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## INFORMATION

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# Integrability, Conservation Laws by Variational Analysis and Lump Solitons for (2+1) Fourth-Order Biharmonic Equations with Quantum Field Applications

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## ABSTRACT

We study the integrability via conservation laws and discuss the non-linearity of the fourth-order biharmonic equations in  $(2 + 1)$  dimensions related to quantum field models based on the potential functions  $h(u)$ . Lie symmetry reduction is performed, and the forms of the invariant solutions are presented, including travelling wave solutions. Variational analysis has been performed based on the various potential functions  $h(u)$ . Corresponding Euler-Lagrange equations and conservation laws are investigated by Noether's theorem and presented in the form of conserved vectors. The obtained conserved flows define energy, momentum and flow dynamics supporting the system integrability. Furthermore, detailed lump and breather solutions are presented for each potential  $h(u)$  using Bilinear forms illustrating various localized and oscillatory field characteristics.

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## 1 Introduction

The study of non-linear partial differential equations (NLPDEs) particularly those of higher order has attracted the attention of researchers from a wide range of scientific disciplines. Particularly in the domains of beam and plate theories, fluid dynamics, image processing, phase field models, quantum mechanics, elasticity, and shell theory. Fourth-order partial differential equations (PDEs) are extensively employed (see [1,2]) to study several important physical phenomenon. To aid in the detailed analysis and solution of these equations a wide variety of analytical and numerical techniques have

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been discussed and presented in the scientific literature. The non-linear biharmonic partial differential equation is given by

$$u_{tt} - \Delta^2 u + h'(u) = 0, \tag{1}$$

equivalently

$$u_{tt} - u_{xxxx} - u_{yyyy} - 2u_{xxyy} + h'(u) = 0, \tag{2}$$

describes the growth of a scalar field  $u(x, y, t)$  controlled by high-order spatial diffusion and a non-linear restorative force, where  $\Delta^2 u = u_{xxxx} + 2u_{xxyy} + u_{yyyy}$  denotes the biharmonic operator in two dimensions containing fourth-order spatial derivatives. This operator frequently appears in simulations of higher-order diffusion effects and the non-linear term  $h'(u) = k(u)$  introduces potential gradient representing phenomena such as phase transitions or non-local interactions depending upon the selection of the function  $h(u)$ . The equation can be understood as a non-linear biharmonic wave equation with combining affect of dispersive elasticity and non-linear field theory. These models are pertinent in diverse physical contexts encompassing non-linear elasticity and pattern generation in thin films and phase-field models within materials science. The Lagrangian  $\mathcal{L}$  for this fourth-order (2 + 1) dimensional biharmonic equation is

$$\mathcal{L}(u, u_t, \Delta u) = \frac{1}{2}u_t^2 - \frac{1}{2}(\Delta u)^2 + h(u). \tag{3}$$

equivalently

$$\mathcal{L}(u, u_t, \Delta u) = \frac{1}{2}u_t^2 - \frac{1}{2}u_{xx}^2 - \frac{1}{2}u_{yy}^2 - u_{xy}^2 + h(u). \tag{4}$$

The variational analysis of (1) via Energy and Momentum Conservation is studied in [3] for arbitrary potential function  $h(u)$ .

The non-linear potential functions  $h(u)$  including quadratic to higher order describe fundamental physical processes in quantum mechanics and quantum field theory. The quadratic potential  $h(u) = \frac{1}{2}m^2u^2$  has a linear mass term represents free scalar fields. The quartic expression  $h(u) = \frac{1}{4}au^4$  signifies the self-interaction fundamental to  $\phi^4$  theory. The function values  $h(u) = \frac{1}{2}m^2u^2 - \frac{1}{4}anu^4$  integrates both elements to represent spontaneous symmetry breaking with degenerate vacua, a fundamental mechanism in Higgs field theory. Furthermore  $h(u) = -\frac{1}{2}m^2u^2 + \frac{1}{4}anu^4$  characterizes metastable vacuum states pertinent to quantum tunnelling and cosmological phase transitions. The potential function  $h(u) = \frac{1}{6}cu^6$  incorporates higher-order non-linearity facilitating intricate field configurations like Q-balls and non-topological solitons. The potential function considered in this study are presented in Table 1.

**Table 1:** Potential functions  $h(u)$  and  $k(u)$  and their physical role

$h(u)$	$h'(u) = k(u)$	Physics context	Field theory role
$\frac{1}{2}m^2u^2$	$m^2u$	Free scalar field	Linear Klein–Gordon
$\frac{1}{4}au^4$	$au^3$	Self-interaction	$\phi^4$ theory

(Continued)

**Table 1 (continued)**

$h(u)$	$h'(u) = k(u)$	Physics context	Field theory role
$\frac{1}{2}m^2u^2 - \frac{1}{4}au^4$	$m^2u - au^3$	Spontaneous symmetry breaking	Higgs-like mechanism
$-\frac{1}{2}m^2u^2 + \frac{1}{4}au^4$	$-m^2u + au^3$	False vacuum	Quantum tunnelling, metastability
$\frac{1}{6}cu^6$	$cu^5$	High non-linearity	$\phi^6$ theory

These function values when employed in a (2+1)-dimensional fourth-order biharmonic wave Eq. (1) provide a variety of localized and topological structures modelling quantum coherent states, symmetry-breaking dynamics and soliton phenomena.

Menglian Li, Omid Nikan, Wenlin Qiu, and Da Xu investigated the two-dimensional Burgers-type equation relevant to turbulent fluid flows [4]. Bluman et al. presented time-independent Lie symmetries of the fourth-order linear equation [5], whereas Masood et al. [6] examined a non-linear and non-homogeneous fourth-order equation via Lie symmetries approach, Noether theorem and discussed conservation laws by association. Additionally, in [7], the Lie symmetries of the equation  $\frac{d^4y}{dx^4} = h(y)$  are examined. We examine the fourth-order time-dependent partial differential equation and determined its Lie and Noether symmetries along with the associated conservation laws and presented solitons based on the function values  $h(u)$ .

Variational symmetries or the invariance of an underlying variational functional are thoroughly investigated in the cited references [8–10]. The invariance technique is used to simplify the system of equations into an equivalent more manageable form. Additionally double reductions of the differential equation are derived by Noether symmetries [11–14]. Conserved vectors are computed with the renowned Noether theorem [13,14].

The biharmonic equation has been extensively studied in elasticity theory, especially for simulating the deformation of thin plates, with seminal contributions by Love [15] and Timoshenko and Woinowsky-Krieger [16]. Variational approaches have been examined for non-linear biharmonic problems, employing critical point theory, Sobolev embeddings, and energy minimization techniques to establish the existence and qualitative characteristics of solutions [17]. These methodologies generally focus on existence, multiplicity, and regularity instead than closed-form solutions. The latest study indicate symmetry analysis has attracted interest for studying higher-order dispersive systems [18,19] however numerous findings are still numerical or dependent on perturbation techniques.

This study examines Lie symmetries, Noether symmetries through the Noether method and travelling wave transformations to obtain soliton solutions of non-linear biharmonic-type partial differential equations derived from variational principles related to the function values of  $h(u)$ . Exact solutions are derived encompassing quartic, modulated and cubic travelling waves. These solutions provide a direct physical interpretation and enhance the current literature by offering manageable interpretable models for wave propagation, elastic deformation and associated phenomena.

Analytical and numerical methods [20–22] are important and play main role for the study of non-linear PDEs in physics and engineering. Techniques such as the exp-function method, inverse

scattering transform, and sub-ODE approaches enable construction of exact and soliton solutions for models including KdV, non-linear Schrödinger, biharmonic and optical media equations [23,24]. Lie symmetry analysis provide systematic approach to invariant solutions and to understanding wave propagation and transport in fluids and dispersive media. Recent progress [25–28] extends these approaches to higher-dimensional and generalized non-linear equations, integrating stability, sensitivity, and computational analysis with exact solution retrieval.

Further studies highlight non-linear wave phenomena via soliton interactions, lumps, rogue waves and higher-dimensional structures [26,27] in physically important partial differential equations. Advanced tools including modified F-expansion, generalized Kudryashov, auxiliary equation, Painlevé analysis and Hirota’s direct method yield [28] solutions for Hirota-Satsuma-Ito, Fokas, Zoomeron, Bogoyavlenskii, KP-type and non-linear Schrödinger equations. These [29–32] extend the classical multi-soliton concept from foundational theory to wave interactions, bifurcations, and chaos.

Lump, breather, and rogue wave solutions [33] along with their interactions dominate studies of higher-dimensional non-linear evolution equations. Kadomtsev-Petviashvili work [34] on two dimensional solitons extends to Burgers-type, breaking soliton and generalized KP equations uncovering rich solitary-rogue interactions. Recent analyses [35,36] reveal lumps, breathers and rational solutions in stochastic DNA and liquid crystals. Lump-line soliton interactions, bifurcations, chaos and sensitivity further support the study of multidimensional non-linear wave dynamics.

This paper presents the study of fourth dimensional non-linear biharmonic equation including its Lie symmetry, variational and soliton analyses. Exact invariant solutions and integrability is discussed using variational symmetries and conservation laws. In Sections 2 and 3, the invariance properties are presented based on Lie and Noether symmetries. In Sections 3 and 4, the conservation laws and corresponding integrability properties are investigated based on the potential functions  $h(u)$ . Soliton solutions including lump, breather solitons are depicted in Section 5, with graphical illustrations based on potential  $h(u)$ .

## 2 Analysis of the NLPDE via Lie Symmetry Approach

The Lagrangian  $\mathcal{L}$  (3) that yields a NLPDE (1) can be expressed [8–10] in the following form

$$u_{tt} - \Delta^2 u + k(u) = u_{tt} - u_{xxxx} - 2u_{xyy} - u_{yyy} + k(u) = 0, \text{ where } h'(u) = k(u). \quad (5)$$

The symmetry generator admitted by PDE (1) defined as follows

$$\mathcal{V} = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}. \quad (6)$$

has the corresponding one-parameter Lie group of point transformations

$$\bar{x} = x + \varepsilon \xi^1 + O(\varepsilon^2),$$

$$\bar{y} = y + \varepsilon \xi^2 + O(\varepsilon^2),$$

$$\bar{t} = t + \varepsilon \tau + O(\varepsilon^2),$$

$$\bar{u} = u + \varepsilon \eta + O(\varepsilon^2).$$

The higher-order prolongations of the vector field  $\mathcal{V}$  in (6) have been defined as follows

$$\mathcal{V}^{[4]} = \text{pr}^{(4)} \mathcal{V} = \mathcal{V} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_u \frac{\partial}{\partial u_{tt}} + \zeta_{xxxx} \frac{\partial}{\partial u_{xxxx}} + \zeta_{xyyy} \frac{\partial}{\partial u_{xyyy}} + \zeta_{yyyy} \frac{\partial}{\partial u_{yyyy}} + \dots \quad (7)$$

which has the corresponding Lie invariance condition given by

$$\mathcal{V}^{[4]} (u_{xxxx} + 2u_{xyyy} + u_{yyyy} - u_{tt} - k(u))_{u_{xxxx}+2u_{xyyy}+u_{yyyy}=u_{tt}+k(u)} = 0. \quad (8)$$

and take us to the expanded form under the action of prolongation defined by

$$(\zeta_{xxxx} + 2\zeta_{xyyy} + \zeta_{yyyy} - \zeta_{tt} - \eta k'(u))_{u_{xxxx}+2u_{xyyy}+u_{yyyy}-u_{tt}=k(u)} = 0. \quad (9)$$

If  $k$  depends on  $u$  arbitrarily then from Eq. (9) then we obtain the following symmetry generators

$$\mathcal{V}_1 = \frac{\partial}{\partial x}, \mathcal{V}_2 = \frac{\partial}{\partial y}, \mathcal{V}_3 = \frac{\partial}{\partial t}, \mathcal{V}_4 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \quad (10)$$

A travelling wave reduction arise from taking a one parameter linear combination of translation symmetries

$$\mathcal{V}_1 = \frac{\partial}{\partial x}, \quad \mathcal{V}_2 = \frac{\partial}{\partial y}, \quad \mathcal{V}_3 = \frac{\partial}{\partial t}. \quad (11)$$

These symmetries span a three dimensional abelian Lie algebra with commutators

$$[\mathcal{V}_i, \mathcal{V}_j] = 0.$$

An optimal system classifies one dimensional sub-algebras based on the general element  $\mathcal{V} = a_1 \mathcal{V}_1 + a_2 \mathcal{V}_2 + a_3 \mathcal{V}_3$  up to equivalence under the adjoint action yielding representatives

$$\mathcal{V}^1 = \mathcal{V}_1 = \frac{\partial}{\partial x}, \quad (12)$$

$$\mathcal{V}^2 = \mathcal{V}_1 + \lambda \mathcal{V}_2 = \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y}, \quad (\lambda \in \mathbb{R}) \quad (13)$$

$$\mathcal{V}^3 = \mathcal{V}_1 + \mu \mathcal{V}_3 = \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial t}. \quad (\mu \in \mathbb{R}) \quad (14)$$

Reduction of (5) under  $\mathcal{V} = a_1 \mathcal{V}_1 + a_2 \mathcal{V}_2 + a_3 \mathcal{V}_3$  for which the respective invariant variables are

$$\alpha = \frac{ta_2 - xa_1}{a_2}, \beta = -\frac{xa_3 - ya_2}{a_2}. \quad (15)$$

Invariant solution is thus of the form

$$w(x, y, t) = F\left(-\frac{xa_3 - ya_2}{a_2}, \frac{ta_2 - a_1x}{a_2}\right), \quad (16)$$

is the solution of the Eq. (5) if  $F(\alpha, \beta)$  satisfies the fourth order reduced ODE

$$\begin{aligned} & - (a_2^2 + a_3^2)^2 \left( \frac{\partial^4}{\partial \beta^4} F(\alpha, \beta) \right) - 2a_1^2 (a_2^2 + 3a_3^2) \left( \frac{\partial^4}{\partial \beta^2 \partial \alpha^2} F(\alpha, \beta) \right) - 4a_1 a_3 (a_2^2 + a_3^2) \left( \frac{\partial^4}{\partial \beta^3 \partial \alpha} F(\alpha, \beta) \right) \\ & - a_1^4 \left( \frac{\partial^4}{\partial \alpha^4} F(\alpha, \beta) \right) - 4a_3 a_1^3 \left( \frac{\partial^4}{\partial \beta \partial \alpha^3} F(\alpha, \beta) \right) + a_2^4 \left( \frac{\partial^2}{\partial \alpha^2} F(\alpha, \beta) + h(F) \right) = 0, \end{aligned} \quad (17)$$

where  $F = F(\alpha, \beta)$ . Similarly,  $\mathcal{V}^2 = \mathcal{V}_1 + \lambda\mathcal{V}_2$  has invariant solution

$$w(x, y, t) = F(-\lambda x + y, t), \quad (18)$$

which is the invariant solution of (5) provided it satisfies the reduced ODE

$$\frac{\partial^2}{\partial \beta^2} F(\alpha, \beta) - \left( \frac{\partial^4}{\partial \beta^4} F(\alpha, \beta) \right) \lambda^4 - \left( \frac{\partial^4}{\partial \alpha^4} F(\alpha, \beta) \right) - 2 \left( \frac{\partial^4}{\partial \beta^2 \partial \alpha^2} F(\alpha, \beta) \right) \lambda^2 + k(F(\alpha, \beta)) = 0, \quad (19)$$

where the invariants are

$$\alpha = t, \quad \beta = -\lambda x + y. \quad (20)$$

The invariant solution corresponding to  $\mathcal{V}^3 = \mathcal{V}_1 + \mu\mathcal{V}_3$  has the following form

$$w(x, y, t) = F(y, -\lambda x + t), \quad \text{where } \alpha = y, \beta = -\lambda x + t, \quad (21)$$

and satisfies

$$\frac{\partial^2}{\partial \beta^2} F(\alpha, \beta) - \left( \frac{\partial^4}{\partial \beta^4} F(\alpha, \beta) \right) \lambda^4 - \left( \frac{\partial^4}{\partial \alpha^4} F(\alpha, \beta) \right) - 2 \left( \frac{\partial^4}{\partial \beta^2 \partial \alpha^2} F(\alpha, \beta) \right) \lambda^2 + k(F) = 0. \quad (22)$$

### 3 Analysis of the NLPDE via Variational Symmetry Approach

The Lagrangian  $\mathcal{L}$  (3) and the corresponding vector field  $\mathcal{V}$  given in (6) is often known [5,10–12] as a variational symmetry if the functional  $\mathcal{J}$  given by

$$\mathcal{J} = \iiint \mathcal{L}(u, u_t, \Delta u) dt dx dy \quad (23)$$

is invariant under the vector field  $\mathcal{V}$ . The vector field  $\mathcal{V}$  with zero gauge satisfies the following invariance criterion

$$\mathcal{V}^{[2]} \mathcal{L} + (D_t \tau + D_x \xi^1 + D_y \xi^2) \mathcal{L} = 0. \quad (24)$$

The Noether symmetries for the case in which the gauge is taken as zero are referred to as variational symmetries. The vector field  $\mathcal{V}$  prolonged to the second-order is written as follows

$$\mathcal{V}^{[2]} = \mathcal{V} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_y \frac{\partial}{\partial u_y} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \zeta_{xy} \frac{\partial}{\partial u_{xy}} + \zeta_{yy} \frac{\partial}{\partial u_{yy}}. \quad (25)$$

For arbitrary  $k(u)$  we obtain the following Noether symmetries

$$\mathcal{V}_1 = \frac{\partial}{\partial x}, \mathcal{V}_2 = \frac{\partial}{\partial y}, \mathcal{V}_3 = \frac{\partial}{\partial t}. \quad (26)$$

The conservation laws are expressed in the notion of conserved vectors, written as follows

$$D_t T^t + D_x T^x + D_y T^y = 0. \quad (27)$$

holds on solutions of the Euler Lagrange equation

$$\mathcal{E}(L) = \frac{\delta L}{\delta u} = 0, \quad (28)$$

subject to  $\mathcal{E}(t, x, y, u, u_t, u_x, u_y, u_{xx}, \dots) = 0$ . Next we present the Noether theorem and use it to calculate the conserved vectors. The operator  $\mathcal{V}$  in (6) is a variational symmetry, in particular with zero gauge, if it optimizes the following function

$$\mathcal{J} = \iiint \mathcal{L} dt dx dy. \quad (29)$$

Moreover, the vector  $T^x$ ,  $T^y$  and  $T^t$  are written as follows

$$\begin{aligned} T^t &= L\tau + w \frac{\partial L}{\partial u_t}, \\ T^x &= L\xi + w \left( \frac{\partial L}{\partial u_x} - D_x \frac{\partial L}{\partial u_{xx}} - D_y \frac{\partial L}{\partial u_{xy}} \right) + D_x w \frac{\partial L}{\partial u_{xx}} + D_y w \frac{\partial L}{\partial u_{xy}}, \\ T^y &= L\zeta + w \left( \frac{\partial L}{\partial u_y} - D_x \frac{\partial L}{\partial u_{xy}} - D_y \frac{\partial L}{\partial u_{yy}} \right) + D_x w \frac{\partial L}{\partial u_{xy}} + D_y w \frac{\partial L}{\partial u_{yy}}, \end{aligned} \quad (30)$$

where  $w = \eta - u_x \xi^1 - u_y \xi^2 - u_t \tau$  is the characteristic of  $\mathcal{V}$ .

### Conservation Laws: For Free Scalar Field $h(u) = \frac{1}{2}m^2u^2$

This section presents the conservation laws associated with the free scalar field characterized by the potential  $h(u) = \frac{1}{2}m^2u^2$ .

i.  $w = -u_x$  linear momentum in  $x$

$$\begin{aligned} T^x &= -\frac{1}{2}u_t^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2 - \frac{1}{2}m^2u^2 + \frac{1}{2}u_x u_{xyy} + u_x u_{xxx}, \\ T^y &= -\left( u_{xy} u_{yy} + \frac{1}{2}u_{xx} u_{xy} - u_x u_{yyy} - \frac{1}{2}u_x u_{xxy} \right), \\ T^t &= u_t u_x. \end{aligned} \quad (31)$$

ii.  $w = -u_y$  linear momentum in  $y$

$$\begin{aligned} T^x &= \frac{1}{2}u_t^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2 + \frac{1}{2}m^2u^2 - u_y u_{yyy} - \frac{1}{2}u_y u_{xxy}, \\ T^y &= -\left( \frac{1}{2}u_{xy} u_{yy} + u_{xx} u_{xy} - \frac{1}{2}u_y u_{xyy} - u_y u_{xxx} \right), \\ T^t &= u_t u_y. \end{aligned} \quad (32)$$

iii.  $w = -u_t$  energy conservation law

$$\begin{aligned} T^x &= u_{yy} u_{yt} + \frac{1}{2}u_{xy} u_{xt} - u_t u_{yyy} - \frac{1}{2}u_t u_{xxy}, \\ T^y &= -\left( \frac{1}{2}u_{xy} u_{yt} + u_{xx} u_{xt} - \frac{1}{2}u_t u_{xyy} - u_t u_{xxx} \right), \\ T^t &= \frac{1}{2} (u_t^2 + u_{xx}^2 + u_{yy}^2 + u_{xy}^2 - m^2u^2). \end{aligned} \quad (33)$$

### Conservation Laws: For Self-Interaction $\phi^4$ -Theory $h(u) = \frac{1}{4}au^4$

This section outlines the conservation rules related to the self-interacting scalar field characterized by the  $\phi^4$ -theory, wherein the potential is defined as  $h(u) = \frac{1}{4}au^4$ .

i.  $w = -u_x$  linear momentum in  $x$

$$\begin{aligned} T^x &= -\frac{1}{2}u_t^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2 - \frac{1}{4}au^4 + \frac{1}{2}u_x u_{xyy} + u_x u_{xxx}, \\ T^y &= -\left(u_{xy}u_{yy} + \frac{1}{2}u_{xx}u_{xy} - u_x u_{yyy} - \frac{1}{2}u_x u_{xxy}\right), \\ T^t &= u_t u_x. \end{aligned} \quad (34)$$

ii.  $w = -u_y$  linear momentum in  $y$

$$\begin{aligned} T^x &= \frac{1}{2}u_t^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2 + \frac{1}{4}au^4 - u_y u_{yyy} - \frac{1}{2}u_y u_{xxy}, \\ T^y &= -\left(\frac{1}{2}u_{xy}u_{yy} + u_{xx}u_{xy} - \frac{1}{2}u_y u_{xyy} - u_y u_{xxx}\right), \\ T^t &= u_t u_y. \end{aligned} \quad (35)$$

iii.  $w = -u_t$  energy conservation law

$$\begin{aligned} T^x &= u_{yy}u_{yt} + \frac{1}{2}u_{xy}u_{xt} - u_t u_{yyy} - \frac{1}{2}u_t u_{xxy}, \\ T^y &= -\left(\frac{1}{2}u_{xy}u_{yt} + u_{xx}u_{xt} - \frac{1}{2}u_t u_{xyy} - u_t u_{xxx}\right), \\ T^t &= \frac{1}{2}\left(u_t^2 + u_{xx}^2 + u_{yy}^2 + u_{xy}^2 - \frac{1}{2}au^4\right). \end{aligned} \quad (36)$$

**Conservation Laws: For Higgs-Like Mechanism  $h(u) = \frac{1}{2}m^2u^2 - \frac{1}{4}au^4$**

This section provides the conservation laws associated with a Higgs-like process characterized by the scalar field potential  $h(u) = \frac{1}{2}m^2u^2 - \frac{1}{4}au^4$  which signifies spontaneous symmetry breaking.

i.  $w = -u_x$  linear momentum in  $x$ :

$$\begin{aligned} T^x &= -\frac{1}{2}u_t^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2 - \frac{1}{2}m^2u^2 + \frac{1}{4}au^4 + \frac{1}{2}u_x u_{xyy} + u_x u_{xxx}, \\ T^y &= -\left(u_{xy}u_{yy} + \frac{1}{2}u_{xx}u_{xy} - u_x u_{yyy} - \frac{1}{2}u_x u_{xxy}\right), \\ T^t &= u_t u_x. \end{aligned} \quad (37)$$

ii.  $w = -u_y$  linear momentum in  $y$ :

$$\begin{aligned} T^x &= \frac{1}{2}u_t^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2 + \frac{1}{2}m^2u^2 - \frac{1}{4}au^4 - u_y u_{yyy} - \frac{1}{2}u_y u_{xxy}, \\ T^y &= \frac{1}{2}u_{yy}u_{xy} + u_{xx}u_{xy} - \frac{1}{2}u_y u_{xyy} - u_y u_{xxx}, \\ T^t &= u_t u_y. \end{aligned} \quad (38)$$

iii.  $w = -u_t$  energy conservation law:

$$\begin{aligned} T^x &= -\frac{1}{2}u_{xy}u_{yt} - u_{xx}u_{xt} + \frac{1}{2}u_t u_{xyy} + u_t u_{xxx}, \\ T^y &= -\left(u_{yy}u_{yt} + \frac{1}{2}u_{xy}u_{xt} - u_t u_{yyy} - \frac{1}{2}u_t u_{xxy}\right), \\ T^t &= \frac{1}{2}(u_t^2 + u_{xx}^2 + u_{yy}^2 + u_{xy}^2) - \frac{1}{2}m^2 u^2 + \frac{1}{4}a u^4. \end{aligned} \quad (39)$$

**Conservation Laws: For Quantum Tunnelling  $h(u) = -\frac{1}{2}m^2 u^2 + \frac{1}{4}a u^4$**

This section summarizes the conservation laws pertinent to quantum tunnelling processes represented by a scalar field with a double-well potential  $h(u) = -\frac{1}{2}m^2 u^2 + \frac{1}{4}a u^4$  which defines metastable states and barrier penetration effects.

i.  $w = -u_x$  linear momentum in  $x$ :

$$\begin{aligned} T^x &= -\frac{1}{2}u_t^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2 + \frac{1}{2}m^2 u^2 - \frac{1}{4}a u^4 + \frac{1}{2}u_x u_{xyy} + u_x u_{xxx}, \\ T^y &= -\left(u_{xy}u_{yy} + \frac{1}{2}u_{xx}u_{xy} - u_x u_{yyy} - \frac{1}{2}u_x u_{xxy}\right), \\ T^t &= u_t u_x. \end{aligned} \quad (40)$$

ii.  $w = -u_y$  linear momentum in  $y$ :

$$\begin{aligned} T^x &= -\frac{1}{2}u_{yy}u_{xy} - u_{xx}u_{xy} + \frac{1}{2}u_y u_{xyy} + u_y u_{xxx}, \\ T^y &= -\frac{1}{2}u_t^2 + \frac{1}{2}u_{xx}^2 - \frac{1}{2}u_{yy}^2 + \frac{1}{2}m^2 u^2 - \frac{1}{4}a u^4 + u_y u_{yyy} + \frac{1}{2}u_y u_{xxy}, \\ T^t &= u_t u_y. \end{aligned} \quad (41)$$

iii.  $w = -u_t$  energy conservation law:

$$\begin{aligned} T^x &= -\frac{1}{2}u_{xy}u_{yt} - u_{xx}u_{xt} + \frac{1}{2}u_t u_{xyy} + u_t u_{xxx}, \\ T^y &= -u_{yy}u_{yt} - \frac{1}{2}u_{xy}u_{xt} + u_t u_{yyy} + \frac{1}{2}u_t u_{xxy}, \\ T^t &= \frac{1}{2}(u_t^2 + u_{xx}^2 + u_{yy}^2 + u_{xy}^2) + \frac{1}{2}m^2 u^2 - \frac{1}{4}a u^4. \end{aligned} \quad (42)$$

**Conservation Laws: For High Non-Linearity  $\phi^6$ -Theory  $h(u) = \frac{1}{6}c u^6$**

This section finds the conservation laws for a highly non-linear scalar field model characterized by the  $\phi^6$  theory with the potential  $h(u) = \frac{1}{6}c u^6$  which encapsulates more strong self-interaction effects than lower-order models.

i.  $w = -u_x$  linear momentum in  $x$ :

$$\begin{aligned} T^x &= -\frac{1}{2}u_t^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2 + \frac{1}{2}m^2u^2 - \frac{1}{6}cu^6 + \frac{1}{2}u_xu_{xyy} + u_xu_{xxx}, \\ T^y &= -\left(u_{yy}u_{xy} + \frac{1}{2}u_{xx}u_{xy} - u_xu_{yyy} - \frac{1}{2}u_xu_{xxy}\right), \\ T^t &= u_tu_x. \end{aligned} \quad (43)$$

ii.  $w = -u_y$  linear momentum in  $y$ :

$$\begin{aligned} T^x &= -\frac{1}{2}u_{yy}u_{xy} - u_{xx}u_{xy} + \frac{1}{2}u_yu_{xyy} + u_yu_{xxx}, \\ T^y &= -\frac{1}{2}u_t^2 + \frac{1}{2}u_{xx}^2 - \frac{1}{2}u_{yy}^2 + \frac{1}{2}m^2u^2 - \frac{1}{6}cu^6 + u_yu_{yyy} + \frac{1}{2}u_yu_{xxy}, \\ T^t &= u_tu_y. \end{aligned} \quad (44)$$

iii.  $w = -u_t$  energy conservation law:

$$\begin{aligned} T^x &= -\frac{1}{2}u_{xy}u_{yt} - u_{xx}u_{xt} + \frac{1}{2}u_tu_{xyy} + u_tu_{xxx}, \\ T^y &= -u_{yy}u_{yt} - \frac{1}{2}u_{xy}u_{xt} + u_tu_{yyy} + \frac{1}{2}u_tu_{xxy}, \\ T^t &= \frac{1}{2}(u_t^2 + u_{xx}^2 + u_{yy}^2 + u_{xy}^2) + \frac{1}{2}m^2u^2 - \frac{1}{6}cu^6. \end{aligned} \quad (45)$$

The conserved vectors  $(T^t, T^x, T^y)$  with corresponding characteristics  $w$  satisfies  $D_tT^t + D_xT^x + D_yT^y = 0$  on solutions of the underlying PDE. For example, for the case  $k(u) = cu^5$  with  $w = -u_x$  a direct differentiation gives the factorization

$$D_tT^t + D_xT^x + D_yT^y = u_x(u_{tt} - u_{xxxx} - 2u_{xxyy} - u_{yyyy} + k(u)).$$

Hence on the solution set of the given PDE, we get

$$u_{tt} - u_{xxxx} - 2u_{xxyy} - u_{yyyy} + k(u) = 0 = D_tT^t + D_xT^x + D_yT^y.$$

So the divergence expression is satisfied.

#### 4 Integrability via Conservation Laws

Conservation laws play an important role in establishing integrability [5,10,12] of non-linear PDEs by providing infinitely many independent quantities preserved along solutions. For equations like the KdV, a hierarchy of conservation laws derived from Noether theorem describes complete integrability as their densities and fluxes generate a Lax pair that confirms the integrability. In this context, the conserved vector  $(T^t, T^x, T^y)$  with characteristic  $w = -u_x$  reduced the PDE (5) into a simpler form, which is integrable and easier to solve.

##### **Integrability: For Free Scalar Field $k(u) = m^2u$**

This section presents the integrability associated with potential  $k(u) = m^2u$ .

**i.**  $w = -u_x$ : For this case PDE (5) under invariance characteristic  $u_x = 0$  simplifies to

$$u_{tt} - u_{yyyy} + k(u) = 0, \quad (46)$$

reducing the model into the space  $u = u(t, y)$ . The invariant solutions corresponding to Noether symmetries  $\mathcal{V}_2$  and  $\mathcal{V}_3$  are given, respectively

$$u(x, y, t)_{\mathcal{V}_2} = F_1(x) \sin(\sqrt{m}y) + F_2(x) \cos(\sqrt{m}y) + F_3(x) e^{\sqrt{m}y} + F_4(x) e^{-\sqrt{m}y}, \quad (47)$$

$$u(x, y, t)_{\mathcal{V}_3} = F_5(x) \sin(mt) + F_6(x) \cos(mt). \quad (48)$$

**ii.**  $w = -u_y$ : Invariant solutions under  $y$ -translations  $\mathcal{V}_2 = \partial_y$  satisfy  $u_y = 0$ . The substitution  $w = -u_y = 0$  reduces the PDE (5) to

$$u_{tt} - u_{xxxx} + k(u) = 0. \quad (49)$$

The invariant solutions corresponding to Noether symmetries  $\mathcal{V}_1$  and  $\mathcal{V}_3$  are given, respectively

$$u(x, y, t)_{\mathcal{V}_1} = F_3(y) e^{(-\frac{i}{2})2^{3/4}\sqrt{m}x} + F_4(y) e^{(\frac{-1}{2})2^{3/4}\sqrt{m}x} + F_5(y) e^{(\frac{i}{2})2^{3/4}\sqrt{m}x} + F_6(y) e^{\frac{1}{2}2^{3/4}\sqrt{m}x}, \quad (50)$$

$$u(x, y, t)_{\mathcal{V}_3} = F_5(x) \sin(mt) + F_6(x) \cos(mt). \quad (51)$$

### Integrability: For Higgs-Like Mechanism $k(u) = m^2u - au^3$

This section presents the integrability associated with potential  $k(u) = m^2u - au^3$ .

**i.**  $w = -u_x$ : For this case PDE (5) under invariance characteristic  $u_x = 0$  simplifies to

$$u_{tt} - u_{yyyy} + k(u) = 0, \quad (52)$$

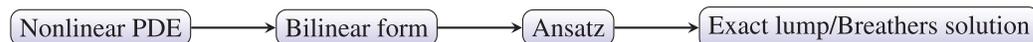
reducing the model into the space  $u = u(t, y)$ . The invariant solutions corresponding to Noether symmetries  $\mathcal{V}_1$  and  $\mathcal{V}_2$  are given, respectively

$$u(x, y, t)_{\mathcal{V}_1} = 1/4 \frac{\sqrt{5}\sqrt{6}m}{\left(\cosh\left(i/4\sqrt{5}mt + 1/4\sqrt{m}y + i/4y\sqrt{m} - c_1\right)\right)^2 \sqrt{a}}, \quad (53)$$

$$u(x, y, t)_{\mathcal{V}_2} = \frac{F_1(x) \sqrt{2}m}{\sqrt{(F_1(x))^2 a + 2m^2 - a}} \text{JacobiSN} \left( 1/2 \frac{\left(i\sqrt{2}\sqrt{-2m^2 + at} + 2F_2(x)\right) \sqrt{2}m}{\sqrt{(F_1(x))^2 a + 2m^2 - a}}, \right. \\ \left. \frac{iF_1(x) \sqrt{a}}{\sqrt{-2m^2 + a}} \right). \quad (54)$$

## 5 Soliton Solution Analysis

In this section we derive lump and breather solutions for various functions values  $h(u)$  discussed above using Hirota bilinear method.



The Hirota bilinear method provides a powerful framework for constructing lump solutions to non-linear PDEs transforming complex non-linear dynamics into manageable bilinear forms. By employing the logarithmic derivative transformation  $u = 2(\ln f)_x$  the method systematically linearises

the equation, enabling the use of polynomial ansatz that guarantee spatial localization and rational decay.

**Lump Solution: For Free Scalar Field**  $h(u) = \frac{1}{2}m^2u^2$

The following ansatz is carried out in the non-linear partial differential Eq. (1), as illustrated in [32,33], in the succeeding form

$$u = 2(\ln f(x, t))_x, \tag{55}$$

and obtain the following form

$$\begin{aligned} & m^2 f^4 f_x + 2f^2 f_t f_x - f^3 f_{tt} f_x - 24f_y^4 f_x + 36ff_y^2 f_{yy} f_x - 6f^2 f_{yy}^2 f_x - 8f^2 f_{yy} f_{yy} f_x + f^3 f_{yyy} f_x - 48f_y^2 f_x^3 + 12ff_y f_x^3 \\ & - 24f_x^5 - 2f^3 f_{xt} f_x + f^4 f_{xtt} + 24ff_y^3 f_{xy} - 24f^2 f_{yy} f_{yy} f_{xy} + 4f^3 f_{yyy} f_{xy} + 72ff_y f_x^2 f_{xy} - 24f^2 f_x f_x^2 - 12f^2 f_y^2 f_{xyy} \\ & + 6f^3 f_{yy} f_{xyy} - 12f^2 f_x^2 f_{xyy} + 4f^3 f_{xy} f_{xyy} - f^4 f_{xyyy} + 36f_y^2 f_x f_{xx} - 12f^2 f_{yy} f_x f_{xx} + 60ff_x^3 f_{xx} - 24f^2 f_y f_{xy} f_{xx} \\ & + 6f^3 f_{xy} f_{xxx} - 30ff_x^2 f_{xx}^2 - 24f^2 f_{yy} f_x f_{xxx} + 12f^3 f_{xy} f_{xxx} + 6f^3 f_x f_{xxx} - 4f^2 f_y^2 f_{xxx} + 2f^3 f_{yy} f_{xxx} \\ & - 20f^2 f_x^2 f_{xxx} + 10f^3 f_{xx} f_{xxx} + 4f^3 f_y f_{xxx} - 2f^4 f_{xxx} + 5f^3 f_x f_{xxx} - f^4 f_{xxxx}. \end{aligned} \tag{56}$$

We now assess lump solutions utilizing this bilinear form. For LS, we assume a general quadratic function  $f$  as

$$\begin{cases} f = \psi_1^2 + \psi_2^2 + k_9, \\ \psi_1 = k_1 x + k_2 y + k_3 t + k_4, \\ \psi_2 = k_5 x + k_6 y + k_7 t + k_8, \end{cases} \tag{57}$$

where  $k_i, 1 \leq i \leq 9$ , are the real parameters to be found. Utilizing Eqs. (57) into (56) to obtain the particular equations that produce values for the coefficients.

When  $k_1 = k_6 = 0$ , the following solutions are obtained:

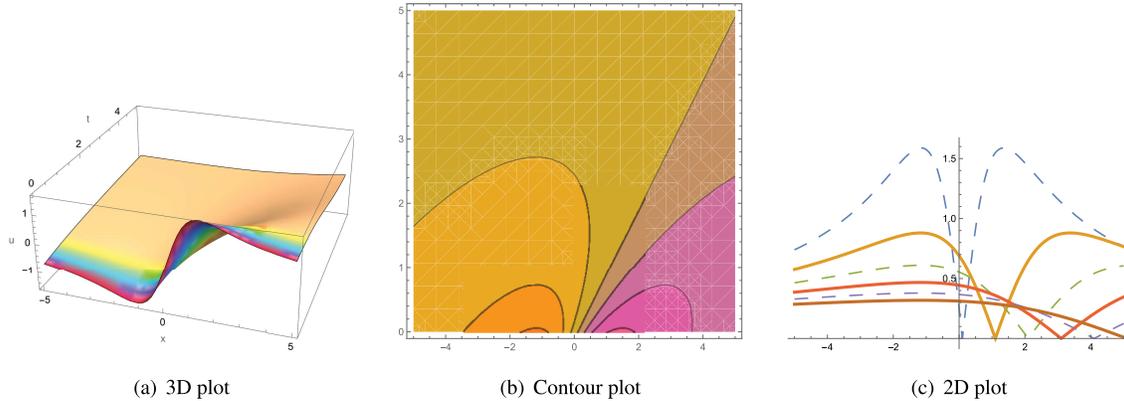
$$k_4 = \sqrt{\frac{4}{3}k_8^2 m^2 + \frac{1}{6}k_7^2}, k_9 = -\frac{25}{3}k_8^2.$$

By using these parameters in Eq. (56), yields

$$u = \frac{4k_5(k_8 + k_7 t + k_5 x)}{-\left(\frac{25k_8^2}{3}\right) + (k_8 + k_7 t + k_5 x)^2 + \left(\sqrt{\frac{k_7^2/6 + (4k_8^2 m^2)/3}{m}} + k_3 t + k_2 y\right)^2}. \tag{58}$$

The solution given in (58) is presented in Fig. 1.

The parameters  $k_2 = 0.5, k_3 = 5, k_5 = -5, k_7 = 5, k_8 = 0.5, m = 0.1$  control the shape, localization, direction and dynamics of the wave profiles shown in the 3D contour and 2D plots. These parameters collectively govern the interplay of non-linearity, dispersion, amplitude and direction dictating whether solutions manifest as smooth travelling waves, localized lumps or interacting structures.



**Figure 1:** The wave profiles for analytical solution of Eq. (58), when  $k_2 = 0.5, k_3 = 5, k_5 = -5, k_7 = 5, k_8 = 0.5, m = 0.1$ : (a) 3D plot; (b) Contour plot; (c) 2D plot

**Breather Solution: For Free Scalar Field  $h(u) = \frac{1}{2}m^2u^2$**

For breather solution, we choose  $f$  as:

$$\begin{cases} f(x, y, t) = m_1 e^{q\psi_1(x,y,t)} + e^{-q\psi_1(x,y,t)} + m_2 \cos(q_1\psi_2(x, y, t)) + a_8, \\ \psi_1 = a_1x + a_2y + a_3t + a_4, \\ \psi_2 = a_5x + a_6y + a_7t, \end{cases} \tag{59}$$

where  $m_1, m_2,$  and  $a_i$  are real parameters to be found. By inserting Eq. (59) into Eq. (56), some equations that provide coefficient values are found.

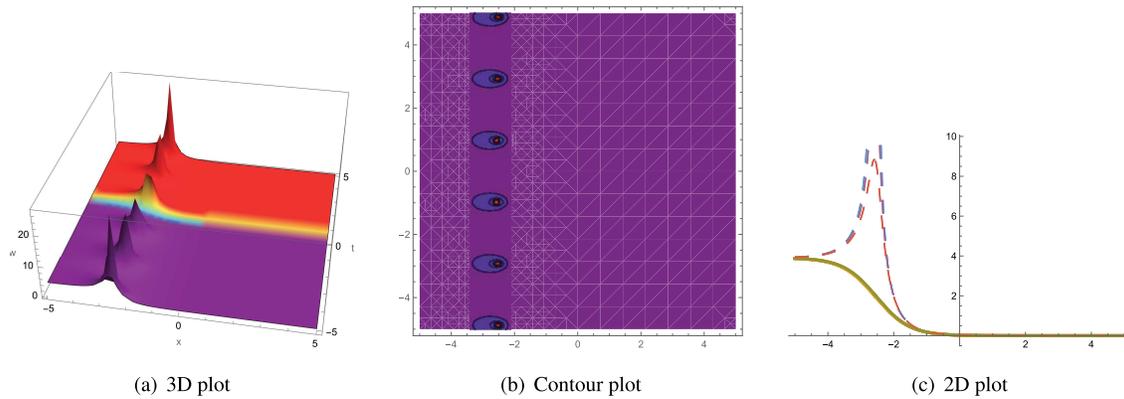
When  $a_2 = a_6 = 0$ , the subsequent solutions are obtained:

$$a_1 = \frac{\left(\frac{1}{12}a_7^2q_1^2 + \frac{7}{12}m^2\right)^{\frac{1}{4}}}{q}, a_3 = \frac{\sqrt{-\left(\frac{11}{3}a_7^2q_1^2 + \frac{5}{3}m^2\right)}}{2q}, a_5 = 0, m_1 = 0.$$

By inserting the above values in Eq. (55), yields

$$u = -\frac{2e^{-q\left(a_4 + \frac{\sqrt{\left(\frac{5m^2}{3} + \frac{11a_7^2q_1^2}{3}\right)}t + \frac{\left(\frac{7m^2}{12} + \frac{a_7^2q_1^2}{12}\right)^{1/4}}{q}x\right)}}{\left(\frac{7m^2}{12} + \frac{a_7^2q_1^2}{12}\right)^{1/4}}}{a_8 + e^{-q\left(a_4 + \frac{\sqrt{\left(\frac{5m^2}{3} + \frac{11a_7^2q_1^2}{3}\right)}t + \frac{\left(\frac{7m^2}{12} + \frac{a_7^2q_1^2}{12}\right)^{1/4}}{q}x\right)}} + m_2 \cos(a_7q_1t)} \tag{60}$$

Fig. 2 displays the solution obtained in (60).



**Figure 2:** The wave profiles for analytical solution of Eq. (60), when  $q = 5.5, m_2 = 1, q_1 = 0.2, a_4 = 0.5, a_7 = 1, a_8 = 8.5, m = 5$ : (a) 3D plot; (b) Contour plot; (c) 2D plot

**Lump Solution: For Self-Interaction  $\phi^4$ -Theory  $h(u) = \frac{1}{4}au^4$**

For second function, we have bilinear form

$$\begin{aligned}
 & 2f^2f_t^2f_x - f^3f_{tt}f_x - 24f_y^4f_x + 36ff_y^2f_{yy}f_x - 6f^2f_{yy}^2f_x - 8f^2f_{yy}f_{yy}f_x + f^3f_{yyy}f_x + 4af^2f_x^3 - 48f_y^2f_x^3 + 12ff_{yy}f_x^3 \\
 & - 24f_x^5 - 2f^3f_{xt} + f^4f_{xt} + 24ff_y^3f_{xy} - 24f^2f_{yy}f_{yy}f_{xy} + 4f^3f_{yyy}f_{xy} + 72ff_y^2f_x^2f_{xy} - 24f^2f_x^2f_{xy}^2 - 12f^2f_y^2f_{xy}^2 \\
 & + 6f^3f_{yy}f_{xy} - 12f^2f_x^2f_{xy}^2 + 4f^3f_{yy}f_{xy} - f^4f_{xy}^2 + 36f_y^2f_x^2f_{xx} - 12f^2f_{yy}f_x^2f_{xx} + 60ff_x^3f_{xx} - 24f^2f_y^2f_x^2f_{xx} \\
 & + 6f^3f_{xy}f_{xx} - 30ff_x^2f_{xx}^2 - 24f^2f_y^2f_x^2f_{xx} + 12f^3f_{xy}f_{xx} + 6f^3f_x^2f_{xy} - 4f^2f_y^2f_{xxx} + 2f^3f_{yy}f_{xxx} - 20f^2f_x^2f_{xxx} \\
 & + 10f^3f_{xx}f_{xxx} + 4f^3f_y^2f_{xxx} - 2f^4f_{xxx} + 5f^3f_x^2f_{xxx} - f^4f_{xxxx}
 \end{aligned} \tag{61}$$

Utilizing Eqs. (57) into (61) to obtain the particular equations that produce values for the coefficients.

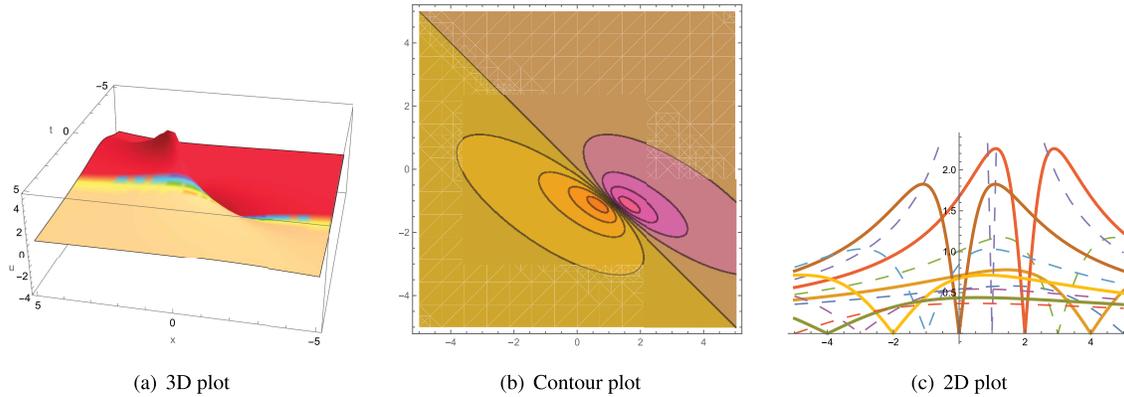
When  $k_1 = k_6 = 0$ , the following solutions are obtained:

$$k_3 = \frac{\sqrt{7}}{3}k_7, a = -\frac{160k_7^2}{297k_5^2}, k_8 = 0.$$

By using these parameters in Eq. (56), yields

$$u = \frac{4k_5(k_7t + k_5x)}{k_9 + (k_7t + k_5x)^2 + \left(k_4 + \frac{\sqrt{7}}{3}k_7t + k_2y\right)^2}. \tag{62}$$

The graphical representation of the solution (62) is shown in Fig. 3.



**Figure 3:** The wave profiles for analytical solution of Eq. (62), when  $k_2 = 5, k_4 = 5, k_5 = 5, k_7 = 5, k_9 = 5$ : (a) 3D plot; (b) Contour plot; (c) 2D plot

**Breathers Solution: For Self-Interaction  $\phi^4$ -Theory  $h(u) = \frac{1}{4}au^4$**

$$\begin{cases} f(x, y, t) = m_1 e^{q\psi_1(x,y,t)} + e^{-q\psi_1(x,y,t)} + m_2 \cos(q_1\psi_2(x, y, t)) + a_8, \\ \psi_1 = a_1x + a_2y + a_3t + a_4, \\ \psi_2 = a_5x + a_6y + a_7t, \end{cases} \quad (63)$$

where  $m_1, m_2$ , and  $a_i$  are real parameters to be found. By inserting Eqs. (63) into (61), some equations that provide coefficient values are found.

When  $a_2 = a_6 = 0$ , the subsequent solutions are obtained:

$$a = \frac{11a_1^4q^2 + a_3^2}{4a_1^2}, m_1 = 0.$$

By inserting the above values in Eq. (55), yields

$$u = \frac{2(-a_1qe^{-q(a_4+a_3t+a_1x)} - a_5m_2q_1 \sin(q_1(a_7t + a_5x)))}{a_8 + e^{-q(a_4+a_3t+a_1x)} + m_2 \cos(q_1(a_7t + a_5x))}. \quad (64)$$

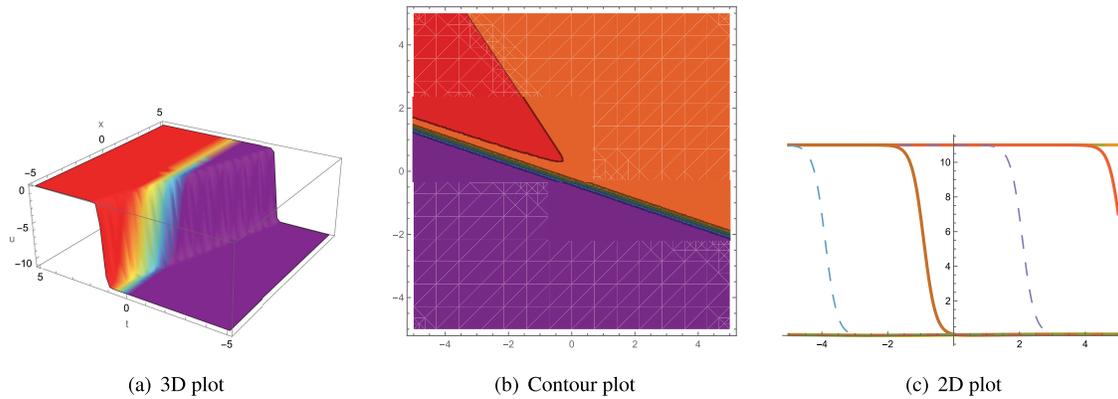
In Fig. 4 we depict the solution corresponding to solution (64).

**Lump Solutions: For Higgs-Like Mechanism  $h(u) = \frac{1}{2}m^2u^2 - \frac{1}{4}au^4$**

For third function, we have bilinear form

$$\begin{aligned} & -m^2f^4f_x - 2f^2f_t^2f_x + f^3f_t f_x + 24f_{y,x}^4f_x - 36ff_{y,y}^2f_{y,x}f_x + 6f^2f_{y,y}^2f_x + 8f^2f_{y,y}f_{y,y}f_x - f^3f_{y,y}f_x + 4af^2f_x^3 + 48f_y^2f_x^3 \\ & - 12ff_{y,y}f_x^3 + 24f_x^5 + 2f^3ff_{x,t} - f^4f_{x,t} - 24ff_{y,y}^3f_x + 24f^2f_{y,y}f_{y,y}f_x - 4f^3f_{y,y}f_{y,y} - 72ff_{y,x}f_x^2f_{y,y} + 24f^2f_x^2f_{y,y}^2 \\ & + 12f^2f_y^2f_{y,y} - 6f^3f_{y,y}f_{y,y} + 12f^2f_x^2f_{y,y} - 4f^3f_{y,y}f_{y,y} + f^4f_{y,y}f_{y,y} - 36ff_{y,y}^2f_{x,x} + 12f^2f_{y,y}f_{x,x} - 60ff_x^3f_{x,x} \\ & + 24f^2f_{y,y}f_{x,x} - 6f^3f_{y,y}f_{x,x} + 30f^2f_x^2f_{x,x} + 24f^2f_x f_{x,x} - 12f^3f_{x,x}f_{x,x} - 6f^3f_x f_{x,x} + 4f^2f_y^2f_{x,x} - 2f^3f_{y,y}f_{x,x} \\ & + 20f^2f_x^2f_{x,x} - 10f^3f_{x,x}f_{x,x} - 4f^3f_{y,y}f_{x,x} + 2f^4f_{x,x}f_{y,y} - 5f^3f_x f_{x,x} + f^4f_{x,x}f_{x,x} = 0. \end{aligned} \quad (65)$$

Utilizing Eqs. (57) into (65) to obtain the particular equations that produce values for the coefficients.



**Figure 4:** The wave profiles for analytical solution of Eq. (64), when  $q = 5.5, m_2 = 1, q_1 = 0.2, a_1 = 1, a_3 = 3, a_4 = 0.5, a_5 = 1.5, a_7 = 1, a_8 = 8.5$ : (a) 3D plot; (b) Contour plot; (c) 2D plot

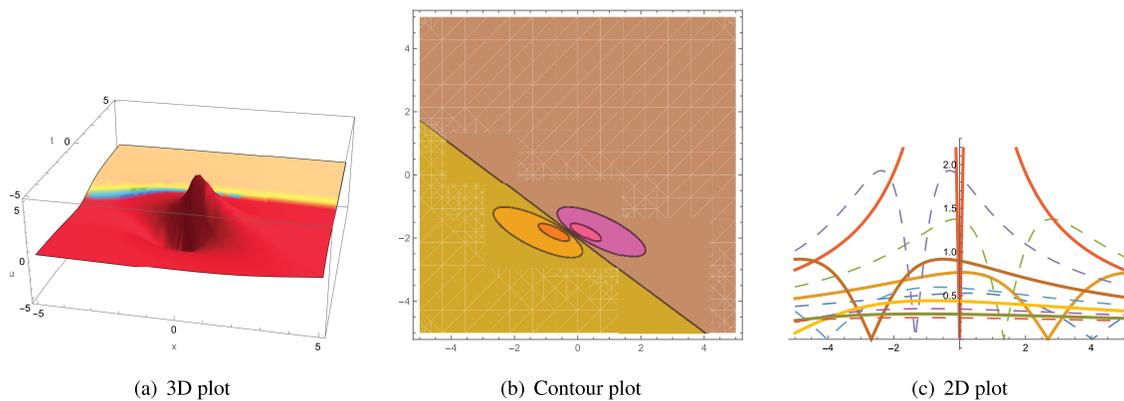
When  $k_1 = k_6 = 0$ , the following solutions are obtained:

$$k_3 = \frac{\sqrt{7}}{3}k_7, k_5 = \sqrt{\frac{10}{9a}}k_7, m = 0.$$

By using these parameters in Eq. (56), yields

$$u = \frac{4\sqrt{10}\sqrt{\frac{1}{a}}k_7 \left( k_8 + k_7t + \frac{\sqrt{10}}{3}\sqrt{\frac{1}{a}}k_7x \right)}{3 \left( k_9 + \left( k_8 + k_7t + \frac{\sqrt{10}}{3}\sqrt{\frac{1}{a}}k_7x \right)^2 + \left( k_4 + \frac{\sqrt{7}}{3}k_7t + k_2y \right)^2 \right)}. \tag{66}$$

The behaviour of the solution (66) is visualized in Fig. 5.



**Figure 5:** The wave profiles for analytical solution of Eq. (66), when  $k_2 = 5, k_4 = 4, k_7 = 2.5, k_8 = 5, k_9 = 0.5, a = 2$ : (a) 3D plot; (b) Contour plot; (c) 2D plot

**Breathers Solution: For Higgs-Like Mechanism**  $h(u) = \frac{1}{2}m^2u^2 - \frac{1}{4}au^4$

$$\begin{cases} f(x, y, t) = m_1 e^{q\psi_1(x,y,t)} + e^{-q\psi_1(x,y,t)} + m_2 \cos(q_1\psi_2(x, y, t)) + a_8, \\ \psi_1 = a_1x + a_2y + a_3t + a_4, \\ \psi_2 = a_5x + a_6y + a_7t, \end{cases} \quad (67)$$

where  $m_1, m_2$ , and  $a_i$  are real parameters to be found. By inserting Eqs. (67) into (65), some equations that provide coefficient values are found.

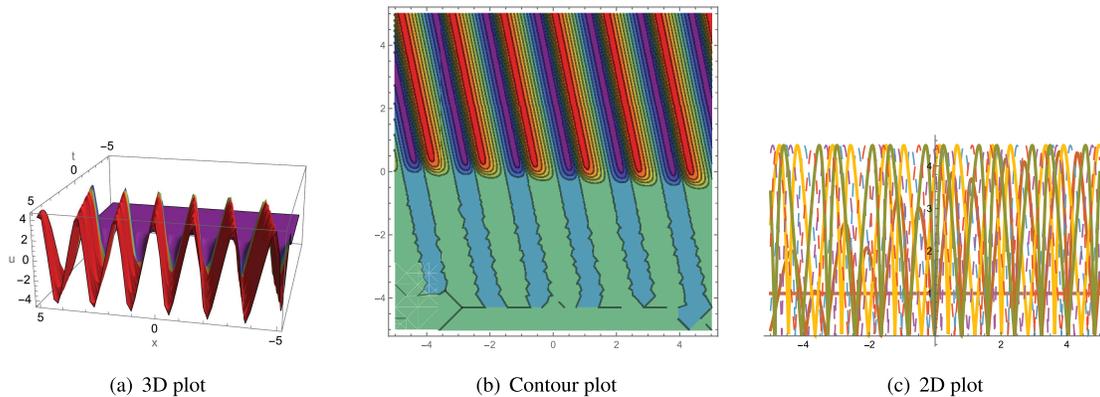
When  $a_2 = a_6 = 0$ , the subsequent solutions are obtained:

$$4a_3 = \frac{\sqrt{-11a_1^4q^4 - 4aa_1^2q^2 + 6m^2}}{q}, m_1 = 0.$$

By inserting the above values in Eq. (55), yields

$$u = \frac{2 \left( -a_1 e^{-q \left( a_4 + \frac{\sqrt{6m^2 - 4aa_1^2q^2 - 11a_1^4q^4}t}{q} + a_1x \right)} q - \frac{m_2 \sqrt{6m^2 - 4aa_1^2q^2 - 11a_1^4q^4} q_1 \sin \left[ q_1 \left( a_7t + \frac{\sqrt{6m^2 - 4aa_1^2q^2 - 11a_1^4q^4}x}{q} \right) \right]}{q} \right)}{a_8 + e^{-q \left( a_4 + \frac{\sqrt{6m^2 - 4aa_1^2q^2 - 11a_1^4q^4}t}{q} + a_1x \right)} + m_2 \cos \left[ q_1 \left( a_7t + \frac{\sqrt{6m^2 - 4aa_1^2q^2 - 11a_1^4q^4}x}{q} \right) \right]}. \quad (68)$$

Fig. 6 illustrates the solution derived in (68).



**Figure 6:** The wave profiles for analytical solution of Eq. (68), when  $q = 5.5, m_2 = 1, q_1 = 0.2, a_1 = 1, a_3 = 3, a_4 = 0.5, a_5 = 1.5, a_7 = 1, a_8 = 8.5$ : (a) 3D plot; (b) Contour plot; (c) 2D plot

**Lump Solutions: For Quantum Tunnelling**  $h(u) = -\frac{1}{2}m^2u^2 + \frac{1}{4}au^4$

For fourth function, we have bilinear form

$$\begin{aligned} & m^2f_x^4f_x - 2f_y^2f_x^2f_x + f_y^3f_xf_x + 24f_y^4f_x - 36ff_y^2f_yf_x + 6f_y^2f_y^2f_x + 8f_y^2f_yf_yf_x - f_y^3f_yf_x - 4af_y^2f_x^3 \\ & + 48f_y^2f_x^3 - 12ff_yf_x^3 + 24f_x^5 + 2f_y^3f_xf_x - f_x^4f_xf_x - 24ff_y^3f_xf_y + 24f_y^2f_yf_yf_x - 4f_y^3f_yf_xf_y - 72ff_yf_x^2f_y \end{aligned}$$

$$\begin{aligned}
 &+ 24f^2 f_x f_{xy}^2 + 12f^2 f_y^2 f_{xyy} - 6f^3 f_{yy} f_{xyy} + 12f^2 f_x^2 f_{xyy} - 4f^3 f_{xy} f_{xyyy} + f^4 f_{xyyyy} - 36ff_y^2 f_x f_{xx} + 12f^2 f_{yy} f_x f_{xx} \\
 &+ -60ff_x^3 f_{xx} 30f^2 f_x f_{xx}^2 + 24f^2 f_y f_x f_{xyx} - 12f^3 f_{xy} f_{xyx} - 6f^3 f_x f_{xyy} + 4f^2 f_y^2 f_{xxx} - 2f^3 f_{yy} f_{xxx} + 20f^2 f_x^2 f_{xxx} \\
 &- 10f^3 f_{xx} f_{xxx} - 4f^3 f_{xy} f_{xyyy} + 2f^4 f_{xyyy} - 5f^3 f_x f_{xxxx} + f^4 f_{xxxxx} = 0
 \end{aligned} \tag{69}$$

Utilizing Eqs. (57) into (69) to obtain the particular equations that produce values for the coefficients.

When  $k_1 = k_6 = 0$ , the following solutions are obtained:

$$k_5 = \sqrt{-\frac{3k_3 k_4^2 m^2 - 3k_3 k_8^2 m^2 - 6k_4 k_7 k_8 m^2 - k_3^3 + 3k_3 k_7^2}{2ak_3}}, \tag{70}$$

$$k_9 = \frac{-42k_3 k_4^2 m^2 - 18k_3 k_8^2 m^2 - 36k_4 k_7 k_8 m^2 + 9k_3^3 - 7k_3 k_7^2}{6k_3 m^2}. \tag{71}$$

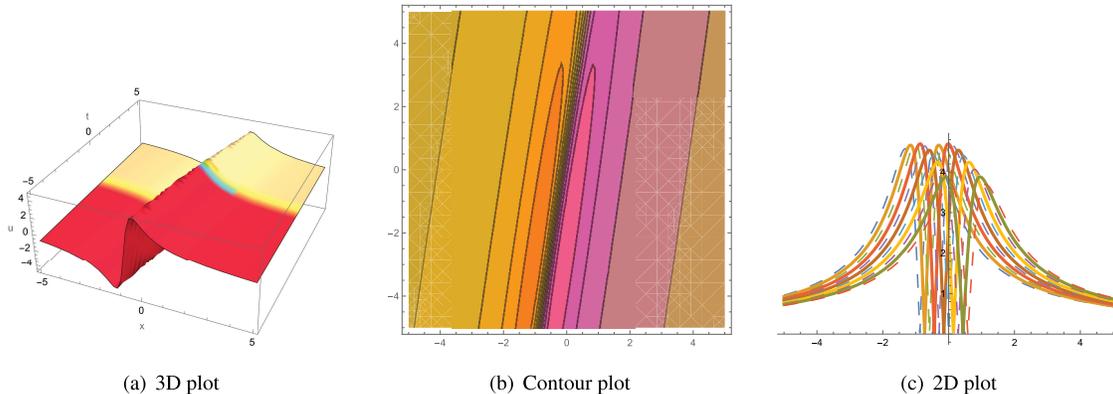
By using these parameters in Eq. (56), yields

$$u = \frac{2\sqrt{2}C (k_8 + k_7 t + \frac{Cx}{\sqrt{2}})}{\frac{9k_3^3 - 7k_3 k_7^2 - 42k_3 k_4^2 m^2 - 36k_4 k_7 k_8 m^2 - 18k_3 k_8^2 m^2}{6k_3 m^2} + (k_8 + k_7 t + \frac{Cx}{\sqrt{2}})^2 + (k_4 + k_3 t + k_2 y)^2}, \tag{72}$$

where

$$C = \sqrt{-\frac{-k_3^3 + 3k_3 k_7^2 + 3k_3 k_4^2 m^2 - 6k_4 k_7 k_8 m^2 - 3k_3 k_8^2 m^2}{ak_3}}.$$

The solution obtained from (72) is plotted in Fig. 7.



**Figure 7:** The wave profiles for analytical solution of Eq. (72), when  $k_2 = 5, k_3 = 1.5, k_4 = 4, k_7 = -5, k_8 = 5, m = 5, a = -4$ : (a) 3D plot; (b) Contour plot; (c) 2D plot

**Breathers Solution: For Quantum Tunnelling**  $h(u) = -\frac{1}{2}m^2 u^2 + \frac{1}{4}au^4$

$$\begin{cases}
 f(x, y, t) = m_1 e^{q\psi_1(x,y,t)} + e^{-q\psi_1(x,y,t)} + m_2 \cos(q_1 \psi_2(x, y, t)) + a_8, \\
 \psi_1 = a_1 x + a_2 y + a_3 t + a_4, \\
 \psi_2 = a_5 x + a_6 y + a_7 t,
 \end{cases} \tag{73}$$

where  $m_1, m_2$ , and  $a_i$  are real parameters to be found. By inserting Eqs. (73) into (69), some equations that provide coefficient values are found.

When  $a_2 = a_6 = 0$ , the subsequent solutions are obtained:

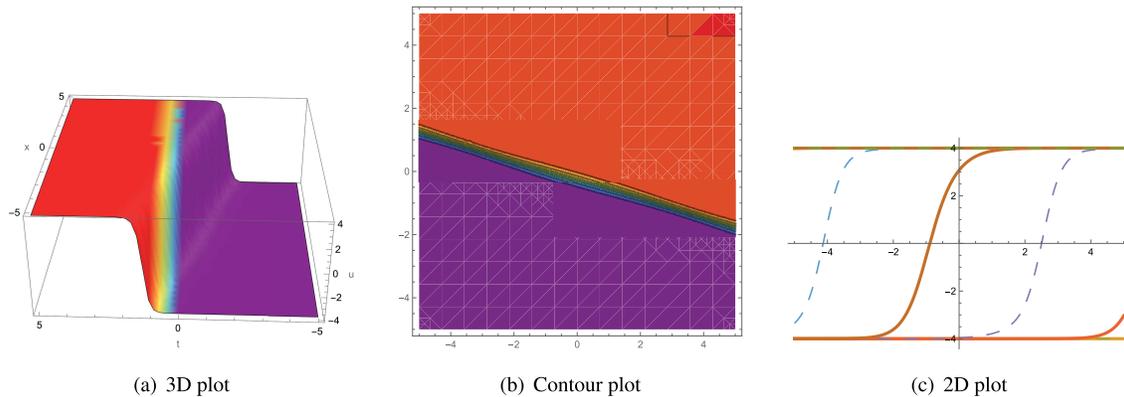
$$a_3 = \frac{\sqrt{a_1^4 q^4 + 8aa_1^2 q^2 - 4m^2}}{q},$$

$$m_1 = -\frac{2(6a_1^4 q^4 + 2aa_1^2 q^2 + m^2) a_8^2}{-24a_1^4 q^4 + 68aa_1^2 q^2 - 29m^2}.$$

By inserting the above values in Eq. (55), yields

$$u = \frac{2(-a_1 e^{-q(a_4 + \frac{\sqrt{\zeta} t}{q} + a_1 x)})_q - \frac{2a_1 a_8^2 e^{q(a_4 + \frac{\sqrt{\zeta} t}{q} + a_1 x)}}{-29m^2 + 68aa_1^2 q^2 - 24a_1^4 q^4} (m^2 + 2aa_1^2 q^2 + 6a_1^4 q^4) - a_5 m_2 q_1 \sin[q_1(a_7 t + a_5 x)]}{a_8 + e^{-q(a_4 + \frac{\sqrt{\zeta} t}{q} + a_1 x)} - \frac{2a_8^2 e^{q(a_4 + \frac{\sqrt{\zeta} t}{q} + a_1 x)}}{-29m^2 + 68aa_1^2 q^2 - 24a_1^4 q^4} (m^2 + 2aa_1^2 q^2 + 6a_1^4 q^4) + m_2 \cos[q_1(a_7 t + a_5 x)]}. \tag{74}$$

where  $\zeta = -4m^2 + 8aa_1^2 q^2 + a_1^4 q^4$ . A visualization of the solution (74) is provided in Fig. 8.



**Figure 8:** The wave profiles for analytical solution of Eq. (74), when  $q = 0.5, m_2 = 0.5, q_1 = 0.5, a_1 = 4, a_7 = 0.4, a_5 = 2, a_4 = 3, a_8 = 1, a = 1, m = 1$ : (a) 3D plot; (b) Contour plot; (c) 2D plot

**Lump Solutions: For High Non-Linearity  $\phi^6$ -Theory  $h(u) = \frac{1}{6}cu^6$**

For fifth function, we have bilinear form

$$2f^2 f_t^2 f_x - f^3 f_t f_x - 24f_y^4 f_x + 36ff_y^2 f_{yy} f_x - 6f^2 f_{yy}^2 f_x - 8f^2 f_{y\ y} f_{y\ y} f_x + f^3 f_{y\ y\ y\ y} f_x - 48f_y^2 f_x^3 + 12ff_{y\ y} f_x^3 - 2f^3 f_{x\ t} + f^4 f_{x\ t} + 24ff_y^3 f_{xy} - 24f^2 f_{y\ y} f_{xy} + 4f^3 f_{y\ y} f_{xy} + 72ff_{y\ y} f_{xy}^2 - 24f^2 f_{x\ y} f_{xy}^2 - 12f^2 f_{y\ y} f_{xy}^2 + 6f^3 f_{y\ y} f_{xy} - 12f^2 f_{x\ y} f_{xy} + 4f^3 f_{y\ y} f_{xy} - f^4 f_{xy}^2 + 36ff_y^2 f_{x\ x} - 12f^2 f_{y\ y} f_{x\ x} + 60ff_{x\ x} f_{x\ x} - 24f^2 f_{y\ y} f_{x\ x} + 6f^3 f_{x\ y} f_{x\ x} - 30f^2 f_{x\ x} f_{x\ x} - 24f^2 f_{y\ y} f_{x\ y} + 12f^3 f_{x\ y} f_{x\ y} + 6f^3 f_{x\ y} f_{x\ y} - 4f^2 f_{x\ y}^2 + 2f^3 f_{y\ y} f_{x\ x} - 20f^2 f_{x\ y} f_{x\ x} + 10f^3 f_{x\ y} f_{x\ x} + 4f^3 f_{y\ y} f_{x\ y} - 2f^4 f_{x\ y} f_{x\ y} + 5f^3 f_{x\ y} f_{x\ x} - f^4 f_{x\ x} f_{x\ x} - 24f_x^5 + 16cf_x^5 = 0. \tag{75}$$

Utilizing Eqs. (57) into (75) to obtain the particular equations that produce values for the coefficients.

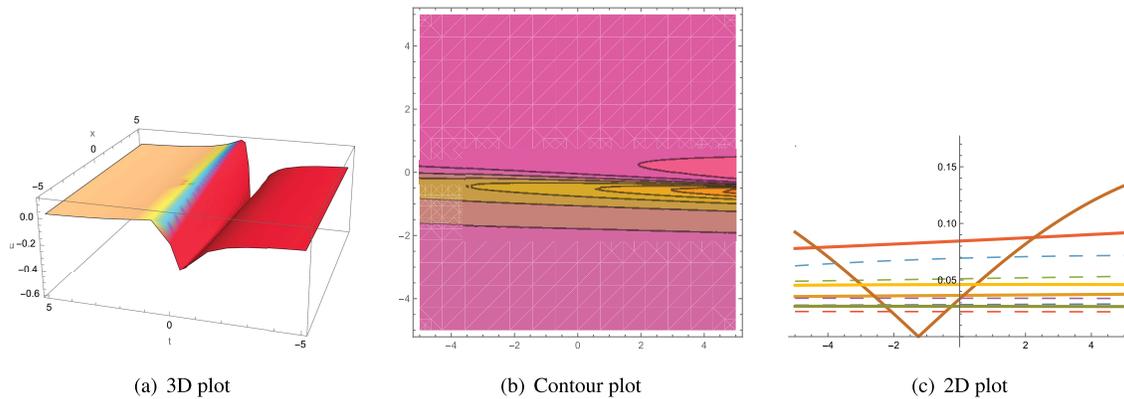
When  $k_1 = k_6 = 0$ , the following solutions are obtained:

$$k_3 = \sqrt{\frac{3}{5}}k_7, k_9 = -\frac{5(12k_2^4 - 24k_2^2k_5^2 + k_3^2k_8^2 + 12k_5^4)}{k_3^2}.$$

By using these parameters in Eq. (56), yields

$$u = \frac{4k_5(k_8 + k_7t + k_5x)}{\left(\frac{25(12k_2^4 - 24k_2^2k_5^2 + 12k_5^4 + \frac{3k_7^2k_8^2}{5})}{3k_7^2}\right) + (k_8 + k_7t + k_5x)^2 + \left(k_4 + \sqrt{\frac{3}{5}}k_7t + k_2y\right)^2}, \quad (76)$$

The lump solution in (76) is presented graphically in Fig. 9.



**Figure 9:** The wave profiles for analytical solution of Eq. (76), when  $k_2 = 0.5, k_4 = 5, k_5 = 0.4, k_7 = 8, k_8 = 0.5$ : (a) 3D plot; (b) Contour plot; (c) 2D plot

### Breathers Solution For High Non-Linearity $\phi^6$ -Theory $h(u) = \frac{1}{6}cu^6$

$$\begin{cases} f(x, y, t) = m_1 e^{q\psi_1(x,y,t)} + e^{-q\psi_1(x,y,t)} + m_2 \cos(q_1\psi_2(x, y, t)) + a_8, \\ \psi_1 = a_1x + a_2y + a_3t + a_4, \\ \psi_2 = a_5x + a_6y + a_7t, \end{cases} \quad (77)$$

where  $m_1, m_2$ , and  $a_i$  are real parameters to be found. By inserting Eqs. (77) into (69), some equations that provide coefficient values are found.

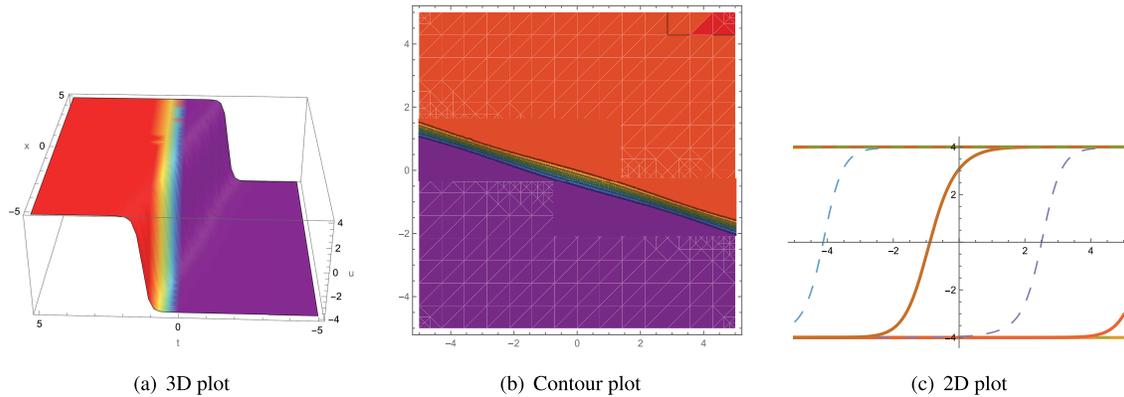
When  $a_2 = a_6 = 0$ , the subsequent solutions are obtained:

$$m_1 = -\frac{1}{8} \frac{a_8^2(11a_1^4q^2 + a_3^2)}{10a_1^4cq^2 - 4a_1^4q^2 + a_3^2}, m_2 = 0.$$

By inserting the above values in Eq. (55), yields

$$u = \frac{2 \left( -a_1 e^{-q(a_4+a_3t+a_1x)} q - \frac{a_1 a_8^2 e^{q(a_4+a_3t+a_1x)} q (a_3^2 + 11a_1^4 q^2)}{8 (a_3^2 - 4a_1^4 q^2 + 10a_1^4 c q^2)} \right)}{a_8 + e^{-q(a_4+a_3t+a_1x)} - \frac{a_8^2 e^{q(a_4+a_3t+a_1x)} (a_3^2 + 11a_1^4 q^2)}{8 (a_3^2 - 4a_1^4 q^2 + 10a_1^4 c q^2)}}. \tag{78}$$

Fig. 10 shows the solution corresponding to Eq. (78).



**Figure 10:** The wave profiles for analytical solution of Eq. (78), when  $q = 0.5, m_2 = 0.5, q_1 = 0.5, a_1 = 4, a_3 = 0.4, a_4 = 3, a_8 = 1, c = -10$ : (a) 3D plot; (b) Contour plot; (c) 2D plot

Singular solutions in non-linear differential equations represent special solutions that often arise from envelopes of the general solution family or points where standard derivation fails such as division by zero. They capture physically critical behaviours like blow-up phenomena, shock formation or caustics in wave propagation which regular solutions fail to produce.

## 6 Conclusions

The study developed an aligned framework for the analysis of (2 + 1)-dimensional fourth-order biharmonic equations via variational analysis and Noether theorem. Conservation laws were established for various potentials  $h(u)$  validating the model integrability. Detailed lump and breather solutions were derived for quadratic, quartic and higher order non-linearities, demonstrating complex localized and oscillatory phenomena. The findings connect Noether symmetry with conserved flows through the variational principle and soliton dynamics illustrating lump and breathers type solutions incorporating non-linearity with quantum field applications based on potential function  $h(u)$  values. For the future, more potential functions  $h(u)$  could be studied for the biharmonic equation based on their physical properties.

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Visualization, Writing Original Draft, Writing Review Editing. All authors reviewed the results and approved the final version of the manuscript.

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**Ethics Approval:** Not applicable.

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