INTERNATIONAL CENTER FOR NUMERICAL METHODS IN ENGINEERING

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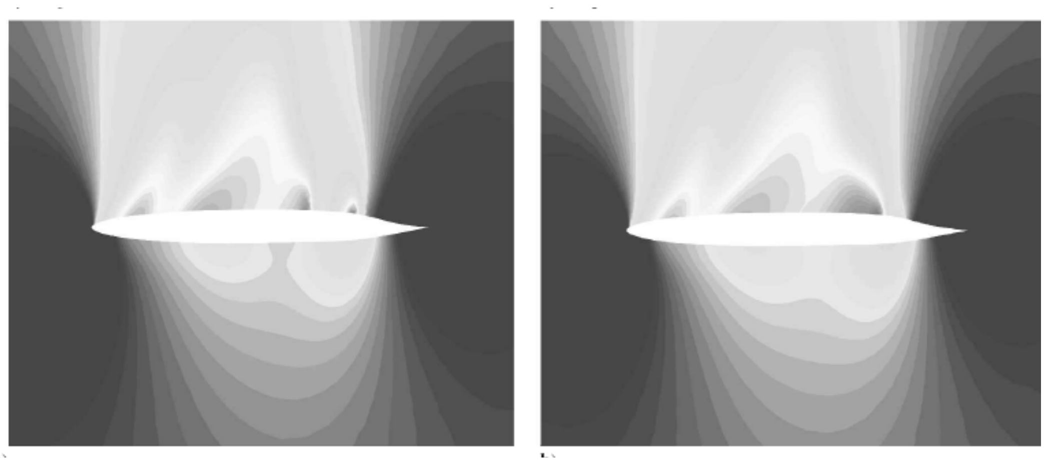

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Abstract:

This document is an introduction to some important methodologies that have been developed in robust design in aerospace engineering. After describing the concept of robustness and uncertainty, multipoint, minimax, expected value, second order second moment and Taguchi methods are mentioned. At the end of this report, Game Theory, as one of the approach for multi objective optimization problems has been introduced.

Contents

Chapter 1

Introduction

1.1 Robustness:

In order to design and manufacture high quality products at a minimum of costs, techniques are needed which are able to find those designs which meet the requirements usually specified by objectives (goal functions) at the beginning of the design process. Provided that the general system design has been fixed (e.g., the type of product and its desired basic properties are given), it is the engineer's task to choose the design parameters *x* according to an (or some) objective function(s) *f(x)*. These objective functions may be given by verbal descriptions, mathematical models, simulation models, or physical models. The process of finding the right design parameters is usually referred to as optimization. Typically, the optimization has also to account for design constraints imposed on the design parameters *x*. Such constraints can be modelled by inequalities and/or equalities restricting the design space (search space). In mathematical terms a general optimization task can be stated *as*:

$$\left.\begin{array}{llll} \textit{optimize}: & f(x) & & (a) \\ \textit{subject to}: & g_i(x) \le 0 & i = 1,\dots,I & (b) \\ & h_j(x) = 0 & j = 1,\dots,J & (c) \end{array}\right\} \qquad (1)$$

where (b) represents the set of inequality constraints and (c) the set of equality constraints.

There are principle problems that might prevent us from identifying the optimum of *f(x)* in (1), like NP-hardness in discrete search spaces or multi-modality in continuous search spaces. However, one might also ask whether the formulation of the optimization problem in (1) is as general and practical as it seems. The question arises whether it is desirable to locate isolated, singular design points with a high precision:

1- The global optimal design clearly depends on the goal (objective) function(s) and constraints in (1), however, these functions always represent models and/or approximations of the real world. As long as one does not have detailed knowledge of the error function of the model, one cannot be certain the model optimum can be mapped to the true optimum. Thus, being too precise in the model might waste resources, which could be better used at a later design stage.

2- Even if one were able to map the model optimum to the true optimum, one might not be able to build the true optimum either because of manufacturing uncertainties or because the required precision during the manufacturing stage would be too costly. There is always an economical trade-off between a potentially more complex manufacturing process and the performance gain by the new design.

3- The formulation of the optimization problem in (1) is inherently static. Reality is dynamic: environmental parameters fluctuate (temperature, Reynolds number for gas turbine design), materials wear down, parts of a complete system might be replaced. Since the constraints on which the original design process was based change, (1) is only correct for a limited time span.

4- Life cycle costs have to be taken into account for many engineering designs. Life cycle engineering focuses on the whole life span of a design, e.g., easier maintenance (system design to enable a cheap disassembly and assembly process, e.g., for gas turbines), longer maintenance intervals, effect of attrition during operation, or environmentally friendly disposal, e.g., recycling capability.

Systems (1) optimized in the classical sense can be very sensitive to small changes which are likely to occur as we have just argued. A better target for a design is one that provides a high degree of robustness. Marczyk writes ‘‘*Optimization is actually just the opposite of robustness*’’ [1]. Although there is some truth in this statement, it does make sense to reconsider the current optimization algorithm philosophy and the test functions and instances used to evaluate these in the framework of robustness. We will come back to Marczyk’s statement later in the paper. As a result, again one will search for optimal solutions, however, for robust solutions. The procedure of finding such solutions is referred to as robust design optimization. The appeal of robust design optimization is that its solutions and performance results remain relatively unchanged when exposed to uncertain conditions.

Robust design and optimization has even deeper roots in engineering. There it is inextricably linked with the name of Taguchi [2] who initiated a highly influential design philosophy (see Section 2.5). Due to the advent of high-speed computers and its exponentially increasing FLOPS-rates (floating point operations per second), robust design optimization has gained increasing interest in the past few years. One of these cases is aerospace engineering.

1.2 Aerospace shape optimization:

The definition of the aerodynamic shapes of aircraft relies heavily on computational simulation to enable the rapid evaluation of many alternative designs. Wind tunnel testing is then used to confirm the performance of designs that have been identified by simulation as promising to meet the performance goals. In the case of wing design and propulsion system integration, several complete cycles of computational analysis followed by testing of a preferred design may be used in the evolution of the final configuration. Wind tunnel testing also plays a crucial role in the development of the detailed loads needed to complete the structural design, and in gathering data throughout the flight envelope for the design and verification of the stability and control system. The use of computational simulation to scan many alternative designs has proved extremely valuable in practice, but it still suffers the limitation that it does not guarantee the identification of the best possible design. Generally one has to accept the best so far by a given cut-off date in the program schedule. To ensure the realization of the true best design, the ultimate goal of computational simulation methods should not just be the analysis of prescribed shapes, but the automatic determination of the true optimum shape for the intended application. This is the underlying motivation for the combination of computational fluid dynamics with numerical optimization methods.

Recently, there has been significant progress in airfoil shape optimization (see Anderson and Bonhaus 1999; Anderson and Venkatakrishnan 1997; Drela 1998; Nielsen and Anderson 1998 and references therein) [3,4,5]. These papers demonstrate impressive shape optimization using high-fidelity CFD codes, reliable grid generation, and numerically efficient sensitivity calculations. Equally impressive progress has been made in optimization of 3-D wings (Elliott and Peraire 1997, 1998; Reuther et al. 1999; Nielsen and Anderson 2001) [6,7,8,9] and in coupled structural- aerodynamic optimization (Gumbert et al. 2001) [10]. These traditional aerospace shape optimization methods are based on deterministic parameters. These methods do not involve uncertainty (fluctuated) parameters. However, with a few exceptions such as (Drela 1998) and (Reuther et al. 1999), these aerodynamic shape optimization projects all find optimal shapes based on one fixed operating condition. Nowadays, thanks to presence of powerful computing systems, it is possible to have some uncertainty parameters in the optimization process. In this report, we study airfoil shape optimization problem using more operating conditions including uncertainty in the operating conditions for the airfoil shape optimization.

The simplest approach to optimization is to define an objective function *f* which might, for example, be the drag coefficient or the lift to drag ratio, and *f* is regarded as a function of the parameters *a* and *b* of the form $F(a,b): A \otimes B \rightarrow R$ where:

$a \in A$ represents decision variables, designs controlled by the engineer (for example shape of airfoil).

$b \in B$ represents uncertainty parameters, inputs not controlled by the engineer (for example Mach number).

Our (unattainable) goal is to find $a^* \in A$ such that, for every $b \in B$,

$$f(a^*, b) < f(a, b) \quad \forall a \in A \tag{2}$$

In following, we explain uncertainty parameters b in engineering optimization problems.

1.3 Uncertainty parameters:

Both computational results and experimental measurements in aerodynamic applications are subject to considerable uncertainty as illustrated by the scatter in the solutions submitted by various researchers. For these reasons, the current certification process relies heavily on full-scale flight testing. Dramatic savings would be achieved if the need for expensive full-scale flight tests could be reduced through improved reliability (or dependability) of modelling results. In a deterministic approach, the performance of a design is typically assessed for a limited number of design or operating conditions. Uncertainty associated with the operating conditions (like variable payload, atmospheric conditions) and manufacturing uncertainty (like fluctuations in the geometry and smoothness of the skin) may have substantial impact on the results. Techniques for propagating these uncertainties require additional computational effort but are well established. An example application is described in [11]. Exact and approximate uncertainty assessment methods are applied to an airfoil optimization problem in [12], where a dramatic improvement of the robustness of the design was achieved.

At this point in the discussion it is appropriate to spend some time on nomenclature, especially the difference between "error", "scatter" and "uncertainty". Unfortunately, these terms have been used to denote somewhat different things in different fields of application.

We adopt the following definitions, which are commonly used by statisticians:

• Error is a deterministic concept and is defined as the difference between the true or exact answer to a problem and the answer, computed or measured using a faulty or simplified theory.

• Scatter measures the range or spread of the data, but gives no information about the potential bias due to systematic measurement error or due to missing terms in the CFD code.

• Uncertainty indicates that the result can be only known with a limited amount of confidence for a given level of precision. This uncertainty is an inherent property of the measurement technique or model description and is due to lack of knowledge.

The distinction between the deterministic and stochastic nature is apparent in the AIAA definitions as well [13]:

• Error is a recognizable deficiency in any phase or activity of modelling and simulation that is not due to lack of knowledge.

• Uncertainty is a potential deficiency in any phase or activity of modelling and simulation that is due to lack of knowledge.

Since error is a recognizable deficiency, all errors are in principle at least correctable and therefore deterministic.

Since uncertainty is caused by a fundamental lack of knowledge, it cannot be eliminated. If a higher confidence level (or level of credibility) of the prediction is required, the result can only be given with less precision. Uncertainty forces a fundamental trade-off between confidence and precision.

Aerospace design has to face different kinds of uncertainties which are usually beyond the (direct) control of the designer:

(A) *Changing environmental and operating conditions*. Examples are the angle of attack and Mach number in airfoil design, operating temperature, pressure, humidity, changing material properties and drift, etc.

(B) *Production tolerances and actuator imprecision*. The design parameters of a product can be realized only to a certain degree of accuracy. High precision machinery is expensive; therefore, a design less sensitive to manufacturing tolerances reduces costs.

(C) *Uncertainties in the system output*. These uncertainties are due to imprecision in the evaluation of the system output and the system performance. This kind of uncertainty includes measuring errors and all kinds of approximation errors due to the use of models instead of the real physical objects (model errors).

(D) *Feasibility uncertainties*. Uncertainties concerning the fulfilment of constraints the design variables must obey. This kind of uncertainty is different to (A)–(C) in that it does not consider the uncertainty effects on f but on the design space. In real-world applications it often appears together with the uncertainty types (A) and (B).

(E) *Discretization and solution*. These uncertainties are truncation error (spatial and temporal), iterative convergence, discrete geometry representation, etc.

There are different possibilities to quantify the uncertainties subsumed under (A)–(E) mathematically. Basically, the uncertainties can be modelled deterministically, probabilistically, or possibilistically:

(1) The deterministic type defines parameter domains in which the uncertainties. can vary.

(2) The probabilistic type defines probability measures describing the likelihood by which a certain event occurs.
(3) The possibilistic type defines fuzzy measures describing the possibility or membership grade by which a certain event can be plausible or believable.

The analysis and quantification of uncertainty and error due to these different sources have been studied by several authors. Particularly, discretization error has been studied extensively and a number of techniques have been proposed for modelling of this error [2]. Grid adoption techniques (see for example [2]) have been proposed to control this error. The influence of geometrical uncertainty has been addressed in [2], whereas uncertainty related to the turbulence modelling has been studied in [2]. Uncertainty analysis related to the operating conditions is mentioned in [2].

Some source of error can be made negligible thanks to the increase of computational resource, such as discretization error, iterative convergence error and etc. However, other sources of error and uncertainty remain significant and motivate further research in this field.

There are tow approaches regarded as an optimization method which tries to account five uncertainties that have been defined in the previous section. These tow approaches will be mentioned in the next section.

1.4 Aerospace shape optimization under uncertainty:

The two major classes of uncertainty-based design problems are robust design problems and reliability-based design problems. A robust design problem seeks a design that is relatively insensitive to small changes in the uncertain quantities. A reliability-based design seeks a design that has a probability of failure that is less than some acceptable (invariably small) value. The same abstract mathematical formulation can be used to describe both robust design and reliability-based design. However, their domains of applicability are rather different.

Figure 1 illustrates these domains. The two major factors are the frequency of the event and the impact of the event. No system is viable if everyday fluctuations can lead to catastrophe. Instead, one would like the system to be designed such that the performance is insensitive, i.e., robust, to everyday fluctuations. On the other hand, one would like to ensure that the events that lead to catastrophe are extremely unlikely. This is the domain of reliability-based design. In both cases, the design risk is a combination of the likelihood of an undesired event and the consequences of that event. An example of risk in the robust design context is the likelihood that the aircraft design will fail to meet the aerodynamic performance targets and will consequently lose sales and perhaps even go bankrupt. An example of risk in the reliability-based design context is the probability that a critical structural component will fail, which could lead to the loss of the vehicle or spacecraft, payload, and passengers, and to potential lawsuits.

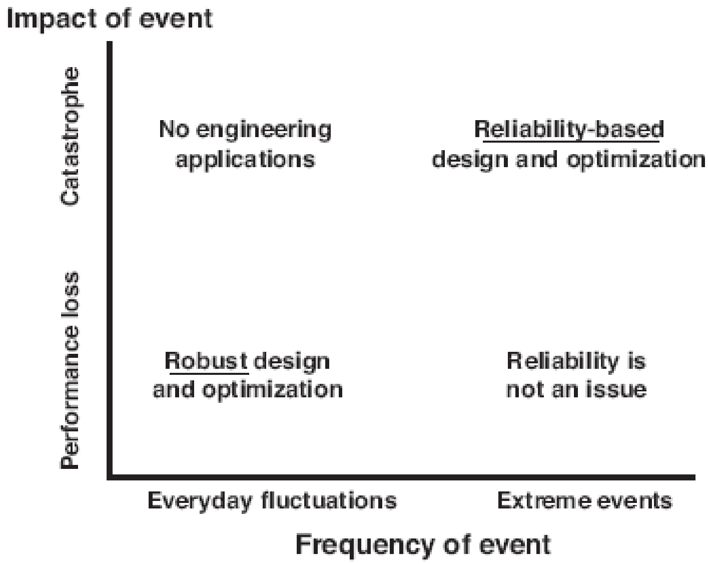

Figure 1. Uncertainty-based design domains (from Huyse 2001 [12]).

As figure 2 illustrates, robust design is concerned with the event distribution near the mean of the probability density function, whereas reliability-based design is concerned with the event distribution in the tails of the probability density function. Obviously, it is much more difficult to accurately characterize the tail of a distribution than the center of the distribution. An additional consideration in distinguishing between robustness and reliability is that the mathematical techniques used for solving robust design problems are considerably different from those used for solving reliability-based design problems. The mathematical methods for robust design procedures are less well developed than those for reliability based design procedures. Certainly, the aerodynamic design procedures in use in industry are exclusively deterministic. There has been considerable work on "robust controls". Although the robust design principles of Taguchi (1987) are used in aerospace engineering, these are not necessarily the best or even appropriate methods for many robust design problems.

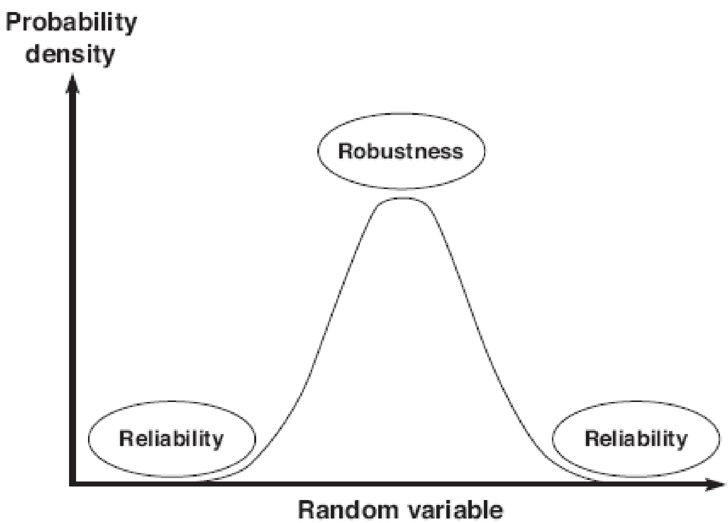

Figure 2. Reliability versus robustness in terms of the PDF (Zang [16]).

Traditional design procedures for aerospace vehicle structures are based on combinations of factors of safety and knockdown factors, as illustrated in figure 3.

Factors of safety are numbers greater than 1.0 that are applied to the loads. Knockdown factors are numbers less than 1.0 that are applied to the strengths. Both factors are intended to account for uncertainties. They have proven useful during nearly six decades of design for conventional metal airframes.

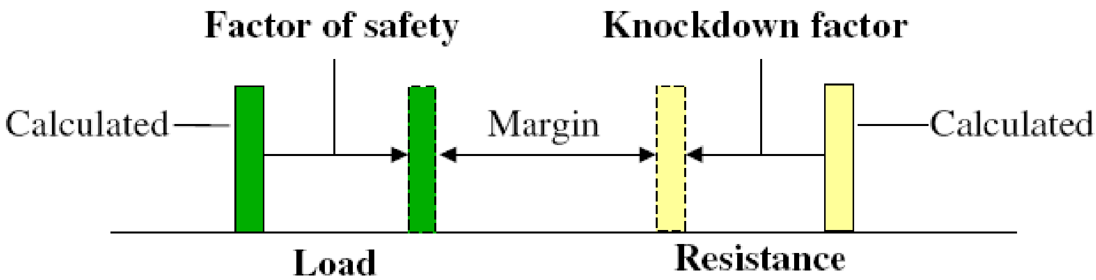

Figure 3. Factor of safety approach (Huyse [12]).

Newly emerging uncertainty-based design procedures will help to overcome the shortcomings of the traditional design procedures. In particular, measures of reliability and robustness will be available during the design process and for the final design. This information will allow the designer to produce a consistent level of reliability and performance throughout the vehicles with no unnecessary over designs in some areas. As a result, designers will be able to save weight while maintaining reliability. In addition, with an uncertainty-based design procedure it will be possible to determine the sensitivity of the reliability to design changes that can be linked to changes in cost. As a result, it will be possible to make trade-offs between reliability and cost. For the same cost, airframes can be made safer than with traditional design approaches or, for the same reliability; the airframe can be made at a lower cost.

Some principal barriers to the adoption of uncertainty-based design methods for aerospace vehicles are as follows:

B1. Industry feels comfortable with traditional design methods.
B2. Few demonstrations of the benefits of uncertainty-based design methods are available.
B3. Current uncertainty-based design methods are more complex and much more computationally expensive than deterministic methods.

This report consists of a survey of the state of the art in robust design. In particular, section 2 provides a generic overview of current robust design methods. Section 3 focuses on game theory, as one of the most useful methods for multi objective optimization problems, and its application with Nash equilibrium and Pareto front.

Chapter 2

Overview of Available Robust Design Methods

In most industrial applications, some design operative parameters are not fixed or it is impossible to set a constant value. For example, some uncertainties could characterize some geometric entities (lengths, relative positions, angles) that are related to the problem studied. Many times the operative conditions are not fixed, but there is the presence of fluctuations: in turbo machinery, it is the case of the mass flow rate and the inlet pressure, while, in aeronautics, it is the case of the flight speed, the angle of attack, the air temperature, etc.

For engineering design problems, an optimal design is usually obtained under some explicit/implicit assumptions. This leads to a design that works well under ideal operating conditions but may perform inadequately under non-ideal (off-design) conditions. The problem is that the optimal design does not consider the uncertainty or variability of some parameters/data that will affect the actual performance of the design in a real-world situation. Therefore, it is necessary to include uncertainties in a practical design optimization process.

For these reasons, in all these cases the design parameters can be specified by the mean value and their variance, following the classic Gaussian theory. In the case of fluctuations of the operative conditions, it is important to achieve the stability of the solution, because a traditional optimization approach (single point method) could tend to a problem of "over-optimization" (figure 4), giving high performances in correspondence of the design point, but giving poor off-design characteristics.

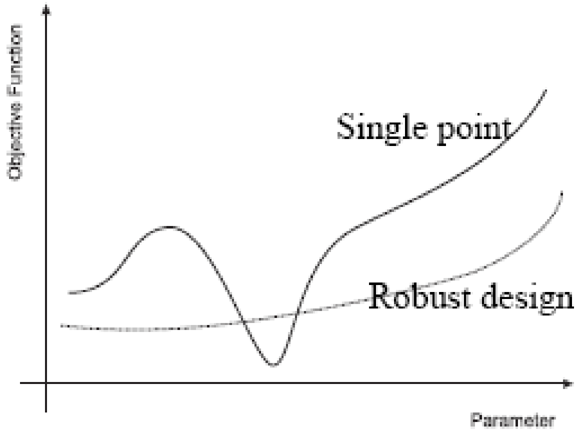

Figure 4: Comparison of robust design optimisation with single point optimization

With reference to figure 5, it is possible to note that the function presents an absolute extreme and a relative extreme respectively corresponding to the x_1 and x_2 value of the parameter *x*; in this case the operative uncertainties could be represented by the tolerance δ of the input parameter *x*. Obviously a standard optimization, that does not consider the fluctuations, would find out the point x_1. On the contrary, a robust design optimization would find both the point x_1, which corresponds to the highest mean value of *f(x)*, and the point x_2, that corresponds to the highest value of stability of the function inside the tolerance range δ.

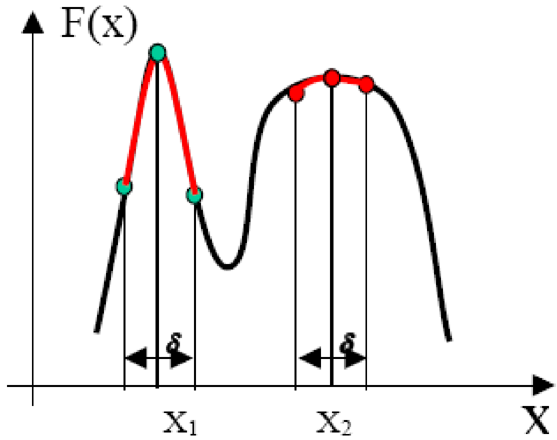

Figure 5: Selection of the best out put design parameter with robust design optimization

In a traditional (deterministic) context, aerodynamic shape optimization of airfoils is concerned with obtaining the most aerodynamically favourable geometry for fixed, either known or assumed, operating or design conditions. Consider the practical case where the drag C_d is to be minimized with lift-constrained $C_l \geq C_l^*$ at a given, fixed free flow Mach number:

$$\begin{cases} \min_{a \in A} C_d(a, M) \\ \text{subject to } C_l(a, M) \geq C_l^* \end{cases} \tag{3}$$

where *a* is the vector of design variables, *A* is the design space and C_l^* is the minimal lift value.

This deterministic, single-point optimization model is not necessarily an accurate reflection of the reality. The formulation in (3) contains no information regarding off-design condition performance. It is documented by other researchers [5] that, with formulation (3), the drag reduction is attained only over a narrow range of Mach numbers (see Figure 6a). We will refer to this in the remainder as "localized optimization". Drela explains that the optimizer creates a "bump" on the airfoil to fill the transitional separation bubble (see Figure 6b). This effectively reduces the drag penalty which occurs when a bubble undergoes transition and reattachment [5]. However, the location of this bubble varies with M and this explains the poor behaviour in off-design conditions for this "locally optimized" design.

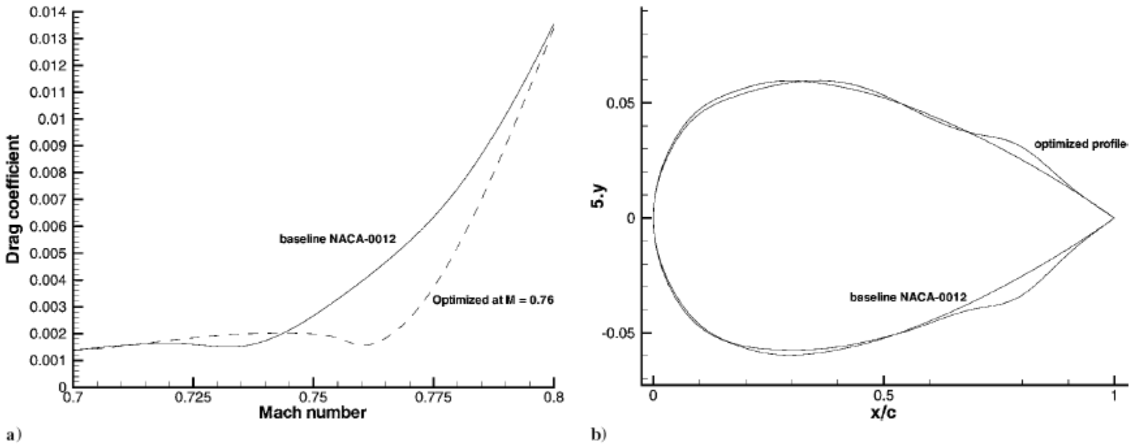

Fig. 6 Single-point optimization using $C_l^* = 0.2$ **, drag profile and aigrfoil geometry [17].**

It can be concluded that the real problem is not with the optimization code, which is likely to perform well, but with the problem formulation of (3). The local optimization effect is particularly worrisome if substantial variability is associated with the operating conditions. Explicit tradeoffs between different design conditions should be considered in the problem formulation.

Figure 7 shows that choice of *M* design dramatically affects the design performance. This method is not clear which point to select as design point. The mean value is not a good choice for the design point when the model is highly non-linear and also the highest Mach number ($M = 0.8$) is not necessarily the best design point either.

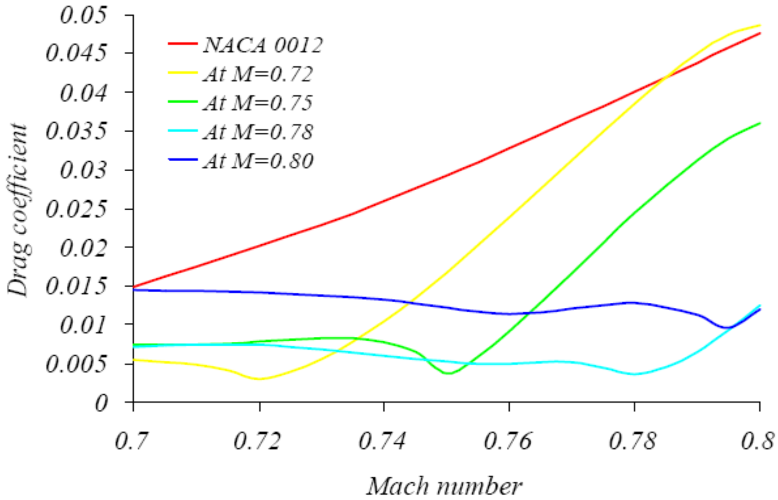

Fig. 7 Different choices of *M* design and their effects on performance [17].

Now, we mention some robust design methods which regard fluctuated (uncertain) parameters. Developing optimization methods that result in more robust designs and regard fluctuated (uncertain) parameters sounds appealing. There are five popular methods of robust design: multipoint, minimax, expected value, second order second moment and Taguchi method.

2.1 Multipoint method:

A straightforward, but heuristic, approach to avoid localized optimization is to consider different Mach numbers and to generalize the objective in (3) to a linear combination of flight conditions (*m* in total):

$$\begin{cases} \min_{a\in A} \sum_i^m w_i\, C_d(a, M_i) \\ subject\ to\ C_{l,i}(a, M_i) \geq C_l^* \quad for\ i = 1,\ldots,m \end{cases} \tag{4}$$

where w_i ’s are positive weights and M_i ’s are the Mach numbers.

Drela [5] studied the behaviour of the optimization solutions of (3) in two-dimensional viscous flow when the number of free-design variables is relatively large. He concluded that increasing the number of geometric design variables requires a corresponding increase in the number of design conditions (Mach numbers) used in the multipoint optimization problem (4). He also suggested that the number of design points must be greater than the number of free-design variables to achieve a smooth airfoil geometry. Other notable conclusions made by Drela (1998) are as follows:

(A) Near-continuous sampling of the operating space (i.e., in the range of Mach numbers) may be required in the theoretical limit of a general airfoil design problem with a very large number of degree of freedom (for geometric variables), a very expensive proposition.

(B) The most suitable operating points to be actually sampled in multipoint airfoil optimization (i.e., M_1, M_2, . . . , M_m) are not apparent a priori. From limited experience, sampling somewhat beyond the expected operating range appears to be best.

(C) The point weights (i.e., w_1, w_2, . . . , w_m) used in multipoint airfoil optimization are arbitrary, and their appropriate values can not be easily estimated without prior experience.

(D) Optimized aerodynamic shapes are usually "noisy" and require a posteriori smoothing.

It is noticeable that W. Li has given a mathematical argument to validate Drela's hypothesis [14].

Practical problems arise with the selection of the flight conditions M_i and with the specification of the weights w_i. There are no clear theoretical principles to guide the selection, which is, in fact, largely left up to the designer's discretion (see, for example, [5,6,7,15]).

With the multipoint formulation of (4), an improved C_d can be realized over a wider range of Mach numbers M [5]. However, this formulation is still unable to avoid localized optimization. In fact, multiple bumps might appear on the airfoil, one associated with each flight conditions M_i. In the transonic regime, each bump occurs at the shock foot location for each of the sampled Mach numbers.

The two-point optimization results shown in Fig. 8 illustrate the shortcomings of this method, which were also reported by Drela [5]. Optimization at selected Mach numbers results in clearly distinguishable drag troughs at each of the design Mach numbers (Fig. 8). Drela leaves it up to the designer to determine which Mach numbers to include in the objective in (4) and which weights to choose. Three reasonable selections are compared with each other in Fig. 8: the endpoints of the Mach interval and selected interior points. It is clear that, at least in this case, the selection of the design conditions has an important effect on the final results. In particular, the selection of the endpoints of the Mach range can lead to troubling results. We observed this in both two-point (Fig. 8) and four-point (Fig. 9) optimization.

Because the drag is higher in the upper part of the Mach range, one may want to include more design Mach numbers from the upper part than from the lower part in the multipoint objective function in (4). This procedure does not require any additional function calls, keeping the computational cost under control, and it should result in lower drag at the upper end of the Mach range. Figure 9 shows the result of such a multipoint optimization. The selected Mach numbers are indicated on the chart. The maximum drag is reduced, but a penalty is paid near the lower end of the Mach range. In fact, the drag has increased from 0.0044 to 0.0055 compared with the result for evenly spaced Mach numbers. Also, the drag trough at each sample Mach number persists, which indicates localized optimization, as explained earlier. In his study, Drela applies larger weights to the upper part of the Mach range to ensure that the upper part is not compromised excessively by the less important lower part.

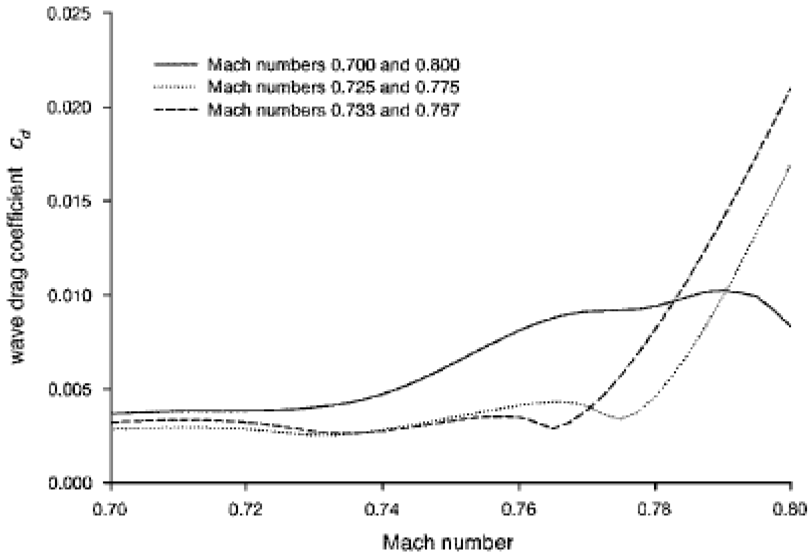

Fig. 8 Optimal drag profiles obtained using different two-point optimization Strategies [12] ($w_1 = w_2$ = 0.5 and C_l^* = 0.6).

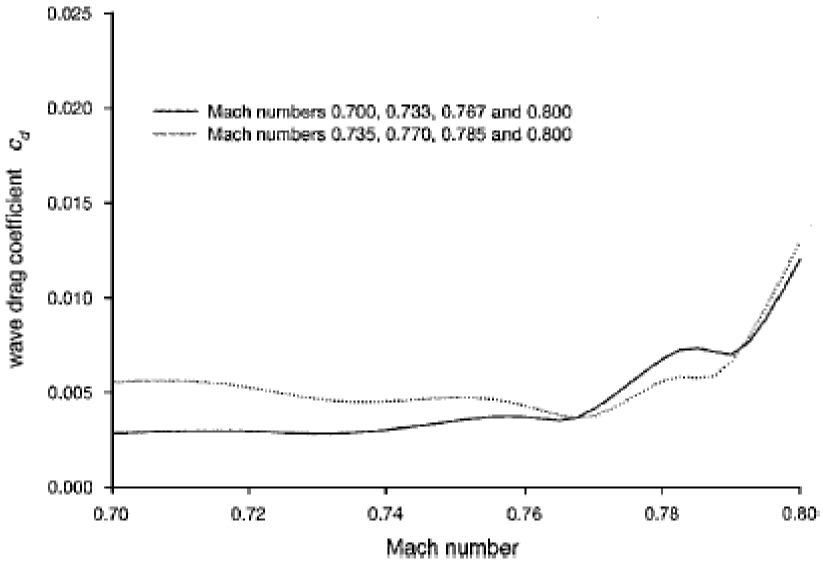

Fig. 9 Drag profile obtained using different four-point optimization Strategies [12] ($w_1 = w_2 = w_3 = w_4$ = 0.25 and C_l^* = 0.6).

Figure 10 explains this in more detail using two contour plots of the local Mach number. The operating conditions (free stream Mach numbers) are very similar, but the flow solutions (local Mach number) are very different. The multipoint optimization process introduces geometric features to the airfoil that lock the shock waves in place. Because we used four-point optimization, we have four shocks (see also [5]). In Fig. 10a, four shocks can be distinguished along the top surface. In Fig. 10b, the most-aft shock has basically disappeared. The optimizer has locally modified the geometry and eliminated the shock associated with Mach 0.8, which was one of the design conditions.

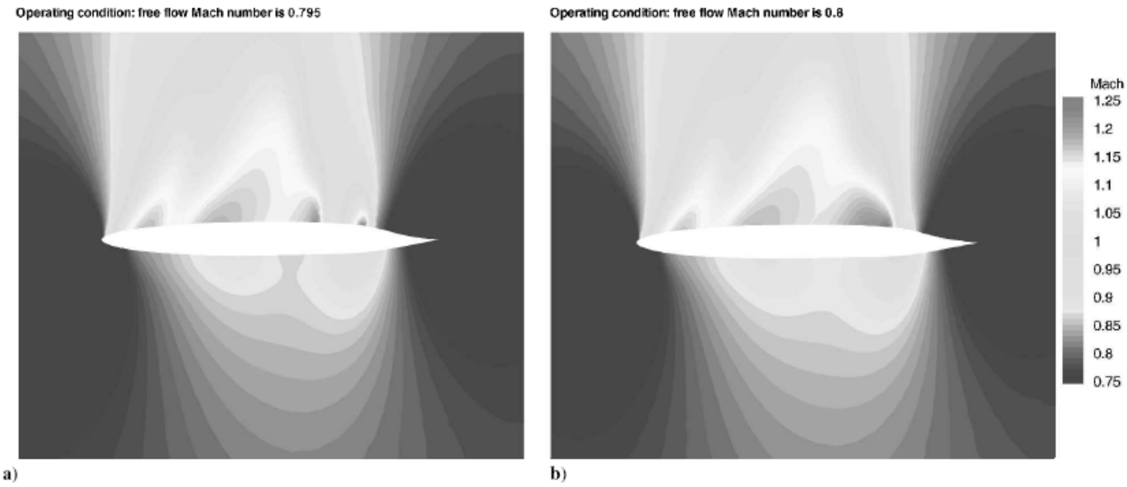

Fig. 10 Local Mach number for two free-flow conditions after four-point optimization using C_l^* = 0.4 [17].

These results indicate that a multipoint optimization achieves a better overall drag reduction than single-point optimization for both sets of design Mach numbers. However, a drag trough may form at or near each of the discrete design points, in which case the drag increases rapidly away from the design points. This effect becomes more pronounced near the high end of the Mach range.

In order to avoid arbitrariness of weights, some new methods have been introduced. In [19] adaptive weights are also proposed in conjunction with a new scheme that permits to automatically add new points to prevent from off-design loss. Another method, called profile optimization method that relies on adaptive adjustments of the weights, has been introduced to achieve a consistent reduction of the cost function f over a range of operating conditions [14].

Alternative Strategies

Because derivatives have to be calculated for each variable at various operating conditions, the computational cost associated with multipoint optimization using CFD is substantial. Alternate methods, which incorporate some common-sense engineering knowledge into the optimization process, may be a valuable and cheaper alternative. One such example is the method of the weighted average of geometries (WAG) [15].

In the WAG method, the optimal design is obtained as a weighted average of n single-point optimal designs, each one of them corresponding to one of the n chosen operating conditions. The weighting factors depend on the relative importance of each operating conditions. However, a key feature of this optimization method is that it still requires the formulation of some aggregate objective function, which describes the overall, or aggregate, performance of the design as a function of the variable Mach number.

The method requires the designer to select appropriate weights before the optimization process is started. As a result, the quality of the optimum solution is directly related to the actual choice of the selected weights. Sometimes such a choice may be quite difficult to make, and this selection introduces some arbitrariness in the design process.

In the analytic hierarchy process [21], the weights can be changed during the optimization process itself. For each optimization step, a pair wise comparison matrix can be defined that indicates the relative importance of each of the design conditions at the current iteration step in the optimization process. The method monitors the optimization progress and gives the designer the opportunity to adjust the relative weight accordingly.

The need to adjust the weights is eliminated altogether in Messac's physical programming method [19]. In this method, the designer expresses preferences concisely using a classification function. Examples thereof are smaller-is-better, range-is-better, and must-be larger.

In addition a degree of desirability is associated with each of these classes, ranging from unacceptable to highly desirable. By the use of these preference functions, the different design metrics are all mapped onto a dimensionless scale on which the actual optimization is performed. The method seems quite well suited for problems where various objectives have different dimensions, such as range and speed.

2.2 Minimax method:

We can also use the following minimax optimization formulation for robust optimization under uncertainty (Ben-Tal and Nemirovski 1997) [22]:

$$\begin{cases} \min\limits_{a \in A} \max\limits_{M_{\min} \le M \le M_{\max}} \rho(M) C_d(a, M) \\ subject\ to\ C_l(a, M) \ge C_l^* \quad for\ M_{\min} \le M \le M_{\max} \end{cases} \tag{5}$$

Here $\rho(M) > 0$ is a positive weighting function of M.Since we can not compute C_l and C_d for all M in the range [$M_{\min}, M_{\max}$], computationally tractable approximations of (5) must be used. The simplest approximation scheme is to replace [$M_{\min}, M_{\max}$] by a finite subset of [$M_{\min}, M_{\max}$], say {$M_1, M_2, \ldots, M_m$}. Then minimax formulation (5) can be discretized as follows:

$$\begin{cases} \min\limits_{a \in A} \max\limits_{1 \le i \le m} \rho_i C_d(a, M_i) \\ subject\ to\ C_{l,i}(a, M_i) \ge C_l^* \quad for\ 1 \le i \le m \end{cases} \tag{6}$$

Here $\rho_i > 0$ is determined by $\rho(M)$ and M_i.

The objective of minimax strategies is to mitigate the detrimental effects of the worst-case performance, which can be used to find a design with the optimal worst-case performance. This problem also was studied by Huyse and Lewis (2001)[14] ; Huyse (2001)[18] and Tang (2005)[23].

It seems that (4) and (6) are completely different. However, under the strict complementarity condition (i.e., Lagrange multipliers are nonzero for all active constraints), (4) is mathematically equivalent to (6). W. Li and L. Huyse have shown this in their research [20].

2.3 Expected value-based method:

Reconsider the basic problem in (3). If we minimize the drag C_d over a range of free-flow Mach numbers *M* while maintaining the lift $C_l > C_l^*$, the optimization problem (3) is now interpreted as a statistical decision making problem. Note that *M* is now treated as a random variable.

In the presence of uncertainty, a designer is forced, in effect, to take a gamble. Under such circumstances, rather than naively hoping for the best or conservatively focusing on the worst, the right decision consists of the best possible choice of the design, whether favourable or unfavourable operating conditions occur. All decision problems have two essential characteristics [24]:
1) A choice, or sequence of choices, must be made among various possible designs.
2) Each of these choices corresponds to a performance, but the designer cannot be sure a priori what this performance will be. The exact performance also depends in part on unpredictable events, in this case the operating conditions.

In this example, only one initial choice regarding the design needs to be made; the airfoil geometry must be selected. According to the Von Neumann–Morgenstern statistical decision theory (see Ref. 24), the best course of action in the presence of uncertainty is to select the design airfoil that leads to the lowest expected drag. This is commonly known as the maximum (or minimum) expected value criterion. The risk ρ, associated with a particular design *a*, is identified as the expected value of the perceived loss associated with the design. The best design or decision, which minimizes the overall risk, is referred to as Bayes's decision. In our problem formulation, Bayes's risk ρ^* and Bayes's decision a^* are given by Eqs. (7) and (8), respectively:

$$\begin{cases} \rho^* = \min_{a \in A} \int C_d(a, M) f_M(M) d(M) \\ \textit{subject to } C_l(a, M) \geq C_l^* \quad \textit{for all } \; M \end{cases} \tag{7}$$

or

$$\rho^* = \int C_d(a^*, M) f_M(M) d(M) \tag{8}$$

where $f_M(M)$ is the PDF of the free-flow Mach number *M*.

The practical problem with formulation (7) is that integration is required in each of the optimization steps. Because the objective function C_d is computationally expensive to evaluate, this approach, although theoretically sound, becomes prohibitively expensive. Therefore, a computational scheme that minimizes the number of function calls is desirable.

In addition, the physical and mathematical models themselves used for the objective function will not be error free. Each of these model errors can be treated as a random variable. Their effect on the optimal solution is readily assessed by extending the integration over these additional random variables.

Note that in this problem we are not concerned with rapid Mach number variations. Only slowly varying Mach numbers (steady states) are considered. Because the Mach number is constant for a certain length of time, the angle of attack can be adjusted to reach the required lift C_l^* . Consequently, the lift constraint in (7) is not probabilistic but remains deterministic.

The integration with respectto *M* in (7) can also be performed numerically. Irrespective of the chosen integration scheme, integral (7) can formally be written as (*m* integration points):

$$\rho^* = \min\left[\sum_{k=1}^{m} w_k C_d\left(a, M_k\right) + \varepsilon(m)\right] \tag{9}$$

where the integration error $\varepsilon(m) \to 0$ when $m \to \infty$.

Formulation (9) is strikingly similar to (4). It is, therefore, interesting to analyze how Bayes's decision a^* compares with the multipoint solution and exactly how localized optimization is avoided. In the multipoint approach, the design condition Mach numbers and weights need to be selected by the designer. In the statistical approach, the Mach numbers are determined by the integration scheme. The weights are directly related to the relative importance of each Mach number through the integration over the probability density. In short, the statistical approach removes the arbitrariness from the weighting process. Comparison of (4) with (9) reveals the shortcoming in the multipoint formulation that causes localized optimization. Numerical integration of (7) results in Eq. (9) and includes a random, zero-mean error term $\varepsilon(m)$, which decreases as the number of sampling points increases. The multipoint optimization (4) differs from (9) only in the sense that this error term is not explicitly considered in the objective function. However, omitting this term alters the structure of the problem at hand.

The multipoint optimization looks for the design, which minimizes the weighted sum of the goal function C_d, evaluated in the m specified points M_k. There is no control over the objective function C_d in the neighbourhood around these *m* sampling points. Experience by other researchers [25] and the preceding deterministic examples have indicated that significant troughs in plots of C_d vs M are introduced near the sampling points. In effect, multipoint optimization will prefer a design a_1 over a design a_2 even when design a_1 is considerably worse than design a_2 in all but the m specified sampling points. The multipoint formulation allows the optimizer to mold the goal function C_d to its own advantage. What was originally a random integration error is no longer random, and the discrete sum in (4) no longer approximates the integral in (7) at all.

This undesirable behaviour is avoided if we can prevent the optimizer from exploiting the approximation error in (9) to its own advantage. We need to make sure that the discrete sum in (9) remains a good approximation of the integral in (7) throughout the optimization process. An elegant solution is to randomize the sampling points M_k in the evaluation of the integral. This ensures that the optimizer maximizes the performance not just for *m* specific values of M_k, but for any set of values M_k ; k =1, . . . , *m*. To minimize the loss of accuracy in the integration due to random location of the integration points, stratified sampling can be used to generate the values of M_k. Our experience with the spline-based integration also suggests that the sampling points should not be allowed to be arbitrarily close to each other.

However, with this scheme, a repeated evaluation of the objective function C_d for identical values of the design parameters *a* will lead to different results. This makes it hard to identify whether a new design is really better than a preceding one, or if the improvement should be attributed to random fluctuations instead. When a trial solution *a* is still far away from the optimal solution a^*, large improvements ΔC_d can be expected. This means that a very crude integration, which requires very few function evaluations, will suffice in the early stages of the optimization. The improvement of the goal function is expected to be smaller closer to the optimal solution, and more sampling points M_k will be required to keep the integration error small enough. Current research focuses on the development of a strategy that takes maximum advantage of this effect.

Relatively small random perturbations can be used to change the integration points, but other, more adaptive strategies are currently being researched. The number of integration points must be sufficiently high so that the integration error caused by the change of integration points between optimization steps is smaller than the decrease of drag in that particular optimization step.

Figure 11 shows that EV optimization strategy with normal distribution for *M* using random sampling points M_k in the evaluation of the integral results in a much smoother drag profile over the entire Mach range. The resulting airfoil geometry is somewhat smoother as well. It may be concluded that, for the same computational effort as multipoint algorithm, the EV scheme results in a superior design.

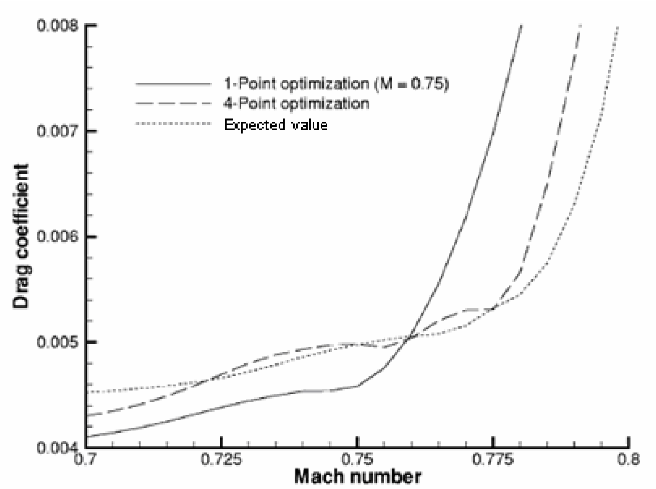

Fig. 11. Drag profile obtained using different optimization strategies (results obtained using C_l = 0.175) [12].

Impact of PDF: In most practical cases the Mach number will not be uniformly distributed over a given range. A key advantage of the explicitly statistical approach is that the relative importance of each operating condition is automatically accounted for through the PDF.

Quite often a cruise Mach number is set and the assumption of a (truncated) "normal" or Gaussian distribution around this mean value seems appropriate. The truncated normal PDF is very well approximated by a Beta distribution. The Beta distribution is always bounded within an interval [a, b] and has shape parameters α_1 and α_2, both greater than zero:

$$Beta(x,a,b,\alpha_1,\alpha_2)=\frac{\left(\frac{x-a}{x-b}\right)^{(\alpha_1-1)}\left(\frac{b-x}{b-a}\right)^{(\alpha_2-1)}}{(b-a)\times B(\alpha_1,\alpha_2)} \tag{10}$$

where $B(\alpha_1,\alpha_2)$=, (α_1), (α_2)/, (α_1 + α_2)is the Beta-function.

Depending on the value of α_1 and α_2, the Beta PDF can assume a variety of shapes (see Figure 12). When $\alpha_1 = \alpha_2$ the distribution is symmetric and can be used as an approximation for the truncated normal, especially for $\alpha_i \geq 5$. When $\alpha_1 = \alpha_2 = 1$, the distribution becomes uniform. For $\alpha_1 \neq \alpha_2$ the distribution is skewed towards either left or right. Bath-tub distributions are obtained when both parameters in the PDF are less than 1 (but greater than 0).

Figure 13 compares the optimal drag-profile obtained using three different Beta-distributions bounded within [0.7, 0.8]. When a PDF Beta (3, 1) is assumed for the Mach number, the higher Mach numbers are more likely to occur. Their impact is easily discerned from Figure 13. The greater weight of the higher Mach numbers translates into a lower drag in the high Mach range. The trade-off is a higher drag at the lower end of the Mach range. For a Beta (l, 3)-distribution, a lower drag is obtained in the lower part of the Mach range at the expense of much faster drag increase for higher Mach numbers. When a Beta (5, 5) PDF is used, the higher likelihood of Mach numbers near the mean value *M* = 0.75 results in the lowest drag of all three curves near the middle of the Mach range.

This illustrative comparison reveals the importance of an accurate quantification of the PDF of the Mach range. In general, careful data analysis is required when the PDF is selected for each of the uncertain variables. Practical experience has revealed that especially the tails of the distributions need to be modelled very carefully because they tend to have a large impact [26]. Cut-off values should most definitely not be chosen arbitrarily!

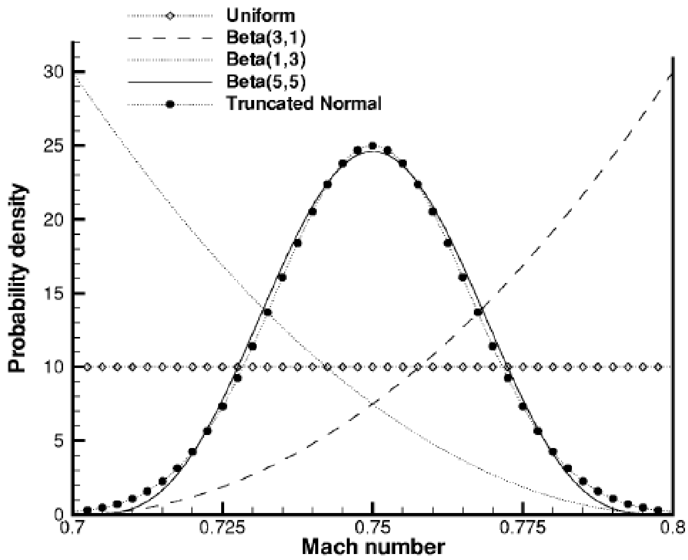

Fig. 12. Sample Beta distributions, and comparison with truncated Normal [12]

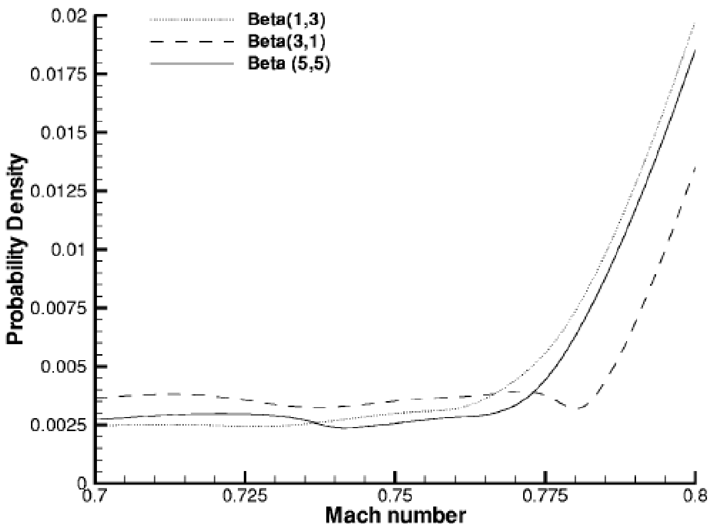

Fig. 13. Impact of assumed PDF for *M* on optimal drag profile [12].

2.4 Second order second moment method (SOSM):

When the variability of the free-flow Mach number *M* is not too large, a second-order Taylor series expansion of C_d around the mean value $\overline{M}$ may be a sufficiently accurate model of the variation of the drag C_d with respect to *M*:

$$C_d(a,M) = C_d(a,\overline{M}) + \nabla_M C_d.(M-\overline{M}) + \frac{1}{2}\nabla_M^2 C_d.(M-\overline{M})^2 \tag{11}$$

When substituted in Bayes's risk expression (7), the linear term disappears after integration over *M* because the Taylor series is built around the mean value $\overline{M}$. Bayes's risk Eq. (4) can be approximated by

$$\begin{aligned} &\rho^* = \min\left[C_d(a,\overline{M}) + Var(M)\nabla_M^2 C_d(a,\overline{M})\right] \\ &subject\ to\ C_l(a,M) \ge C_l^* \quad for\,all\ \ M \end{aligned} \tag{12}$$

where $Var(M)$ denotes the variance of the Mach number *M*.

It seems that we have substituted the integration with an almost equally expensive computation of a second-order derivative. However, this theoretical result provides additional insight into the problem. It follows from (11) that the variability of *M* can affect the optimal design only if the objective function C_d is highly nonlinear in this parameter.

In mathematical terms, the advantage of working with expected utilities is that the minimum is second-order accurate with respect to variations in the parameters. This ensures a more global solution and localized optimization will be avoided. This can also be explained in an intuitive manner: The second-order derivative is a measure for the curvature. Because this curvature is now a part of the objective function, a design that results in a drag trough or cusp at $\overline{M}$ as found in the optimal solution will not be accepted by the optimizer. The high curvature of the cusp at $\overline{M}$ would increase the objective in (11), and excessive localized optimization will be avoided.

Equation (11) indicates that first order sensitivities of the drag C_d with respect to the uncertain variable *M* do not affect the expected value of the design. The second order information represents the curvature of the $C_d(M)$ curve, and a large value of curvature near a design Mach number is indicative of localized optimization. In the SOSM formulation, the weighting between the drag and the curvature is determined by the variance of the Mach number. Figure 14 shows a considerable reduction in curvature of C_d vs *M* using SOSM.

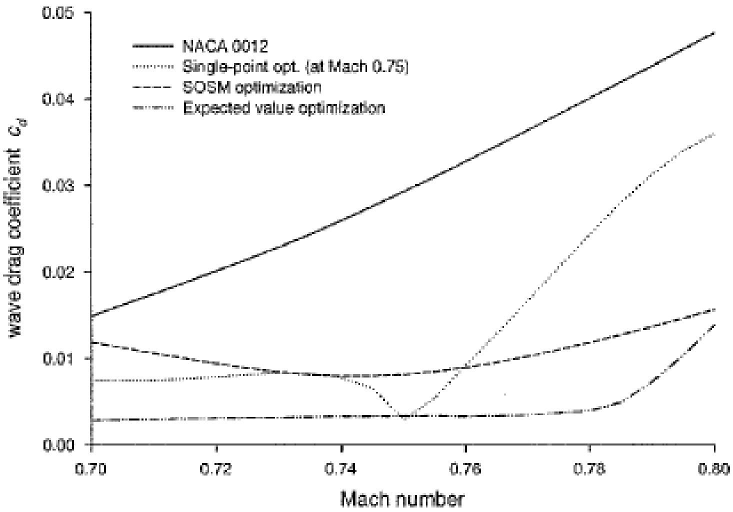

Fig. 14 Comparison of SOSM result with single-point and EV optimization [17].

The overall drag reduction is not as good as obtained using explicit numerical integration (Fig. 11), but the computational effort is a lot smaller. The method is particularly useful if higher-order derivatives are available (and numerically reliable) and several uncertain variables are present in the problem. Table 1 compares the relative computational cost for each optimization step. SOSM scales linearly with the number of random variables (one additional second derivative is required for each additional random variable in the optimization), whereas full numerical integration rapidly becomes expensive. Because the method is based on a second-order Taylor series approximation of the objective function, SOSM will give the best results if the variance of the random variables is relatively small.

Table 1. Number of function/derivative evaluations required per optimization step (Note: SOSM requires less if analytic derivatives are available) [12].

Optimization Method	1 Random Variable	3 Random Variables
Single Point	1	1
SOSM	3	7
Expected Value (using 4 points)	4	64

2.5 Taguchi Method:

Early attempts to account for design uncertainties in the framework of quality engineering are closely connected with Taguchi, the ‘‘father of robust design’’ who envisioned a three-stage design methodology comprising [27]:

(1) Systems design: determines the basic performance parameters of the product and its general structure.
(2) Parameter design: optimizes the design parameters in order to meet the quality requirements.
(3) Tolerance design: fine-tuning of the design parameters obtained in the second stage.

The system design is a step where new ideas are generated to provide products to customers. Within the parameter design step, the designer determines the optimum setting for control factors using orthogonal arrays and SN ratios. The manufacturing cost will not be affected by the parameter design, since the tolerances are fixed. The final goal of the parameter design is that the designer makes products insensitive to noise factors without eliminating them. The tolerance design is implemented to improve quality at a minimum cost. It should be used when the sensitivity of responses resulting from the parameter design is not satisfactory. In particular, the parameter design scheme of the Taguchi method is adopted for robust design.

From viewpoint of mathematical optimization, the differentiation between the second and the third stage seems superfluous since both stages differ only in the granularity by which design parameters are treated (of course practically the classification might be important because stage two and three can occur under very different constraints, e.g., design time vs. operation time). That is why we will only concentrate on the second stage.

Taguchi argued that one should design a product in such a way as to make its performance insensitive to variation in variables beyond the designer's control. His methods for robust design distinguish two types of inputs to a system: control parameters (or control factors) are the inputs that can be easily controlled or manipulated by the designer, hence the inputs that constitute optimization variables *x*; noise variables (or noise factors) are the inputs that are difficult or expensive to control, hence the inputs *b* to whose variation product performance is desired to be insensitive. For example, *x* might specify the design of a photocopier and *b* might specify the environment in which it must operate.

The main difference of Taguchi's method compared to ordinary optimization lies in the accounting for performance variations due to noise factors beyond the control of the designer. That is, there are two kinds of parameters entering the objective function: control parameters *x*, which are to be tuned to optimality, and noise factors *b*, such as environmental conditions (e.g., temperature, pressure, etc.) and production tolerances (e.g., weight and length variations, purity of material used, etc.) difficult to be controlled by the designer.

In the statistical approach, one consider the fluctuating operating conditions $\mathbf{b} = (b_i)_{i=1,\dots,N}$ as samples of random variables $\mathbf{B} = (B_i)_{i=1,\dots,N}$, whose statistical characteristics are known (mean $\mu(\mathbf{B}) = (\mu_B^i)_{i=1,\dots,N}$, variance $\sigma^2(\mathbf{B}) = (\sigma_B^{2\,i})_{i=1,\dots,N}$ etc). One also suppose for the sake of simplicity that the random variables $\mathbf{B} = (B_i)_{i=1,\dots,N}$ are independent. The statistical characteristics of operating conditions can be determined by experimental measurements or engineering experience. Gaussian Probability Density Functions (PDFs) or truncated Gaussian PDFs are often used in practice (see [12] for instance).

The main consequence of this assumption is that the cost function of the problem is also a random variable *f*. According to the Von Neumann-Morgenstern decision theory, the best choice is then to select the design which leads to the best expected fitness. This is known as the Maximum Expected Values (MEV) criterion (Sec. 2.3). The decision or design that minimizes the risk is known as the Bayes' decision and is solution of the following problems:

$$\text{Minimize} \quad \mu_f = \int_{\Omega(B)} f(x,b)\rho_B(b)d(b) \tag{13}$$

However, problem (13) does not address the variability of the fitness. The mean value of the fitness is the only criterion that is considered in the Bayes' decision. For engineering problems, one also would like to select a design for which the fitness is not subject to large variation as operating conditions fluctuate. Then, a second criterion is often joined to the MEV criterion that relies on the minimization of the variance σ_f^2 of the fitness:

$$\text{Minimize} \begin{cases} \mu_f = \int_{\Omega(B)} f(x,b)\rho_B(b)d(b) \\ \sigma_f^2 = \int_{\Omega(B)} \left(f(x,b) - \mu_f\right)^2 \rho_B(b)d(b) \end{cases} \tag{14}$$

This approach aims at determining a trade-off between the expected fitness and the expected fitness variation as operating conditions randomly fluctuate. Although this approach is satisfactory from theoretical and practical viewpoints, its application is not straightforward. Particularly, the estimation of the mean and variance can be tedious from complex CFD applications. This issue is detailed below.

2.5.1 Deterministic viewpoint:

For a given design determined by *x*, the statistical mean μ_f and variance σ_f^2 are defined by:

$$\begin{aligned} \mu_f &= \int_{\Omega(B)} f(x,b)\rho_B(b)d(b) \\ \sigma_f^2 &= \int_{\Omega(B)} \left(f(x,b) - \mu_f\right)^2 \rho_B(b)d(b) \end{aligned} \tag{15}$$

These integrals can not be analytically evaluated since no analytical formulation of *f* is available. To estimate them, one should either provide an analytical approximation of the cost function or discretize the integrals. The latter solution is quite similar to the multipoint approach, since the integrals become weighted sums of the cost functions evaluated for some prescribed operating conditions. Then, the former approach using an approximation $\tilde{f}$ is usually preferred:

$$\tilde{\mu}_f = \int_{\Omega(B)} \tilde{f}(x,b)\rho_B(b)d(b)$$
$$\tilde{\sigma}_f^2 = \int_{\Omega(B)} \left(\tilde{f}(x,b) - \tilde{\mu}_f\right)^2 \rho_B(b)d(b) \quad (16)$$

2.5.2 Stochastic viewpoint:

To estimate the mean and variance of the random variable f, one can simply use statistical estimators in a classical Monte-Carlo approach. A sample of operating conditions is $(b_i)_{i=1,\dots,N}$ if size N is generated according to the PDF. Then, unbiased estimators of the mean and variance are:

$$M_f = \sum_{i=1}^{N} f(x,b_i)$$
$$S_f^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(f(x,b_i) - M_f\right)^2 \quad (17)$$

This approach does not suffer from point-optimization effect since the sample $(b_i)_{i=1,\dots,N}$ is generated randomly according to the PDF ρ_B. However, it is well known that this stochastic approach requires a large sample to provide an accurate estimation of the statistics. For CFD applications, a direct Monte-Carlo method is not conceivable presently. Nevertheless, a cheaper approximation $\tilde{f}$ of the cost function can be used in a Monte-Carlo approach to estimate the mean and variance, as suggested in [28]:

$$\tilde{M}_f = \sum_{i=1}^{N} \tilde{f}(x,b_i)$$
$$\tilde{S}_f^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(\tilde{f}(x,b_i) - \tilde{M}_f\right)^2 \quad (18)$$

Chapter 3

Game Strategies for Multi objective Optimization

Multi objective optimization can be described as a methodology for the design of systems where the interaction between several objectives must be considered, and where the designer is free to significantly influence the system performance with more than one objective.

Although single objective optimization problems may have a unique optimal solution, multi-objective optimization problems offer a possibly uncountable set of solutions, which when evaluated produce vectors whose components represent trade-offs in decision space. A decision maker then implicitly chooses an acceptable solution by selecting one of these vectors. Mathematically speaking, an multi objective optimization problem minimizes (or maximizes) the components of a vector $f(x)$ where x is an n-dimensional decision variable vector, or in general

$$\begin{aligned} &\textit{Minimize} \quad f(x) = \{f_1(x),\dots, f_p(x)\} \\ &\textit{subject to} \quad g_i(x) \le 0 \;\; i = 1,\dots, m \end{aligned} \tag{19}$$

A multi objective optimization problem then consists of n variables, m constraints, and p objectives ($p \geq 2$), of which any or all of the objective functions may be nonlinear (Hwang and Masud 1979 [29]). Multi objective optimization problems are often characterized by measures of performance (objectives) which may be (in) dependent and/or non-commensurable; the multiple objectives being optimized almost always conflict. These opposing objectives place a partial, rather than total, ordering on the search space. In order to successfully deal with these characteristics, several EC-based methods (Fonseca and Fleming 1995 [30]) were developed to determine optimal solutions given a multi-objective optimization problems objectives and constraints. One of the best approaches to many of these methods, however, is the use of Game Theory.

Game Theory [31], formulated mathematically by J.F.Nash in the early 50s, has found their first applications in economics, in particular to solve the problems concerning the decisions that have some effects on different and often competitive fields.

These strategies may however been adopted also in the industrial design, and in particular they can be combined with Evolutionary Algorithms, in order to optimise a product following several criteria and objectives, with the great advantage to save a lot of computational time, that is perhaps the first need in industrial field.

We shortly describe the basic ideas of three main Game Strategies, while in the following chapters we will show how these strategies can be combined with Evolutionary Algorithms and become a robust algorithm for the multi-objective optimisation.

Currently, there are tow different Game strategies to solve the above problem:
1) Cooperative Games (Pareto Front)
2) Competitive Games (Nash Game)

3.1 Cooperative Games (Pareto Front):
In a problem of minimisation of two functions $f(A)$ and $f(B)$, we define the variables space $(x, y) \in A \times B$ as the set of rational strategies. In fact, if we consider *A* and *B* as two players, each pair (x,y) represents a combination of the strategies played by the two players.

The Pareto front may be seen as the result of a cooperative game, in which the two players *A* and *B* try to minimize both the two functions; in other words, each strategy played by the players is paid by the fitness of the two functions, it means how much the solution satisfies the objectives of minimization of the two functions.

Not a single solution is found, but instead a set of solutions that is called as *Pareto front*. This set is characterised by the fact that there does not exist a solution such that both the two functions have a better fitness of any point of the front. In mathematical terms:

$\left(x^*, y^*\right) \in A \times B$ to Pareto front if and only if:

$$\exists (X,Y) \in A \times B : \quad \begin{cases} f_A(X,Y) \leq f_A\left(x^*, y^*\right) \\ f_B(X,Y) \leq f_B\left(x^*, y^*\right) \end{cases} \tag{20}$$

The analytical Pareto front solution of above optimization problem is computed by introducing a weighting function $\lambda \in [0,1]$, then make a linear combination such as $F(x) = \lambda f_1 + (1-\lambda) f_2$. Thus, a parametric representation of the optimal Pareto set is obtained by solving:

$$\begin{cases} \dfrac{\partial F}{\partial x} = 0 \\ \dfrac{\partial F}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} x = x(\lambda) \\ y = y(\lambda) \end{cases} \tag{21}$$

So, on the Pareto front, there exist $f_1(x,y) = f_1(\lambda)$ and $f_2(x,y) = f_2(\lambda)$ which are function of weighting constants, the Pareto front pairs are ($f_1(\lambda_i)$, $f_2(\lambda_i)$), where $\lambda_i \in [0,1]$. The zero gradient of linear combined function $F(x)$ implies that the minimum solution of function $F(x)$ at a given λ is on the Pareto front.

3.2 Competitive Games (Nash Game):

Nash optima define a non-cooperative (competitive) multiple objectives optimization approach firstly proposed by J.F. Nash [31]. Since it originated in Games Theory and Economics, the notion of player is often used and we kept it. For an optimization problem with N objectives defined. A Nash strategy consists in having *N* players, each optimizing his own criterion. However, each player has to optimize his criterion given that all the other criteria are fixed by the rest of the players. When no player can further improve his criterion, it means that the system has reached a state of equilibrium called *Nash Equilibrium*.

Let X_i be the search space for the ith criterion, $X_i \subset X = X_1 \otimes ... \otimes X_i \otimes ... \otimes X_N$. An strategy pair $(x_1^*, x_2^*, ..., x_N^*) \in X$ is said to be a *Nash Equilibrium* if and only if:

$$\begin{aligned} f_i(x_1^*, x_2^*, ..., x_N^*) &= \inf_{x_i \in X_i} f_i(x_1^*, x_2^*, ..., x_{i-1}^*, x_i^*, x_{i+1}^*, ..., x_N^*) \\ & for\ i = 1,2,...,N \end{aligned} \tag{22}$$

Now, we present how 2 players try to optimize 2 different objectives. Of course, it is possible to have *n* players optimizing *n* criteria as presented in the definition. But to make thing as clear as possible, we will restrict ourselves to *n* = 2.

Let's assume that we have two design targets f_1 and f_2, whatever they are conflict or not, these two targets are functions of design variables $x = (x_1, x_2)$, where $x \in X$ and $X = X_1 \otimes X_2$, $x_1 \in X_1$ and $x_2 \in X_2$. We further assume that two targets are both minimization problem for convenience.

Player1 is responsible for f_1 by modifying x_1, *Player2* is responsible for f_2 by modifying x_2, so the design problem can be explained as follows:

$$\begin{aligned} & Player1: \min_{x_1} f_1(x_1, x_2) \\ & Player2: \min_{x_2} f_2(x_1, x_2) \end{aligned} \tag{23}$$

Where x_1 is the free design variable of cost function f_1, x_2 is fixed in *Player1* and comes from the result of *Player2*, similarly x_2 is the free design variable of cost function f_2, x_1 is fixed in *Player2* and comes from the result of *Player1*.

Let $x^{m-1} = (x_1^{m-1}, x_2^{m-1})$ is the best design variables in $m-1$ design iterations, where x_1^{m-1} is the best design found by *Player1* at $m-1$ step, and x_2^{m-1} is the best design found by *Player2* at $m-1$ step. At *m* step, *Player1* optimizes x_1 from x_1^{m-1} to achieve a better value x_1^m while using x_2^{m-1} in order to evaluate f_1. At the same time, *Player2* optimizes x_2 from x_2^{m-1} to achieve a better value x_2^m while using x_2^{m-1} in order to evaluate f_2. In this case

$$
\begin{aligned}
f_1(x_1^m, x_2^{m-1}) &= \inf_{x_1 \in X_1} f_1\,(x_1, x_2^{m-1}) \\
f_2(x_1^{m-1}, x_2^m) &= \inf_{x_2 \in X_2} f_2\,(x_1^{m-1}, x_2)
\end{aligned}
\tag{24}
$$

After the optimization process, *Player1* sends the best value x_1^m to *Player2* who will use it at *m* + 1 step, Similarly, *Player2* sends the best value x_2^m to *Player1* who will use it at *m* + 1 step. So the best solution at the end of step *m* is $x^m = (x_1^m, x_2^m)$. Nash equilibrium is reached when neither *Player1* nor *Player2* can further improve their criteria.

In fig.15 we reproduce the comparison of the Pareto front with Nash.

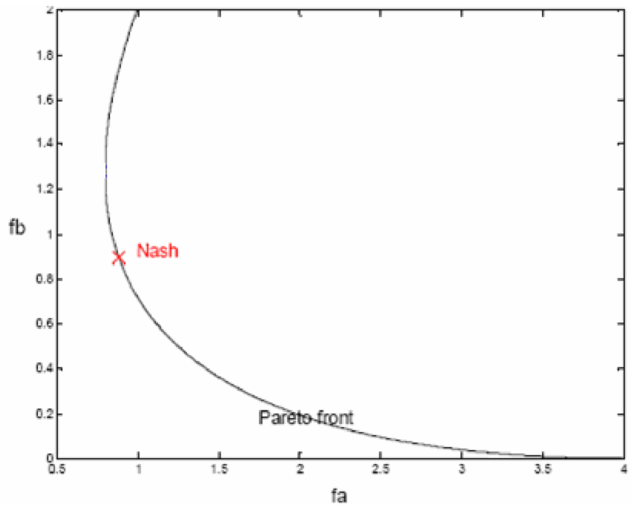

Fig. 15. Comparison of Pareto and Nash solutions.

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