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Radially convergent flow in heterogeneous porous media

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Abstract. We present an analytical solution for apparent effective transmissivity under radially convergent steady state flow conditions, produced by constant pumping from a single well of finite radius, r_w. Apparent effective transmissivity, T_e, is defined as the value that relates the expected values of flow and head gradient at a certain location. The domain is two-dimensional, of annular shape, and the size of the pumping well is explicitly taken into account. The solution for the steady state heads is obtained by solving the perturbed flow equation and substituting it into Darcy's law to obtain a consistent second-order expansion for T_e. We show that apparent effective transmissivity is a scalar for any choice of isotropic covariance model, with an expression given in integral form. Our main result is that T_e in a heterogeneous, statistically isotropic random field, under radial steady state flow conditions, is a monotonic increasing function of r (distance from the well) that rises from the harmonic mean of the point transmissivity values (close to the well) and tends asymptotically towards the geometric mean (far from the well). The asymptotic value is reached at a distance of a few integral scales (1.5–2 for the Gaussian model and 3–5 for the exponential one). The apparent effective transmissivity versus normalized r curves are in excellent agreement with previously published numerical work carried out using Monte Carlo method.

1. Introduction

The mathematical treatment of groundwater flow and solute transport in heterogeneous media is usually casted in a geostatistical framework. A large number of studies consider the variables that appear in these equations to be regionalized variables, characterized by probability distributions (rather than deterministic values as in the classical approach). The theory of random variables recognizes the impossibility of knowing (or testing) the aquifer in detail, but a few statistical moments of the input variables may be estimated from data at many locations. Under certain simplifying assumptions, this suffices to derive the statistical moments of the output variables (heads, concentrations, etc.). A large body of literature has been devoted to the study of effective hydraulic conductivity, defined as the tensor that relates the expected values of flow and head gradient for given flow conditions, but most of this work has been carried out under uniform, parallel in the mean, flow. Recent reviews on this topic have been given by *Neuman and Orr* [1993] and *Sánchez-Vila et al.* [1996], among others.

Nonuniform flow in saturated porous media has not been frequently addressed in the stochastic hydrogeology literature. This is so despite most field methods to obtain hydraulic parameter values that rely on flow towards a single point (pump tests). Effective conductivity values under radially converging flow were first studied by *Shvidler* [1964] and *Matheron* [1967], who carried out a formal first-order expansion of flow in a two-dimensional annular domain with deterministic constant heads imposed at the inner and outer radii and calculated the expected value and variance of the flux. Their most important conclusion is that a single effective transmissivity value depending only on the statistical properties of the T field cannot be derived (this is a most important difference with the uniform flow case). However, they obtained "apparent" effective transmissivity values depending on the distance to the pumping well. "Apparent" effective transmissivity is defined in this context as the homogeneous T value which would provide, on the average, the same discharge as the heterogeneous formation under the same boundary conditions. In a sense, this definition corresponds to an equivalent parameter rather than to an effective one. Apparent values range from the arithmetic mean, when the well radius tends to zero, to the harmonic mean, when the external radius tends to infinite.

Naff [1991] carries out a complete perturbation solution of a quasi-three-dimensional radial flow towards a well in heterogeneous, statistically anisotropic media, assuming random discharge at the well. He uses a different definition of apparent effective conductivity, taken as the value that relates the expected values of flow and head gradient at a certain location. This a local definition, essentially different to that of *Matheron* [1967]. In the work by Naff, when choosing the statistical structure so as to emulate a two-dimensional flow, T_e becomes the arithmetic mean near the well and the harmonic mean distant from the well.

One of the problems when analyzing radially convergent flow is that it is strongly influenced by the stochasticity associated with the choice of boundary conditions. *Dagan* [1989, equation 5.4.3] finds that close to the well, the effective transmissivity should be equal to the harmonic mean of the point transmissivity values. This is obtained assuming constant deterministic pumping rate, as opposed to *Matheron*'s [1967] boundary conditions (prescribed deterministic head values at two different radii). The analysis by Dagan is based upon the observation that close to the well, we may employ the solution for a homogeneous formation; that is,

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$$q_r = -T\frac{dh}{dr} = -\frac{Q}{2\pi r} \tag{1}$$

where q_r is radial specific discharge, T is (local) transmissivity, h is piezometric head, r is the distance from the well, and Q is the pumping rate. Assuming deterministic Q, dividing (1) by T, and ensemble averaging, we get

$$\frac{d\langle h\rangle}{dr} = \frac{Q}{2\pi r}\left\langle\frac{1}{T}\right\rangle = \frac{Q}{2\pi r T_H} = -\frac{q_r}{T_H} \tag{2}$$

so that the effective transmissivity becomes the harmonic mean (T_H) of the point T values. If, instead, we assume deterministic head and random discharge, we get directly from (1)

$$\langle q_r\rangle = -\langle T\rangle\frac{dh}{dr} = -T_A\frac{dh}{dr} \tag{3}$$

and T_e now becomes the arithmetic mean (T_A). This analysis is only valid close to the well, that is, $r \to r_w$. *Dagan* [1989] further argues that very far from the well, T_e should be the same as obtained for uniform mean flow (i.e., the geometric mean, T_G, in two-dimensional isotropic correlation structures).

Neuman and Orr [1993], while investigating the existence of effective transmissivities under radially converging mean flow in bounded domains, carried out Monte Carlo simulations and analyzed under prescribed deterministic flow the evolution of T_e with respect to distance from the pumping well. They used the same definition for T_e as *Dagan* [1989] and *Naff* [1991]. Their curves (Figures 13 and 14 of *Neuman and Orr* [1993]) clearly agree with the limiting values suggested by Dagan, even for variances of log transmissivity (σ_Y^2) as high as 4.

Other authors use somewhat different approaches in the study of radial flow in heterogeneous media. *Desbarats* [1992] addresses upscaling of transmissivities in a combined analytical-numerical approach. *Durlofsky* [1992] looks at different upscaling techniques for assigning block T values. *Desbarats* [1993] considers flow between an injection and a pumping well. *Gómez-Hernández and Gorelick* [1989] use a power-averaging (empirical) approach to assign T_e values in a complex groundwater flow system, including several wells. *Ababou and Wood* [1990] proposed a modified form of *Matheron*'s [1967] T_e formula in exponential form. *Indelman and Abramovich* [1994] study the nonlocal structure of nonuniform average flows in a very general case, including two- and three-dimensional infinite domains and any type of source term; working in Fourier space, they provide effective conductivity values, as convolution integrals of the conductivity spatial moments, and they find again that the limiting values for T_e in the case of a single point source in a two-dimensional domain should be the ones suggested by *Dagan* [1989].

The objective of this paper is to provide an analytical expression in the real space for the apparent effective transmissivities under radially convergent flow in a finite (annular) domain. We carry out a perturbation analysis of the groundwater flow equation, taking into account the singularity at the well location, and find the solution for T_e as a function of the distance to the well, up to second order in the expansion. In this approach we consider a constant deterministic pumping rate. We shall prove analytically that the expected tangential component of the flow is equal to zero. The radial component, instead, is a monotonic increasing function of r, that rises from the harmonic mean (for small r values) up to the geometric mean (large r) of the point T values. This radial component depends on the well radius (r_w), the distance to the pumping well (r), and the parameters characterizing the heterogeneity of the medium: variance of log T, integral scale (d), and covariance model. In most situations, d is much larger than r_w; then, the radial component of T_e depends only upon the ratio r/d and the variance and covariance models.

Our methodology is tested via two different ways. First, limiting values agree with those obtained by *Dagan* [1989]. Second, we compare the shape of the apparent effective transmissivity versus normalized distance curves with those given by *Neuman and Orr* [1993] for two different values of log T variance and one particular covariance model. Agreement can be considered excellent.

2. The Groundwater Flow Equation

We consider a two-dimensional annular domain, D, with an internal radius, r_w, and an external one, R_{ext}. Saturated, steady state flow is governed by

$$\text{In } D \qquad \nabla\cdot[T(\mathbf{x})\nabla h(\mathbf{x})] = 0 \tag{4}$$

with two Dirichlet-type boundary conditions

$$\text{At } r^* = r_w \qquad h(x, y) = h_w \tag{5}$$

$$\text{At } r^* = R_{\text{ext}} \qquad h(x, y) = h_R \tag{6}$$

where r^* stands for radial distance from the center of the well, and h_w and h_R are head values in the inner and outer radii, respectively. An important parameter is the pumping rate, Q, which is equal to the total flow crossing any cylinder (of radius r); that is,

$$Q = r\int_0^{2\pi} T(r, \theta)\frac{\partial h}{\partial r}(r, \theta)\, d\theta \tag{7}$$

This relationship is simpler if applied at the well. There it becomes

$$Q = 2\pi r_w T_w\left(\frac{\partial h}{\partial r}\right)_{r=r_w} \tag{8}$$

It is important to notice that owing to the randomness of T, it is not possible to impose deterministically h_w, h_R, and Q at the same time. On the basis of the considerations given in the introduction, we decided to prescribe deterministically one head and the total flow, thus following the work by *Dagan* [1989], *Neuman and Orr* [1993], and *Indelman and Abramovich* [1994], among others. The remaining head becomes, then, a random space function (RSF). For convenience, we select h_w as the random value. In a single realization h_w will be a constant value (and we can use it as a boundary condition), but it will vary from one realization to another. These same boundary conditions are used in the works by *Dagan* [1989], *Desbarats* [1992], and *Indelman and Abramovich* [1994]. *Neuman and Orr* [1993] instead assign deterministic head at the inner boundary and constant flux, uniformly distributed, at the outer one; in this case it means considering h_R as an RSF. This difference does not affect the results in T_e; it will be seen later that the important value in the derivations is the difference between h_R and h_w and not their values independently. Another way to show this is by comparing the results of *Desbarats* [1992] and *Neuman and Orr* [1993]. Despite their difference in selecting which head was random, the effective T values computed far from the outer boundary are very similar.

By defining $Y = \ln T$, (4) can be written as

$$\nabla^2 h + \nabla Y \nabla h = 0 \quad (9)$$

As Y is not fully defined at every specific location, (9) cannot be solved directly. Instead, we follow the traditional stochastic approach and regard $Y(\mathbf{x})$ as a stationary RSF. Consequently, (9) becomes a stochastic partial differential equation, $h(\mathbf{x})$ becomes also an RSF, and the solution is sought using a perturbation expansion. For this purpose we decompose $Y(\mathbf{x})$ into its expected value ($\langle Y \rangle$, deterministic, assumed constant) plus a random component (Y', with zero mean). So, we have $Y = \langle Y \rangle + Y'$. Now, we expand formally h in an asymptotic sequence: $h = h^{(0)} + h^{(1)} + h^{(2)} + \cdots$. Substituting into (9), we get

$$\nabla^2 h^{(0)} + \nabla^2 h^{(1)} + \nabla^2 h^{(2)} + \nabla Y' \nabla h^{(0)} + \nabla Y' \nabla h^{(1)} + \cdots = 0 \quad (10)$$

We can select $h^{(i)}$ so that they are the solution of the following system of equations, which can be solved iteratively:

$$\text{In } D \qquad \nabla^2 h^{(0)} = 0 \quad (11)$$

$$\nabla^2 h^{(i)} + \nabla Y' \nabla h^{(i-1)} = 0 \qquad i = 1, n \quad (12)$$

In the selection of the boundary conditions we find an important difference with respect to the parallel flow case. In the classical stochastic approach, all nonhomogeneous boundary conditions are associated with $h^{(0)}$, which in our case would lead to a nondeterministic $h^{(0)}$ function. This would lead to considerable inconveniences in the mathematical developments. Instead, it is much better to work with a deterministic $h^{(0)}$, just by passing the stochasticity in the boundary conditions to the $h^{(1)}$ term. Thus we can select the $h^{(i)}$ functions to satisfy the following boundary conditions:

$$\text{At } r = 1 \qquad h^{(0)} = \langle h_w \rangle$$
$$\text{At } r = R \qquad h^{(0)} = h_R \quad (13)$$

$$\text{At } r = 1 \qquad h^{(1)} = h_w - \langle h_w \rangle$$
$$\text{At } r = R \qquad h^{(1)} = 0 \quad (14)$$

$$\text{At } r = 1 \qquad h^{(i)} = 0$$
$$\text{At } r = R \qquad h^{(i)} = 0 \qquad i = 2, n \quad (15)$$

where we have switched to adimensional polar coordinates (r, θ), with $r = r^*/r_w$ (so that $r = 1$ for $r^* = r_w$). R is equal to R_{ext}/r_w. The solution to (11) under the conditions in (13) is a deterministic function given by

$$\text{In } D \qquad h^{(0)} = \langle h_w \rangle + \frac{h_R - \langle h_w \rangle}{\ln R} \ln r \quad (16)$$

To get the next term in the expansion of h, we consider (12) for $i = 1$. From (16), and noticing that $h^{(0)}$ is just a function of r, we have

$$\text{In } D \qquad \nabla^2 h^{(1)} = -\frac{A}{r} \frac{\partial Y'}{\partial r} \quad (17)$$

In (17), $A = (h_R - \langle h_w \rangle)/\ln R$ is a deterministic constant. The boundary conditions associated with (17) are nonhomogeneous and are given by (14). The solution to this stochastic partial differential equation is

$$h^{(1)}(r, \theta) = \frac{(h_w - \langle h_w \rangle)}{\ln R} \ln (R/r) + A \int \int_D \frac{\partial Y'(\rho, \phi)}{\partial \rho} G(r, \theta, \rho, \phi)\, d\rho\, d\phi \quad (18)$$

The first term in (18) is related to the nonhomogeneous boundary conditions; the second, corresponds to the solution of the Poisson's equation with homogeneous boundary conditions. G is the Green's function for a Poisson's type equation in an annulus of inner radius equal to 1 and outer radius equal to R, which can be expressed as an infinite series [*Weinberger*, 1965],

$$G(r, \theta, \rho, \phi) = \frac{\ln \rho \ln (R/r)}{2\pi \ln R} + \sum_{n=1}^{\infty} \frac{(\rho^n - \rho^{-n}) \cos n(\theta - \phi)}{2\pi n(R^n - R^{-n})} \left[\left(\frac{R}{r}\right)^n - \left(\frac{r}{R}\right)^n \right] \quad (19)$$

valid for $\rho < r$ with a similar expression for $\rho > r$, but with r and ρ exchanged. From now on, we consider only the case of large R; that is, the external radius is much larger than the well radius, which is a common hypothesis in groundwater. Expanding (19), we have

For $\rho < r$

$$G = \frac{\ln \rho \ln (R/r)}{2\pi \ln R} + \sum_{n=1}^{\infty} \frac{\rho^n - \rho^{-n}}{2\pi n} \cdot \cos n(\theta - \phi) \left(\frac{1}{r^n} - \frac{r^n}{R^{2n}} + \cdots \right) \quad (20)$$

For $\rho > r$

$$G = \frac{\ln r \ln (R/\rho)}{2\pi \ln R} + \sum_{n=1}^{\infty} \frac{r^n - r^{-n}}{2\pi n} \cdot \cos n(\theta - \phi) \left(\frac{1}{\rho^n} - \frac{\rho^n}{R^{2n}} + \cdots \right) \quad (21)$$

where the remaining terms inside the series are dropped as they involve higher order terms in R^{-n}. Equations (20) and (21) verify all the conditions required for a Green's function and, furthermore, the infinite series are uniformly convergent. By substituting (20) and (21) into (18), we get the final expression for $h^{(1)}$:

$$h^{(1)}(r, \theta) = \frac{(h_w - \langle h_w \rangle)}{\ln R} \ln (R/r) + A \int_0^{2\pi} \int_1^r \frac{\partial Y'}{\partial \rho} \frac{\ln \rho}{2\pi} \left(1 - \frac{\ln r}{\ln R} \right) d\rho\, d\phi$$
$$+ A \int_0^{2\pi} \int_1^r \frac{\partial Y'}{\partial \rho} \sum_{n=1}^{\infty} \frac{\rho^n - \rho^{-n}}{2\pi n} \cos n(\theta - \phi) \left(\frac{1}{r^n} - \frac{r^n}{R^{2n}} \right) d\rho\, d\phi$$
$$+ A \int_0^{2\pi} \int_r^R \frac{\partial Y'}{\partial \rho} \frac{\ln r}{2\pi} \left(1 - \frac{\ln \rho}{\ln R} \right) d\rho\, d\phi$$
$$+ A \int_0^{2\pi} \int_r^R \frac{\partial Y'}{\partial \rho} \sum_{n=1}^{\infty} \frac{r^n - r^{-n}}{2\pi n} \cos n(\theta - \phi) \left(\frac{1}{\rho^n} - \frac{\rho^n}{R^{2n}} \right) d\rho\, d\phi \quad (22)$$

Equations (16) and (22) together constitute the first order expansion of the solution for the piezometric heads, $h(\mathbf{x})$. In the next section we will see how we can use them to get a consistent second order expansion for the effective transmissivity.

3. Darcy's Equation

Darcy's law at the local scale can be written as

$$\mathbf{q}(\mathbf{x}) = -T(\mathbf{x})\nabla h(\mathbf{x}) \tag{23}$$

where $\mathbf{q}$ is flow integrated along the vertical. In (23) we consider T to be a scalar at the local scale. Our goal is to get a similar expression which would be valid in terms of expected values and would look like

$$\langle \mathbf{q} \rangle = -\mathbf{T}_e \nabla \langle h \rangle \tag{24}$$

where $\mathbf{T}_e$ is, in general, a second-order symmetric tensor. We now carry out a small perturbation expansion of Darcy's equation. We can expand $\mathbf{q}$ as

$$\mathbf{q} = -e^Y \nabla h = -T_G\left(1 + Y' + \frac{Y'^2}{2} + \cdots\right) \cdot \nabla(h^{(0)} + h^{(1)} + h^{(2)} + \cdots) \tag{25}$$

where $T_G = \exp(\langle Y \rangle)$. By retaining terms up to second order in perturbations and taking expected values, we get

$$\langle \mathbf{q} \rangle = -T_G[\nabla\langle h^{(0)}\rangle + \langle Y'\nabla h^{(0)}\rangle + \tfrac{1}{2}\langle Y'^2 \nabla h^{(0)}\rangle + \nabla\langle h^{(1)}\rangle + \langle Y'\nabla h^{(1)}\rangle + \nabla\langle h^{(2)}\rangle] \tag{26}$$

From (22), $\langle h^{(1)}\rangle = 0$, and so $\nabla\langle h^{(1)}\rangle = 0$; similarly, as $h^{(0)}$ is deterministic, we have $\langle Y'\nabla h^{(0)}\rangle = 0$ and $\langle Y'^2 \nabla h^{(0)}\rangle = \sigma_Y^2 \nabla h^{(0)}$. On the other hand, we will show later that $\langle h^{(2)}\rangle = O(\sigma_Y^2)$. Then, in (24), we have $\langle h \rangle = h^{(0)} + \langle h^{(2)}\rangle$ up to order σ_Y^2. This is, again, an important difference with respect to the parallel flow case, where $\langle h \rangle = h^{(0)}$. We will see later that although $\langle h^{(2)}\rangle \neq 0$, it is not necessary to derive the last term in (26) explicitly.

The only particular term to analyze in (26) is $\langle Y'\nabla h^{(1)}\rangle$. In polar coordinates, and using $h' = h^{(1)}$ (change of notation),

$$\langle Y'\nabla h'\rangle = \left(\left\langle Y' \frac{\partial h'}{\partial r}\right\rangle, \left\langle Y' \frac{1}{r}\frac{\partial h'}{\partial \theta}\right\rangle\right) \tag{27}$$

We will start with the tangential component, as it is the simplest one. Deriving (22) with respect to θ, multiplying by Y'/r, and taking expected value, we get

$$\left\langle \frac{Y'}{r}\frac{\partial h'}{\partial \theta}\right\rangle = \frac{-A}{r}\int_0^{2\pi}\int_1^r \frac{\partial C(\mathbf{r}, \boldsymbol{\rho})}{\partial \rho} \sum_{n=1}^{\infty} \frac{\rho^n - \rho^{-n}}{2\pi} \cdot \sin n(\theta - \phi)\left(\frac{1}{r^n} - \frac{r^n}{R^{2n}}\right) d\rho\, d\phi - \frac{A}{r}\int_0^{2\pi}\int_r^R \frac{\partial C(\mathbf{r}, \boldsymbol{\rho})}{\partial \rho} \sum_{n=1}^{\infty} \frac{r^n - r^{-n}}{2\pi} \cdot \sin n(\theta - \phi)\left(\frac{1}{\rho^n} - \frac{\rho^n}{R^{2n}}\right) d\rho\, d\phi \tag{28}$$

where $C(\mathbf{r}, \boldsymbol{\rho})$ is the covariance function of Y. If we consider now any type of isotropic, second order stationary correlation structure, we can prove (section A.1) that

$$\left\langle Y'(\mathbf{r}) \frac{1}{r}\frac{\partial h'}{\partial \theta}(\mathbf{r})\right\rangle = 0 \tag{29}$$

Next, we compute the radial component in (27), that is, $\langle Y'(\mathbf{r})\, \partial h'(\mathbf{r})/\partial r\rangle$. By taking the derivative of (22) with respect to r (applying Leibniz's rule), multiplying by Y', and taking expected values, we get

$$\left\langle Y' \frac{\partial h'}{\partial r}\right\rangle = \langle Y'(\mathbf{r})(\langle h_w\rangle - h_w)\rangle \frac{1}{r}\frac{1}{\ln R} - \frac{A}{2\pi r}\int_0^{2\pi}\int_1^r \frac{\partial C(\mathbf{r}, \boldsymbol{\rho})}{\partial \rho}\sum_{n=1}^{\infty}(\rho^n - \rho^{-n}) \cdot \cos n(\theta - \phi)\left(\frac{1}{r^n} + \frac{r^n}{R^{2n}}\right) d\rho\, d\phi + \frac{A}{2\pi r}\int_0^{2\pi}\int_r^R \frac{\partial C(\mathbf{r}, \boldsymbol{\rho})}{\partial \rho}\sum_{n=1}^{\infty}(r^n + r^{-n}) \cdot \cos n(\theta - \phi)\left(\frac{1}{\rho^n} - \frac{\rho^n}{R^{2n}}\right) d\rho\, d\phi + \frac{A}{2\pi}\int_0^{2\pi}\int_r^R \frac{\partial C(\mathbf{r}, \boldsymbol{\rho})}{\partial \rho} d\rho\, d\phi - \frac{A}{2\pi r}\int_0^{2\pi}\int_1^R \frac{\partial C(\mathbf{r}, \boldsymbol{\rho})}{\partial \rho}\frac{\ln \rho}{\ln R} d\rho\, d\phi \tag{30}$$

The first term in (30) is equal to zero for large r, as Y' and h_w are independent. For small r, it can be proven numerically that the term $\langle Y'h_w\rangle$ is finite, and there is a term in $\ln R$ in the denominator, so the first term drops again. The last term in (30) is also equal to zero for large R. Integration of the three remaining terms in (30) can be performed numerically for any choice of covariance model. For any isotropic covariance function, these terms depend only on the radial distance to the pumping well (r), and not on θ. In the next section we evaluate (30) for two different isotropic covariance models.

4. Evaluation of the Radial Term

In principle, any type of acceptable covariance model is suitable to be input in (30). In this section we will consider the isotropic Gaussian and the isotropic exponential models and compare the values obtained.

4.1. The Isotropic Gaussian Covariance Model

With this model, part of the integration can be performed analytically. The covariance function can be written as

$$C(\mathbf{r}, \boldsymbol{\rho}) = \sigma_Y^2 \exp\left\{-\left(\frac{\rho^2 + r^2 - 2r\rho\cos(\theta - \phi)}{a^2}\right)\right\} \tag{31}$$

where a is the range in the covariance function divided by r_w owing to the original change of coordinates. The normalized integral scale, d, can be related to a by $d = \pi^{1/2}a/2$. By substituting (31) into (30), we have (see section A.2 for the derivation)

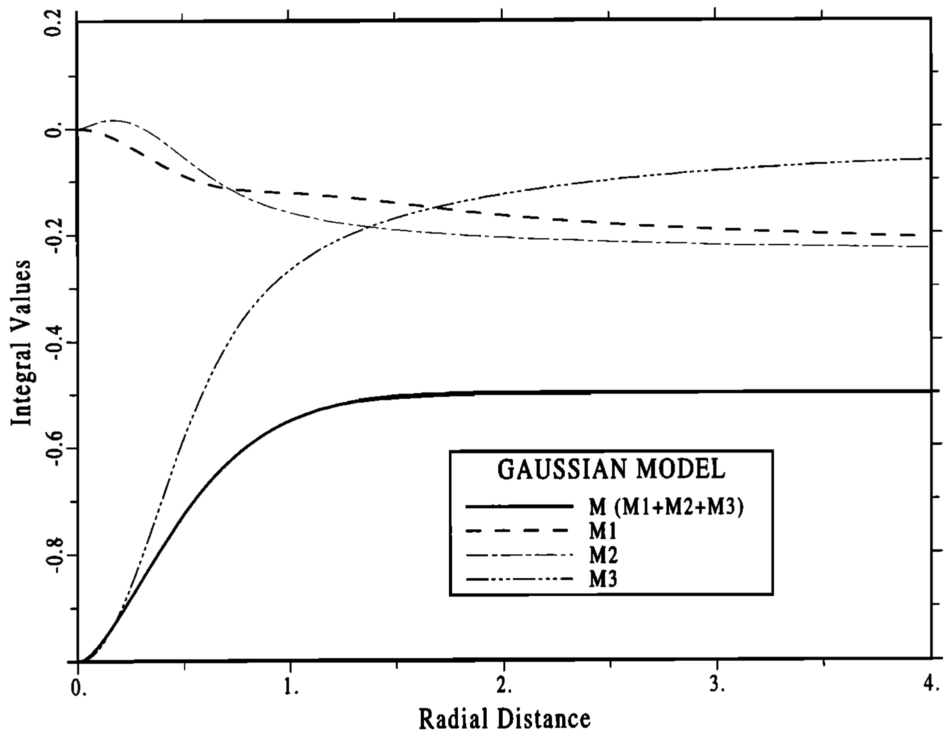

Figure 1. Values of M_1, M_2, M_3, and M (equation (34)) versus r/d for an isotropic Gaussian covariance model.

$$\left\langle Y' \frac{\partial h'}{\partial r} \right\rangle = \frac{2A\sigma_Y^2}{ra^2} \left\{ \int_1^r \exp\left(\frac{-r^2 - \rho^2}{a^2} \right) \cdot \sum_{n=1}^{\infty} (\rho^n - \rho^{-n}) \Lambda_n(r, \rho) \left(\frac{1}{r^n} + \frac{r^n}{R^{2n}} \right) d\rho - \int_r^R \exp\left(\frac{-r^2 - \rho^2}{a^2} \right) \sum_{n=1}^{\infty} (r^n + r^{-n}) \Lambda_n(r, \rho) \left(\frac{1}{\rho^n} - \frac{\rho^n}{R^{2n}} \right) d\rho - \int_r^R \exp\left(\frac{-r^2 - \rho^2}{a^2} \right) \Lambda_0(r, \rho)\, d\rho \right\} \tag{32}$$

where the Λ_n functions are combinations of Bessel functions

$$\Lambda_n(r, \rho) = -\frac{r}{2} I_{n-1}\left(\frac{2\rho r}{a^2} \right) + \rho I_n\left(\frac{2\rho r}{a^2} \right) - \frac{r}{2} I_{n+1}\left(\frac{2\rho r}{a^2} \right) \tag{33}$$

For small a, the size of the well affects the results in (32). In any case, r_w is usually small with respect to the integral scale of transmissivities, so that in general $a \gg 1$. For this case we perform the integration of (32) numerically, using the following notation:

$$\left\langle Y' \frac{\partial h'}{\partial r} \right\rangle = \frac{A\sigma_Y^2}{r} (M_1 + M_2 + M_3) = \frac{A\sigma_Y^2}{r} M \tag{34}$$

where each M_i is one of the integral functions in (32). Figure 1 shows the values for M_1, M_2, M_3, and M as a function of r (normalized by the integral scale, d). We see that M is a monotonic increasing function rising from $M = -1$ when $r/d \to 0$, up to $M = -0.5$ when $r/d \to \infty$. Notice that a value very close to $M = -0.5$ is achieved after a distance equal to 1.5–2 integral scales.

4.2. The Isotropic Exponential Covariance Model

In this case, the covariance function can be written as

$$C(\mathbf{r}, \boldsymbol{\rho}) = \sigma_Y^2 \exp\left(-\frac{[\rho^2 + r^2 - 2r\rho \cos(\theta - \phi)]^{1/2}}{a} \right) \tag{35}$$

where now the integral scale is $d = a$. Using this model, there is not a simple way to avoid the double integration in (30). Instead, it is possible to get rid of the infinite sum, by using a new function F defined as

$$F(z) = -\sum_{n=1}^{\infty} z^n \cos n\theta = \frac{z^2 - z\cos\theta}{z^2 + 1 - 2z\cos\theta} \tag{36}$$

which is valid for $|z| < 1$. From (35) and (36), (30) can be written

$$\left\langle Y' \frac{\partial h'}{\partial r} \right\rangle = \frac{A\sigma_Y^2}{2\pi r} \int_0^{2\pi} \int_1^r P(r, \rho) \left(-F\left(\frac{\rho}{r} \right) + F\left(\frac{1}{\rho r} \right) - F\left(\frac{r\rho}{R^2} \right) + F\left(\frac{r}{\rho R^2} \right) \right) d\rho\, d\phi + \frac{A\sigma_Y^2}{2\pi r} \cdot \int_0^{2\pi} \int_r^R P(r, \rho) \left(F\left(\frac{r}{\rho} \right) + F\left(\frac{1}{\rho r} \right) - F\left(\frac{r\rho}{R^2} \right) - F\left(\frac{\rho}{rR^2} \right) \right) d\rho\, d\phi + \frac{A\sigma_Y^2}{2\pi r} \int_0^{2\pi} \int_r^R P(r, \rho)\, d\rho\, d\phi \tag{37}$$

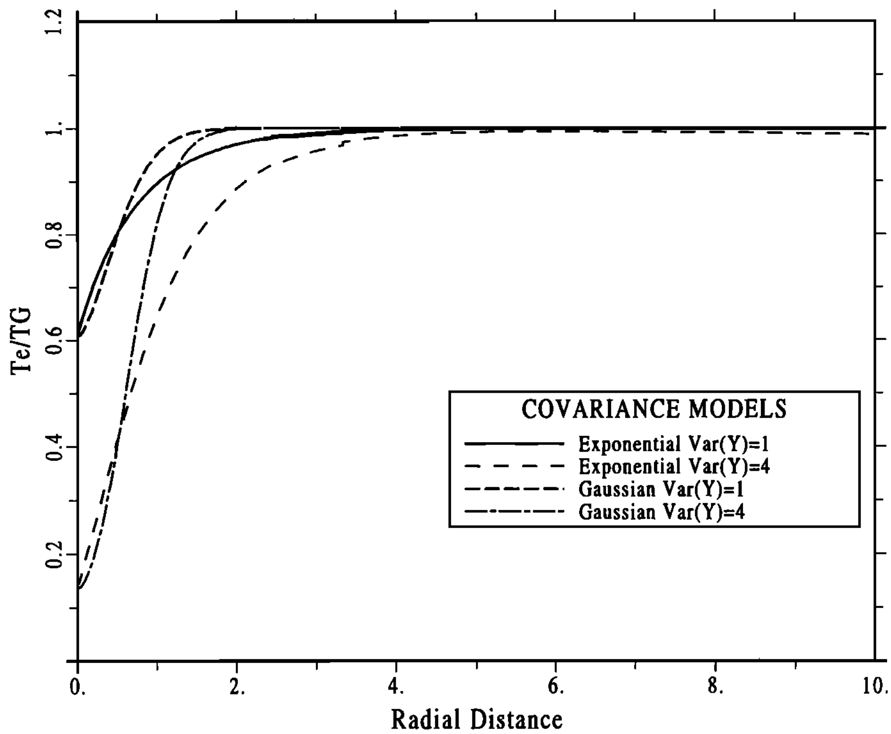

Figure 2. Apparent effective transmissivity (equation (43)) normalized by T_G versus r/d for the isotropic Gaussian and exponential covariance models and two different values of σ_Y^2 ($\sigma_Y^2 = 1.0$ and $\sigma_Y^2 = 4.0$).

where $P(r, \rho)$ stands for the derivative with respect to ρ of the exponencial autocorrelation function, which is given by

$$P(r, \rho) = -\frac{1}{a} \exp\left(\frac{-(\rho^2 + r^2 - 2\rho r \cos \phi)^{1/2}}{a}\right) \cdot \frac{\rho - r \cos \phi}{(\rho^2 + r^2 - 2\rho \cos \phi)^{1/2}} \quad (38)$$

Writing (37) in the same form as (34), and evaluating the integrals numerically, we obtain M as a function of r/d. Again, M is a monotonic increasing function of r, going from $M = -1.0$ for small r to $M = -0.5$ for large r.

5. Effective Transmissivity

The last term to evaluate in (26) is $\nabla\langle h^{(2)}\rangle$. From (12), and after some manipulations, we can write an equation for $\langle h^{(2)}\rangle$,

$$\nabla^2\langle h^{(2)}\rangle = -\nabla \cdot \langle Y'\nabla h'\rangle \quad (39)$$

with homogeneous boundary conditions. It is clear from (29) that only the radial term in $\nabla\langle h^{(2)}\rangle$ is different from zero. Then, from (26) we have

$$\langle \mathbf{q}\rangle = (\langle q_r\rangle, \langle q_\theta\rangle) = (\langle q_r\rangle, 0) \quad (40)$$

so we get a very logical result: The expected value of the transversal flow is equal to zero (in other words, in the mean the flow is radial). Furthermore, from (39) and (34), it is possible to see that the radial term, $\partial\langle h^{(2)}\rangle/\partial r$ is of the order σ_Y^2.

Now, we use all the previous derivations to obtain the apparent effective transmissivity values. Setting the radial terms in the right-hand sides of equations (24) and (26) to be equal, and knowing that $\partial h^{(0)}/\partial r = A/r$ and $\partial\langle h^{(2)}\rangle/\partial r = (1/r)O(\sigma_Y^2)$, we can write

$$T_G\left[\left(1 + \frac{\sigma_Y^2}{2}\right)\frac{A}{r} + \frac{A\sigma_Y^2}{r}M + \frac{1}{r}O(\sigma_Y^2)\right] = T_e\left[\frac{A}{r} + \frac{1}{r}O(\sigma_Y^2)\right] \quad (41)$$

and so we find an expression for T_e up to first order in σ_Y^2:

$$T_e = T_G[1 + (\tfrac{1}{2} + M)\sigma_Y^2] \quad (42)$$

If we consider the term in brackets as the first two terms in a series expansion of an exponential, we can write

$$T_e = T_G \exp[(\tfrac{1}{2} + M)\sigma_Y^2] \quad (43)$$

and using the relationship between T_G and T_H in a log T Gaussian field, we have

$$T_e = T_H \exp[(1 + M)\sigma_Y^2] \quad (44)$$

In Figure 2 we plot apparent effective transmissivity (equation (43)) versus normalized distance to the pumping well for the two covariance models considered in this paper. The differences in the curves appear in their shapes, in the behavior at the origin, and in the number of integral scales needed to achieve the asymptotic value. It is worth mentioning here the discussion given by *Indelman and Abramovich* [1994, p. 3388] regarding the different behavior between parallel and nonuniform flow: In parallel flow, T_e is invariant to the correlation model at first order in σ_Y^2, while in nonuniform flow it is not. The curve corresponding to the exponential model increases rapidly, while the Gaussian curve has a zero derivative for $r \rightarrow$

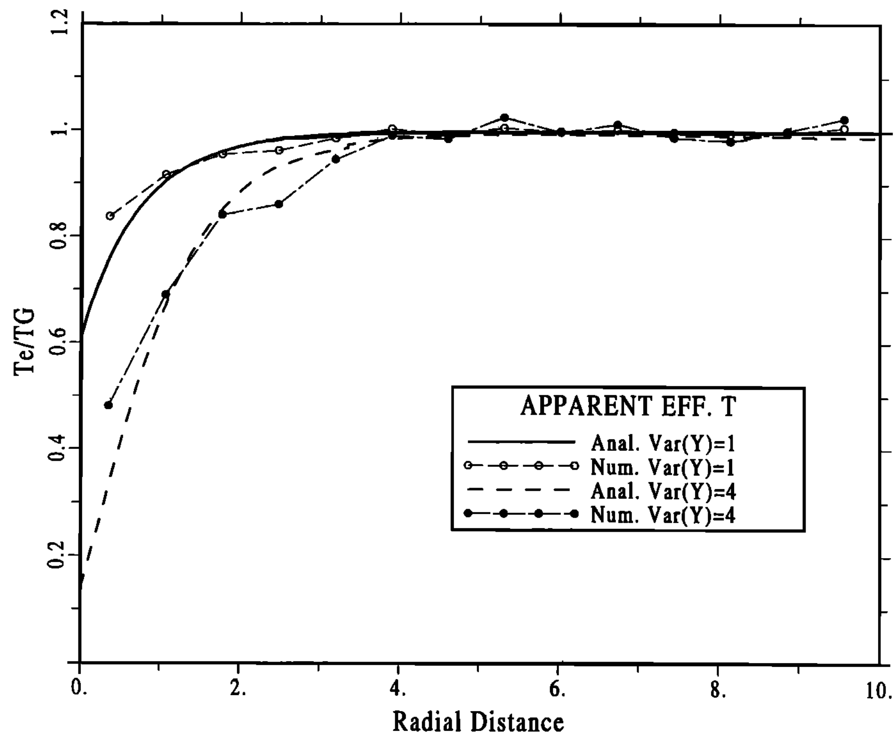

Figure 3. T_e/T_G versus r/d for two different σ_Y^2 values. Analytical curves are given by (43). Numerical curves are adapted from *Neuman and Orr* [1993].

0; this is due to the different continuity behavior at the origin in the two models. A second difference is the number of integral scales which are necessary to achieve the asymptotic value. For the Gaussian case it is achieved after 1.5–2 integral scales, while it is 3–5 integral scales for the exponential model. The limiting values are the same in both cases,

$$\lim_{r/d\to 0} T_e = T_H \qquad \lim_{r/d\to\infty} T_e = T_G \tag{45}$$

and coincide with the values suggested by *Dagan* [1989]. Equations (43) and (44) can be compared with numerical simulations by *Neuman and Orr* [1993]. They simulate two-dimensional flow to a point sink in a statistically homogeneous, lognormal T field with isotropic exponential covariance. They plotted T_e/T_G and T_e/T_H versus distance (their Figures 13 and 14) for both $\sigma_Y^2 = 1$ and $\sigma_Y^2 = 4$. We compare their results with our solution in the case of exponential covariance. Figures 3 and 4 compare the curves that can be drawn from (43) and (44) with Neuman and Orr's numerical curves. From all the curves in those figures, we choose only the one representing a larger domain, and we plot it only up to about half of the total radius, so that the artificial influence of the outer boundary is not felt (as discussed by Neuman and Orr in their paper). Agreement in both figures is remarkably good, except for very short distances, where, in any case, Neuman and Orr discuss that T_e should be equal to the harmonic mean (that is, their numerical results should be closer to the analytical curve).

6. Summary and Conclusions

The study of effective parameters is one of the most frequently addressed problem in stochastic hydrogeology. In spite of this, nonuniform flow has been treated by only a few authors.

In this paper we address the problem of finding the evolution of T_e with r. T_e is defined as the average between the expected values of flow and head gradient at a certain location. For this purpose we first expand the flow equation to find a relationship between the perturbations of heads and log transmissivities, written in terms of a Green's function, which appears from solving Poisson's equation in an annular domain. This relation is used later to evaluate the expected value of the product of perturbations of log T and ∇h, a term coming from the expansion of Darcy's equation. Regarding this term, we first prove that the tangential component is equal to 0 for any type of isotropic covariance model, and second, we find a general expression for the radial component (equation (30)), which is given in terms of the distance to the well (r), log T integral scale (a), log T variance (σ_Y^2), and the covariance model. For large integral scales with respect to the well radius, the relationship can be established in terms of r/a. In this last case we work with a model continuous at the origin (Gaussian) and a discontinuous one (exponential). In both cases, after substituting the values obtained from Darcy's equation, we obtain T_e as a monotonic increasing function of r, going from the harmonic mean of the point T values (for small r), up to the geometric mean (large r). The actual shape of the curve depends on the covariance model selected. The asymptotic value (T_G) is obtained at a distance equal to 1.5–2 integral scales for the Gaussian model and 3–5 for the exponential one.

There is an important point to mention here. As presented throughout the paper, T_e is not a value that can be derived solely from the statistical parameters of the random function T, but rather it depends upon the choice of boundary conditions. What we have found is that the situation is not as bad as

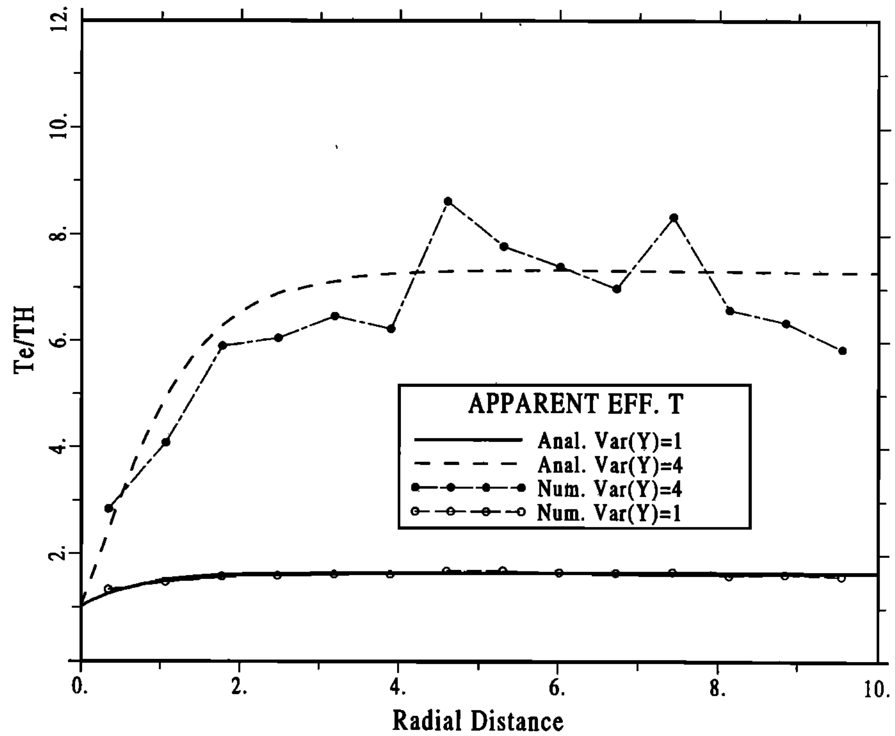

Figure 4. T_e/T_H versus r/d for two different σ_Y^2 values. Analytical curves are given by (44). Numerical curves are adapted from *Neuman and Orr* [1993].

it seems, as T_e is approximately equal to T_G almost everywhere (except near the boundaries). This casts a lot of additional significance to the concept of effective transmissivity, which seems to be a concept applicable beyond uniform, parallel in the mean, flow conditions.

Verification of the methodology is achieved by two means. First, the limiting values obtained for T_e (r small, r large) can be derived by other methods [*Dagan*, 1989] and are well-known results in the stochastic hydrogeology literature. Second, the shape of the analytical curves T_e versus r for the exponential covariance model and for different σ_Y^2 values compare excellently to results published by *Neuman and Orr* [1993], obtained by Monte Carlo simulations.

Appendix: Derivation of $\langle Y'\nabla h'\rangle$

A.1. Derivation of the Tangential Term

Any type of isotropic, second-order stationary correlation structure can be written in a general form

$$C(\mathbf{r}, \boldsymbol{\rho}) = C(|\mathbf{r} - \boldsymbol{\rho}|) = f\{[r^2 + \rho^2 - 2\rho r \cos(\theta - \phi)]^{1/2}\} \quad \text{(A1)}$$

where f can be any type of admissible covariance function. Then

$$\frac{\partial C(\mathbf{r}, \boldsymbol{\rho})}{\partial \rho} = f'(|\mathbf{r} - \boldsymbol{\rho}|) \frac{[\rho - r\cos(\theta - \phi)]}{|\mathbf{r} - \boldsymbol{\rho}|} \quad \text{(A2)}$$

and we can write (28) as

$$\left\langle Y'(\mathbf{r}) \frac{1}{r} \frac{\partial h'(\mathbf{r})}{\partial \theta} \right\rangle = \frac{-A}{r} \int_0^{2\pi} \int_1^r f'(|\mathbf{r} - \boldsymbol{\rho}|) \frac{[\rho - r\cos(\theta - \phi)]}{|\mathbf{r} - \boldsymbol{\rho}|}$$

$$\cdot \sum_{n=1}^{\infty} \frac{\rho^n - \rho^{-n}}{2\pi} \sin[n(\theta - \phi)]\left(\frac{1}{r^n} - \frac{r^n}{R^{2n}}\right) d\rho\, d\phi$$

$$- \frac{A}{r} \int_0^{2\pi} \int_r^R f'(|\mathbf{r} - \boldsymbol{\rho}|) \frac{[\rho - r\cos(\theta - \phi)]}{|\mathbf{r} - \boldsymbol{\rho}|}$$

$$\cdot \sum_{n=1}^{\infty} \frac{r^n - r^{-n}}{2\pi} \sin n(\theta - \phi)\left(\frac{1}{\rho^n} - \frac{\rho^n}{R^{2n}}\right) d\rho\, d\phi \quad \text{(A3)}$$

Now, noticing that integration in ϕ is over $[0, 2\pi]$, we see that both integrals cancel out, owing to the presence of the sine terms (odd functions in ϕ integrated over a symmetrical domain). Then finally,

$$\left\langle Y'(\mathbf{r}) \frac{1}{r} \frac{\partial h'}{\partial \theta}(\mathbf{r}) \right\rangle = 0 \quad \text{(A4)}$$

which corresponds to (29).

A.2. Derivation of the Radial Term Expression for the Gaussian Model

The starting point is (30). Substituting (31) into (30) leads to

$$\left\langle Y' \frac{\partial h'}{\partial r} \right\rangle = \frac{A\sigma_Y^2}{\pi r a^2}$$

$$\cdot\left[\int_0^{2\pi}\int_1^r \exp\left(\frac{-\rho^2 - r^2 + 2\rho r\cos(\theta-\phi)}{a^2}\right)\right.$$

$$\cdot[\rho - r\cos(\theta-\phi)]\sum_{n=1}^{\infty}(\rho^n - \rho^{-n})\cos n(\theta-\phi)$$

$$\cdot\left(\frac{1}{r^n} + \frac{r^n}{R^{2n}}\right)d\rho\, d\phi$$

$$-\int_0^{2\pi}\int_r^R \exp\left(\frac{-\rho^2 - r^2 + 2\rho r\cos(\theta-\phi)}{a^2}\right)$$

$$\cdot(\rho - r\cos(\theta-\phi))$$

$$\sum_{n=1}^{\infty}(r^n + r^{-n})\cos n(\theta-\phi)\left(\frac{1}{\rho^n} - \frac{\rho^n}{R^{2n}}\right)d\rho\, d\phi$$

$$-\int_0^{2\pi}\int_r^R \exp\left(\frac{-\rho^2 - r^2 + 2\rho r\cos(\theta-\phi)}{a^2}\right)$$

$$\left.\cdot(\rho - r\cos(\theta-\phi))\, d\rho\, d\phi\right] \quad \text{(A5)}$$

Integration with respect to variable ϕ can be carried out by taking into account two formulae from calculus; first

$$\cos(\theta-\phi)\cos n(\theta-\phi) = \tfrac{1}{2}\{\cos[(n+1)(\theta-\phi)] + \cos[(n-1)(\theta-\phi)]\} \quad \text{(A6)}$$

and second, the definition of the modified Bessel function $I_n(z)$,

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\psi}\cos n\psi\, d\psi \quad \text{(A7)}$$

From (A6) and (A7) we can integrate (A5) over ϕ and after defining

$$\Lambda_n(r, \ ) = -\frac{r}{2}I_{n-1}\left(\frac{2\rho r}{a^2}\right) + \rho I_n\left(\frac{2\rho r}{a^2}\right) - \frac{r}{2}I_{n+1}\left(\frac{2\rho r}{a^2}\right) \quad \text{(A8)}$$

we finally get (32).

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