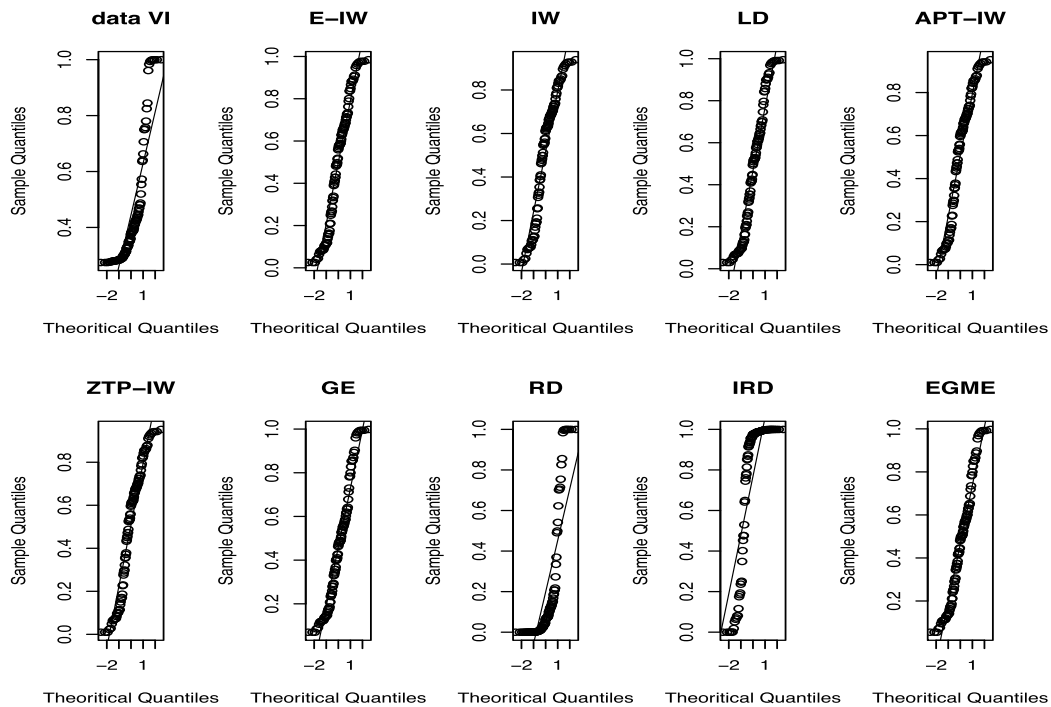


Figure 28: Fitted Quantile-Quantile curves for data III

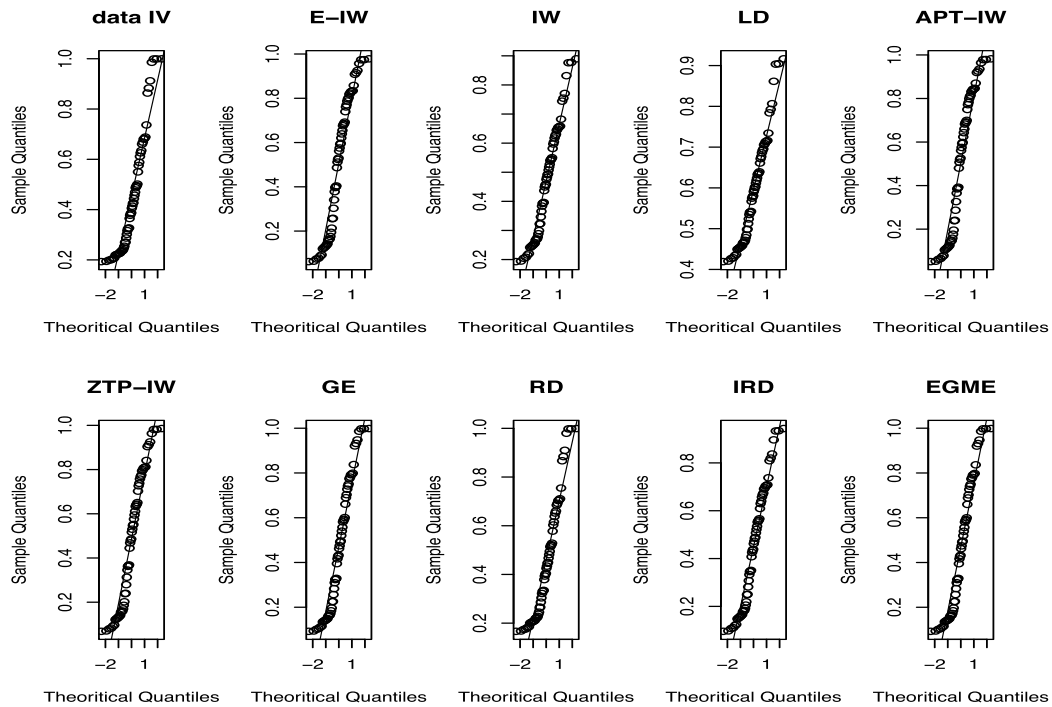


Figure 29: Fitted Quantile-Quantile curves for data IV

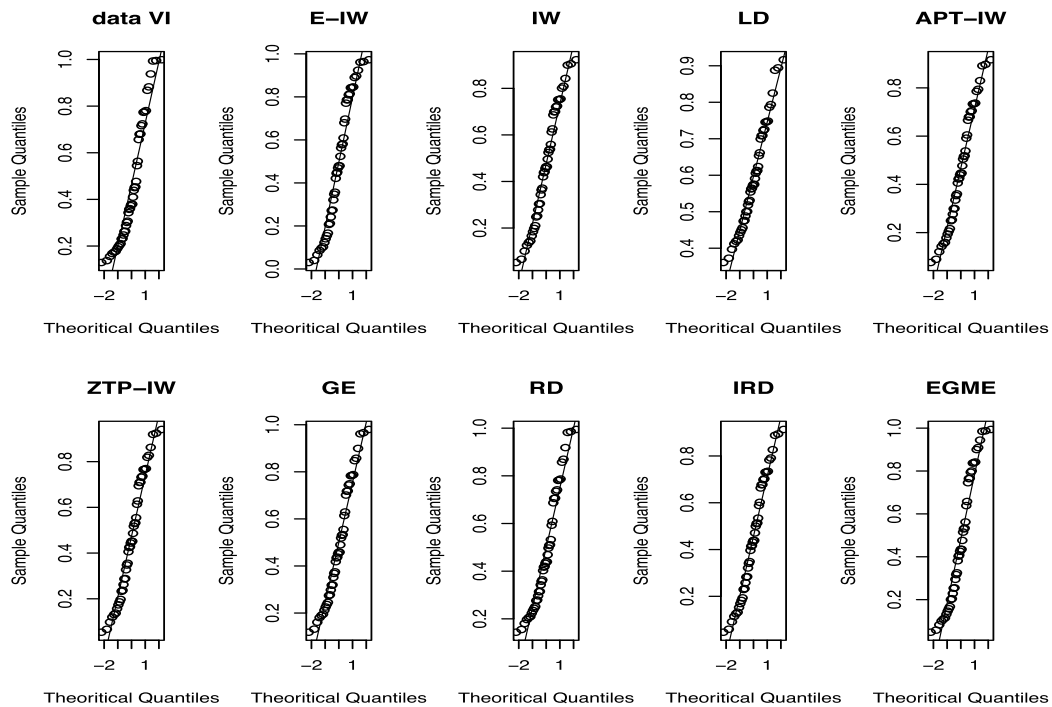


Figure 30: Fitted Quantile-Quantile curves for data V

Finally, we computed the final estimates of the E-IW distribution using the proposed estimation procedures. The numerical findings are given in Table 17. Additionally, Figs. 31–35 display several diagnostic plots of the E-IW model for the MCMC methodology.

Table 17: Estimation values of α , β and θ of the E-IW by applying several estimation techniques

	Data	LSE	WLSE	MPS	Bayes
I	θ	46.297	46.972	25.499	45.374
	α	2.9430	2.8637	2.2044	2.2108
	β	0.2837	0.2819	0.2773	0.2770
II	θ	707.647	707.328	709.388	709.146
	α	4.7494	4.7413	4.7152	4.7134
	β	0.4537	0.4685	0.4164	0.4148
III	θ	29.519	29.578	29.6274	29.751
	α	2.2153	2.2537	2.2311	2.1729
	β	0.2473	0.2491	0.2245	0.2157
IV	θ	2.6115	2.2852	2.5794	2.8327
	α	962.99	962.95	962.91	962.64
	β	1.9213	1.9352	1.9246	1.9197
V	θ	5.8419	5.7701	5.7852	5.8149
	α	142.821	144.798	146.263	148.522
	β	1.2744	1.2936	1.3382	1.3247

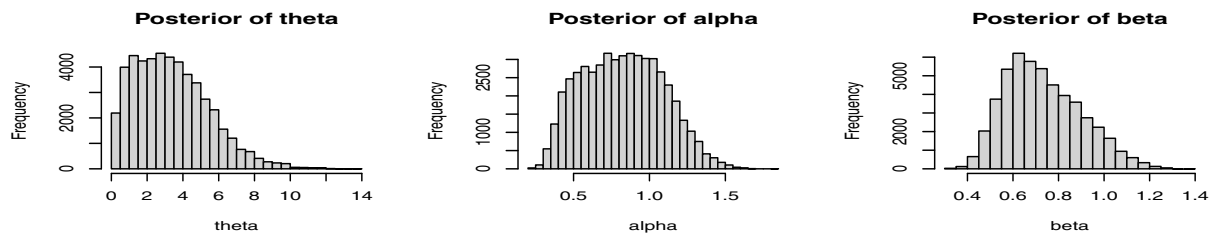


Figure 31: (Continued)

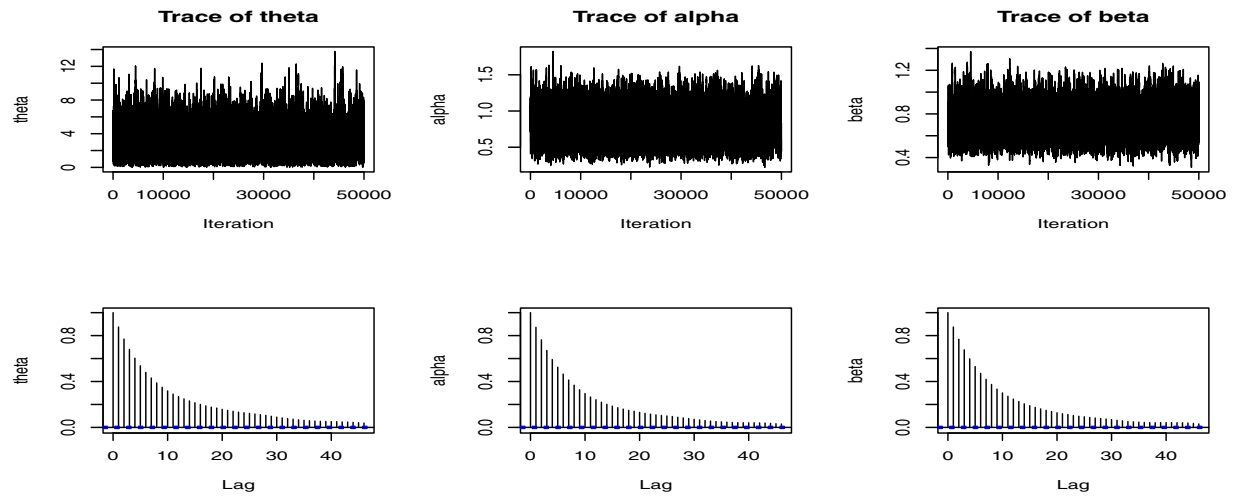


Figure 31: MCMC convergence results using data I

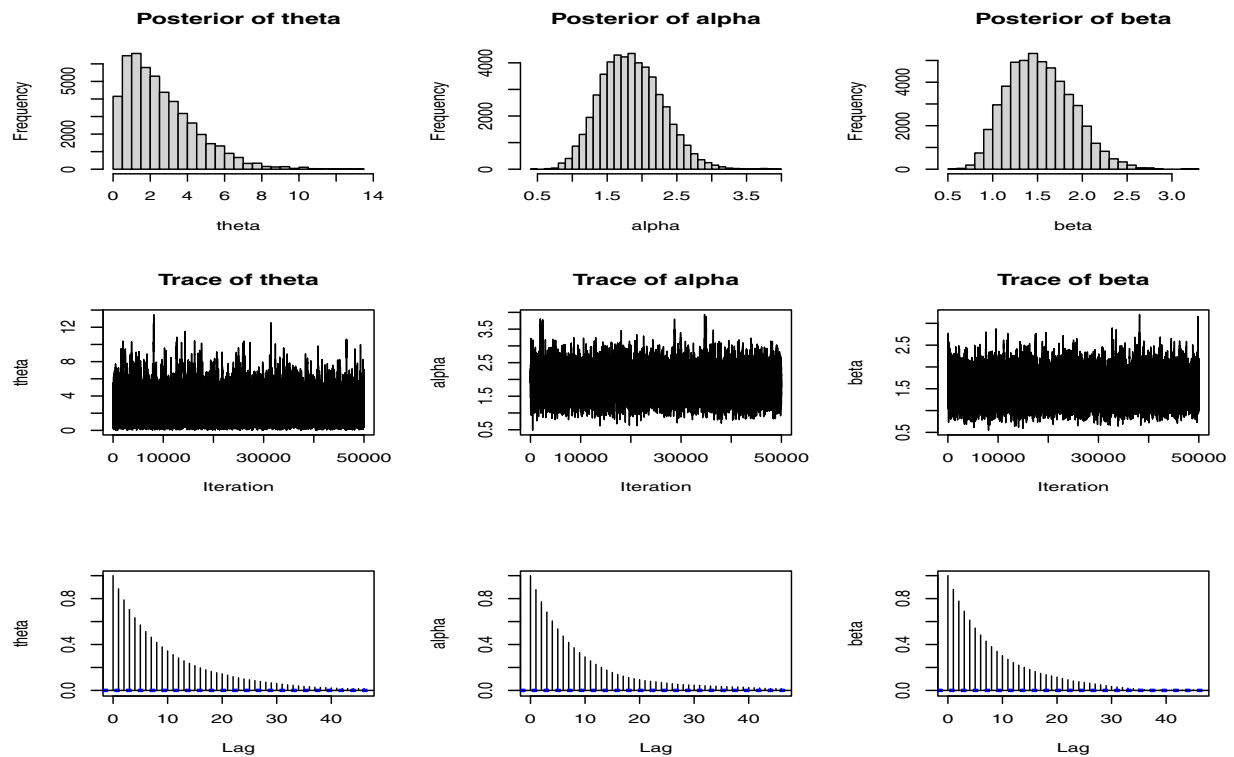


Figure 32: MCMC convergence results using data II

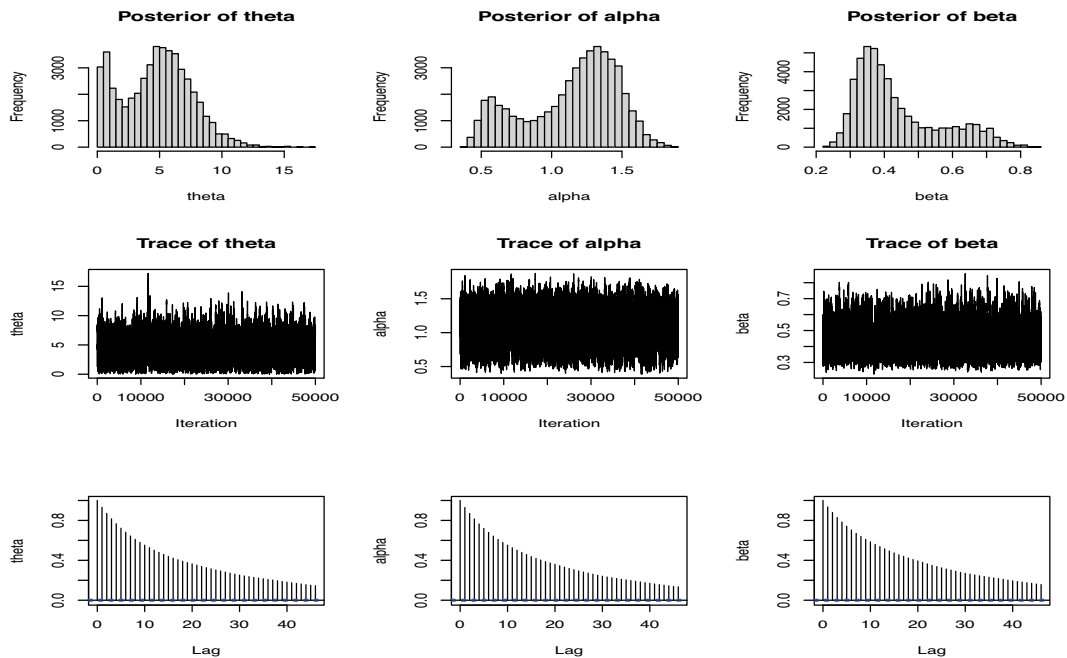


Figure 33: MCMC convergence results using data III

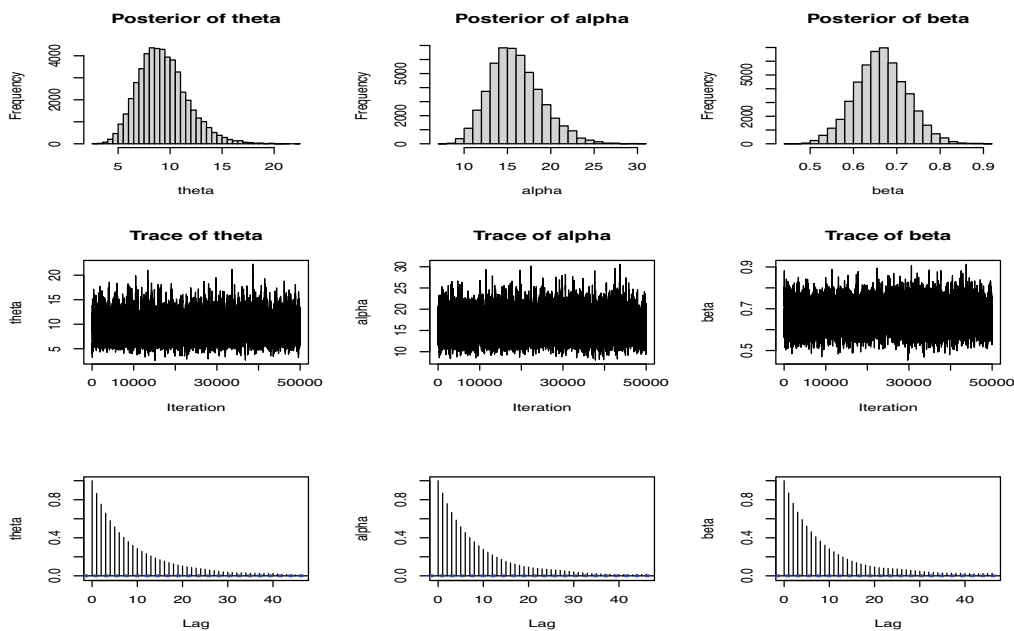


Figure 34: MCMC convergence results using data IV

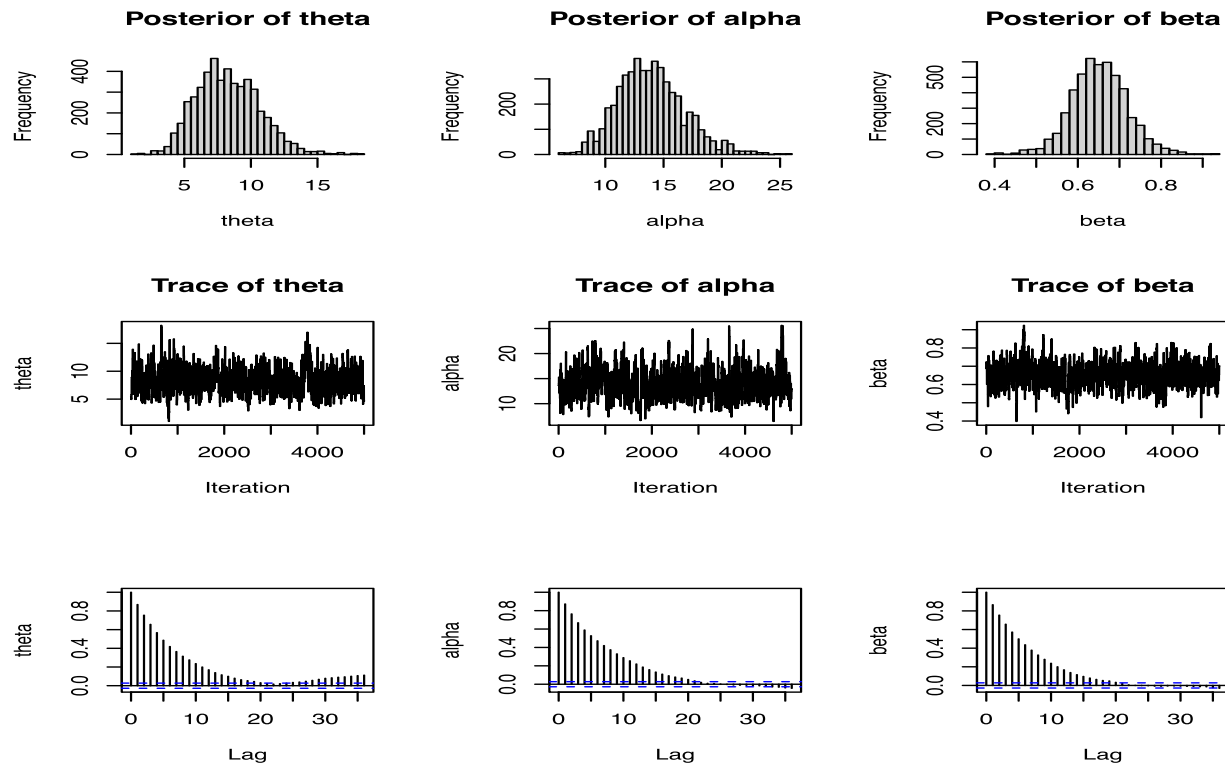


Figure 35: MCMC convergence results using data V

8 Conclusion

Using the E-X methodology, this paper defines a flexible model based on the inverse Weibull distribution. The proposed model was termed the E-IW distribution. The paper investigates several distributional and mathematical characterizations of the proposed E-IW model. The estimation of model parameters is specified using various techniques, such as classical and Bayesian methods. Further, several indicator risk measures are calculated for the recommended distribution. The parameters of the novel probability model are estimated using classical and Bayesian estimation techniques, and the performance of these tools is evaluated through an extensive simulation study. The applicability of the E-IW model is demonstrated using several datasets of radiation and financial data. The results illustrate that the E-IW model is the best-fit model compared to the well-known distributions of competitors in fitting the two radiation data sets.

For future studies, we can extend the proposed E-IW distribution to a bivariate model and apply it to survival, engineering, and biology data. Future research could also focus on improving the theoretical analysis of its tail behavior and stochastic ordering, as well as exploring machine learning-based estimation techniques.

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