

Advanced Reliability Analysis with Applications of Hjorth Constant-Stress Normal-Operating Setting Using Newly Progressive Censored Data

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ABSTRACT

Modern high-reliability products fail rarely, so researchers rely on accelerated life testing to obtain failure information within practical time limits. This study presents a useful framework integrating constant-stress accelerated life tests with an improved adaptive progressive Type-II censoring plan to estimate the reliability function under normal operating conditions of the Hjorth model. The Hjorth model is chosen because its hazard rate can be constant, increasing, decreasing, or bathtub-shaped, which reduces errors due to incorrect hazard assumptions. Stress affects lifetime through a log-linear relationship applied to the scale and the shape parameter. We derive the full likelihood for the proposed censoring plan across several stress levels, obtain maximum likelihood estimates with confidence intervals based on the observed information matrix, and develop a Bayesian analysis with informative prior distributions and sampling by Markov chain Monte Carlo technique. We then estimate reliability at normal operating conditions together with its interval estimates by both approaches. Extensive Monte Carlo simulations demonstrate the superior accuracy of the Bayesian estimators, especially when the number of observed failures is small or censoring is heavy, while maintaining interval coverage close to the nominal level. The practical utility of the proposed methodology is demonstrated through its application to real-world accelerated lifetime data sets. Applications to real-world data sets show that the proposed model fits the data well and yields reliable estimates of reliability at normal operating conditions.

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1 Introduction

Many products are engineered to function for years, but testing their lifespan under normal conditions is difficult because failures rarely occur within a limited testing period. To address this, accelerated life tests (ALTs) are used. In ALTs, products are exposed to higher stress levels (e.g., increased heat, voltage, or pressure) than normal to cause failures more quickly. Engineers then use

statistical models to extrapolate the data from these extreme conditions and estimate how the products would perform under regular use. This includes predicting key metrics like the reliability function (RF) and hazard rate function. Escobar and Meeker [1] provide a detailed review of ALTs and explain how different stress factors relate to accelerated failure times. ALTs typically follow two methods: constant-stress ALTs (CSALTs), where products are tested at a fixed high-stress level until failure or the test ends, and step-stress testing, where stress is increased gradually at predefined times during the test to induce failures faster. The CSALTs plans have been considered for numerous lifetime distributions., see for example, Guan et al. [2] for generalized exponential distribution, Dey and Nassar [3] for exponentiated Lindley distribution, Wu et al. [4] for two-parameter exponential distribution under Type-II censoring, Du and Gui [5] for Gompertz distribution, and Feng and Tang [6] for Weibull distribution. On the other hand, one may refer to the works of Lee and Pan [7], and Kateri and Nikolov [8], among others, for more detail about the step-stress ALTs.

In reliability studies, the integration of ALTs with censoring plans has become a widely adopted practice. This approach allows experiments to conclude more quickly, avoiding the need to observe failures in all test units. Various censoring methods are employed in reliability studies, each offering distinct advantages. Traditional approaches include Type-I (time-based), Type-II (failure-based), and hybrid censoring schemes, where no units are removed until the conclusion of the test, see for more detail Balakrishnan and Kundu [9]. In contrast, modern reliability experiments often utilize multi-stage censoring methods that permit the removal of units at different stages. Notable examples include progressive Type-II censoring (PTIIC) plan, in which test units are systematically withdrawn during the experiment, and adaptive PTIIC (APTIIC) plan, introduced by Ng et al. [10], which dynamically adjusts the removal process based on real-time observations. These flexible methods improve efficiency by reducing test duration and resource use while retaining statistical accuracy. In recent years, these censoring plans have been extensively utilized by researchers. For instance, Wu and Gui [11], El-Sherpieny et al. [12], and Yao and Gui [13] have conducted significant work on the PTIIC plan, while Elshahhat and Nassar [14], Dutta et al. [15], and Anakha and Chacko [16] have concentrated on the APTIIC plan.

The APTIIC scheme is designed to ensure a predetermined number of failures by the conclusion of the test while permitting researchers to cease the removal of operational units once a specified time limit is reached. Ng et al. [10] observed that the APTIIC plan is most effective in scenarios where strict time constraints on testing are not imposed. Nevertheless, in the context of highly reliable products, this methodology may result in excessively lengthy testing durations. To address this limitation, Yan et al. [17] proposed an enhanced version referred to as the Improved APTIIC (IAPTIIC) scheme. This approach integrates concepts from PTIIC, APTIIC, and various other censoring methodologies, ensuring that the test is completed within a defined deadline while still achieving the target number of failures. Several studies that have considered the IAPTIIC scheme include Dutta and Kayal [18], who explored it within the context of a competing risks model; Zhang and Yan [19], who examined its application to the Chen distribution; Swaroop et al. [20], who investigated its relevance to the generalized inverted exponential distribution; and Irfan et al. [21], who analyzed its implications for the Kumaraswamy-G family of distributions. See also for more detail Swaroop et al. [20] and Alotaibi et al. [22]. In the following section, we detail the operational framework of the IAPTIIC scheme within the context of the CSALTs, which constitutes the primary focus of this study.

In this study, we apply the IAPTIIC plan under CSALTs to analyze failure times modeled using the Hjorth distribution proposed by Hjorth [23]. To our knowledge, this distribution has rarely been studied in ALTs, especially in CSALTs. The Hjorth distribution is uniquely flexible, modeling four hazard rate trends: constant, decreasing, increasing, and bathtub-shaped. Despite having only two

parameters, a scale and a shape, it outperforms other two-parameter models like the Weibull and gamma in flexibility. Hjorth [23] showed that its RF equals the product of the RFs of the Rayleigh and Lomax distributions, making it ideal for scenarios where multiple failure causes compete. However, few works explore its use in reliability analysis, particularly for estimating parameters or metrics like RF and the hazard rate function. Exceptions include Yadav et al. [24], Elshahhat and Nassar [14] and Alotaibi et al. [25]. In the following section, we define the key functions of the Hjorth distribution. This study is driven by three primary motivations:

1. The integration of the IAPTIIC scheme with CSALTs: This combination allows earlier test termination while still delivering accurate estimates of reliability at normal operating conditions. This is valuable when failures are rare, as is typical for modern high-reliability products.
2. The flexibility of the Hjorth distribution in modeling lifetime data: The Hjorth distribution can represent constant, decreasing, increasing, and bathtub-shaped hazards using only two parameters. This flexibility reduces the risk of model misspecification and supports more reliable extrapolation from accelerated stress to normal use.
3. The lack of prior applications of the IAPTIIC scheme in CSALT settings: Despite its advantages, only one known study by Nassar et al. [26] has been employed, who used it for the Weibull distribution. This gap highlights the novelty of applying the IAPTIIC plan to the Hjorth model in the presence of CSALTs.

These points emphasize the originality and real-world value of our work. This paper aims to develop methods to estimate the Hjorth distribution's parameters as well as the reliability under normal operating conditions under the IAPTIIC plan in CSALTs. We assume the Hjorth distribution's scale and shape parameters vary with stress levels and follow a log-linear relationship with stress. The contributions of this study are:

1. Formulation of CSALTs with the IAPTIIC plan, and derivation of the full joint likelihood across multiple stress levels.
2. Development of classical maximum likelihood estimates (MLEs) because it is consistent, asymptotically efficient, and enables direct construction of confidence intervals from the observed information; interval estimates for both parameters and use-level reliability are derived via the delta method.
3. Derivation of maximum likelihood estimators (MLEs) together with approximate confidence intervals (ACIs) for the unknown parameters and the reliability at normal operating conditions. The Maximum likelihood estimation is developed because it yields consistent, asymptotically efficient estimates and a direct route to confidence intervals via the observed information.
4. Development of a Bayesian estimation framework because it handles small samples and heavy censoring, allows incorporation of prior information, and provides full posterior uncertainty through Bayes credible intervals (BCIs). The Bayes estimates (BEs) are acquired using the squared error loss function and the Markov Chain Monte Carlo (MCMC) technique.
5. Comparative evaluation of estimators efficiency in a simulation study, and demonstration of practical relevance through two real-world applications.

The paper is organized as follows: [Section 2](#) introduces the Hjorth distribution model and key assumptions. [Section 3](#) details the process of obtaining the MLEs and constructing ACIs for the parameters and RF under normal operating conditions. [Section 4](#) discusses the BEs and BCIs using

the MCMC procedure. Section 5 describes the simulation study designed to test the accuracy of the proposed methods. Section 6 applies the methods to two real-world data sets. Some practical and theoretical implications are presented in Section 7, while final remarks are given in Section 8.

2 Model Description and Assumptions

In this study, we assume product lifetimes follow the Hjorth distribution. This study is conducted under two key assumptions:

1. Under the stress level $s_i, i = 1, \dots, k$, the lifetimes of the test units are modeled using the Hjorth distribution, with the following probability density function (PDF) and cumulative distribution function (CDF):

$$g_i(y; \theta_i, \mu_i) = (1 + y)^{-(\mu_i+1)} [\mu_i + \theta_i y (1 + y)] e^{-0.5\theta_i y^2}, y > 0, \theta_i, \mu_i > 0, \quad (1)$$

and

$$G_i(y; \theta_i, \mu_i) = 1 - (1 + y)^{-\mu_i} e^{-0.5\theta_i y^2}, \quad (2)$$

respectively, where θ_i and μ_i are the scale and shape parameters, respectively.

2. We assume that the scale parameter θ_i and shape parameter μ_i follow a log-linear relationship with stress. This means their natural logarithms are modeled as linear functions of the applied stress level s_i , expressed as:

$$\log(\theta_i) = a_0 + a_1 s_i \text{ and } \log(\mu_i) = b_0 + b_1 s_i, i = 1, \dots, k, \quad (3)$$

here, a_0, a_1, b_0 , and b_1 are unknown coefficients parameters. These parameters depend on the product's characteristics and the testing conditions.

Under the CSALTs, consider k increasing stress levels $s_1 < s_2 < \dots < s_k$ with a normal-use stress level s_u , where $s_u < s_1$. A total of N identical test units are divided into k groups of sizes $n_1, n_2, \dots, n_k$, with $\sum_{i=1}^k n_i = N$. Before testing, the following quantities are predetermined for each stress level $s_i, i = 1, \dots, k$:

- The desired number of failures $m_i < n_i$.
- The progressive censoring plan $(Q_{i1}, \dots, Q_{im_i})$ satisfying $n_i = m_i + \sum_{j=1}^{m_i} Q_{ij}$.
- Two time thresholds T_{i1} and T_{i2} ($0 < T_{i1} < T_{i2} < \infty$).

The IAPT2C procedure under CSALTs operates as follows: At each stress level s_i , where $i = 1, \dots, k$, upon the occurrence of the first failure, denoted by Y_{i1} , Q_{i1} surviving units are randomly removed from the test. Similarly, at the time of the second failure Y_{i2} , Q_{i2} units are randomly withdrawn from the remaining survivors, and this process continues. In this context, we consider three distinct cases, which are outlined as follows:

1. **Case I:** If the m_i -th failure occurs before T_{i1} ($Y_{im_i} < T_{i1}$), terminate the test at Y_{im_i} and remove all remaining $Q_{im_i} = n_i - m_i - \sum_{j=1}^{m_i-1} Q_{ij}$ units.
2. **Case II:** If T_{i1} falls between the r_i -th and $(r_i + 1)$ -th, i.e., $Y_{ir_i} < T_{i1} < Y_{ir_i+1}$, and $Y_{im_i} < T_{i2}$, then removals are suspended after T_{i1} by setting $Q_{ir_i+1} = \dots = Q_{im_i-1} = 0$. The test is then terminated at Y_{im_i} , and the final removal is set as $Q_{im_i} = n_i - m_i - \sum_{j=1}^{r_i} Q_{ij}$.
3. **Case III:** If the m_i -th failure occurs after T_{i2} , i.e., $Y_{im_i} > T_{i2}$, then testing is stopped at T_{i2} with $r_i^* < m_i$ observed failures, with the understanding that no units will be discarded from the

experiment when the test time passes the threshold T_{i1} . Then, at T_{i2} , all remaining units are removed, where $Q_i^* = n_i - r_i^* - \sum_{j=1}^{r_i} Q_{ij}$.

Let $a = (a_0, a_1)$ and $b = (b_0, b_1)$, then based on the observed IAPTIIC data, denoted by $\underline{y}$, we can write the likelihood function of a and b as given below, without constant terms,

$$L(a, b; \underline{y}) = \prod_{i=1}^k \left\{ \prod_{j=1}^{\mathfrak{J}_i} g_i(y_{ij}) \prod_{j=1}^{\mathfrak{D}_i} [1 - G_i(y_{ij})]^{Q_{ij}} [1 - G_i(\tau_i)]^{Q_i^*} \right\}, \quad (4)$$

where

$$\mathfrak{J}_i = \begin{cases} m_i, & \text{for Cases I and II;} \\ r_i^*, & \text{for Case III,} \end{cases} \quad \mathfrak{D}_i = \begin{cases} m_i - 1, & \text{for Case I;} \\ r_i, & \text{for Cases II and III,} \end{cases}$$

$$Q_i^* = \begin{cases} n_i - m_i - \sum_{j=1}^{m_i-1} Q_{ij}, & \text{for Case I;} \\ n_i - r_i - \sum_{j=1}^{r_i} Q_{ij}, & \text{for Case II;} \\ n_i - r_i^* - \sum_{j=1}^{r_i} Q_{ij}, & \text{for Case III,} \end{cases}$$

and $\tau_i = x_{im_i}$ for Cases I and II, and T_{i2} for Case III.

3 Likelihood Inference

In this section, we discuss the MLEs of the parameters a and b using IAPTIIC in conjunction with CSALT data. Under normal operating conditions s_u , the MLEs for the scale parameter θ_u , shape parameter μ_u and the RF at time t , denoted by $R_u(t)$, are also derived. Furthermore, interval bounds for the various parameters are constructed based on the asymptotic properties of the MLEs.

3.1 Point Estimation

Let $y_{ij}^* = 1 + y_{ij}$, $\tau_i^* = 1 + \tau_i$, then the likelihood function can be obtained using (1)–(4), as

$$L(a, b; \underline{y}) = \exp \left[\sum_{i=1}^k \sum_{j=1}^{\mathfrak{J}_i} \log(e^{b_0+b_1 s_i} + y_{ij} y_{ij}^* e^{a_0+a_1 s_i}) - \sum_{i=1}^k \psi_i e^{b_0+b_1 s_i} - \sum_{i=1}^k \varphi_i e^{a_0+a_1 s_i} \right], \quad (5)$$

where

$$\psi_i = \sum_{j=1}^{\mathfrak{J}_i} \log(y_{ij}^*) + \sum_{j=1}^{\mathfrak{D}_i} Q_{ij} \log(y_{ij}^*) + Q_i^* \log(\tau_i^*)$$

and

$$\varphi_i = 0.5 \left(\sum_{j=1}^{\mathfrak{J}_i} y_{ij}^2 + \sum_{j=1}^{\mathfrak{D}_i} Q_{ij} y_{ij}^2 + Q_i^* \tau_i^2 \right).$$

Using (5), the associated log-likelihood function, denoted by $\mathfrak{L}(a, b; \underline{y})$, is given by

$$\mathfrak{L}(a, b; \underline{y}) = \sum_{i=1}^k \sum_{j=1}^{\mathfrak{J}_i} \log(e^{b_0+b_1 s_i} + y_{ij} y_{ij}^* e^{a_0+a_1 s_i}) - \sum_{i=1}^k \psi_i e^{b_0+b_1 s_i} - \sum_{i=1}^k \varphi_i e^{a_0+a_1 s_i}. \quad (6)$$

To obtain the MLEs of the parameters a, b , denoted by $\hat{a} = (\hat{a}_0, \hat{a}_1)$ and $\hat{b} = (\hat{b}_0, \hat{b}_1)$, we compute the partial derivatives of (6) with respect to each parameter. These derivatives are then set to zero, and the resulting system of equations is solved as follows

$$\frac{\partial \mathcal{L}(a, b; \underline{y})}{\partial a_0} = \sum_{i=1}^k \sum_{j=1}^{\mathfrak{J}_i} \frac{y_{ij} y_{ij}^* e^{a_0 + a_1 s_i}}{e^{b_0 + b_1 s_i} + y_{ij} y_{ij}^* e^{a_0 + a_1 s_i}} - \sum_{i=1}^k \varphi_i e^{a_0 + a_1 s_i} = 0, \quad (7)$$

$$\frac{\partial \mathcal{L}(a, b; \underline{y})}{\partial a_1} = \sum_{i=1}^k \sum_{j=1}^{\mathfrak{J}_i} \frac{s_i y_{ij} y_{ij}^* e^{a_0 + a_1 s_i}}{e^{b_0 + b_1 s_i} + y_{ij} y_{ij}^* e^{a_0 + a_1 s_i}} - \sum_{i=1}^k s_i \varphi_i e^{a_0 + a_1 s_i} = 0, \quad (8)$$

$$\frac{\partial \mathcal{L}(a, b; \underline{y})}{\partial b_0} = \sum_{i=1}^k \sum_{j=1}^{\mathfrak{J}_i} \frac{e^{b_0 + b_1 s_i}}{e^{b_0 + b_1 s_i} + y_{ij} y_{ij}^* e^{a_0 + a_1 s_i}} - \sum_{i=1}^k \psi_i e^{b_0 + b_1 s_i} = 0 \quad (9)$$

and

$$\frac{\partial \mathcal{L}(a, b; \underline{y})}{\partial b_1} = \sum_{i=1}^k \sum_{j=1}^{\mathfrak{J}_i} \frac{s_i e^{b_0 + b_1 s_i}}{e^{b_0 + b_1 s_i} + y_{ij} y_{ij}^* e^{a_0 + a_1 s_i}} - \sum_{i=1}^k s_i \psi_i e^{b_0 + b_1 s_i} = 0. \quad (10)$$

Eqs. (7)–(10) do not have closed-form solutions. To solve them numerically, iterative methods like the Newton-Raphson algorithm can be applied. Once the MLEs $\hat{a}$ and $\hat{b}$ of the parameters a and b are obtained, the MLE of the RF $R_u(t)$ under normal conditions s_u can be calculated using the invariance property of MLEs. Specifically, for a given time t , the MLE of $R_u(t)$ is derived as follows: First, compute the MLEs of the scale and shape parameters under normal conditions, respectively, as $\hat{\theta}_u = e^{\hat{a}_0 + \hat{a}_1 s_u}$ and $\hat{\mu}_u = e^{\hat{b}_0 + \hat{b}_1 s_u}$.

Then, the MLE of $R_u(t)$ under normal operating settings is given by

$$\hat{R}_u(t) = (1 + t)^{-\hat{\mu}_u} e^{-0.5 \hat{\theta}_u t^2}.$$

3.2 Interval Estimation

To construct the ACIs for the unknown parameters or any function of them, we rely on the asymptotic properties of the MLEs. As the sample size increases, the MLEs exhibit asymptotic normality, implying that they approximately follow a normal distribution. This property enables the construction of confidence intervals and the performance of hypothesis testing using normal distribution theory. However, because the exact distributions of the MLEs are generally intractable, we approximate the required variance-covariance matrix by inverting the observed Fisher information matrix, as follows

$$\Sigma(\hat{a}, \hat{b}) = \begin{bmatrix} \frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial a_0^2} & \frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial a_0 \partial a_1} & \frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial a_0 \partial b_0} & \frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial a_0 \partial b_1} \\ \frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial a_1^2} & \frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial a_1 \partial b_0} & \frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial a_1 \partial b_1} & \\ \frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial b_0^2} & \frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial b_0 \partial b_1} & & \\ \frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial b_1^2} & & & \end{bmatrix}_{(a,b)=(\hat{a},\hat{b})}^{-1}$$

$$= \begin{bmatrix} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} & \hat{\Sigma}_{13} & \hat{\Sigma}_{14} \\ & \hat{\Sigma}_{22} & \hat{\Sigma}_{23} & \hat{\Sigma}_{24} \\ & & \hat{\Sigma}_{33} & \hat{\Sigma}_{34} \\ & & & \hat{\Sigma}_{44} \end{bmatrix}, \quad (11)$$

where, the second derivatives of the log-likelihood function are as follows

$$\frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial a_0^2} = - \sum_{i=1}^k \sum_{j=1}^{\mathfrak{I}_i} \frac{y_{ij} y_{ij}^* e^{a_0 + b_0 + (a_1 + b_1) s_i}}{(e^{b_0 + b_1 s_i} + y_{ij} y_{ij}^* e^{a_0 + a_1 s_i})^2} - \sum_{i=1}^k \varphi_i e^{a_0 + a_1 s_i},$$

$$\frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial a_1^2} = - \sum_{i=1}^k \sum_{j=1}^{\mathfrak{I}_i} \frac{s_i^2 y_{ij} y_{ij}^* e^{a_0 + b_0 + (a_1 + b_1) s_i}}{(e^{b_0 + b_1 s_i} + y_{ij} y_{ij}^* e^{a_0 + a_1 s_i})^2} - \sum_{i=1}^k s_i^2 \varphi_i e^{a_0 + a_1 s_i},$$

$$\frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial b_0^2} = - \sum_{i=1}^k \sum_{j=1}^{\mathfrak{I}_i} \frac{y_{ij} y_{ij}^* e^{a_0 + b_0 + (a_1 + b_1) s_i}}{(e^{b_0 + b_1 s_i} + y_{ij} y_{ij}^* e^{a_0 + a_1 s_i})^2} - \sum_{i=1}^k \psi_i e^{b_0 + b_1 s_i},$$

$$\frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial b_1^2} = - \sum_{i=1}^k \sum_{j=1}^{\mathfrak{I}_i} \frac{s_i^2 y_{ij} y_{ij}^* e^{a_0 + b_0 + (a_1 + b_1) s_i}}{(e^{b_0 + b_1 s_i} + y_{ij} y_{ij}^* e^{a_0 + a_1 s_i})^2} - \sum_{i=1}^k s_i^2 \psi_i e^{b_0 + b_1 s_i},$$

while the mixed partial derivatives are given by

$$\frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial a_0 \partial a_1} = - \sum_{i=1}^k \sum_{j=1}^{\mathfrak{I}_i} \frac{s_i y_{ij} y_{ij}^* e^{a_0 + b_0 + (a_1 + b_1) s_i}}{(e^{b_0 + b_1 s_i} + y_{ij} y_{ij}^* e^{a_0 + a_1 s_i})^2} - \sum_{i=1}^k s_i \varphi_i e^{a_0 + a_1 s_i},$$

$$\frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial a_0 \partial b_0} = - \sum_{i=1}^k \sum_{j=1}^{\mathfrak{I}_i} \frac{y_{ij} y_{ij}^* e^{a_0 + b_0 + (a_1 + b_1) s_i}}{(e^{b_0 + b_1 s_i} + y_{ij} y_{ij}^* e^{a_0 + a_1 s_i})^2},$$

$$\frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial a_0 \partial b_1} = \frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial a_1 \partial b_0} = - \sum_{i=1}^k \sum_{j=1}^{\mathfrak{I}_i} \frac{s_i y_{ij} y_{ij}^* e^{a_0 + b_0 + (a_1 + b_1) s_i}}{(e^{b_0 + b_1 s_i} + y_{ij} y_{ij}^* e^{a_0 + a_1 s_i})^2},$$

$$\frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial a_1 \partial b_1} = - \sum_{i=1}^k \sum_{j=1}^{\mathfrak{I}_i} \frac{s_i^2 y_{ij} y_{ij}^* e^{a_0 + b_0 + (a_1 + b_1) s_i}}{(e^{b_0 + b_1 s_i} + y_{ij} y_{ij}^* e^{a_0 + a_1 s_i})^2},$$

$$\frac{\partial^2 \mathcal{L}(a, b; \underline{y})}{\partial b_0 \partial b_1} = - \sum_{i=1}^k \sum_{j=1}^{\mathfrak{I}_i} \frac{s_i y_{ij} y_{ij}^* e^{a_0 + b_0 + (a_1 + b_1) s_i}}{(e^{b_0 + b_1 s_i} + y_{ij} y_{ij}^* e^{a_0 + a_1 s_i})^2} - \sum_{i=1}^k s_i \psi_i e^{b_0 + b_1 s_i}.$$

Employing the asymptotic property of the MLEs that $(\hat{a}, \hat{b})$ asymptotically follow the normal distribution with mean (a, b) and estimated variance-covariance matrix $\Sigma(\hat{a}, \hat{b})$, as given by (11). The $100(1 - \alpha)\%$ ACIs for a and b are given by

$$\hat{a}_0 \pm z_{\alpha/2} \sqrt{\hat{\Sigma}_{11}}, \quad \hat{a}_1 \pm z_{\alpha/2} \sqrt{\hat{\Sigma}_{22}}, \quad \hat{b}_0 \pm z_{\alpha/2} \sqrt{\hat{\Sigma}_{33}}, \quad \hat{b}_1 \pm z_{\alpha/2} \sqrt{\hat{\Sigma}_{44}},$$

where $z_{\alpha/2}$ is the upper $(\alpha/2)$ -th percentile of the standard normal distribution.

Constructing the ACI for the RF under normal operating conditions requires estimating the variance of the associated MLE $\hat{R}_u(t)$. A practical solution is the delta method, which approximates this variance using the following steps:

1. Compute the first-order partial derivatives of the RF with respect to the various parameters, evaluated at their MLEs as

$$\Delta_{R_u} = \left(\frac{\partial R_u(t)}{\partial a_0}, \frac{\partial R_u(t)}{\partial a_1}, \frac{\partial R_u(t)}{\partial b_0}, \frac{\partial R_u(t)}{\partial b_1} \right) \bigg|_{(a,b)=(\hat{a},\hat{b})}$$

2. Apply the delta method formula to estimate the variance as follows

$$\hat{\Sigma}_{R_u} \approx \Delta_{R_u} \Sigma(\hat{a}, \hat{b}) \Delta_{R_u}^\top,$$

where $\Sigma(\hat{a}, \hat{b})$ is given by (11).

The first-order partial derivatives of $R_u(t)$ are given by

$$\begin{aligned} \frac{\partial R_u(t)}{\partial a_0} &= -0.5t^2 e^{a_0+a_1 s_u} (1+t)^{-e^{b_0+b_1 s_u}} e^{-0.5t^2 e^{a_0+a_1 s_u}}, \\ \frac{\partial R_u(t)}{\partial a_1} &= -0.5t^2 s_u e^{a_0+a_1 s_u} (1+t)^{-e^{b_0+b_1 s_u}} e^{-0.5t^2 e^{a_0+a_1 s_u}}, \\ \frac{\partial R_u(t)}{\partial b_0} &= -\ln(1+t) e^{b_0+b_1 s_u} (1+t)^{-e^{b_0+b_1 s_u}} e^{-0.5t^2 e^{a_0+a_1 s_u}}, \\ \frac{\partial R_u(t)}{\partial b_1} &= -\ln(1+t) s_u e^{b_0+b_1 s_u} (1+t)^{-e^{b_0+b_1 s_u}} e^{-0.5t^2 e^{a_0+a_1 s_u}}. \end{aligned}$$

Accordingly, the $100(1 - \alpha)\%$ ACI for the RF under the designed stress s_u is then calculated as

$$\hat{R}_u(t) \pm z_{\alpha/2} \sqrt{\hat{\Sigma}_{R_u}}.$$

Likelihood-based inference for model parameters and the RF at normal operating conditions may have important limitations. The MLEs may be unstable and imprecise in small samples or under heavy progressive censoring, which increases uncertainty. Misspecification of the Hjoth distribution or of the life–stress relationship can introduce systematic bias in use-level reliability. Moreover, the ACIs based on large-sample theory for both the model parameters and the RF may not perform well and can exhibit poor coverage.

4 Bayesian Estimation

Classical estimation methods often produce accurate results when dealing with large data sets or complete data. However, in scenarios with small sample sizes or censored observations, these methods can become unreliable. Bayesian estimation addresses these limitations by integrating prior knowledge

into the analysis. This approach improves estimation accuracy, especially under challenging scenarios. In this section, we present Bayesian methods to estimate: (1) The model parameters a and b , (2) The RF under the designed stress level. The Bayesian framework provides both BEs and BCIs for these quantities.

4.1 Prior Knowledge and Posterior Distribution

The parameters a and b are assumed to be independent. Since the likelihood function in (5) has a complex form, there are no natural conjugate priors for these parameters. Additionally, deriving Jeffrey's priors is impractical due to the complicated structure of the Fisher information matrix. Given these challenges, we assign gamma prior distributions to the parameters a_1 and b_1 . The choice of gamma priors is motivated by their flexibility, which allows them to accommodate a wide variety of prior beliefs. Moreover, gamma priors typically do not introduce significant computational complexity in posterior evaluations or calculations, especially when utilizing the MCMC procedures. On the other hand, the parameters a_0 and b_0 may take positive or negative values, depending on the characteristics of the product under consideration. To reflect this, we assume that a_0 and b_0 follow normal prior distributions. This choice aligns with the nature of these parameters and ensures compatibility with their potential range of values. Based on the above assumptions, the joint prior distribution of a and b can be expressed as:

$$w(a, b) \propto a_1^{c_1-1} b_1^{c_2-1} \exp \left[-v_1 a_1 - v_2 b_1 - \frac{(a_0 - \vartheta_1)^2}{2\sigma_1^2} - \frac{(b_0 - \vartheta_2)^2}{2\sigma_2^2} \right], a_1, b_1 > 0, -\infty < a_0, b_0 < \infty, \quad (12)$$

where $c_h, v_h, \sigma_h > 0$ and $-\infty < \vartheta_h < \infty, h = 1, 2$, are the hyper-parameters.

Combining the observed data represented by the likelihood function in (5) with prior information given by the joint prior distribution in (12), the posterior distribution of the unknown parameters (a, b) can be written as

$$W(a, b | \underline{y}) = \frac{1}{A} a_1^{c_1-1} b_1^{c_2-1} \exp \left[\sum_{i=1}^k \sum_{j=1}^{3_i} \log (e^{b_0+b_1 s_i} + y_{ij} y_{ij}^* e^{a_0+a_1 s_i}) - \sum_{i=1}^k \psi_i e^{b_0+b_1 s_i} \right. \\ \left. - \sum_{i=1}^k \varphi_i e^{a_0+a_1 s_i} - v_1 a_1 - v_2 b_1 - \frac{(a_0 - \vartheta_1)^2}{2\sigma_1^2} - \frac{(b_0 - \vartheta_2)^2}{2\sigma_2^2} \right], \quad (13)$$

where A is given by

$$A = \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty w(a, b) L(a, b; \underline{y}) da_0 db_0 da_1 db_1.$$

Under the squared error loss function, the BE of a parameter (or function of parameters) corresponds to the posterior mean. Let $\varpi(a, b)$ denote any function of the unknown parameters (a, b) , such as a parameter itself, e.g., a_0, a_0, b_0 , or b_1 , as well as the RF under normal operating conditions. The BE of $\varpi(a, b)$ is given by the expectation of $\varpi(a, b)$ with respect to its posterior distribution

$$\tilde{\varpi}(a, b) = \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \varpi(a, b) W(a, b | \underline{y}) da_0 db_0 da_1 db_1. \quad (14)$$

The BE cannot be derived directly due to the complexity of the posterior distribution, making it necessary to apply the MCMC techniques to approximate BEs and construct credible intervals.

The MCMC methods are fundamental to Bayesian analysis, providing a practical means for numerically approximating and sampling from complex posterior distributions. Among these methods, the Metropolis-Hastings (MH) algorithm plays a key role by using a proposal distribution to efficiently explore the posterior space. The following subsection details the use of the MCMC procedures to generate samples from the posterior distribution and to obtain the required BEs and BCIs.

4.2 MCMC Procedures and Posterior Analysis

To apply MCMC methods, we first need to derive the full conditional distributions for each parameter. These distributions are derived directly from the joint posterior distribution defined in (13) as given below

$$W(a_0|a_1, b, \underline{y}) \propto \exp \left[\sum_{i=1}^k \sum_{j=1}^{\mathfrak{J}_i} \log (e^{b_0+b_1 s_i} + y_{ij} y_{ij}^* e^{a_0+a_1 s_i}) - \sum_{i=1}^k \varphi_i e^{a_0+a_1 s_i} - \frac{(a_0 - \vartheta_1)^2}{2\sigma_1^2} \right], \quad (15)$$

$$W(a_1|a_0, b, \underline{y}) \propto a_1^{c_1-1} \exp \left[\sum_{i=1}^k \sum_{j=1}^{\mathfrak{J}_i} \log (e^{b_0+b_1 s_i} + y_{ij} y_{ij}^* e^{a_0+a_1 s_i}) - \sum_{i=1}^k \varphi_i e^{a_0+a_1 s_i} - v_1 a_1 \right], \quad (16)$$

$$W(b_0|a, b_1, \underline{y}) \propto \exp \left[\sum_{i=1}^k \sum_{j=1}^{\mathfrak{J}_i} \log (e^{b_0+b_1 s_i} + y_{ij} y_{ij}^* e^{a_0+a_1 s_i}) - \sum_{i=1}^k \psi_i e^{b_0+b_1 s_i} - \frac{(b_0 - \vartheta_2)^2}{2\sigma_2^2} \right], \quad (17)$$

and

$$W(b_1|a, b_0, \underline{y}) \propto b_1^{c_2-1} \exp \left[\sum_{i=1}^k \sum_{j=1}^{\mathfrak{J}_i} \log (e^{b_0+b_1 s_i} + y_{ij} y_{ij}^* e^{a_0+a_1 s_i}) - \sum_{i=1}^k \psi_i e^{b_0+b_1 s_i} - v_2 b_1 \right]. \quad (18)$$

Examining the conditional distributions in (15)–(18), we find they do not follow standard probability forms. This lack of conjugate priors makes direct sampling methods (e.g., inverse transform sampling) impractical. To overcome this challenge we use the MH algorithm to iteratively generate samples from these complex posterior distributions. This algorithm utilizes a proposal distribution to produce candidate samples. In this analysis, the proposal distribution is taken to be a normal distribution, with its mean and variance initialized using the MLEs of the parameters. Through an iterative process, a sequence of dependent samples is generated, which asymptotically converges to the target posterior distribution. Below are the detailed steps for generating the MCMC samples:

Step 1: Set $(a^{(0)}, b^{(0)}) = (\hat{a}, \hat{b})$.

Step 2: Put $v = 1$.

Step 3: Derive $a_0^{(v)}$ from (15) based on the following MH algorithm:

(i) Obtain a proposal value a_0^* using $N(\hat{a}_0, \hat{\Sigma}_{11})$.

(ii) Obtain:

$$\zeta_{a_0} = \min \left\{ 1, \frac{W(a_0^*|a_1^{(v-1)}, b^{(v-1)}, \underline{y})}{W(a_0^{(v-1)}|a_1^{(v-1)}, b^{(v-1)}, \underline{y})} \right\}.$$

(iii) Simulate u from unit uniform distribution.

(iv) Put $a_0^{(v)} = a_0^*$ if $u \leq \zeta_{a_0}$; and set $a_0^{(v)} = a_0^{(v-1)}$, otherwise.

Step 4: Generate $a_1^{(v)}$ from (16) using the MH algorithm in step 3.

Step 5: Obtain $b_0^{(v)}$ from (17) using the MH algorithm in step 3.

Step 6: Simulate $b_1^{(v)}$ from (18) using the MH algorithm in step 3.

Step 7: At iteration v , obtain the RF under the designed stress level as:

$$R_u^{(v)} = (1 + t)^{-\mu_u} e^{-0.5\theta_u t^2},$$

where

$$\theta_u^{(v)} = e^{a_0^{(v)} + a_1^{(v)} s_u} \quad \text{and} \quad \mu_u = e^{b_0^{(v)} + b_1^{(v)} s_u}.$$

Step 8: Set $v = v + 1$.

Step 9: Carry out Steps 3–8 for $\mathcal{N}$ iterations to generate the MCMC samples:

$$\{a_0^{(v)}, a_1^{(v)}, b_0^{(v)}, b_1^{(v)}, R_u^{(v)}\}, \quad v = 1, \dots, \mathcal{N}.$$

Once MCMC samples are generated, calculating BEs and BCIs becomes straightforward. Under the squared error loss function, the BEs of the unknown parameters a, b and the RF under normal operating settings are

$$\begin{aligned} \tilde{a}_0 &= \frac{1}{\mathcal{N} - \mathcal{M}} \sum_{v=\mathcal{M}+1}^{\mathcal{N}} a_0^{(v)}, \quad \tilde{a}_1 = \frac{1}{\mathcal{N} - \mathcal{M}} \sum_{v=\mathcal{M}+1}^{\mathcal{N}} a_1^{(v)}, \quad \tilde{b}_0 = \frac{1}{\mathcal{N} - \mathcal{M}} \sum_{v=\mathcal{M}+1}^{\mathcal{N}} b_0^{(v)}, \\ \tilde{b}_1 &= \frac{1}{\mathcal{N} - \mathcal{M}} \sum_{v=\mathcal{M}+1}^{\mathcal{N}} b_1^{(v)}, \quad \text{and} \quad \tilde{R}_u(t) = \frac{1}{\mathcal{N} - \mathcal{M}} \sum_{v=\mathcal{M}+1}^{\mathcal{N}} R_u^{(v)}, \end{aligned}$$

where $\mathcal{M}$ is the burn-in period. To compute the BCIs, we first sort the acquired MCMC samples in ascending order as:

$$\begin{aligned} \{a_0^{[\mathcal{M}+1]} < \dots < a_0^{[\mathcal{N}]}\}, \quad \{a_1^{[\mathcal{M}+1]} < \dots < a_1^{[\mathcal{N}]}\}, \quad \{b_0^{[\mathcal{M}+1]} < \dots < b_0^{[\mathcal{N}]}\}, \\ \{b_1^{[\mathcal{M}+1]} < \dots < b_1^{[\mathcal{N}]}\}, \quad \text{and} \quad \{R_u^{[\mathcal{M}+1]} < \dots < R_u^{[\mathcal{N}]}\}. \end{aligned}$$

The $100(1 - \alpha)\%$ BCIs are obtained by computing the $\alpha/2$ and $(1 - \alpha/2)$ percentiles from the sorted samples as

$$\begin{aligned} \{a_0^{[\alpha(\mathcal{N}-\mathcal{M})/2]}, a_0^{[(1-\alpha/2)(\mathcal{N}-\mathcal{M})]}\}, \quad \{a_1^{[\alpha(\mathcal{N}-\mathcal{M})/2]}, a_1^{[(1-\alpha/2)(\mathcal{N}-\mathcal{M})]}\}, \quad \{b_0^{[\alpha(\mathcal{N}-\mathcal{M})/2]}, b_0^{[(1-\alpha/2)(\mathcal{N}-\mathcal{M})]}\}, \\ \{b_1^{[\alpha(\mathcal{N}-\mathcal{M})/2]}, b_1^{[(1-\alpha/2)(\mathcal{N}-\mathcal{M})]}\}, \quad \text{and} \quad \{R_u^{[\alpha(\mathcal{N}-\mathcal{M})/2]}, R_u^{[(1-\alpha/2)(\mathcal{N}-\mathcal{M})]}\}. \end{aligned}$$

Finally, it is important to mention that the Bayesian estimation method can be computationally expensive and time-consuming, especially with Markov chain Monte Carlo, which often requires long runs and substantial memory to stabilize estimates. Results are sensitive to prior specification; with small samples or heavy censoring the prior can materially affect parameter and use-level reliability estimates. In addition, the MCMC technique may exhibit poor mixing, convergence failures, or multimodality.

5 Monte Carlo Comparisons

To comprehensively assess and compare the efficiency of the proposed estimation techniques for the model parameters a_i, b_i (for $i = 0, 1$), and $R_u(t)$, we conduct several setups of a Monte Carlo simulation study. To achieve this goal, a total of 1000 IAPT2C samples using CSALTs from

Hjorth's lifespan model when $(a_0, a_1, b_0, b_1) = (0.2, 0.5, 0.4, 0.3)$ are generated. At the same time, when $s_u(=t) = 0.1$, we evaluate all estimates of $R_u(t)$ when its true value is set to be 0.87804. The evaluation is performed under varying configurations of effective sample sizes $n_i(m_i)$, dual inspection time points T_{i1} and T_{i2} , and a variety of progressive censoring patterns Q_{ij} for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, m_i$. To perform a comprehensive numerical illustration, in Table 1, we present several experimental designs to assess the associated behaviors of the offered estimation frameworks. To simplify notation, we symbolize a censoring plane such as $(2, 0, 0, 0, 5)$ by $(2, 0 * 3, 5)$.

Table 1: Various tests in Monte Carlo comparisons

(n_1, m_1)	(n_2, m_2)	$(Q_{11}, Q_{12}, \dots, Q_{1m_1})$	$(Q_{21}, Q_{22}, \dots, Q_{2m_2})$	Design
(30, 15)	(40, 20)	$(3 * 5, 0 * 10)$	$(4 * 5, 0 * 15)$	[1]
		$(0 * 5, 3 * 5, 0 * 5)$	$(0 * 7, 4 * 5, 0 * 8)$	[2]
		$(0 * 10, 3 * 5)$	$(0 * 15, 4 * 5)$	[3]
(30, 25)	(40, 35)	$(1 * 5, 0 * 20)$	$(1 * 5, 0 * 30)$	[4]
		$(0 * 10, 1 * 5, 0 * 10)$	$(0 * 15, 1 * 5, 0 * 15)$	[5]
		$(0 * 20, 1 * 5)$	$(0 * 30, 1 * 5)$	[6]
(50, 20)	(60, 30)	$(5 * 6, 0 * 14)$	$(5 * 6, 0 * 24)$	[1]
		$(0 * 7, 6 * 5, 0 * 7)$	$(0 * 12, 6 * 5, 0 * 12)$	[2]
		$(0 * 10, 3 * 5)$	$(0 * 15, 4 * 5)$	[3]
(50, 40)	(60, 50)	$(1 * 10, 0 * 30)$	$(1 * 20, 0 * 30)$	[4]
		$(0 * 15, 1 * 10, 0 * 15)$	$(0 * 15, 1 * 20, 0 * 15)$	[5]
		$(0 * 30, 1 * 10)$	$(0 * 30, 1 * 20)$	[6]
(90, 50)	(80, 40)	$(5 * 8, 0 * 42)$	$(5 * 8, 0 * 32)$	[1]
		$(0 * 21, 5 * 8, 0 * 21)$	$(0 * 16, 5 * 8, 0 * 16)$	[2]
		$(0 * 42, 5 * 8)$	$(0 * 32, 5 * 8)$	[3]
(90, 80)	(80, 60)	$(1 * 10, 0 * 70)$	$(1 * 20, 0 * 40)$	[4]
		$(0 * 35, 1 * 10, 0 * 35)$	$(0 * 20, 1 * 20, 0 * 20)$	[5]
		$(0 * 70, 1 * 10)$	$(0 * 40, 1 * 20)$	[6]

Briefly, to conduct a life-test study using an improved adaptive progressive Type-II censored Hjorth constant-stress model, we offer the following methodology:

Step 1. Assign values of $k, n_i, m_i, T_{i1}, T_{i2}$, and Q_{ij} for $i = 1, 2, \dots, k$, and $j = 1, 2, \dots, m_i$.

Step 2. Assign values of $a_i, b_i, \theta_i, \mu_i$ (for $i = 0, 1$).

Step 3. Generate k independent Uniform $(0, 1)$ samples: $\mathcal{U}_{ij}$ for $j = 1, \dots, m_i$.

Step 4. Compute $\mathcal{W}_{ij}$ (for $i = 1, \dots, k, j = 1, \dots, m_i$) as:

$$\mathcal{W}_{ij} = \mathcal{U}_{ij}^{\left(j + \sum_{q=m_i-j+1}^{m_i} Q_{iq}\right)^{-1}}.$$

Step 5. Compute

$$\mathcal{U}_{ij}^* = 1 - \left[\prod_{Q=m_i-j+1}^{m_i} \mathcal{W}_{iQ} \right].$$

Step 6. Generate IAPT2C observations from the Hjorth (θ_i, μ_i) model of size m_i as

$$Y_{ij} = G_i^{-1}(\mathcal{U}_{ij}; \theta_i, \mu_i), \quad i = 1, 2, \dots, k, \quad j = 1, 2, \dots, m_i.$$

Step 7. Use Step 6, determine the number of observed failures r_i and r_i^* at T_{i1} and T_{i2} , respectively.

Step 8. At T_{i1} , remove the remaining sample Y_{ij} , $i = 1, 2, \dots, k$, $j = r_i + 2, \dots, m_i$.

Step 9. Generate the first $m_i - r_i - 1$ order statistics as $Y_{r_i+2}, \dots, Y_{m_i}$ from a truncated distribution $g_i(y)/[1 - G_i(y_{r_i+1})]$, $i = 1, 2, \dots, k$.

Step 10. Specify the case type of the proposed scheme such that

- If $Y_{im_i} < T_{i1}$ (Case I), the experiment stops at m_i^{th} failure. Here, the remaining live units $Q_{im_i} = n_i - m_i - \sum_{j=1}^{m_i-1} Q_{ij}$ are removed.
- If $Y_{ir_i} < T_{i1} < Y_{ir_i+1}$ (Case II), the experiment stops at Y_{im_i} with failure data Y_{ij} for $i = 1, \dots, k$, $j = 1, \dots, m_i$ and progressive censoring $(Q_{i1}, Q_{i2}, \dots, Q_{ir_i}, 0, \dots, 0, Q_{im_i})$. Here, the remaining live units $Q_{im_i} = n_i - m_i - \sum_{j=1}^{r_i} Q_{ij}$ are removed.
- If $Y_{im_i} > T_{i2}$ (Case III), the experiment stops at T_2 with failure data Y_{ij} for $i = 1, \dots, k$, $j = 1, \dots, r_i^*$, where $r_i^* < m_i$, with the same progressive censoring of Case II. Here, the remaining live units $Q_i^* = n_i - r_i^* - \sum_{j=1}^{r_i} Q_{ij}$ are removed.

In a Bayesian protocol, the major challenge is specifying hyperparameter values. It is well known that if inadequate prior information on the unknown parameters a_i and b_i (for $i = 0, 1$) is given, the objective posterior distribution diminishes to its likelihood function. As a result, while the Bayes technique is computationally exhausting, it is better to figure out the unknown parameters using any frequentist approach. The values of c_i and v_i for $i = 1, 2$ are chosen so that the means of previous distributions are precisely identical to the true values of the parameters. Without loss of generality, we pick $(c_1, c_2) = (5, 3)$ and $v_i = 10$, $i = 1, 2$, for a_1 and b_1 . We also take $(\vartheta_1, \vartheta_2) = (0.2, 0.4)$ and $\sigma_i = 1$, $i = 1, 2$, for a_0 and b_0 .

In Bayesian settings, specifying hyperparameters is crucial. Improper priors lead to posteriors coinciding with the likelihood function, encouraging frequentist methods. To maintain consistency, hyperparameters $(c_1, c_2) = (8, 2)$ and $w_i = 10$ are chosen such that prior means equal the true values. For β_0 , a normal prior with $(\mu, \sigma) = (0.5, 1)$ is used. Alternative prior settings may be guided by historical data. We now run the MH algorithm in MCMC with $\mathcal{N} = 12,000$ draws (the first $\mathcal{M} = 2000$ discarded), initialized using the corresponding frequentist estimates of a_i and b_i (for $i = 0, 1$), in turn to obtain the Bayes point and credible estimates. All required computational procedures are encoded in R (version 4.2.2) through three packages, namely, ‘maxLik’, ‘coda’, and ‘GoFKernel’ by Henningsen and Toomet [27], Plummer et al. [28], and Pavia [29], respectively.

Computationally, the average point estimates (APEs) of a_i , b_i (for $i = 0, 1$), or $R_u(t)$ (say) is given by

$$APE(\check{\Lambda}_i) = \frac{1}{1000} \sum_{j=1}^{1000} \check{\Lambda}_i^{(j)}, \quad i = 1, \dots, 5,$$

where $\check{\Lambda}_i^{(j)}$ denotes the calculated estimate of ω_i at the j th dataset, and $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5) = (a_0, a_1, b_0, b_1, R_u(t))$.

For the performance of point estimators, the RMSE and MAB are given by

$$\text{RMSE}(\check{\Lambda}_t) = \sqrt{\frac{1}{1000} \sum_{j=1}^{1000} (\check{\Lambda}_t^{(j)} - \Lambda_t)^2}, \quad t = 1, \dots, 5,$$

and

$$\text{MRAB}(\check{\Lambda}_t) = \frac{1}{1000} \sum_{j=1}^{1000} |\check{\Lambda}_t^{(j)} - \Lambda_t|, \quad t = 1, \dots, 5,$$

respectively.

To assess interval estimates, we calculate:

$$\text{ACL}_{\Lambda_t}(1 - \alpha)\% = \frac{1}{1000} \sum_{j=1}^{1000} \left(\mathcal{U}(\check{\Lambda}_t^{(j)}) - \mathcal{L}(\check{\Lambda}_t^{(j)}) \right), \quad t = 1, \dots, 5,$$

and

$$\text{CP}_{\Lambda_t}(1 - \alpha)\% = \frac{1}{1000} \sum_{j=1}^{1000} \mathbf{1} \left(\Lambda_t \in (\mathcal{L}(\check{\Lambda}_t^{(j)}), \mathcal{U}(\check{\Lambda}_t^{(j)})) \right), \quad t = 1, \dots, 5,$$

respectively.

Simulation outcomes for all considered parameters are summarized in [Tables 2](#) and [11](#). Specifically, [Tables 2–6](#) display the APEs, RMSEs, and MABs in the first, second, and third columns, respectively, whereas [Tables 7–11](#) summarize AILs and CPs (at $\alpha = 5\%$) in the first and second columns, respectively.

Table 2: Point estimates of a_0 .

(s_1, s_2)	(n_1, n_2)	Design	MLE						Bayes					
			(T_{11}, T_{12})						(T_{21}, T_{22})					
(0.5, 1.0)	(30, 40)	[1]	0.3068	1.1838	0.3883	0.3834	0.1753	0.1701	0.2130	0.1588	0.1546	0.2098	0.1269	0.0573
		[2]	0.2077	1.0646	0.3363	0.2596	0.1609	0.1550	0.2601	0.1555	0.1511	0.3118	0.1157	0.0520
		[3]	0.0901	0.9715	0.3134	0.1126	0.1555	0.1529	0.2376	0.1487	0.1437	0.3766	0.0994	0.0333
		[4]	0.1352	0.8267	0.3047	0.1690	0.1517	0.1463	0.2713	0.1465	0.1418	0.3380	0.0861	0.0283
		[5]	0.2824	0.8520	0.3081	0.3530	0.1524	0.1489	0.1925	0.1465	0.1421	0.3400	0.0956	0.0310
		[6]	0.1496	0.6993	0.2919	0.1870	0.1512	0.1438	0.2448	0.1435	0.1404	0.4832	0.0708	0.0275
	(50, 60)	[1]	0.2056	0.6592	0.2816	0.2570	0.1512	0.1417	0.1615	0.1410	0.1386	0.4070	0.0689	0.0274
		[2]	0.1405	0.6385	0.2641	0.1756	0.1468	0.1412	0.1744	0.1390	0.1369	0.4955	0.0666	0.0265
		[3]	0.0854	0.5563	0.2382	0.1068	0.1463	0.1410	0.1893	0.1374	0.1369	0.5929	0.0642	0.0253
		[4]	0.1366	0.5434	0.2302	0.1707	0.1418	0.1374	0.2146	0.1373	0.1345	0.6556	0.0634	0.0250
		[5]	0.1316	0.5500	0.2304	0.1645	0.1460	0.1397	0.1642	0.1373	0.1365	0.6185	0.0637	0.0253
		[6]	0.1569	0.5235	0.2201	0.1962	0.1412	0.1373	0.2331	0.1353	0.1338	0.6732	0.0616	0.0250
	(90, 80)	[1]	0.1199	0.5068	0.2196	0.1499	0.1403	0.1366	0.1112	0.1351	0.1317	0.4889	0.0605	0.0244
		[2]	0.0960	0.4788	0.2153	0.1200	0.1378	0.1358	0.1569	0.1310	0.1305	0.5692	0.0566	0.0234
		[3]	0.0839	0.3246	0.2022	0.1049	0.1366	0.1355	0.1120	0.1299	0.1294	0.6019	0.0527	0.0232
		[4]	0.1002	0.2689	0.1592	0.1252	0.1339	0.1323	0.1674	0.1277	0.1257	0.5844	0.0521	0.0207
		[5]	0.1095	0.3088	0.1762	0.1369	0.1365	0.1335	0.1141	0.1297	0.1272	0.5257	0.0524	0.0209
		[6]	0.1053	0.2162	0.1568	0.1317	0.1329	0.1311	0.1201	0.1255	0.1251	0.5844	0.0501	0.0206

(Continued)

Table 2 (continued)

(s_1, s_2)	(n_1, n_2)	Design	MLE				Bayes			MLE				Bayes	
			(T_{11}, T_{12})						(T_{21}, T_{22})						
(1.0, 1.5)	(30, 40)	[1]	0.3352	0.2153	0.1559	0.4190	0.1309	0.1304	0.5212	0.1220	0.1191	0.1949	0.0453	0.0199	
		[2]	0.2993	0.2134	0.1536	0.3742	0.1283	0.1277	0.3315	0.1195	0.1178	0.2750	0.0400	0.0188	
		[3]	0.3792	0.2053	0.1534	0.4740	0.1274	0.1249	0.2061	0.1194	0.1166	0.4393	0.0367	0.0186	
		[4]	0.2269	0.1914	0.1451	0.2836	0.1230	0.1217	0.2823	0.1109	0.1102	0.2850	0.0342	0.0181	
		[5]	0.2794	0.1997	0.1464	0.3493	0.1259	0.1222	0.3082	0.1152	0.1110	0.2595	0.0354	0.0184	
		[6]	0.3020	0.1846	0.1396	0.3776	0.1162	0.1141	0.2720	0.1107	0.1079	0.4258	0.0340	0.0174	
	(50, 60)	[1]	0.2298	0.1764	0.1391	0.2872	0.1150	0.1122	0.1751	0.1074	0.1052	0.3293	0.0313	0.0174	
		[2]	0.2077	0.1612	0.1287	0.2596	0.1146	0.1115	0.1573	0.1019	0.1014	0.4511	0.0211	0.0168	
		[3]	0.1904	0.1599	0.1265	0.2381	0.1129	0.1105	0.2210	0.0988	0.0953	0.5727	0.0198	0.0146	
		[4]	0.2293	0.1500	0.1217	0.2866	0.1107	0.1059	0.2385	0.0920	0.0876	0.5827	0.0191	0.0143	
		[5]	0.2652	0.1517	0.1221	0.3315	0.1107	0.1098	0.1930	0.0927	0.0880	0.6107	0.0192	0.0144	
		[6]	0.1482	0.1456	0.1175	0.1852	0.1076	0.1030	0.2140	0.0867	0.0859	0.6090	0.0189	0.0141	
	(90, 80)	[1]	0.1422	0.1453	0.1166	0.1778	0.1050	0.1010	0.1303	0.0864	0.0824	0.4549	0.0183	0.0139	
		[2]	0.1325	0.1430	0.1099	0.1656	0.1017	0.0981	0.1199	0.0816	0.0770	0.5283	0.0178	0.0135	
		[3]	0.0693	0.1411	0.1040	0.0866	0.0996	0.0949	0.1224	0.0790	0.0750	0.5585	0.0169	0.0133	
		[4]	0.1000	0.1278	0.0938	0.1250	0.0890	0.0844	0.1323	0.0666	0.0620	0.5361	0.0163	0.0130	
		[5]	0.1150	0.1308	0.1018	0.1438	0.0921	0.0873	0.1294	0.0764	0.0719	0.5523	0.0166	0.0132	
		[6]	0.1036	0.1030	0.0729	0.1295	0.0816	0.0768	0.1304	0.0638	0.0590	0.5469	0.0155	0.0126	

Table 3: Point estimates of a_1

(s_1, s_2)	(n_1, n_2)	Design	MLE				Bayes		MLE				Bayes	
			(T_{11}, T_{12})						(T_{21}, T_{22})					
(0.5, 1.0)	(30, 40)	[1]	0.6428	1.0977	0.7299	0.5563	0.2634	0.2567	0.7915	1.8891	1.1334	0.5367	0.2749	0.2695
		[2]	0.7748	0.9478	0.3476	0.5147	0.2580	0.2519	0.7995	1.8181	1.0685	0.5270	0.2634	0.2577
		[3]	0.8155	0.9550	0.4730	0.5573	0.2606	0.2539	0.8376	1.8813	1.1148	0.5585	0.2669	0.2610
		[4]	0.6301	0.8251	0.3395	0.4229	0.2530	0.2468	0.6438	1.7484	1.0029	0.4032	0.2579	0.2521
		[5]	0.8341	0.8127	0.3164	0.4173	0.2424	0.2112	0.4172	1.3000	0.9832	0.3981	0.2469	0.2178
		[6]	0.7354	0.7530	0.2958	0.5614	0.2240	0.1804	0.4619	1.2050	0.9185	0.5554	0.2232	0.1860
	(50, 60)	[1]	0.5643	0.7513	0.2874	0.6935	0.2131	0.1734	0.5941	1.1839	0.9171	0.6716	0.2231	0.1809
		[2]	0.5697	0.6606	0.2336	0.7324	0.2123	0.1668	0.6820	1.1284	0.8780	0.6554	0.2179	0.1733
		[3]	0.5607	0.6614	0.2658	0.6738	0.2125	0.1692	0.4170	1.1761	0.8947	0.6843	0.2192	0.1789
		[4]	0.5370	0.5657	0.2274	0.7163	0.2115	0.1656	0.5378	1.0771	0.8640	0.6963	0.2122	0.1719
		[5]	0.6964	0.5375	0.2240	0.6850	0.2084	0.1644	0.6930	1.0638	0.8572	0.6973	0.2050	0.1711
		[6]	0.7402	0.5166	0.2134	0.6942	0.2047	0.1599	0.5411	1.0265	0.8551	0.6954	0.2050	0.1643
	(90, 80)	[1]	0.4404	0.5011	0.2110	0.6067	0.2035	0.1572	0.8864	1.0041	0.7762	0.5891	0.2038	0.1636
		[2]	0.4179	0.4842	0.1729	0.5812	0.1892	0.1548	0.7119	0.9250	0.7594	0.4411	0.2003	0.1618
		[3]	0.4951	0.4948	0.2057	0.5844	0.1931	0.1568	0.7126	0.9256	0.7644	0.5735	0.2030	0.1624
		[4]	0.4304	0.4646	0.1639	0.6649	0.1880	0.1529	0.8850	0.8912	0.7298	0.6023	0.1972	0.1536
		[5]	0.5204	0.4614	0.1606	0.6557	0.1880	0.1504	0.7165	0.8874	0.7287	0.5667	0.1967	0.1528
		[6]	0.4425	0.4606	0.1578	0.6557	0.1877	0.1502	0.7336	0.8522	0.6851	0.5863	0.1943	0.1511

(Continued)

Table 3 (continued)

(s_1, s_2)	(n_1, n_2)	Design	MLE				Bayes				MLE				Bayes			
			(T_{11}, T_{12})								(T_{21}, T_{22})							
(1.0, 1.5)	(30, 40)	[1]	0.6065	0.4555	0.1517	0.5100	0.1840	0.1501	0.4898	0.8446	0.6412	0.4948	0.1901	0.1509				
		[2]	0.5941	0.4266	0.1487	0.4576	0.1779	0.1369	0.6554	0.8051	0.2430	0.5276	0.1857	0.1491				
		[3]	0.7452	0.4287	0.1487	0.4980	0.1790	0.1462	0.5761	0.8400	0.3112	0.4958	0.1884	0.1498				
		[4]	0.6360	0.4159	0.1486	0.4140	0.1776	0.1322	0.6250	0.7320	0.2203	0.3961	0.1846	0.1476				
		[5]	0.6109	0.3389	0.1473	0.4055	0.1770	0.1314	0.4531	0.5406	0.2114	0.3933	0.1832	0.1471				
		[6]	0.5995	0.3210	0.1412	0.5642	0.1766	0.1286	0.7819	0.5250	0.2113	0.5946	0.1809	0.1464				
	(50, 60)	[1]	0.4900	0.2329	0.1393	0.6637	0.1755	0.1251	0.4747	0.5176	0.1931	0.6478	0.1727	0.1462				
		[2]	0.5424	0.1755	0.1381	0.7663	0.1253	0.0961	0.7331	0.4505	0.1840	0.5986	0.1723	0.1445				
		[3]	0.4555	0.1755	0.1385	0.6623	0.1306	0.0987	0.5985	0.4601	0.1879	0.6591	0.1727	0.1457				
		[4]	0.5112	0.1749	0.1374	0.6736	0.1247	0.0889	0.6154	0.4236	0.1804	0.6857	0.1700	0.1407				
		[5]	0.5578	0.1748	0.1369	0.6852	0.1196	0.0857	0.6529	0.4227	0.1769	0.6795	0.1678	0.1377				
		[6]	0.5608	0.1682	0.1367	0.6846	0.1140	0.0857	0.8830	0.4189	0.1687	0.7299	0.1645	0.1368				
	(90, 80)	[1]	0.4293	0.1600	0.1364	0.5983	0.1111	0.0838	0.8420	0.3614	0.1486	0.5577	0.1639	0.1335				
		[2]	0.4036	0.1531	0.1295	0.5690	0.1036	0.0817	0.4824	0.2741	0.1369	0.6078	0.1573	0.1300				
		[3]	0.4045	0.1599	0.1306	0.6108	0.1102	0.0817	0.7927	0.3513	0.1474	0.5861	0.1607	0.1321				
		[4]	0.4253	0.1492	0.1267	0.6351	0.1020	0.0804	0.7515	0.2648	0.1356	0.5571	0.1555	0.1287				
		[5]	0.4258	0.1444	0.1227	0.6307	0.1017	0.0784	0.6287	0.2312	0.1353	0.5491	0.1434	0.1223				
		[6]	0.4261	0.1443	0.1197	0.6162	0.1003	0.0751	0.5901	0.2098	0.1257	0.5453	0.1284	0.1063				

Table 4: Point estimates of b_0

(s_1, s_2)	(n_1, n_2)	Design	MLE				Bayes				MLE				Bayes			
			(T_{11}, T_{12})								(T_{21}, T_{22})							
(0.5, 1.0)	(30, 40)	[1]	0.4217	2.4907	2.8235	0.4286	1.2660	1.4484	0.5170	1.5878	2.0746	0.4920	1.1378	1.2693				
		[2]	0.5193	2.2505	2.2188	0.3780	1.2169	1.4205	0.6057	1.4062	1.8945	0.5891	1.0067	1.0954				
		[3]	0.4987	2.2144	2.1604	0.2260	1.1763	1.4116	0.4992	1.3872	1.6872	0.5745	0.8699	0.9858				
		[4]	0.4486	1.8979	1.8627	0.4589	1.0100	1.3743	0.5381	1.1002	1.0769	0.5381	0.7948	0.8476				
		[5]	0.4806	1.9611	1.9314	0.4763	1.1024	1.3976	0.5049	1.2106	1.1959	0.6177	0.8208	0.9190				
		[6]	0.5058	1.8604	1.8207	0.4220	0.9347	1.3122	0.4954	1.0783	1.0639	0.6072	0.7554	0.8469				
	(50, 60)	[1]	0.4627	1.8064	1.7864	0.3234	0.9283	1.2636	0.5296	1.0708	1.0326	0.5546	0.7289	0.8045				
		[2]	0.7625	1.7923	1.7559	0.1691	0.9135	1.2306	0.4774	1.0383	1.0269	0.4949	0.7186	0.7114				
		[3]	0.6338	1.6186	1.5920	0.0980	0.8963	1.1847	0.4439	0.9794	0.9573	0.5845	0.7064	0.6688				
		[4]	0.4144	1.5340	1.5086	0.1691	0.8532	1.1576	0.4460	0.9695	0.9354	0.6172	0.6669	0.6257				
		[5]	0.6355	1.5791	1.5636	0.2320	0.8601	1.1670	0.4384	0.9726	0.9573	0.4810	0.6707	0.6452				
		[6]	0.3802	1.5093	1.4890	0.1377	0.8443	1.0277	0.4594	0.9674	0.9157	0.5602	0.6542	0.6143				
	(90, 80)	[1]	0.4698	1.4752	1.4376	0.3207	0.8355	1.0263	0.4713	0.9205	0.9065	0.4832	0.6245	0.6116				
		[2]	0.5731	1.4589	1.4335	0.1928	0.8347	1.0024	0.5389	0.8942	0.8792	0.7123	0.6143	0.5774				
		[3]	0.5948	1.3685	1.3396	0.0546	0.8011	0.9820	0.4465	0.8870	0.8574	0.5920	0.6071	0.5069				
		[4]	0.4940	1.3406	1.3123	0.2945	0.7333	0.9616	0.4401	0.8560	0.8307	0.5574	0.5724	0.5019				
		[5]	0.5691	1.3685	1.3396	0.3138	0.7674	0.9735	0.4462	0.8757	0.8490	0.5687	0.5990	0.5066				
		[6]	0.5900	1.2977	1.2886	0.2595	0.7069	0.9607	0.4240	0.8359	0.8196	0.4808	0.5667	0.4666				

(Continued)

Table 4 (continued)

(s_1, s_2)	(n_1, n_2)	Design	MLE				Bayes				MLE				Bayes			
			(T_{11}, T_{12})								(T_{21}, T_{22})							
(1.0, 1.5)	(30, 40)	[1]	0.5343	1.2228	1.2126	0.4556	0.7023	0.9243	0.5701	0.8238	0.8074	0.5984	0.5302	0.3818				
		[2]	0.0084	1.2137	1.1811	0.4688	0.6692	0.9176	0.5344	0.7971	0.7856	0.5343	0.5193	0.3407				
		[3]	0.6041	1.0910	1.0763	0.3284	0.6505	0.8705	0.5294	0.7941	0.7522	0.6090	0.4411	0.3252				
		[4]	0.4202	0.9414	0.8987	0.3883	0.5712	0.8259	0.5205	0.7483	0.7432	0.5741	0.3999	0.2997				
		[5]	0.3632	1.0565	0.9851	0.4692	0.6297	0.8701	0.5264	0.7653	0.7460	0.5295	0.4039	0.3078				
		[6]	0.5920	0.8933	0.8806	0.3424	0.5453	0.7672	0.5090	0.7430	0.7338	0.5741	0.3903	0.2721				
	(50, 60)	[1]	0.4207	0.8914	0.8486	0.3410	0.5364	0.7104	0.4967	0.7194	0.7040	0.4736	0.3759	0.2571				
		[2]	0.5314	0.8636	0.8228	0.2919	0.5357	0.7095	0.4871	0.7142	0.6919	0.4931	0.3634	0.2366				
		[3]	0.4235	0.8625	0.8202	0.2657	0.4991	0.7016	0.4828	0.7082	0.6867	0.6326	0.3634	0.2347				
		[4]	0.5851	0.8285	0.7802	0.1296	0.4323	0.5220	0.4706	0.6298	0.6048	0.5573	0.3205	0.2265				
		[5]	0.6155	0.8346	0.8144	0.2310	0.4601	0.5705	0.4778	0.6974	0.6845	0.6269	0.3575	0.2325				
		[6]	0.4228	0.8216	0.7632	0.1268	0.4287	0.5151	0.5204	0.5838	0.5787	0.5354	0.3011	0.2123				
(90, 80)	[1]	0.6763	0.7784	0.7549	0.3083	0.4260	0.4329	0.5092	0.5767	0.5706	0.5157	0.2980	0.2117					
	[2]	0.6380	0.6401	0.5927	0.3259	0.4204	0.3980	0.5086	0.5757	0.5678	0.4307	0.2668	0.1995					
	[3]	0.8886	0.6330	0.5597	0.0674	0.3765	0.3959	0.5057	0.5725	0.5653	0.3522	0.2056	0.1989					
	[4]	0.7081	0.4396	0.3910	0.4483	0.2467	0.1896	0.4833	0.5293	0.4697	0.0640	0.1874	0.1692					
	[5]	0.7811	0.4600	0.4087	0.4470	0.2891	0.3235	0.5008	0.5703	0.5330	0.1330	0.1952	0.1744					
	[6]	0.6123	0.3285	0.2737	0.3288	0.1879	0.1773	0.4767	0.4527	0.4029	0.0027	0.1830	0.1480					

Table 5: Point estimates of b_1

(s_1, s_2)	(n_1, n_2)	Design	MLE				Bayes				MLE				Bayes			
			(T_{11}, T_{12})								(T_{21}, T_{22})							
(0.5, 1.0)	(30, 40)	[1]	0.4164	0.6188	0.4554	0.3690	0.2943	0.2300	0.4270	0.5372	0.4124	0.4153	0.2912	0.2158				
		[2]	0.4207	0.5262	0.4162	0.3479	0.2555	0.1979	0.4323	0.4414	0.2461	0.4813	0.2519	0.1963				
		[3]	0.3804	0.5747	0.4358	0.4165	0.2819	0.2156	0.5104	0.4516	0.3318	0.4160	0.2614	0.2025				
		[4]	0.4785	0.5037	0.4091	0.4117	0.2420	0.1943	0.4452	0.4055	0.2408	0.3200	0.2426	0.1885				
		[5]	0.4689	0.4958	0.4087	0.3711	0.2403	0.1865	0.3751	0.3630	0.2043	0.3939	0.2424	0.1874				
		[6]	0.4112	0.4651	0.4004	0.4376	0.2364	0.1865	0.3305	0.2803	0.1865	0.3830	0.2339	0.1794				
	(50, 60)	[1]	0.3743	0.4593	0.3914	0.4316	0.2206	0.1741	0.3142	0.2786	0.1861	0.4376	0.2030	0.1737				
		[2]	0.5112	0.4412	0.3584	0.4875	0.2081	0.1641	0.3152	0.2557	0.1679	0.3443	0.1922	0.1526				
		[3]	0.4278	0.4555	0.3819	0.4057	0.2198	0.1721	0.3820	0.2762	0.1813	0.4729	0.1973	0.1703				
		[4]	0.5037	0.4391	0.3565	0.3269	0.2073	0.1629	0.5122	0.2433	0.1646	0.3514	0.1911	0.1510				
		[5]	0.4257	0.4375	0.3533	0.3375	0.1930	0.1569	0.4498	0.2386	0.1638	0.3332	0.1908	0.1485				
		[6]	0.4197	0.4356	0.3517	0.3515	0.1871	0.1552	0.4862	0.2343	0.1548	0.3026	0.1849	0.1336				
	(90, 80)	[1]	0.3840	0.4199	0.3317	0.4403	0.1828	0.1506	0.6976	0.2289	0.1540	0.4668	0.1794	0.1303				
		[2]	0.0768	0.4089	0.3242	0.4468	0.1721	0.1466	0.6040	0.2085	0.1469	0.3530	0.1761	0.1255				
		[3]	0.3241	0.4167	0.3245	0.4302	0.1750	0.1505	0.6655	0.2279	0.1511	0.4325	0.1787	0.1264				
		[4]	0.5242	0.3978	0.3104	0.3305	0.1693	0.1414	0.3603	0.2073	0.1452	0.2901	0.1660	0.1194				
		[5]	0.4794	0.3905	0.3022	0.3214	0.1645	0.1389	0.3189	0.2053	0.1418	0.2953	0.1658	0.1128				
		[6]	0.4400	0.3845	0.3001	0.3214	0.1561	0.1379	0.4931	0.1954	0.1414	0.2879	0.1628	0.1127				

(Continued)

Table 5 (continued)

(s_1, s_2)	(n_1, n_2)	Design	MLE				Bayes			MLE				Bayes		

Table 6: Point estimates of $R_u(t)$

(s_1, s_2)	(n_1, n_2)	Design	MLE				Bayes				MLE				Bayes			
			(T_{11}, T_{12})								(T_{21}, T_{22})							
(0.5, 1.0)	(30, 40)	[1]	0.8558	0.6750	0.5547	0.9168	0.5906	0.4854	0.9390	0.5697	0.4289	0.9586	0.4985	0.3753				
		[2]	0.9789	0.6525	0.5263	0.9789	0.5709	0.4605	0.9366	0.5214	0.3730	0.9579	0.4562	0.3264				
		[3]	0.9300	0.6529	0.5315	0.9542	0.5713	0.4650	0.8637	0.5296	0.3780	0.9211	0.4634	0.3307				
		[4]	0.8411	0.6118	0.4815	0.9099	0.5353	0.4213	0.8655	0.4807	0.3306	0.9221	0.4206	0.2893				
		[5]	0.8757	0.5755	0.4379	0.9272	0.5035	0.3831	0.9379	0.3854	0.2424	0.9583	0.3372	0.2121				
		[6]	0.8735	0.5583	0.4050	0.9265	0.4885	0.3543	0.9235	0.3792	0.2374	0.9514	0.3318	0.2077				
	(50, 60)	[1]	0.7822	0.5330	0.3808	0.8807	0.4664	0.3332	0.8484	0.3500	0.2189	0.9137	0.3062	0.1915				
		[2]	0.9207	0.4933	0.3419	0.9503	0.4317	0.2991	0.9210	0.3053	0.1867	0.9503	0.2671	0.1634				
		[3]	0.7385	0.5164	0.3585	0.8591	0.4518	0.3137	0.8200	0.3213	0.2006	0.8996	0.2811	0.1755				
		[4]	0.8170	0.4925	0.3401	0.8983	0.4309	0.2976	0.9617	0.3021	0.1846	0.9705	0.2644	0.1615				
		[5]	0.7983	0.4810	0.3259	0.8890	0.4209	0.2852	0.9720	0.3015	0.1820	0.9756	0.2638	0.1592				
		[6]	0.8944	0.4408	0.2871	0.9371	0.3857	0.2512	0.9695	0.2914	0.1737	0.9744	0.2549	0.1520				
	(90, 80)	[1]	0.6834	0.4339	0.2851	0.8314	0.3797	0.2494	0.8972	0.2837	0.1729	0.9383	0.2482	0.1513				
		[2]	0.9413	0.3791	0.2377	0.9605	0.3317	0.2080	0.6803	0.2723	0.1624	0.8299	0.2383	0.1421				
		[3]	0.6647	0.3902	0.2492	0.8222	0.3414	0.2181	0.8618	0.2798	0.1655	0.9205	0.2448	0.1448				
		[4]	0.4272	0.3562	0.2204	0.7034	0.3117	0.1929	0.8021	0.2663	0.1616	0.8908	0.2330	0.1414				
		[5]	0.4825	0.3528	0.2172	0.7311	0.3087	0.1900	0.7937	0.2654	0.1586	0.8865	0.2322	0.1388				
		[6]	0.5389	0.3511	0.2162	0.7593	0.3073	0.1892	0.9183	0.2569	0.1564	0.9488	0.2248	0.1368				

(Continued)

Table 6 (continued)

(s_1, s_2)	(n_1, n_2)	Design	MLE				Bayes				MLE				Bayes			
			(T_{11}, T_{12})								(T_{21}, T_{22})							
(1.0, 1.5)	(30, 40)	[1]	0.8249	0.3456	0.2113	0.9013	0.3024	0.1849	0.8765	0.2379	0.1468	0.9272	0.2082	0.1285				
		[2]	0.8590	0.3356	0.2072	0.9192	0.2937	0.1813	0.6097	0.2275	0.1410	0.7941	0.1991	0.1234				
		[3]	0.9093	0.3358	0.2082	0.9438	0.2938	0.1822	0.8729	0.2356	0.1456	0.9255	0.2061	0.1274				
		[4]	0.8221	0.3199	0.1959	0.9002	0.2799	0.1714	0.9289	0.2215	0.1407	0.9536	0.1938	0.1231				
		[5]	0.8103	0.3180	0.1918	0.8943	0.2783	0.1679	0.9162	0.2173	0.1389	0.9474	0.1901	0.1215				
		[6]	0.8113	0.2961	0.1796	0.8953	0.2591	0.1572	0.9402	0.2146	0.1371	0.9597	0.1878	0.1200				
	(50, 60)	[1]	0.7308	0.2908	0.1780	0.8549	0.2544	0.1557	0.8732	0.2019	0.1298	0.9261	0.1766	0.1136				
		[2]	0.9063	0.2672	0.1628	0.9431	0.2338	0.1424	0.6220	0.1941	0.1276	0.8005	0.1699	0.1117				
		[3]	0.6365	0.2805	0.1708	0.8080	0.2455	0.1494	0.8925	0.1979	0.1294	0.9358	0.1732	0.1132				
		[4]	0.8678	0.2507	0.1579	0.9237	0.2193	0.1382	0.9806	0.1842	0.1240	0.9800	0.1611	0.1085				
		[5]	0.9110	0.2388	0.1477	0.9453	0.2089	0.1293	0.9801	0.1809	0.1211	0.9796	0.1583	0.1060				
		[6]	0.9369	0.2254	0.1453	0.9583	0.1972	0.1272	0.9794	0.1803	0.1173	0.9793	0.1578	0.1026				
	(90, 80)	[1]	0.6565	0.2252	0.1398	0.8179	0.1970	0.1223	0.9072	0.1413	0.1138	0.9432	0.1236	0.0996				
		[2]	0.9127	0.2131	0.1326	0.9461	0.1865	0.1161	0.5374	0.1211	0.1073	0.7584	0.1059	0.0939				
		[3]	0.5689	0.2188	0.1394	0.7742	0.1915	0.1220	0.9039	0.1336	0.1090	0.9416	0.1169	0.0954				
		[4]	0.3895	0.1825	0.1244	0.6846	0.1597	0.1089	0.8998	0.1026	0.1026	0.9395	0.0897	0.0897				
		[5]	0.4186	0.1816	0.1208	0.6991	0.1589	0.1057	0.8733	0.1021	0.1021	0.9263	0.0893	0.0893				
		[6]	0.6047	0.1010	0.1009	0.7921	0.0883	0.0883	0.9168	0.1014	0.1013	0.9481	0.0888	0.0887				

Table 7: Interval estimates of a_0

(s_1, s_2)	(n_1, n_2)	Design	95% ACI		95% BCI		95% ACI		95% BCI	
			(T_{11}, T_{12})				(T_{21}, T_{22})			
(0.5, 1.0)	(30, 40)	[1]	2.216	0.868	1.395	0.876	1.915	0.878	1.350	0.882
		[2]	2.020	0.869	1.349	0.877	1.821	0.879	1.271	0.882
		[3]	1.907	0.869	1.323	0.879	1.795	0.881	1.264	0.885
		[4]	1.817	0.871	1.210	0.880	1.706	0.885	1.208	0.891
		[5]	1.903	0.870	1.312	0.881	1.714	0.882	1.251	0.889
		[6]	1.776	0.871	1.201	0.883	1.679	0.888	1.164	0.893
	(50, 60)	[1]	1.527	0.872	1.104	0.882	1.471	0.884	1.097	0.894
		[2]	1.480	0.873	1.097	0.885	1.357	0.883	1.086	0.899
		[3]	1.376	0.874	1.089	0.888	1.354	0.887	1.076	0.904
		[4]	1.292	0.875	1.056	0.895	1.257	0.891	0.978	0.911
		[5]	1.358	0.875	1.075	0.891	1.335	0.890	1.032	0.908
		[6]	1.276	0.877	1.042	0.899	1.133	0.889	0.907	0.912
	(90, 80)	[1]	1.121	0.878	0.905	0.894	1.017	0.886	0.848	0.914
		[2]	1.099	0.879	0.886	0.898	0.993	0.889	0.846	0.923
		[3]	1.089	0.882	0.886	0.904	0.960	0.891	0.710	0.917
		[4]	1.044	0.891	0.821	0.911	0.933	0.899	0.647	0.927
		[5]	1.047	0.885	0.873	0.909	0.947	0.895	0.675	0.921
		[6]	0.996	0.894	0.814	0.919	0.930	0.905	0.613	0.933

(Continued)

Table 7 (continued)

(s_1, s_2)	(n_1, n_2)	Design	95% ACI		95% BCI		95% ACI		95% BCI	
			(T_{11}, T_{12})		(T_{11}, T_{12})		(T_{21}, T_{22})		(T_{21}, T_{22})	
(1.0, 1.5)	(30, 40)	[1]	0.987	0.901	0.814	0.923	0.924	0.912	0.565	0.937
		[2]	0.974	0.907	0.810	0.926	0.920	0.921	0.479	0.932
		[3]	0.863	0.909	0.794	0.928	0.833	0.921	0.456	0.940
		[4]	0.830	0.925	0.741	0.933	0.787	0.928	0.393	0.952
		[5]	0.846	0.915	0.755	0.930	0.794	0.926	0.412	0.942
		[6]	0.814	0.927	0.674	0.936	0.785	0.930	0.387	0.939
	(50, 60)	[1]	0.653	0.933	0.453	0.941	0.621	0.933	0.367	0.944
		[2]	0.612	0.934	0.444	0.942	0.557	0.937	0.349	0.947
		[3]	0.546	0.937	0.440	0.945	0.504	0.938	0.277	0.948
		[4]	0.432	0.942	0.367	0.951	0.431	0.942	0.256	0.951
		[5]	0.514	0.939	0.435	0.948	0.461	0.940	0.272	0.949
		[6]	0.428	0.944	0.240	0.953	0.428	0.945	0.172	0.955
	(90, 80)	[1]	0.411	0.947	0.210	0.956	0.401	0.948	0.153	0.958
		[2]	0.401	0.950	0.204	0.958	0.399	0.952	0.134	0.961
		[3]	0.401	0.952	0.152	0.961	0.391	0.955	0.082	0.964
		[4]	0.387	0.957	0.096	0.966	0.372	0.959	0.077	0.969
		[5]	0.399	0.955	0.104	0.964	0.383	0.957	0.079	0.967
		[6]	0.369	0.959	0.064	0.968	0.357	0.961	0.038	0.971

Table 8: Interval estimates of a_1

(s_1, s_2)	(n_1, n_2)	Design	95% ACI		95% BCI		95% ACI		95% BCI	
			(T_{11}, T_{12})		(T_{11}, T_{12})		(T_{21}, T_{22})		(T_{21}, T_{22})	
(0.5, 1.0)	(30, 40)	[1]	1.097	0.910	0.855	0.912	1.069	0.910	0.923	0.918
		[2]	0.960	0.912	0.791	0.916	0.900	0.914	0.873	0.920
		[3]	1.004	0.911	0.812	0.914	0.971	0.912	0.886	0.919
		[4]	0.882	0.913	0.787	0.918	0.836	0.915	0.854	0.920
		[5]	0.868	0.914	0.766	0.919	0.801	0.918	0.815	0.922
		[6]	0.845	0.914	0.736	0.922	0.799	0.920	0.757	0.923
	(50, 60)	[1]	0.733	0.917	0.713	0.922	0.705	0.921	0.674	0.926
		[2]	0.694	0.920	0.656	0.924	0.643	0.923	0.622	0.927
		[3]	0.717	0.919	0.700	0.923	0.671	0.922	0.650	0.926
		[4]	0.685	0.921	0.625	0.925	0.634	0.923	0.611	0.927
		[5]	0.644	0.922	0.608	0.926	0.626	0.923	0.510	0.928
		[6]	0.638	0.922	0.527	0.926	0.622	0.923	0.488	0.928

(Continued)

Table 8 (continued)

(s_1, s_2)	(n_1, n_2)	Design	95% ACI		95% BCI		95% ACI		95% BCI	
			(T_{11}, T_{12})		(T_{11}, T_{12})		(T_{21}, T_{22})		(T_{21}, T_{22})	
(1.0, 1.5)	(90, 80)	[1]	0.635	0.922	0.480	0.927	0.616	0.925	0.485	0.929
		[2]	0.619	0.924	0.461	0.928	0.603	0.926	0.431	0.929
		[3]	0.635	0.923	0.473	0.927	0.608	0.925	0.460	0.929
		[4]	0.607	0.924	0.460	0.928	0.599	0.926	0.425	0.930
		[5]	0.584	0.924	0.452	0.928	0.592	0.926	0.407	0.930
		[6]	0.584	0.924	0.428	0.929	0.577	0.927	0.404	0.930
	(30, 40)	[1]	0.574	0.925	0.396	0.930	0.576	0.927	0.379	0.931
		[2]	0.568	0.926	0.386	0.932	0.552	0.928	0.379	0.932
		[3]	0.573	0.925	0.389	0.931	0.564	0.928	0.379	0.931
		[4]	0.562	0.926	0.386	0.932	0.546	0.928	0.365	0.932
		[5]	0.535	0.927	0.381	0.932	0.531	0.928	0.363	0.934
		[6]	0.533	0.927	0.374	0.932	0.525	0.929	0.356	0.935
	(50, 60)	[1]	0.531	0.927	0.366	0.935	0.498	0.929	0.346	0.935
		[2]	0.507	0.930	0.341	0.939	0.486	0.930	0.318	0.941
		[3]	0.521	0.928	0.360	0.938	0.487	0.929	0.334	0.936
		[4]	0.483	0.930	0.340	0.941	0.483	0.930	0.315	0.941
		[5]	0.458	0.931	0.292	0.943	0.473	0.931	0.297	0.942
		[6]	0.414	0.932	0.270	0.943	0.451	0.933	0.289	0.944
(1.0, 1.5)	(90, 80)	[1]	0.355	0.935	0.263	0.944	0.431	0.937	0.239	0.947
		[2]	0.321	0.937	0.187	0.947	0.221	0.939	0.175	0.951
		[3]	0.344	0.935	0.246	0.946	0.341	0.937	0.211	0.950
		[4]	0.304	0.942	0.172	0.947	0.220	0.942	0.162	0.952
		[5]	0.292	0.943	0.135	0.948	0.208	0.945	0.151	0.953
		[6]	0.273	0.948	0.040	0.950	0.207	0.950	0.121	0.953

Table 9: Interval estimates of b_0

(s_1, s_2)	(n_1, n_2)	Design	95% ACI		95% BCI		95% ACI		95% BCI	
			(T_{11}, T_{12})		(T_{11}, T_{12})		(T_{21}, T_{22})		(T_{21}, T_{22})	
(30, 40)	(30, 40)	[1]	2.065	0.900	1.620	0.915	2.588	0.883	1.831	0.898
		[2]	2.000	0.900	1.569	0.915	2.479	0.883	1.736	0.898
		[3]	1.962	0.901	1.539	0.916	2.398	0.884	1.658	0.899
		[4]	1.871	0.902	1.467	0.916	2.232	0.884	1.508	0.899
		[5]	1.926	0.901	1.511	0.916	2.271	0.884	1.517	0.899
		[6]	1.768	0.902	1.386	0.916	2.023	0.884	1.311	0.899

(Continued)

Table 9 (continued)

(s_1, s_2)	(n_1, n_2)	Design	95% ACI		95% BCI		95% ACI		95% BCI	
			(T_{11}, T_{12})		(T_{11}, T_{12})		(T_{21}, T_{22})		(T_{21}, T_{22})	
(0.5, 1.0)	(50, 60)	[1]	1.708	0.903	1.340	0.917	1.981	0.885	1.302	0.900
		[2]	1.666	0.904	1.307	0.918	1.916	0.888	1.248	0.903
		[3]	1.620	0.905	1.270	0.919	1.882	0.889	1.239	0.904
		[4]	1.488	0.907	1.167	0.921	1.794	0.890	1.225	0.906
		[5]	1.513	0.906	1.186	0.920	1.813	0.889	1.231	0.904
		[6]	1.422	0.908	1.116	0.922	1.753	0.891	1.221	0.906
	(90, 80)	[1]	1.335	0.910	1.047	0.923	1.662	0.892	1.169	0.906
		[2]	1.313	0.911	1.039	0.924	1.626	0.892	1.154	0.907
		[3]	1.293	0.912	1.024	0.925	1.624	0.893	1.151	0.907
		[4]	1.279	0.912	1.010	0.925	1.522	0.894	1.020	0.908
		[5]	1.288	0.912	1.015	0.925	1.579	0.894	1.095	0.908
		[6]	1.268	0.912	1.006	0.925	1.519	0.894	1.019	0.908
(1.0, 1.5)	(30, 40)	[1]	1.258	0.912	1.006	0.925	1.449	0.896	0.926	0.910
		[2]	1.251	0.913	0.999	0.925	1.443	0.896	0.925	0.910
		[3]	1.248	0.913	0.987	0.926	1.426	0.897	0.915	0.912
		[4]	1.216	0.913	0.931	0.926	1.344	0.898	0.883	0.914
		[5]	1.235	0.913	0.978	0.926	1.402	0.897	0.891	0.914
		[6]	1.157	0.914	0.911	0.926	1.334	0.898	0.872	0.914
	(50, 60)	[1]	1.082	0.916	0.849	0.928	1.282	0.900	0.861	0.916
		[2]	1.008	0.917	0.791	0.929	1.238	0.901	0.860	0.916
		[3]	0.964	0.918	0.756	0.929	1.162	0.901	0.793	0.916
		[4]	0.887	0.921	0.696	0.932	1.079	0.903	0.743	0.917
		[5]	0.922	0.920	0.723	0.931	1.132	0.902	0.787	0.916
		[6]	0.827	0.922	0.649	0.933	0.949	0.904	0.617	0.918
	(90, 80)	[1]	0.775	0.923	0.586	0.934	0.897	0.905	0.610	0.918
		[2]	0.764	0.925	0.582	0.936	0.876	0.909	0.587	0.923
		[3]	0.730	0.927	0.572	0.937	0.869	0.909	0.587	0.924
		[4]	0.715	0.928	0.561	0.938	0.838	0.913	0.557	0.930
		[5]	0.722	0.927	0.566	0.937	0.859	0.912	0.579	0.928
		[6]	0.703	0.930	0.551	0.939	0.825	0.915	0.549	0.932

Table 10: Interval estimates of b_1

(s_1, s_2)	(n_1, n_2)	Design	95% ACI		95% BCI		95% ACI		95% BCI	
			(T_{11}, T_{12})		(T_{11}, T_{12})		(T_{21}, T_{22})		(T_{21}, T_{22})	
(0.5, 1.0)	(30, 40)	[1]	1.407	0.895	0.939	0.901	1.231	0.897	0.906	0.906
		[2]	1.339	0.897	0.799	0.904	1.018	0.899	0.728	0.909
		[3]	1.354	0.896	0.868	0.903	1.122	0.898	0.814	0.908

(Continued)

Table 10 (continued)

(s_1, s_2)	(n_1, n_2)	Design	95% ACI		95% BCI		95% ACI		95% BCI	
			(T_{11}, T_{12})		(T_{11}, T_{12})		(T_{21}, T_{22})		(T_{21}, T_{22})	
(1.0, 1.5)	(50, 60)	[4]	1.287	0.897	0.749	0.905	0.983	0.900	0.725	0.910
		[5]	1.255	0.898	0.731	0.905	0.967	0.901	0.719	0.910
		[6]	1.225	0.898	0.727	0.907	0.946	0.903	0.717	0.912
		[1]	1.217	0.902	0.706	0.909	0.906	0.904	0.654	0.914
		[2]	1.139	0.905	0.676	0.911	0.858	0.907	0.625	0.916
		[3]	1.182	0.902	0.680	0.910	0.876	0.905	0.635	0.914
	(90, 80)	[4]	1.119	0.906	0.660	0.912	0.853	0.907	0.620	0.917
		[5]	1.106	0.906	0.641	0.913	0.843	0.908	0.615	0.917
		[6]	1.088	0.907	0.636	0.913	0.832	0.908	0.611	0.918
		[1]	1.066	0.907	0.632	0.914	0.826	0.908	0.607	0.918
		[2]	1.043	0.908	0.623	0.915	0.817	0.909	0.602	0.919
		[3]	1.059	0.908	0.623	0.914	0.818	0.909	0.605	0.919
	(30, 40)	[4]	1.034	0.909	0.617	0.915	0.807	0.910	0.594	0.920
		[5]	1.023	0.912	0.611	0.915	0.800	0.912	0.590	0.920
		[6]	1.009	0.913	0.602	0.917	0.783	0.914	0.571	0.920
		[1]	0.940	0.913	0.561	0.918	0.748	0.916	0.561	0.921
		[2]	0.818	0.914	0.489	0.920	0.653	0.917	0.492	0.925
		[3]	0.894	0.914	0.534	0.919	0.729	0.916	0.559	0.922
(1.0, 1.5)	(50, 60)	[4]	0.790	0.915	0.472	0.921	0.637	0.917	0.484	0.926
		[5]	0.783	0.915	0.468	0.921	0.627	0.917	0.473	0.926
		[6]	0.766	0.915	0.453	0.922	0.608	0.918	0.458	0.926
		[1]	0.751	0.915	0.448	0.922	0.599	0.918	0.450	0.927
		[2]	0.725	0.916	0.428	0.923	0.579	0.919	0.431	0.928
		[3]	0.745	0.915	0.438	0.923	0.596	0.919	0.446	0.927
	(90, 80)	[4]	0.714	0.916	0.414	0.923	0.557	0.920	0.416	0.928
		[5]	0.661	0.916	0.395	0.924	0.510	0.921	0.370	0.929
		[6]	0.621	0.917	0.371	0.925	0.494	0.922	0.367	0.930
		[1]	0.499	0.921	0.298	0.927	0.398	0.922	0.291	0.932
		[2]	0.481	0.921	0.287	0.928	0.370	0.924	0.267	0.933
		[3]	0.489	0.921	0.292	0.927	0.389	0.923	0.279	0.932
	(30, 40)	[4]	0.468	0.921	0.277	0.929	0.365	0.926	0.260	0.934
		[5]	0.415	0.926	0.266	0.930	0.341	0.926	0.245	0.935
		[6]	0.317	0.931	0.237	0.933	0.284	0.928	0.219	0.938

Table 11: Interval estimates of $R_u(t)$

(s_1, s_2)	(n_1, n_2)	Design	95% ACI		95% BCI		95% ACI		95% BCI	
			(T_{11}, T_{12})				(T_{21}, T_{22})			
(0.5, 1.0)	(30, 40)	[1]	0.856	0.900	0.300	0.907	0.470	0.910	0.277	0.913
		[2]	0.739	0.902	0.272	0.909	0.427	0.910	0.264	0.915
		[3]	0.784	0.902	0.297	0.907	0.467	0.910	0.266	0.914
		[4]	0.592	0.903	0.270	0.909	0.404	0.911	0.242	0.916
		[5]	0.484	0.904	0.255	0.910	0.348	0.911	0.236	0.917
		[6]	0.440	0.904	0.249	0.910	0.336	0.912	0.232	0.918
	(50, 60)	[1]	0.413	0.905	0.244	0.911	0.331	0.912	0.229	0.918
		[2]	0.244	0.907	0.232	0.913	0.232	0.914	0.220	0.920
		[3]	0.305	0.906	0.241	0.912	0.234	0.913	0.227	0.919
		[4]	0.232	0.907	0.215	0.913	0.226	0.915	0.212	0.921
		[5]	0.225	0.909	0.212	0.914	0.218	0.915	0.196	0.923
		[6]	0.217	0.910	0.195	0.915	0.189	0.917	0.185	0.924
	(90, 80)	[1]	0.212	0.911	0.172	0.917	0.181	0.919	0.157	0.925
		[2]	0.208	0.914	0.147	0.922	0.171	0.924	0.143	0.927
		[3]	0.210	0.912	0.154	0.921	0.172	0.922	0.150	0.926
		[4]	0.198	0.915	0.140	0.922	0.169	0.924	0.133	0.928
		[5]	0.193	0.917	0.132	0.923	0.165	0.924	0.126	0.930
		[6]	0.181	0.919	0.124	0.923	0.157	0.925	0.114	0.931
(1.0, 1.5)	(30, 40)	[1]	0.174	0.920	0.113	0.927	0.149	0.926	0.080	0.933
		[2]	0.164	0.921	0.102	0.931	0.147	0.928	0.063	0.936
		[3]	0.169	0.921	0.109	0.930	0.149	0.927	0.070	0.934
		[4]	0.155	0.923	0.096	0.934	0.146	0.928	0.060	0.937
		[5]	0.145	0.923	0.092	0.936	0.136	0.928	0.052	0.939
		[6]	0.138	0.924	0.084	0.937	0.111	0.929	0.046	0.942
	(50, 60)	[1]	0.127	0.926	0.080	0.944	0.100	0.931	0.042	0.947
		[2]	0.099	0.931	0.066	0.950	0.085	0.936	0.032	0.953
		[3]	0.099	0.930	0.076	0.946	0.089	0.935	0.037	0.950
		[4]	0.088	0.932	0.061	0.952	0.080	0.937	0.029	0.956
		[5]	0.083	0.933	0.054	0.954	0.072	0.937	0.025	0.958
		[6]	0.081	0.947	0.051	0.954	0.068	0.952	0.021	0.958
	(90, 80)	[1]	0.076	0.948	0.049	0.955	0.068	0.953	0.017	0.959
		[2]	0.071	0.952	0.039	0.957	0.060	0.957	0.013	0.961
		[3]	0.074	0.951	0.044	0.956	0.062	0.956	0.015	0.960
		[4]	0.069	0.952	0.033	0.957	0.056	0.957	0.010	0.962
		[5]	0.065	0.953	0.028	0.957	0.054	0.958	0.008	0.962
		[6]	0.063	0.953	0.026	0.958	0.051	0.958	0.005	0.962

By focusing on the lowest RMSE, MAB, and AIL values, as well as the highest CP values, several assessments of a_i , b_i (for $i = 0, 1$), and $R_u(t)$ are made as follows:

- The simulation outcomes reveal that both frequentist and Bayesian estimators of a_i , b_i (for $i = 0, 1$), or $R_u(t)$ exhibit satisfactory.
- Increasing the total number of observations, whether via $\sum_{i=1}^k n_i$ or $\sum_{i=1}^k m_i$, leads to a reduction in RMSEs and MABs, indicating consistency and improved efficiency of the estimators under larger sample scenarios.
- Bayesian inference incorporating prior knowledge often yields more efficient estimates. Thus, Bayes point (or 95% BCI) estimates of all unknown quantities outperform other competitive frequentist (or 95% ACI) estimates.
- When the censoring times T_{i1} and T_{i2} are extended, it is noted that:
 - The RMSE and MAB results associated with estimating a_0 , b_i (for $i = 0, 1$), or $R_u(t)$ decrease except for a_1 .
 - The AILs results associated with estimating a_i , b_1 (for $i = 0, 1$), or $R_u(t)$ decrease except for b_0 .
 - The CPs results associated with estimating a_i , b_1 (for $i = 0, 1$), or $R_u(t)$ increase except for b_0 .
- Longer censoring durations (higher T_{i1} and T_{i2} values) yield the estimated CPs that closely approximate the nominal level.
- When the stress levels s_i grow, the RMSEs and MABs associated with a_i , b_i (for $i = 0, 1$), or $R_u(t)$ decrease, demonstrating that higher stress magnitudes improve the information content regarding these parameters. A similar pattern is observed for interval estimation of these specific parameters, such that the AILs tend to narrow down and CPs improve.
- Comparing the proposed progressive censoring patterns (reported in Table 1), in terms of lower RMSE, MAB, and AIL values and higher CP values, it is clear that:
 - The estimates of a_1 , b_1 , and $R_u(t)$ behaved better based on left censoring used in Design[i] for $i = 1, 4$ than others;
 - The estimates of a_0 and b_0 behaved better based on right censoring used in Design[i] for $i = 3, 6$ than others.
- To summarize, simulation to compare the efficiency of the proposed estimation methods is based on distinct precision criteria, namely, RMSE and MAB (for point estimate) and AIL and CP (for interval estimate), demonstrating that the proposed Bayes framework via MH algorithm sampler outperforms other competitive methods strongly.

The comparative study shows that Bayesian estimators consistently outperform maximum likelihood estimators in terms of several metrics of precision, especially under small-sample or heavily censored scenarios. Increasing censoring thresholds and stress levels generally improve estimation accuracy and coverage probabilities, highlighting the role of experimental design in reliability analysis. Moreover, left-censoring designs provide more stable estimates for stress-related parameters and the reliability function, while right-censoring designs perform better for baseline parameters.

6 Accelerated Data Applications

To illustrate the practical adaptability of the proposed estimators, this section presents two real-world applications from the engineering domain, utilizing actual data sets. The results indicate that the estimation methods perform effectively under real-life conditions.

Before presenting the real data applications, it is important to emphasize the practical implications of the proposed methodology. By integrating the Hjorth lifespan model with CSALT under the IAPT2C plan, this study provides a flexible and efficient framework for analyzing the lifetimes of highly reliable products. This approach broadens the applicability of the Hjorth model within the reliability domain and equips practitioners with a robust tool to evaluate accelerated testing experiments, even in the presence of heavy censoring or small sample sizes. The following subsections illustrate these advantages through applications to actual industrial lifetime data sets.

6.1 High-Voltage Transformer Life-Testing

Analyzing the failure times of transformers under high-voltage life-testing (HVLT) offers valuable insights into their durability and reliability when subjected to extreme operating conditions. This information is essential for modeling failure behavior, and improving lifespan estimation. In this example, to demonstrate the practical applicability and effectiveness of the proposed estimation methods, we analyze the failure times (in hours) of HVLT using several stress conditions. For additional details on this dataset, one can see Nelson [30]. In Table 12, the HVLT datasets under two distinct increasing stress levels: 35.4 kV (normal-use condition) and 42.4 kV (stress condition) are provided. For computational efficiency, each original failure time point is scaled by divided by ten. Before addressing our inference, to check whether the Hjorth model provides a significant fit to the HVLT data or not, the Kolmogorov–Smirnov (KS) statistics along its p -value (at $\alpha = 5\%$) significance level are considered. Given that the time to failures follows the population, using the null and alternative hypotheses, respectively, as

- H_0 : The HVLT dataset follows a Hjorth distribution;
- H_1 : The HVLT dataset does not follows a Hjorth distribution.

Table 12: Times to failures in HVLT datasets

Stress		Failure times							
35.4 kV	4.01	5.94	7.12	16.65	20.47	22.97	30.83	53.79	
42.4 kV	0.06	1.34	1.52	1.99	2.50	3.02	3.28	4.44	5.62

Based on all given HVLT datasets, the MLEs (along with their standard errors (Std.Ers)) as well as 95% ACI bounds (along with their interval widths (IW)) of μ and θ are calculated; see Table 13. Since the fitted p -values of the normal-use and stress datasets are greater than the specified 5% significance level, we cannot reject the null hypothesis that the Hjorth model fits the HVLT data adequately.

Again, from Table 12, four visualization fit tools of the Hjorth lifespan model are utilized, namely (i) probability-probability (PP), (ii) quantile-quantile (QQ), RF, and contour of the log-likelihood; see Fig. 1. It reveals that (i) the PP points closely follow the diagonal line, indicating a good agreement between the empirical and theoretical cumulative probabilities; (ii) the empirical quantiles align well with the theoretical quantiles; (iii) the fitted RF line closely tracks its empirical curve; and (iv) the

contour plot indicates stable and reliable frequentist estimates of μ and σ , and both exist and are unique. As a result, Fig. 1 supports the adequacy of the Hjorth model fit.

Table 13: Fitting the Hjorth model from HVLt datasets

Stress	Par.	MLE (Std.Er)	95% ACI [IW]	KS (<i>p</i> -value)
35.4 kV	μ	0.1805 (0.1861)	(0.0041, 0.5452) [0.5411]	0.1497 (0.9698)
	θ	0.1657 (0.0704)	(0.0277, 0.3037) [0.2760]	
42.4 kV	μ	0.0782 (0.1048)	(0.0000, 0.2837) [0.2837]	0.1714 (0.9429)
	θ	0.0025 (0.0010)	(0.0005, 0.0045) [0.0040]	

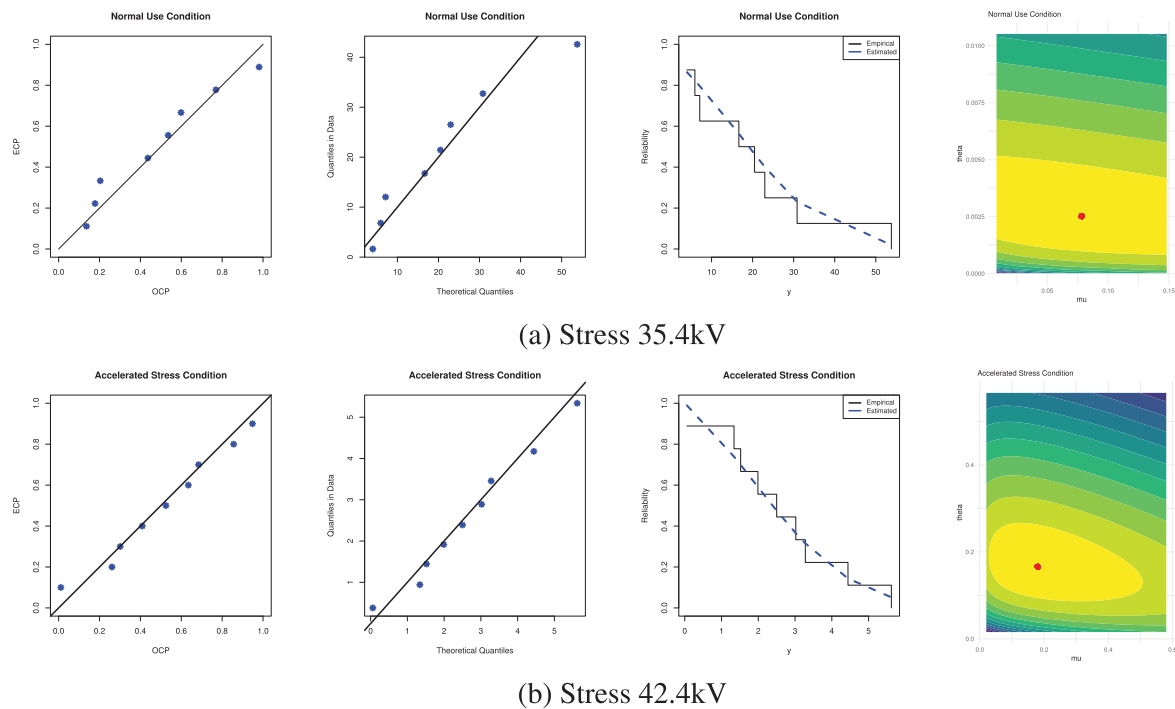


Figure 1: Fitting visualization tools of the Hjorth model from HVLt datasets. (a) Stress 35.4 kV, (b) Stress 42.4 kV

Now, we examine the efficacy of the proposed point and interval estimators of a_i and b_i (for $i = 1, 2$) as well as of μ_i, θ_i (for $i = u, 1, 2$), and $R_u(t)$ through the analysis of several IAPT2C samples using CSALT gathered from the HVLt datasets. From Table 14, by selecting various configurations for the effective sample sizes m_i (for $i = 1, 2$) and corresponding censoring schemes $(Q_{i1}, Q_{i2}, \dots, Q_{im_i})$, three censored datasets are constructed; see Table 14.

For each censored dataset, both offered point estimators—namely, the MLE and MCMC estimates—and 95% interval estimators—including ACI and BCI estimates—are computed for each unknown parameter. Specifically, all evaluations of $R_u(t)$ are done at a specific time point $t = 10$ and under the nominal operational stress level $s_u = 25$. Given the absence of informative prior knowledge regarding a_i and b_i (for $i = 1, 2$), non-informative priors are adopted by setting hyperparameters

$c_i = v_i = \vartheta_i = \sigma_i = 0.001$ (for $i = 1, 2$). This ensures that the posterior distributions closely resemble the likelihood functions. Taking $\mathcal{N} = 50,000$ and $\mathcal{M} = 10,000$, the Bayes' calculations by MCMC sampler are done. The initial values for a_i and b_i (for $i = 1, 2$) are set to their respective fitted frequentist estimates to facilitate convergence. Results, summarized in Table 15, include the point estimates (along with their Std.Ers) and 95% interval estimates (with corresponding IWs).

Table 14: Various IAPT2C samples using CSALT from HVLT data

Sample	Scheme	$T_{11}(r_1)$	$T_{12}(r_1^*)$	τ_1	Q_1^*	Censored Data $((n_1, m_1) = (8, 5))$ at 35.4 kV
S1	(3, 0*4)	5(1)	35(5)	30.83	0	4.01, 7.12, 16.65, 22.97, 30.83
S2	(0*2, 3, 0*2)	8(3)	25(5)	22.97	0	4.01, 5.94, 7.12, 20.47, 22.97
S3	(0*4, 3)	10(3)	17(4)	17	4	4.01, 5.94, 7.12, 16.65
		$T_{21}(r_2)$	$T_{22}(r_2^*)$	τ_2	Q_2^*	Censored Data $((n_2, m_2) = (9, 6))$ at 42.4 kV
S1	(3, 0*5)	0.5(1)	4.5(6)	4.44	0	0.06, 1.52, 1.99, 2.50, 3.02, 4.44
S2	(0*2, 3, 0*3)	1.6(3)	3.5(6)	3.5	0	0.06, 1.34, 1.52, 2.50, 3.02, 3.28
S3	(0*5, 3)	2.1(4)	2.6(5)	2.6	4	0.06, 1.34, 1.52, 1.99, 2.50

Table 15: Point and 95% interval estimates from HVLT data

Sample	Par.	MLE		Bayesian		ACI			BCI		
		Est.	Std.Er	Est.	Std.Er	Low.	Upp.	IW	Low.	Upp.	IW
S1	a_0	-59.233	13.198	-59.233	0.0010	-85.100	-33.366	51.734	-59.235	-59.231	0.0039
	a_1	1.3631	0.3136	1.3631	0.0050	0.7485	1.9778	1.2292	1.3532	1.3728	0.0195
	b_0	-23.890	3.8730	-23.890	0.0050	-31.481	-16.299	15.182	-23.900	-23.880	0.0195
	b_1	0.5271	0.1007	0.5271	0.0050	0.3297	0.7245	0.3949	0.5176	0.5371	0.0195
	μ_u	1.2E-11	4.2E-21	1.2E-11	1.5E-12	1.1E-10	1.4E-10	2.4E-11	9.2E-12	1.5E-11	5.8E-12
	μ_1	1.7E-05	1.6E-09	1.7E-05	3.1E-06	6.0E-05	9.4E-05	3.4E-05	1.2E-05	2.4E-05	1.2E-05
	μ_2	0.2380	0.0627	0.2433	0.0523	0.0005	0.7287	0.7281	0.1511	0.3501	0.1990
	θ_u	2.2E-05	9.5E-10	2.2E-05	2.8E-06	3.8E-05	8.3E-05	4.5E-05	1.7E-05	2.8E-05	1.1E-05
	θ_1	0.0053	0.0000	0.0054	0.0010	0.0006	0.0101	0.0094	0.0037	0.0074	0.0037
	θ_2	0.2141	0.0132	0.2187	0.0469	0.0011	0.4394	0.4383	0.1353	0.3132	0.1779
	$R_u(x_0)$	0.9989	0.0015	0.9989	0.0001	0.9926	0.9990	0.0064	0.9986	0.9991	0.0005
S2	a_0	-71.758	13.120	-71.758	0.0010	-97.472	-46.044	51.428	-71.760	-71.756	0.0039
	a_1	1.6490	0.3102	1.6490	0.0050	1.0410	2.2570	1.2160	1.6391	1.6587	0.0195
	b_0	-21.958	3.5885	-21.958	0.0050	-28.992	-14.925	14.067	-21.968	-21.948	0.0195
	b_1	0.4850	0.0930	0.4850	0.0050	0.3027	0.6673	0.3646	0.4755	0.4950	0.0195
	μ_u	5.5E-14	8.9E-26	5.5E-14	6.9E-15	5.3E-13	6.4E-13	1.1E-13	4.2E-14	6.9E-14	2.7E-14
	μ_1	1.5E-06	1.3E-11	1.6E-06	2.8E-07	5.6E-06	8.7E-06	3.1E-06	1.1E-06	2.1E-06	1.1E-06
	μ_2	0.1591	0.0306	0.1626	0.0350	0.0002	0.5018	0.5016	0.1010	0.2340	0.1330
	θ_u	5.4E-05	4.8E-09	5.4E-05	6.8E-06	8.2E-05	1.9E-04	1.1E-04	4.1E-05	6.7E-05	2.6E-05
	θ_1	0.0083	0.0000	0.0085	0.0015	0.0012	0.0155	0.0143	0.0057	0.0115	0.0058
	θ_2	0.2483	0.0155	0.2537	0.0543	0.0047	0.4920	0.4874	0.1569	0.3632	0.2063
	$R_u(x_0)$	0.9973	0.0035	0.9973	0.0003	0.9905	0.9998	0.0093	0.9966	0.9979	0.0013

(Continued)

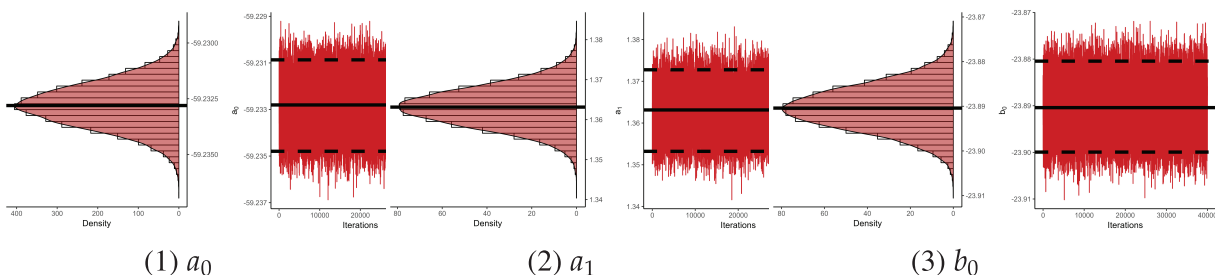
Table 15 (continued)

Sample	Par.	MLE		Bayesian		ACI			BCI		
		Est.	Std.Er	Est.	Std.Er	Low.	Upp.	IW	Low.	Upp.	IW
S3	a_0	-13.556	14.235	-13.556	0.0010	-41.456	14.343	55.799	-13.558	-13.554	0.0039
	a_1	0.2938	0.3470	0.2938	0.0050	-0.3864	0.9740	1.3604	0.2839	0.3034	0.0195
	b_0	-18.231	13.876	-18.231	0.0050	-45.427	8.965	54.392	-18.241	-18.221	0.0195
	b_1	0.3524	0.3775	0.3524	0.0050	-0.3875	1.0923	1.4799	0.3429	0.3624	0.0195
	μ_u	2.0E-03	1.3E-04	2.0E-03	2.5E-04	2.0E-02	2.4E-02	4.0E-03	1.5E-03	2.5E-03	9.8E-04
	μ_1	4.3E-02	8.1E-03	4.3E-02	7.7E-03	1.3E-01	2.2E-01	8.5E-02	2.9E-02	5.9E-02	3.0E-02
	μ_2	0.3335	0.1094	0.3409	0.0733	0.0031	0.9818	0.9787	0.2117	0.4905	0.2788
	θ_u	8.1E-05	1.3E-07	8.2E-05	1.0E-05	6.3E-04	7.9E-04	1.6E-04	6.2E-05	1.0E-04	4.0E-05
	θ_1	0.0032	0.0000	0.0032	0.0006	0.0024	0.0088	0.0063	0.0022	0.0044	0.0022
	θ_2	0.0373	0.0072	0.0381	0.0082	0.0013	0.2039	0.2026	0.0235	0.0545	0.0310
	$R_u(x_0)$	0.9912	0.0189	0.9911	0.0008	0.9541	0.9944	0.0403	0.9896	0.9926	0.0031

The findings demonstrate that the Bayesian (credible interval) approach exhibits superior performance compared to maximum likelihood (asymptotic interval) in terms of yielding lower standard errors and shorter interval lengths. This pattern of improved efficiency is also consistently observed across the computed interval estimates.

To evaluate the performance of the MCMC algorithm, using S1 (for instance), histogram (with Gaussian density kernel) and trace plots based on 40,000 samples are plotted (see Fig. 2). These visual tools confirm that the staying 40,000 Markov iterations achieved proper convergence and mixing behavior. The Bayesian estimates and their credible intervals for key parameters are clearly depicted, with solid and dashed lines, respectively. The burn-in phase is sufficient to remove the influence of initial values. Additionally, posterior distributions exhibited varying shapes such as:

- (i) Symmetrical shape for a_i and b_i (for $i = 1, 2$);
- (ii) Positively skewed shape for μ_i, θ_i (for $i = u, 1, 2$);
- (iii) Negatively skewed shape for $R_u(t)$.


Figure 2: (Continued)

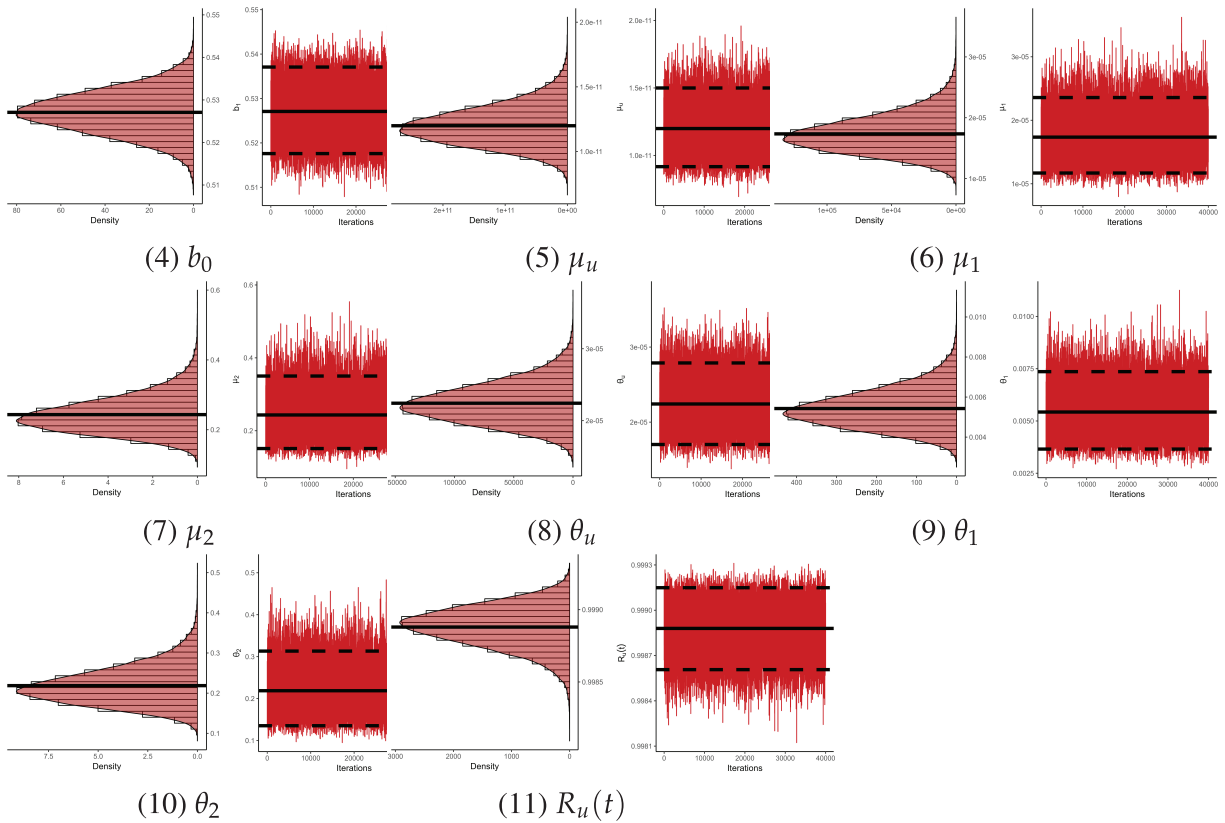


Figure 2: Density (left) and Trace (right) plots from HVLTL data

To better understand the behavior of the estimated reliability function $R_u(t)$ under varying operational conditions, it is essential for reliability analysts to visualize the fitted reliability curves against empirical data. Fig. 3 provides a comparative depiction of the empirical and model-based estimates of the reliability function under two distinct stress scenarios. Fig. 3, using dataset S1 as an illustrative case, compares empirical and estimated reliability functions under normal (35.4 kV) and accelerated (42.4 kV) stress conditions using the Hjorth model. In both cases, the estimated reliability curves (dashed lines) closely follow the empirical step functions, indicating a good model fit across varying stress levels. This visual agreement supports the model's robustness and suitability for analyzing reliability behavior under different operational and stress scenarios.

6.2 Oil of Insulating Fluid

The oil breakdown times (OBTs) are key indicators of how well an insulating fluid withstands electrical stress before failure, directly impacting the reliability of high-voltage equipment. Measuring OBTs under various voltages helps evaluate the fluid's insulation strength and aging behavior. This part examines the OBTs of an insulating fluid subjected to varying levels of high-voltage stress. Following Nelson [30], two sets of OBT observations (in seconds) are analyzed: one collected under a nominal stress level of 30 kV (normal-use condition), and the other under an elevated stress level of 36 kV (stress condition). For computational simplicity, each OBT observation is divided by a factor of 1/10. The transformed data are presented in Table 16. From Table 16, the fitting outcomes

reposted in Table 17 suggest that the Hjoth model adequately fits the OBT data under both stress levels appropriately.

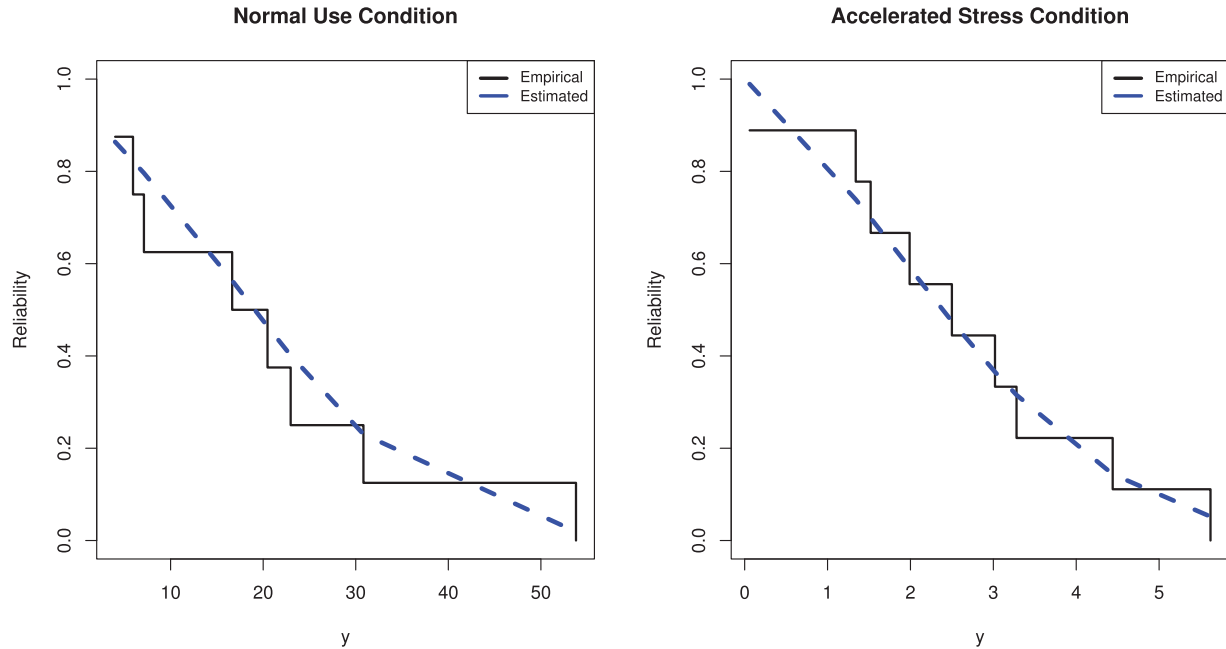


Figure 3: Estimated lines of $R_u(t)$ from HVLt data

Table 16: Times to failures in OBT datasets

Stress	Failure times											
30 kV	1.705	1.774	2.046	2.102	2.266	4.340	4.730	13.907	14.412	17.588	19.490	
36 kV	0.035	0.059	0.096	0.099	0.169	0.197	0.207	0.258	0.271	0.290	0.367	0.399 0.535 1.377 2.550

Table 17: Fitting the Hjoth model from OBT datasets

Stress	Par.	MLE (Std.Er)	95% ACI [IW]	KS (p -value)
30 kV	μ	0.2809 (0.1493)	(0.0000,0.5735) [0.5735]	0.2539 (0.4086)
	θ	0.0092 (0.0051)	(0.0007,0.0192) [0.0185]	
36 kV	μ	3.0997 (0.8980)	(1.3397,4.8597) [3.5201]	0.1513 (0.8335)
	θ	0.0688 (0.4135)	(0.0000,0.8792) [0.8792]	

Fig. 4 shows that (i) the PP plot closely follows the diagonal, (ii) empirical and theoretical quantiles align well, (iii) the fitted RF mirrors the empirical curve, and (iv) the contour plot confirms existence and uniqueness of $\hat{\mu}$ and $\hat{\sigma}$. These facts support the adequacy of the Hjoth model fit.

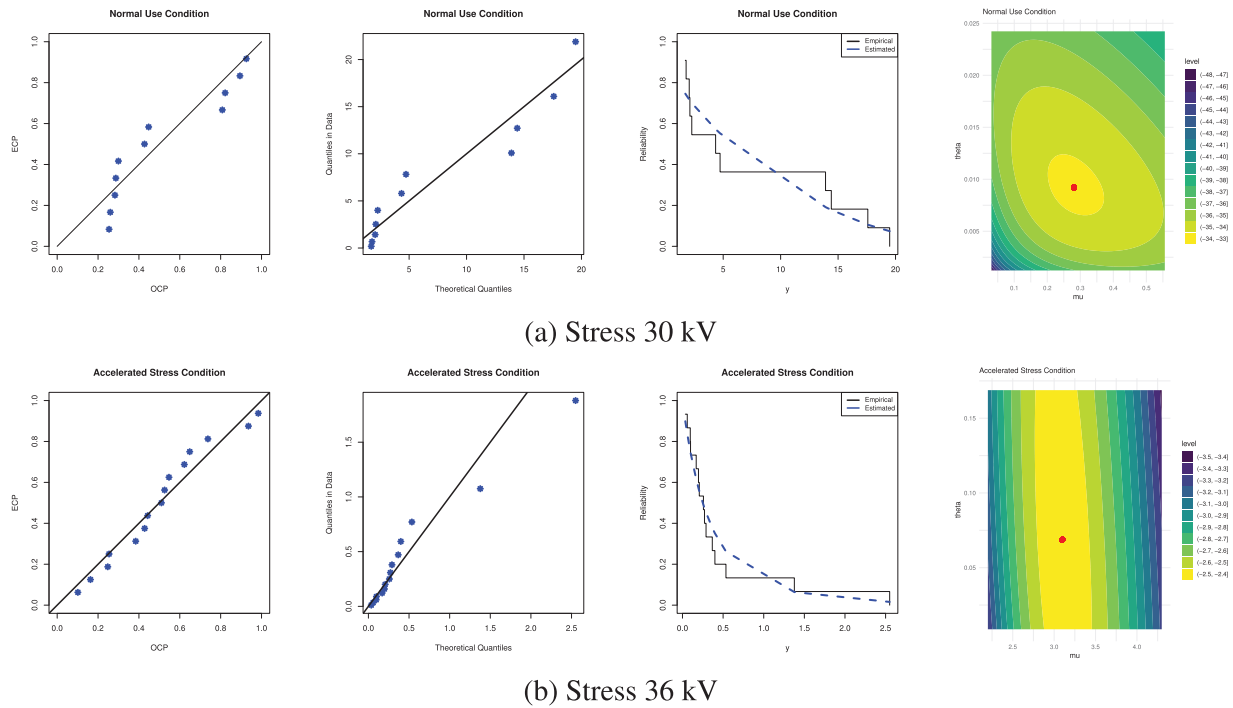


Figure 4: Fitting visualization tools of the Hjorth model from OBT datasets. (a) Stress 30 kV, (b) Stress 36 kV

Now, from OBT data reported in Table 16, we conduct a comprehensive evaluation of the proposed point and interval estimators for the parameters a_i and b_i (for $i = 1, 2$) as well as of μ_i , θ_i (for $i = u, 1, 2$), and $R_u(t)$. To achieve this, several IAPT2C samples are generated through the CSALT procedure, derived from the OBT dataset; see Table 18.

Table 18: Various IAPT2C samples using CSALT from OBT data

Sample	Scheme	$T_{11}(r_1)$	$T_{12}(r_1^*)$	τ_1	Q_1^*	Censored Data $((n_1, m_1) = (11, 8))$ at 30 kV
S1	(3, 0*7)	1.75(1)	14.5(8)	14.412	0	1.705, 1.774, 2.046, 2.102, 2.266, 4.340, 4.730, 14.412
S2	(0*3, 3, 0*4)	2.25(4)	14.5(7)	14.5	1	1.705, 1.774, 2.046, 2.102, 4.340, 4.730, 13.907
S3	(0*7, 3)	3.75(5)	4.5(6)	4.5	5	1.705, 1.774, 2.046, 2.102, 2.266, 4.340
		$T_{21}(r_2)$	$T_{22}(r_2^*)$	τ_2	Q_2^*	Censored Data $((n_2, m_2) = (15, 12))$ at 36 kV
S1	(3, 0*11)	0.15(3)	0.55(12)	0.535	0	0.035, 0.096, 0.099, 0.169, 0.197, 0.207, 0.258, 0.271, 0.290, 0.367, 0.399, 0.535
S2	(0*5, 3, 0*6)	0.22(6)	0.55(10)	0.55	2	0.035, 0.059, 0.096, 0.099, 0.169, 0.197, 0.258, 0.290, 0.367, 0.535
S3	(0*11, 3)	0.07(2)	0.26(8)	0.26	7	0.035, 0.059, 0.096, 0.099, 0.169, 0.197, 0.207, 0.258

For each collected dataset, we computed the MLE/MCMC estimates and corresponding 95% ACI/BCI estimates for each parameter under investigation. Under a designated nominal stress level of $s_u = 25$, all estimates of $R_u(t)$ are obtained at time $t = 1$. The Bayes investigation for each parameter is performed using the same calculation setups depicted in Section 6.1. The results (presented in Table 19) indicate a strong agreement between the frequentist and Bayesian point estimates; however, the Bayesian approach consistently outperforms the frequentist method in terms of yielding smaller Std.Ers and narrower IWhs. This pattern of superiority is also evident in the comparison of interval estimates, reinforcing the efficiency and precision of the Bayesian framework. Using dataset S1 from Table 16, the performance of the MCMC algorithm is evaluated through histogram and trace plots based on 40,000 retained samples (see Fig. 5).

Table 19: Point and 95% interval estimates from OBT data

Sample	Par.	MLE		Bayesian		ACI			BCI		
		Est.	Std.Er	Est.	Std.Er	Low.	Upp.	IW	Low.	Upp.	IW
S1	a_0	-5.6057	29.485	-5.6057	0.0010	-63.395	52.183	115.58	-5.6077	-5.6038	0.0039
	a_1	0.1210	0.9472	0.1210	0.0050	-1.7355	1.9776	3.7131	0.1111	0.1307	0.0195
	b_0	-34.752	5.2477	-34.752	0.0050	-45.037	-24.467	20.571	-34.762	-34.742	0.0195
	b_1	1.0536	0.1524	1.0535	0.0050	0.7548	1.3523	0.5975	1.0441	1.0635	0.0195
	μ_u	0.0758	0.2077	0.0764	0.0096	0.0000	0.9691	0.9691	0.0584	0.0955	0.0371
	μ_1	0.1388	0.0791	0.1404	0.0212	0.0000	0.6899	0.6899	0.1014	0.1831	0.0817
	μ_2	0.2870	2.0482	0.2916	0.0530	0.0000	3.0921	3.0921	0.1952	0.3983	0.2032
	θ_u	0.0002	0.0001	0.0002	0.0001	0.0000	0.0009	0.0009	0.0002	0.0003	0.0001
	θ_1	0.0431	0.0011	0.0435	0.0065	0.0000	0.1069	0.1069	0.0312	0.0565	0.0252
	θ_2	23.959	112.35	24.327	4.4060	3.1843	44.733	41.549	16.274	33.150	16.876
	$R_u(x_0)$	0.9487	0.2996	0.9484	0.0063	0.7615	0.9992	0.2377	0.9359	0.9602	0.0244
S2	a_0	-11.558	5.1842	-11.558	0.0010	-21.718	-1.3968	20.322	-11.560	-11.556	0.0039
	a_1	0.3395	0.1559	0.3394	0.0050	0.0339	0.6450	0.6112	0.3295	0.3491	0.0195
	b_0	-34.937	7.4504	-34.937	0.0050	-49.540	-20.335	29.205	-34.947	-34.927	0.0195
	b_1	1.0169	0.2278	1.0169	0.0050	0.5704	1.4634	0.8930	1.0074	1.0269	0.0195
	μ_u	0.0464	0.0040	0.0467	0.0059	0.0000	0.1697	0.1697	0.0357	0.0584	0.0227
	μ_1	0.2531	0.0310	0.2558	0.0386	0.0000	0.5979	0.5979	0.1849	0.3338	0.1489
	μ_2	1.9401	1.7151	1.9707	0.3580	0.0000	4.5069	4.5069	1.3191	2.6922	1.3731
	θ_u	0.0001	0.0002	0.0001	0.0009	0.0000	0.0003	0.0003	0.0001	0.0001	0.0004
	θ_1	0.0119	0.0001	0.0120	0.0018	0.0000	0.0338	0.0338	0.0086	0.0156	0.0070
	θ_2	5.3248	32.735	5.4066	0.9792	0.0000	16.539	16.539	3.6169	7.3675	3.7506
	$R_u(x_0)$	0.9683	0.0422	0.9681	0.0039	0.8856	0.9995	0.1139	0.9603	0.9755	0.0152
S3	a_0	-61.331	14.376	-61.331	0.0010	-89.507	-33.154	56.353	-61.333	-61.329	0.0039
	a_1	1.7077	0.4086	1.7077	0.0050	0.9068	2.5086	1.6019	1.6978	1.7173	0.0195
	b_0	-27.960	5.3456	-27.960	0.0050	-38.437	-17.483	20.954	-27.970	-27.950	0.0195
	b_1	0.8512	0.1719	0.8512	0.0050	0.5142	1.1882	0.6740	0.8417	0.8612	0.0195
	μ_u	0.0008	0.0001	0.0008	0.0001	0.0000	0.0013	0.0013	0.0006	0.0010	0.0004
	μ_1	0.0004	0.0010	0.0004	0.0006	0.0000	0.0008	0.0008	0.0003	0.0005	0.0002
	μ_2	1.1583	2.7006	1.1765	0.2137	0.0000	4.3791	4.3791	0.7875	1.6073	0.8198
	θ_u	0.0013	0.0002	0.0013	0.0001	0.0000	0.0040	0.0040	0.0010	0.0016	0.0006
	θ_1	0.0886	0.0013	0.0895	0.0135	0.0172	0.1600	0.1429	0.0642	0.1161	0.0519
	θ_2	14.639	187.37	14.864	2.6921	0.0000	41.468	41.468	9.9437	20.255	10.311
	$R_u(x_0)$	0.9994	0.0007	0.9994	0.0001	0.9980	0.9997	0.0017	0.9992	0.9995	0.0003

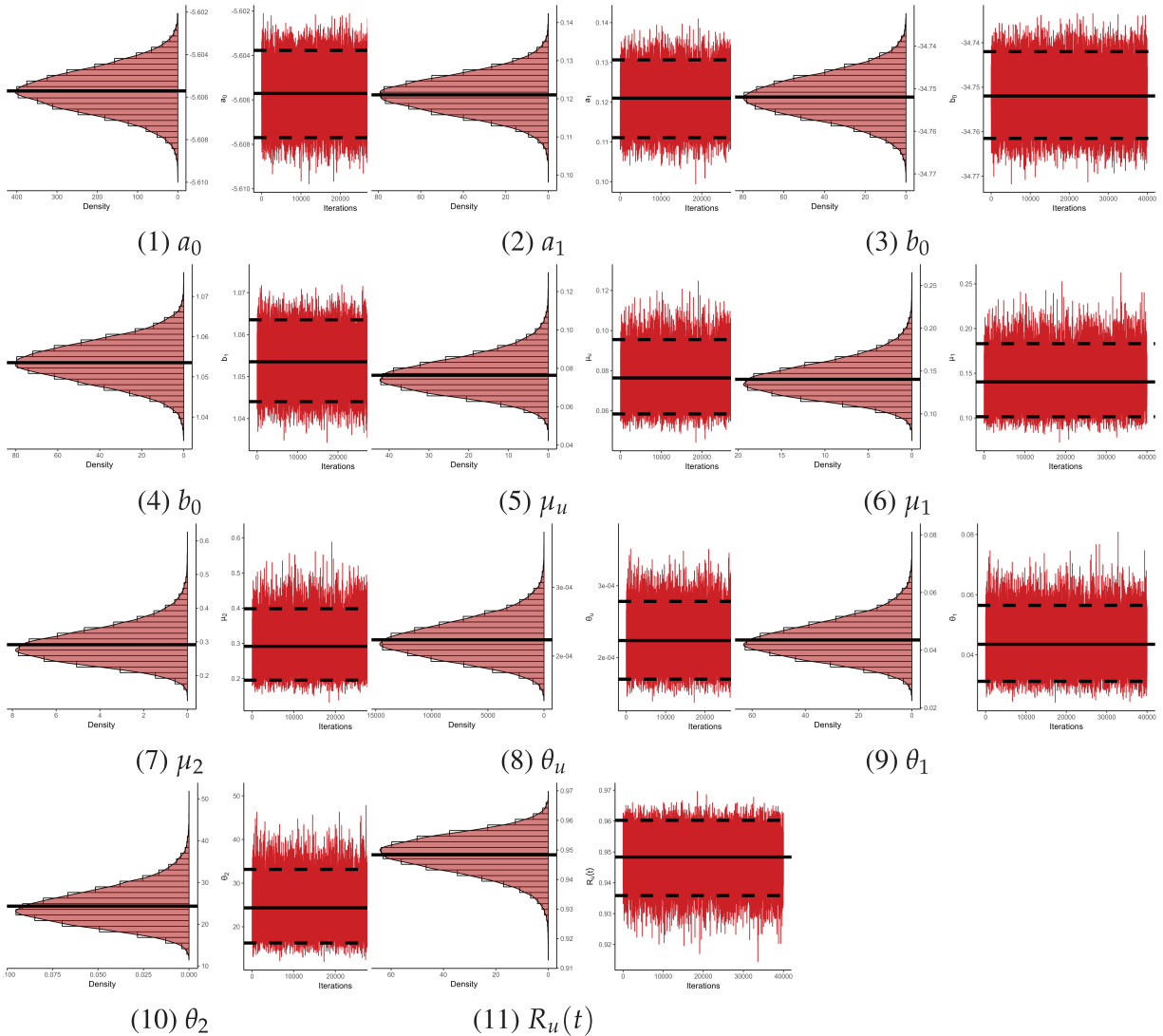


Figure 5: Density (left) and Trace (right) plots from OBT data

The subplots in Fig. 5 confirm that the Markov chains exhibit proper convergence and effective mixing after the burn-in phase. The Bayesian estimates and their credible intervals are visualized, and the burn-in period successfully mitigates the influence of initial values. Fig. 5 exhibits the same behaviors for the distribution of 40,000 MCMC iterations of the parameters as displayed in Fig. 2.

To gain deeper insight into the behavior of the estimated reliability function $R_u(t)$ across different operational conditions, it is crucial to compare the model-based curves with empirical data. Fig. 6, based on dataset S1, illustrates this comparison under two stress levels: normal-use (30 kV) and accelerated (36 kV). The estimated Hjorth reliability curves (dashed lines) align closely with the empirical step functions in both cases, demonstrating a strong model-data agreement. This consistency across stress conditions highlights the model's robustness and reinforces its applicability for reliability assessment in both standard and accelerated environments.

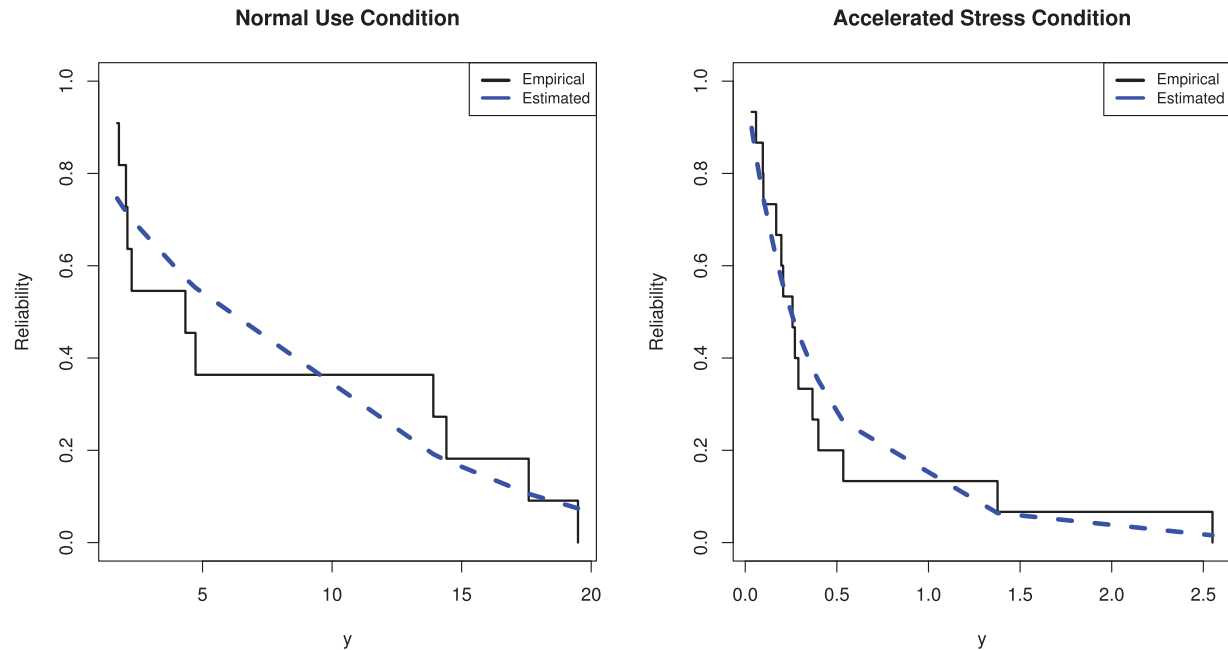


Figure 6: Estimated lines of $R_u(t)$ from OBT data

Finally, the numerical results derived from data on oil breakdown times or high-voltage transformer life-testing failure times showed that the suggested Hjoth model is helpful in solving the engineering problems as well as the suggested inferential issues. Moreover, to place our results in context, it is important to compare them with findings from related studies. The parameter estimates and reliability measures obtained from the two real data sets confirm the flexibility of the Hjoth model under constant-stress accelerated life testing, particularly in capturing increasing and bathtub-shaped hazard rates. The current study is the first effort to integrate the Hjoth lifespan model into the reliability domain using accelerated life tests. In contrast, classical models such as the Weibull, gamma, and generalized exponential distributions are limited in this regard.

7 Practical and Theoretical Implications

The proposed framework carries several important implications for both reliability theory and industrial practice. On the theoretical side, the study demonstrates that the Hjoth distribution, when linked with stress through a log-linear model, offers a flexible alternative to traditional two-parameter lifetime models such as the Weibull and gamma. Its ability to capture constant, increasing, decreasing, and bathtub-shaped hazard rates significantly broadens the class of lifetime distributions available for accelerated life testing, thereby strengthening the statistical foundations of reliability analysis.

From a practical standpoint, the integration of the Hjoth model with constant-stress accelerated life tests under the improved adaptive progressive Type-II censoring scheme provides engineers and reliability practitioners with an efficient tool for decision-making. For instance, in the context of high-voltage transformers, the model can guide the estimation of the reliability function under normal conditions, enabling more accurate scheduling of preventive maintenance and reducing unexpected downtime. In warranty analysis, the ability to model bathtub-shaped hazards supports better prediction of early-life and wear-out failures, allowing firms to design more cost-effective warranty

policies. Likewise, in industries such as aerospace and automotive engineering, where tests often involve small samples and censored data, the Bayesian approach proposed in this study ensures more stable inference, improving risk assessments and long-term planning.

Overall, by combining flexibility in hazard modeling with efficiency in handling censored ALT data, the proposed framework enhances both the theoretical toolkit of reliability analysts and the practical decision-making capabilities of engineers and industrial managers.

8 Concluding Remarks

This study addressed the critical challenge of assessing highly reliable products by proposing an integrated framework that combined constant-stress accelerated life tests with an improved adaptive progressive Type-II censoring scheme. The Hjoth distribution, selected for its exceptional flexibility in modeling four distinct hazard rate patterns (constant, decreasing, increasing, and bathtub-shaped), was applied under a log-linear life-stress relationship. The methodology adopted both classical and Bayesian approaches, deriving maximum likelihood estimates with approximate confidence intervals and Bayesian estimates with credible intervals using Markov Chain Monte Carlo sampling. The estimation covered both the model parameters and the reliability function under normal operating conditions. A Monte Carlo simulation study is conducted to compare the performance of various estimates. The simulation results show that Bayesian estimators outperform maximum likelihood estimators in precision, especially with small samples or heavy censoring. They also show that higher censoring thresholds and a greater number of stress levels generally improve accuracy and coverage, underscoring the influence of experimental design. Moreover, left-censoring yields more stable estimates for stress-related parameters and use-level reliability, whereas right-censoring performs better for baseline parameters. The practical utility of the framework was validated through real-world accelerated lifetime data sets, confirming its effectiveness in reliability applications. By considering the integration of the improved adaptive progressive Type-II censoring scheme with constant-stress accelerated life tests for the Hjoth model, this work filled a significant gap in the reliability literature, where prior studies predominantly focused on conventional distributions. A natural extension is to develop estimation procedures for the Hjoth model, including use-level reliability, for step-stress accelerated life tests using the same censoring scheme. This work is under consideration and will be reported at a later date.

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