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FORCE METHOD CONCEPTION USING TRANSFER MATRIX TO APPLY TO MULTIPHASE FLOW BY ONE-BY-ONE CORRESPONDING PARTICLE-CARTESIAN CELL (*P*/*CC*) MODEL

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Abstract. *This work presents a novel force method for multiphase flow in which displacements and forces progress while strains terminate on surfaces between two different phases, such as the surface between air and liquid. This report is part of a research project to apply Helmholtz decomposition (H-d) to the finite element method. State vectors continues to exist on the surface of multiphase. The conventional scheme uses displacement method and applies a model to set Dirichlet boundary conditions for air on the air-liquid interface. The proposed scheme allows simulating it without such modeling.*

1 INTRODUCTION

1.1 Introduction of a novel force method using the transfer matrix

This work presents a novel force method for multiphase flow in which displacements and forces progress while strains terminate on surfaces between two different phases, such as the surface between air and liquid.

However, because the proposed force method is developed for general continuum mechanics, it differs from the so-called conventional force method in the framework, which is constructed based on the statically determined main structure. †

In the transfer matrix approach, the vector consisting of displacement $\boldsymbol{u}$ and force $\boldsymbol{\sigma}$ is called "state vector."

Let us refer to the overall state vector, which includes components like $\{\boldsymbol{u}, \nabla\boldsymbol{F}, \nabla\nabla\boldsymbol{F}, \cdots\}$ as the "state vector," and $\{\boldsymbol{u}, \nabla\boldsymbol{u}, \nabla\nabla\boldsymbol{u}, \cdots\}$ as the "derivative vector."

The former is represented by the latter as follows: $\{\boldsymbol{u}, \nabla\boldsymbol{F}, \nabla\nabla\boldsymbol{F}, \cdots\} = \{\boldsymbol{u}, \mu\nabla\boldsymbol{u}, \mu\nabla\nabla\boldsymbol{u}, \cdots\}$,

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In conventional "force method,"† the {forces} are independent variables and the equation defining the constraint conditions are satisfied.

where μ is the viscosity coefficient.

The element represented by the state vector will be referred to as the "state vector element." The rationale for this will be explained later but for now let us refer to the proposed approach of employing node parameters $\{\boldsymbol{u}, \nabla \boldsymbol{F}\}_k$ on vertex nodes as "state vector method," which encompasses both the displacement and force methods. ††

The "one-by-one corresponding particle/Cartesian cell (*P/CC*) model" [1] is proposed here as a multiphase flow model. The moving particle and vertex nodes of the neighboring hexahedral cell are independently coupled to the Cartesian cells (CC) in the *P/CC* model.

At the start of the simulation, the particle connects to the initial CC center and the vertex to the CC's vertex nodes. We can image dual grids: one connecting to the neighbor vertices and another to the neighboring particles. We denoted the former as "vertex node cell" and the latter as "particle grid cell." The former cell includes a unique particle, and the latter includes a vertex node.

As the simulation progresses, each particle connects to the nearest CC center, thereby forming particle grid cell. Each vertex node must be in located the particle grid cell individually. Let us position the vertex node on the gravity of the particle grid cell as the initial coordinate, which is corrected to conserve the initial particle cell volume.

At the time step *n*, the current phase of the hexahedral cell can be ascertained using the CC because both the permanent and current addresses are memorized in the CC. The state vector finite element approach is used to calculate the *P/CC* model in this study.

1.2 Discrete Helmholtz decomposition

This report is part of a research project to apply Helmholtz decomposition (*H-d*) to the finite element method (FEM).

To apply *H-d* to FEM, *H-d* is improved, and the improved *H-d* is referred to as "discrete Helmholtz decomposition (*dHd*)," which is the outcome of this study.

An incompressible flow is assumed in this work. The *dHd* expression for incompressible flow is $\boldsymbol{u} = \nabla\boldsymbol{\psi} \cdot \{1,1,1\}^T \ \ (div\boldsymbol{\psi} = 0)$, where $\boldsymbol{u}$ denotes the displacements in the Cartesian coordinates and is represented by the Einstein contraction convention (tensor contraction using the Einstein summation convention) by $\nabla\boldsymbol{\psi}$ in the right hand side.

A new symbol for the $\nabla\boldsymbol{\psi}$ expression is defined as $\nabla\boldsymbol{\psi} \equiv [\nabla_{diag}\boldsymbol{\psi}, \nabla_{offd}\boldsymbol{\psi}]$, which represents the decomposition of $\nabla\boldsymbol{\psi}$ into lateral (longitudinal) and transverse components.

A new vector operator is also defined as $\nabla^1\boldsymbol{\psi} \equiv \nabla\boldsymbol{\psi} \cdot \{1,1,1\}^T$, where $\nabla^1\boldsymbol{\psi} \equiv \nabla_{diag}\boldsymbol{\psi} +$ offd$\boldsymbol{\psi}$, which represents the lateral and transverse components of displacement $\boldsymbol{u}$. Subcontractions are used to represent the latter. Therefore, let us call it "transverse components subcontraction," which is a very important conception to understand that how two-dimensional (2D) model can be obtained from the 3D model. This is explained in the next section.

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In the proposed novel force method††, {displacement, force} are independent variables and the virtual work equation is satisfied.

The compressible component of the flow is represented by $\nabla\varphi$ $(\nabla^2\varphi \neq 0)$. Therefore, $\nabla\varphi$ $(\nabla^2\varphi = 0)$ can be temporarily used for computing the incompressible flow.

The above expressions of *dHd* by potential $\boldsymbol{\psi}$ are also represented in $\nabla^1\boldsymbol{\psi} \equiv u_{i,j}$.

1.3 Proposed relationship between 3D model and 2D model

The *x*-*y* 2D model is a model achieved by the application of the *z*-axis periodic boundary conditions of the 3D model.

In FEM, the number of finite elements in *z*-axis is sufficient to build the system and the node parameters in *x*-*y* planes can be equaled, achieving the 2D model.

The 2D model is represented using displacement $\{u,v\}$ and $\{w\}$, where $\partial w/\partial z = 0$. This perception is important to explain the stream function ψ and recognize mirror images.

The torsion around *z*-axis $\partial w/\partial x$ $(= \partial w/\partial y)$ in 3D are equal to the transverse components in *x*-*y* plane, *i.e.*, $\partial w/\partial x = \partial v/\partial x$ and $\partial w/\partial y = \partial u/\partial y$. Therefore, the off-diagonal components (transverse components) in $\nabla\boldsymbol{u}$ can be contracted from six to three components.

Therefore, the lateral components are represented by scalar potential Ψ, *i.e.*, $\nabla\Psi = \{\partial w/\partial x = \partial v/\partial x,\ \partial w/\partial y = \partial u/\partial y, \partial u/\partial z = \partial v/\partial z\}$, which is similar to the stream function $\nabla\psi = \{v = \partial\psi_3/\partial x = \partial\psi_2/\partial x, u = \partial\psi_3/\partial y = \partial\psi_2/\partial y, w = \partial\psi_1/\partial z\ = \partial\psi_2/\partial z\}$.

The contraction in modeling from 2D to 1D is $\partial w/\partial x = \partial v/\partial x$ $(\partial v/\partial y = 0)$ in the case of the beam model. Therefore, the locking of the so-called Timoshenko's beam is resolved using the variable $\partial v/\partial x$. The locking in Mindlin model is resolved similarly.

1.4 Mapping method

The state vector element is represented in Fig 1 by a quasi-C^1-continuity element, which is an incomplete three-fold third-order function.

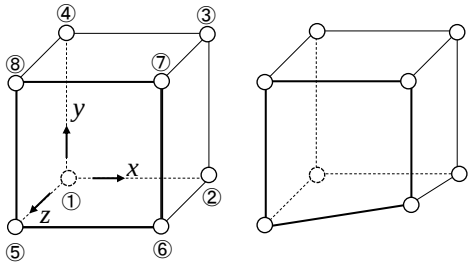

Figure 1: Quasi-C^1-continuity finite element represented by the three-fold third-order function

However, a three-fold first-order function is used to express the shape element.

As a result, the mapping is done using a nonisoparametric technique.

dHd is represented by a normalized cell $\boldsymbol{\chi}$ (*1* × *1* × *1*) and uses nonisoparametric mapping to calculate the dynamic hexahedral cell.

First, FEM is employed to calculate the shape element's $\boldsymbol{x}$ coordinates using the mapping method *f*: $\boldsymbol{x}\rightarrow\boldsymbol{\chi}$.

FEM calculates $\boldsymbol{x}$ using the node parameters $\{\boldsymbol{u}\}_k$: $\boldsymbol{x} = \{\boldsymbol{x}\}_k + \boldsymbol{u}(\boldsymbol{\chi})$, where $\{\boldsymbol{x}\}_k$ denotes the coordinates of the shape element's vertex node and $\boldsymbol{u}$ denotes the displacement.

The displacements $\boldsymbol{u}$ for the state vector element represented by a three-fold third-order function are derived using independent variables $\boldsymbol{x}$, which are calculated above.

(X, Y, Z) represents the global coordinates, and (x, y, z) represents the local coordinates in the hexahedral cell, with origin $(X, Y, Z)_1$, where $(X, Y, Z)_k$ represents the global coordinates of the vertex nodes *k* (*= 1,2,3,⋯;8*).

1.2 Multidirectional finite element method

The most important findings of the study are that the expression of {*u,v*} by stream function is an expression on $\theta = \pi/4$ rotated-coordinates around the *z*-axis, and {*v,w*} and {*w,u*} are rotated individually on $\pi/4$ around the *x*- and *y*-axes and expressed using vector potential $\boldsymbol{\psi}$ in three-dimensions (3D), which is the *H-d* expression.

When and only when the Coulomb gauge is satisfied by a solution of the strong form (analytical solution), that is, when the algebraic solution is guaranteed, the *H-d* equation can represent displacements in arbitrarily rotated coordinates.

However, because FEM is a method for obtaining a solution by week form, therefore, residuals exist between the solutions, for example, solutions between cases $\pi/4$ and $-\pi/4$ or $\theta = 0$.

The simple average of their solutions is used in the multidirectional FEM.

The Cauchy-Riemann equations must be satisfied by the multidirectional FEM to eliminate digit losses. That is, however, general tasks for the robust model.

It is worth noting that Cauchy-Riemann equations are derived in two dimensions (2D). The adoption of the so-called conjugate variable concept enables to apply the conception of Cauchy-Riemann equations to three-dimensional problems applying to the mirror images.

2 FUNDAMENTAL EQUATIONS

In Lagrangian mechanics, the Navier-Stokes (NS) equation is defined as $\rho d\boldsymbol{U}/dt = \mu\nabla^2\boldsymbol{U}$ where $\boldsymbol{U}$ is velocity.

The pressure *P* term is eliminated from the NS equation, since the equation, $\nabla P + 1/3\mu\,\nabla\mathrm{div}\boldsymbol{U} = \boldsymbol{0}$ and $(\nabla p + 1/3\nu\,\nabla\mathrm{div}\boldsymbol{U} = \boldsymbol{0})$.

The velocity $\boldsymbol{U}$ is represented by the velocity of $\nabla^1\boldsymbol{\psi}$ and mapped.

The mapped NS equation is linear in the Lagranian expression. The simultaneous equation appears to be solved without an iterative method based on the implicit method at a galance;

however, iteration is inevitable, because the equations $\mathrm{div}\boldsymbol{\psi} = \mathbf{0}$ and $\nabla\mathrm{div}\boldsymbol{U} = \mathbf{0}$ must be satisfied on *n+1*.

3 FINITE ELEMENTS FOR STATE VECTOR METHOD

The finite element described by state vector $\{\boldsymbol{u}, \nabla\boldsymbol{F}, \nabla\nabla\boldsymbol{F}, \cdots\}$ is represented by an incomplete three-fold third-order function with 32 terms and expressed by node parameter vector $\{u_i, \nabla F_i\}_k$, which has four degrees of freedom per node. This is a quasi-C^1-continuity element.

A scheme in the FEM does not exist to apply this quasi-C^1-continuity element.

The transfer matrix method called reduction method also uses the state vector. The method is developed in 1954–1956 at the universities of (Technischen Hochschulen) Hannover, Braunschweig, and Darmstadt. The method called different respectively.

The "reduction method" means that the state vector on point *a* resolve onto point *b*.

The resolved state vector's, however, half terms can be known (no more parameters) by satisfying the boundary condition on the point *a* and the other half parameters are determined using other side boundary on point *b* or more transferred point. This process is known as the transfer matrix method for the beam.

In the proposed state vector method, the state vector $\{\boldsymbol{u}, \nabla\boldsymbol{F}, \nabla\nabla\boldsymbol{F}, \cdots\}$ is represented with the node parameter vector $\{u_i, \nabla F_i\}_k$ on the vertex nodes. In the case of the beam, their parameters are present on the both ends.

For tree-type framework†, the transfer matrix method has been achieved [2],[3]. However, the research of network-type framework†† has been abandoned [4].

A typical tree-type system is a continuous girder, *i.e.*, a tree-type framework but without branches. Furthermore, the system can be represented using a simultaneous equation with the parameters $\{w, \theta, M, Q\}_k$. This is a state vector method, which is represented by $\{\boldsymbol{u}, \nabla\boldsymbol{F}\}_k$, where $\{\boldsymbol{u}\}_k \equiv \{w, \theta\}_k$ and $\{\nabla\boldsymbol{F}\}_k \equiv \{M, Q\}_k$ as a proper expression for the continuous girder. This expression can be applied to broad continuum mechanics problems represented by finite elements. As a result, the technique that employed the state vector shall be referred to as the "state vector method."

In the finite element, the transfer matrix (reduction matrix) transfers the state vector $\{\boldsymbol{u}, \boldsymbol{F}\}$ (Zustandvektor) from point *a* to point *b* (the reduction matrix deduces the state vector on *a* onto *b*). The coefficient vector of the finite Taylor series is referred to as the "derivative vector," as opposed to the state vector, as previously explained.

Strains are calculated with the former, whereas displacements and forces are calculated

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Tree-type framework† : open-type framework, geschlossene Rahmentrangwerke in German as technical term.

network-type framework††: closed-type framework, geschlossene Rahmentragwerke in German.

vectors from point *a* to point *b*. Rearranging the transfer matrix of the derivative vector yields the transfer matrix of the state vector. In the case of the single phase problem, this is a linear problem on ρ and μ.

The transfer matrix of the derivative vector, $\{u_i^{(jkl)}\}_0$ is the matrix made up of derivatives of the vector $\{x^j y^k z^l / j!\, k!\, l!\}$, which are power series derivatives.

The state vector is obtained by substituting the first-order components of the derivative vector $\{u_i^{(jkl)}\}_0$ with $\{F_i^{(jkl)}\}_0$, where $\{F_i^{(jkl)}\}_0 = \{\mu u_i^{(jkl)}\}_0$. The cell is represented by the node parameter vector $\{\boldsymbol{Z}\}_k$ in Equation (1).

$$\{\boldsymbol{Z}\}_k = \{u_i^{(000)}, F_i^{(100)}, F_i^{(010)}, F_i^{(110)}, F_i^{(001)}, F_i^{(101)}, F_i^{(011)}, F_i^{(111)}\}_k \tag{1}$$

Equation (2) represents the node parameter vector for the quasi-C^1-continuity element.

$$\{\boldsymbol{Z}\}_k = \{u_i^{(000)}, F_i^{(100)}, F_i^{(010)}, F_i^{(001)}\}_k \tag{2}$$

As a result, Equation (3) represents the transfer of the state vector $\{\boldsymbol{Z}\}_0$ on point *0* to point $\boldsymbol{x}$, where $\{\boldsymbol{Z}\}_{\boldsymbol{x}}$ is the state vector on point $\boldsymbol{x}$ and $[\boldsymbol{T}_{\boldsymbol{x}}]$ is the transfer matrix.

$$\{\boldsymbol{Z}\}_{\boldsymbol{x}} = [\boldsymbol{T}_{\boldsymbol{x}}] \cdot \{\boldsymbol{Z}\}_0 \tag{3}$$

The following is a simple example of $[\boldsymbol{T}_{\boldsymbol{x}}]$ for a two-dimensional model (2Dm).

$$\begin{Bmatrix} u^{(00)} \\ F^{(10)} \\ F^{(01)} \\ F^{(11)} \end{Bmatrix}_{\boldsymbol{x}} = [\boldsymbol{T}]_{\boldsymbol{x}} \cdot \{\boldsymbol{Z}\}_{\boldsymbol{0}} = \begin{bmatrix} 1 & \frac{\psi_{1,x}}{\nu} & \frac{\psi_{1,y}}{\nu} & \frac{\psi_{1,x}\psi_{1,y}}{\nu} \\ & 1 & 0 & \psi_{1,y} \\ & & 1 & \psi_{1,x} \\ & & & 1 \end{bmatrix} \cdot \begin{Bmatrix} u^{(00)} \\ F^{(10)} \\ F^{(01)} \\ F^{(11)} \end{Bmatrix}_{\boldsymbol{0}} \tag{4}$$

where $\{\boldsymbol{Z}\}$: state vector (Zustandvektor)
$[\boldsymbol{T}]_{\boldsymbol{x}}$: transfer matrix (Übertragungsmatrix).

The virtual work equation is integrated using $\{\boldsymbol{Z}\}_{\boldsymbol{x}}$ and expressed with node parameters in Equations (3) and (4), which include force parameters $\{\boldsymbol{F}\}_k$. For continuum mechanics, this is the force method.

4 NOVEL Φ-$\boldsymbol{u}$ FINITE ELEMENT HYBRIDIZATION

4.1 Chain rule of Helmholtz decomposition

As previously indicated, *dHd* is the final outcome of the current study, and a novel Φ-$\boldsymbol{u}$ FEM is the outcome of the *dHd* scheme.

H-d can decompose any arbitrary vector field, such as potential, displacement, and strain vector fields, according to the Helmholtz theorem.

Thus, *H-d* is associated with the formula, although it can also be represented using various variables. This is referred to as the "chain rule of *H-d*."

For example, $\nabla^2\varphi - div\boldsymbol{U} = 0$ and $\nabla^2\upsilon - div\boldsymbol{\psi} = 0$. The former is the equation to satisfy the continuity equation and the latter is corresponds to the Coulomb gauge.

4.2 Scalar potentials by the Chain rule

The following equations hold $\nabla^2_{diag}\upsilon = \nabla\varphi, \nabla^2_{diag}\varphi = \nabla\Phi, \nabla^2_{diag}\Phi = \nabla P$, etc., according to the chain rule.

In incompressible flow, the potential P is commonly used instead of the potential φ by weighting with weight factor Δt to offset the lateral components from the velocity.

As discussed later, the potential υ is used to satisfy the Coulomb gauge.

So far, the scalar potential φ has been used to compensate for the residual of the continuity equation.

However, I recently observed that the scalar potential Φ represents the particle's position, which is represented by current coordinates.

As a result, the incremental values of the particle's displacements *u,v,w* must be equal to −x,−y,−z., and their cumulative $\{u, v, w\}$ values must be equal to $\{\Phi_x, \Phi_y, \Phi_z\}$.

4.3 Tensor subcontraction: Equations of components in $\nabla\boldsymbol{u}$

Tensor contraction is used to depict the displacements u_i; u_i is considered as a reflection in three opposing mirrors. The *x-y* mirror is positioned on $z = 0$, and so on.

In the *x-y* mirror, for example, there are three equations between off-diagonal components: $u^{(010)} = w^{(010)}$, $v^{(100)} = w^{(100)}$ in $\nabla\boldsymbol{u}$.

Under the assumption that $curl\boldsymbol{u} = \boldsymbol{0}$, the *dHd* expression of $\nabla^1\boldsymbol{u}$ is expressed as Equation (5). Shear strains are still the transverse components in Equation (5). The variables from Equation (5) will be used in the next sections. However, real values are the average of numerical findings.

$$\nabla^1\boldsymbol{u} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} + \frac{\partial v}{\partial z} + \frac{\partial w}{\partial z} \end{Bmatrix} \left(= \nabla \begin{Bmatrix} \Phi \\ \Phi \\ \Phi \end{Bmatrix}\right) \tag{5}$$

Equation (6) is used to represent the 2Dm. However, $curl\boldsymbol{u}$ is employed in this equation, that is, $(w^{(01)} - w^{(10)})$, since components $\{w^{(10)}, w^{(01)}\}$ are included in 2Dm if, and only if, $w^{(001)} = 0$, representing a 2Dm.

$$\nabla^1 \boldsymbol{u} = \begin{Bmatrix} \frac{\partial u}{\partial x} + (\frac{\partial u}{\partial y} + \frac{\partial w}{\partial y}) \\ (\frac{\partial u}{\partial y} - \frac{\partial w}{\partial x}) + \frac{\partial v}{\partial y} \end{Bmatrix}, where\ \nabla w = \nabla_{shr} \boldsymbol{u} \tag{6}$$

This is the concept of the stream function. Similarly, $curl\boldsymbol{u}$ can be employed in 3D problems based on the following equations.

$$curl\boldsymbol{u} = \{w^{(010)} - w^{(100)}, u^{(001)} - u^{(010)}, v^{(100)} - v^{(001)}\}^T \tag{7}$$

4.4 Conjugate variable conception

Let us refer to a pair of variables $(A + B)$ and $(A - B)$ as "conjugate variables."

When $(A + B)$ includes a loss of digit $|\pm e|$, i.e., $(A + e) + (B - e)$ individually, the conjugate variable must be minimized to minimize $2e$, i.e., by $(A + e) - (B - e) \Longrightarrow 0$, where the converse is also true.

In 2Dm, these are the Cauchy-Riemann equations, which are applied to the mirror images.

4.5 Virtual work equation and Φ-$\boldsymbol{u}$ finite element hybridization for constraint conditions

The following is the virtual work equation using the *dHd* expression:

$$\int_\Omega [\,\delta\boldsymbol{u} \cdot \rho \frac{\partial \boldsymbol{U}}{\partial t} + \delta\nabla\boldsymbol{u} \cdot \nabla \frac{\partial \boldsymbol{F}}{\partial t}] d\Omega = 0, \tag{8}$$

$$where\ \frac{\partial \boldsymbol{U}}{\partial t} = \frac{(\Delta\boldsymbol{u} + \boldsymbol{u}^{m-1})^{n+1} - \boldsymbol{u}^n}{\Delta t^2/2}, \quad \frac{\partial \boldsymbol{F}}{\partial t} = (\frac{\Delta\boldsymbol{F} + \boldsymbol{F}^{m-1}}{\Delta t})^{n+1}$$

Δ: incremental, $\quad \boldsymbol{u}^m = \Delta\boldsymbol{u} + \boldsymbol{u}^{m-1} \quad on\ time\ step\ n+1$

The following section states that the incremental $\Delta\boldsymbol{u}$ is bound by the continuity condition, Cauchy-Riemann equations, and other constraints.

5 CONSTRAINT CONDITIONS FOR VIRTUAL WORK EQUATION

First, let us consider how to satisfy the Coulomb gauge.

The lateral components, i.e., $\{u, v, w\} - \{\Phi, \Phi, \Phi\} = \boldsymbol{0}$, require the Coulomb gauge, as discussed earlier.

When the element volume is conserved, the mass can be conserved to yield an approximate solution under the assumption that the density within the element is constant.

The scalar potential υ (upsilon), which is represented by a three-fold two-order element, is used to satisfy the Coulomb gauge by Equation (9).

$$\left.\begin{aligned}\int_\Omega [\,\delta\nabla^2\upsilon\cdot(\nabla^2\upsilon-(u+v+w))]d\Omega=0\\ \int_\Omega [\,\delta\boldsymbol{u}\cdot(\Delta\boldsymbol{u}+\begin{Bmatrix}\upsilon^{(200)}\\ \upsilon^{(020)}\\ \upsilon^{(002)}\end{Bmatrix})]d\Omega=0\end{aligned}\right\},\ where\ \boldsymbol{u}^n=\boldsymbol{u}^{n-1}+\Delta\boldsymbol{u} \tag{9}$$

The incremental $\Delta\boldsymbol{u}$ offsets the residual of the Coulomb gauge by adding Equqtion (9) as a conditional expression to the vertual work equation.

6 SURFACE TENSION MODEL

The surface tension model is represented in equation (10), where the scalar potential υ is obtained by solving simultaneous equations in which temporarily used and different from the equation (9) under the Dirichlet bounrary at the Ceiling of the simulation domain. P_{g} is the atmospheric and water pressure, where g is gravitational accelration.

$$\left.\begin{aligned}\int_\Omega [\,\delta\boldsymbol{u}\cdot(\rho\frac{\partial\boldsymbol{U}}{\partial t}+\nabla P_{\mathrm{g}})+\delta\nabla\boldsymbol{u}\cdot\nabla\frac{\partial\boldsymbol{F}}{\partial t}]d\Omega=0, where\ P_{\mathrm{g}}=\nabla^1\upsilon\\ \int_\Omega [\,\delta\nabla^2\upsilon\cdot(\nabla^2\upsilon-\rho\mathrm{g})]d\Omega=0\end{aligned}\right\} \tag{10}$$

7 CONCLUSIONS

- The multiphase flows are generally encountered flow systems, for which a multiphase model called *P/CC* model has been proposed.
- State vectors continues to exist on the surface of multiphase. The conventional scheme uses displacement method and applies a model to set Dirichlet boundary conditions for air on the air-liquid interface. The proposed scheme allows simulating it without such modeling.
- Quasi-C^1-continuity ***U***-element method was applied to a 2D cavity problem simulated for 15 years. However, the Cauchy-Riemann equations were not considered. See Appendix-1 for more information.
- Three years prior, the C^1-continuity element method depicted a 2D cavity with a stream function. (Cauchy-Riemann equations not considered.)
- A bending plate scheme based on the C^1-continuity element was recently developed. See Appendix-2 for more information.
- This report results from a long-term study on *H-d* and its application to FEM.
- The most important revelation is that the *H-d* only represents a vector field in one of the rotated coordinates.
- To include Cauchy-Riemann equation to the scheme is very important to construct a robust scheme, i.e., the multidirectional FEM. This results from the long years studies.
- Another difference is that the vector potential $\boldsymbol{\psi}$ represents *dHd* Lagrangian coordinates.
- The vector potential $\boldsymbol{\psi}$ can be represented by stream function ψ of scalar by the contraction of the transvers components. I will present it later.

- A series of investigations have led to the development of a multidirectional FEM concept, which includes Cauchy-Riemann equations in mirror images.
- The scalar potential Φ represents displacement of the particle and φ represents coordinate positions, which is represented by current coordinates, as obtained from the findings of this research.
- The proposed model will be numerically verified in the future.

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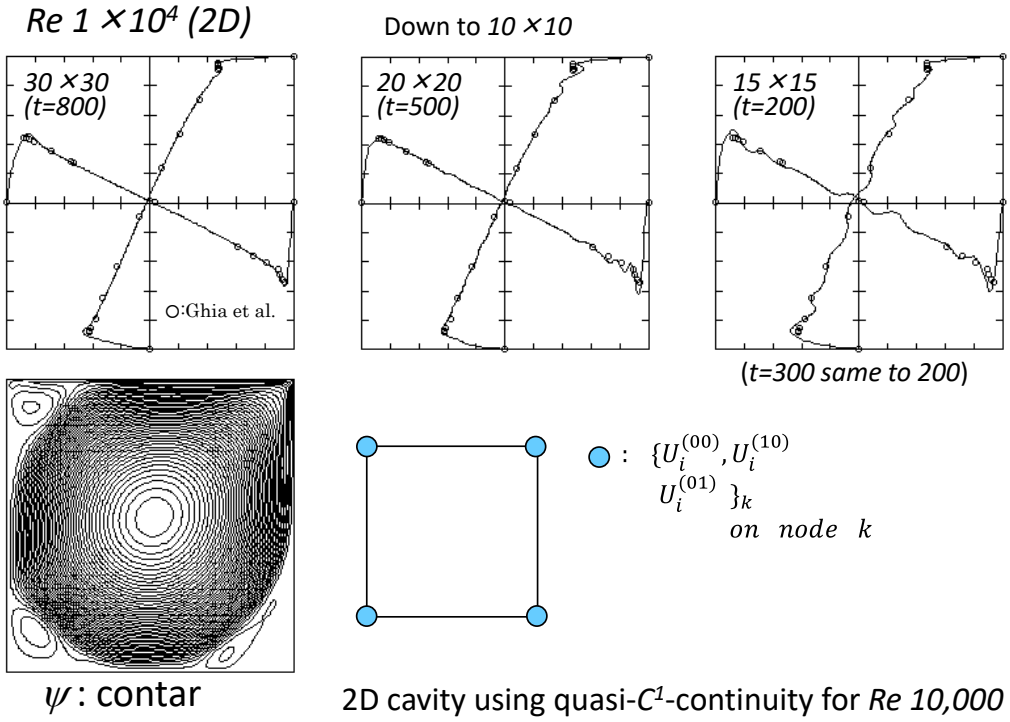
[Appendix-1] 2D cavity flow using quasi-C^1-continuity elements for *Re 10,000*

(Cauchy-Riemann equations was not considered.)



2D cavity using quasi-C^1-continuity for *Re 10,000*

[Appendix-2] The typical example of the force method application: plate bending

The Kirchhoff theory for plate bending represented by biharmonic function in equation (*a*) is widely known as a typical example to be apply the C^1-continuity element, where *D* is plate stiffness, *w* is deflection and *q* is distributed load.

$$D\nabla^2\nabla^2 w = q \tag{a}$$

The C^1-continuity element is represented with node parameter vector $\{\boldsymbol{\varphi}\}_k$ in equation (*b*), where $M^{(11)} = Dw^{(11)}$.

$$\{\boldsymbol{\varphi}\}_k = \{w^{(00)}, w^{(10)}, w^{(01)}, M^{(11)}\}_k \tag{b}$$

A numerical example is shown in Table 1 in case of a peripheral fixing rectangular plate under uniformly distributed load *q*.

Table 1. An application of the proposed force method

Peripheral fixing rectangular thin plate using two-fold third order $\boldsymbol{C^1}$-continuity element

$Exact\ solution\ \ 1.265\ (\times 10^{-3}\ ql^4/D)$

2×2	4×4	6×6	8×8	10×10	20×20	n×n elements
1.303	1.263	1.264	1.265	1.265	1.265	← Print up to 3 decimal place
1.3030	1.2631	1.2643	1.2648	1.2651	1.2653	← Print up to 4 decimal place

Additional cases: 30×30: =1.2653
40×40: =1.2653

The most important point to solve the C^1-continuity element is that the force parameter in the variational formula must be applied as the freedoms to satisfy the Neumann boundary conditions.

That is that the node parameter $\{M^{(11)}\}_k$ must be dedicated for variational formula (*c*) not only on boundaries of the system but also for distribution $M^{(11)}$ in inner domain of the plate to satisfy the stress equilibrium inter element boundaries.

$$\int_\Omega \frac{\partial M^{(11)}}{\partial\{M^{(11)}\}_k} \cdot M^{(11)} d\Omega = 0 \tag{c}$$

The variational in inner domain is to minimize the variation of the residual for the Cauchy-Riemann equation: $\theta_x^{(01)} + \theta_y^{(10)} = 0$ on $\pi/4$ rotated coordinate system.