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Abstract

This work presents the theoretical framework of a new class of constitutive models, which allows to respect five simple physical hypotheses on the mechanical behavior of soils. The most significant of them is that the (effective) stress and the specific volume are the state variables of soils. The well-known Cam-Clay model may be seen as a particular case of the presented class of models. However, in its generality, this class of models allows to describe the progressive accumulation of plastic deformation under cyclic loading.
KEYWORDS: soils, physical hypotheses, state variables, specific volume, mathematical framework, hardening parameter.

1 Introduction

The *Limit State Theory*, LST, originally due to Roscoe, Schofield and Wroth (1958), is today the most widely accepted theory to interpret the observed behavior of soils.

On the basis of the LST, several elasto-plastic constitutive equations have been proposed. Many of them well describe the behavior of soils under monotonic loading. The problem which is still under research studies is a more accurate description of the accumulation of plastic (irreversible) deformations under cyclic loading and the identification of the physical parameters which govern it.

In this paper we present the mathematical frame work for a class of constitutive models, denoted as SUOLO, based on the LST and able to describe in a unified manner the behavior of soils under monotonic and cyclic loading. The fundamental characteristic of this class of models is that the behavior of soils is always governed by the same set of state variables. The basis of the mathematical framework can be found in (De Crescenzo & Fusco, 1993).

2 Stress and Strain Definition

The (symmetric) Cauchy (effective) stress tensor σ_{ij} and the relative deviatoric stress tensor

$$s_{ij} = \sigma_{ij} - \delta_{ij}\frac{\sigma_{kk}}{3}$$

where δ_{ij} is the usual *Kronecker symbol*, are respectively indicated as

$$\begin{aligned} \boldsymbol{\sigma} &= \{\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{12},\sigma_{13},\sigma_{23},\sigma_{21},\sigma_{31},\sigma_{32}\}^T && (1) \\ \mathrm{s} &= \{s_{11},s_{22},s_{33},s_{12},s_{13},s_{23},s_{21},s_{31},s_{32}\}^T && (2) \end{aligned}$$

The principal stresses are indicated with $\sigma_1, \sigma_2, \sigma_3$, where

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

As stress invariant quantities we elect

$$\left\{ \begin{array}{lll} p &= \dfrac{I_1^{(\sigma)}}{3}; & \text{Mean pressure} \\ q &= \left(3J_2^{(\sigma)}\right)^{\frac{1}{2}}; & \text{Equivalent shear stress} \\ \theta &= \frac{1}{3}\arcsin\left[-\dfrac{3\sqrt{3}}{2}\dfrac{J_3^{(\sigma)}}{J_2^{(\sigma)3/2}}\right]; & \text{Angular invariant of stress} \end{array} \right. \qquad (3)$$

where

$$\left\{ \begin{array}{lll} I_1^{(\sigma)} &= \sigma_{ii} &= \widehat{\mathrm{m}}^T\boldsymbol{\sigma} \\ J_2^{(\sigma)} &= \frac{1}{2}s_{ij}s_{ij} &= \frac{1}{2}\mathrm{s}^T\mathrm{s} \\ J_3^{(\sigma)} &= \det\left[s_{ij}\right] & \end{array} \right. \qquad (4)$$

and

$$-\pi/6 \leq \theta \leq \pi/6$$

$$\widehat{\mathrm{m}} = \{1,1,1,0,0,0,0,0,0\}^T$$

With the notations

$$\delta p \ , \ \delta q \ , \ \delta\theta$$

we indicate the infinitesimal variations of p, q, θ.

The (symmetric) linear Lagrangian strain tensor ϵ_{ij} and the relative deviatoric strain tensor

$$e_{ij} = \epsilon_{ij} - \delta_{ij}\frac{\epsilon_{kk}}{3}$$

are respectively indicated as

$$\boldsymbol{\epsilon} = \{\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}, \epsilon_{21}, \epsilon_{31}, \epsilon_{32}\}^T \tag{5}$$

$$\mathbf{e} = \{e_{11}, e_{22}, e_{33}, e_{12}, e_{13}, e_{23}, e_{21}, e_{31}, e_{32}\}^T \tag{6}$$

As strain invariant quantities we elect

$$\left\{\begin{array}{lll} \epsilon_v & = I_1^{(\epsilon)}; & \text{Volumetric strain} \\ \epsilon_s & = 2\left(\dfrac{J_2^{(\epsilon)}}{3}\right)^{\frac{1}{2}}; & \text{Equivalent shear strain} \\ \epsilon_\theta & = \frac{1}{3}\arcsin\left[-\dfrac{3\sqrt{3}}{2}\dfrac{J_3^{(\epsilon)}}{J_2^{(\epsilon)3/2}}\right]; & \text{Angular invariant of strain} \end{array}\right. \tag{7}$$

where

$$\left\{\begin{array}{lll} I_1^{(\epsilon)} & = \epsilon_{ii} & = \mathbf{m}^T\boldsymbol{\epsilon} \\ J_2^{(\epsilon)} & = \frac{1}{2}e_{ij}e_{ij} & = \frac{1}{2}\mathbf{e}^T\mathbf{e} \\ J_3^{(\epsilon)} & = \det\left[e_{ij}\right] & \end{array}\right. \tag{8}$$

and

$$-\pi/6 \leq \epsilon_\theta \leq \pi/6$$

With the notations

$$\delta\epsilon_v \;,\; \delta\epsilon_s \;,\; \delta\epsilon_\theta$$

we indicate the invariants of the incremental strain $\delta\epsilon$. In general, these quantities *do not coincide* with the incremental variations of the strain invariants unless for the case of volumetric strain.

Incidentally, we remark that, according to the above notation convention, deviatoric stress and strain vectors may be respectively calculated as

$$\mathbf{s} = \boldsymbol{\sigma} - \widehat{\mathbf{m}}p \tag{9}$$

$$\mathbf{e} = \boldsymbol{\epsilon} - \widehat{\mathbf{m}}\frac{\epsilon_v}{3} \tag{10}$$

It will be useful to note that, assuming mineral and fluid incompressible compared to the solid skeleton, within the hypothesis of small deformations, the increment of volumetric strain is calculated as

$$\delta\epsilon_v = \frac{\delta v}{v}$$

where v is *the specific volume*, defined as

$$v = \frac{\Omega}{\Omega^{(m)}}$$

and Ω and $\Omega^{(m)}$ are the total volume of a soil sample and the volume of the mineral grains, respectively.

3 The Limit State Theory

Roscoe, Schofield and Wroth (1958) published a paper, which contained the first attempt to explain the behavior of soils in a global way, giving new light to the early intuitions of Hvorslev (1937), Rendulic (1936) and Terzaghi (1943). This paper concerned primarily with the observed pattern of behavior of saturated remoulded claies in triaxial compressions; the main conclusions where:

1. the existence in the (p, q, v) space, p, q effective stress invariants, of a unique *Limit State Boundary Surface*, LSS, Fig. 1a, on which the specific volume value reaches the maximum compatible with assigned p, q,
$$v = v_{max}(p, q)$$
2. the existence on this boundary surface of a unique *Critical State Line*, CSL, where all the (effective) stress paths in triaxial tests terminate. At this state, large shear strains occur with no change in stress and specific volume value, which is the maximum compatible with the stress state (p, q).

In Fig. 1a some parts of the LSS are indicated with dashed lines; this means that in those parts the experimental points are not interpolated by a well defined surface.

The intersection of the LSS with the v, p plane, on which

$$\begin{aligned} q &= 0 \\ \theta &= \text{indeterminate} \end{aligned}$$

is a line, which is usually said *Normal Consolidation Line*, NCL, Fig. 1c. Based on the assumption of the limit state boundary surface, soils may be classified as follows:

1. soils whose state (p, q, v) is on the LSS are defined as *Normally Consolidated Soils*, NCS,
$$v = v_{max}(p, q)$$
2. soils whose state (p, q, v) is inside the LSS are defined as *Overconsolidated Soils*, OCS,
$$v < v_{max}(p, q)$$

From the above definitions it follows that, for a same (p, q) stress state, the specific volume v_{OCS} of a OCS is always smaller than the v_{NCS} of a NCS,

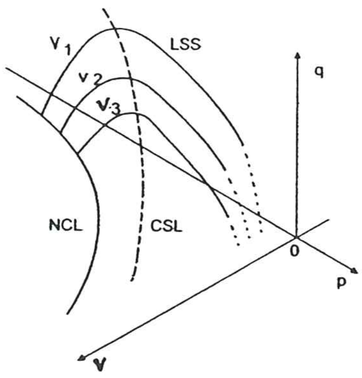
$$v_{OCS} \leq v_{NCS}$$

The typical triaxial responses of a NCS and OCS under the same initial stress condition are shown in Fig. 2.

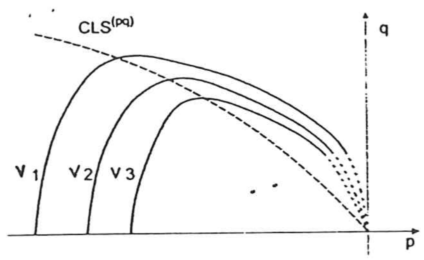
4 Physical Hypotheses

In the following sections we present a mathematical model which attempts to describe the behavior of soil in accordance to the limit state theory. The basic physical hypotheses of this model are:

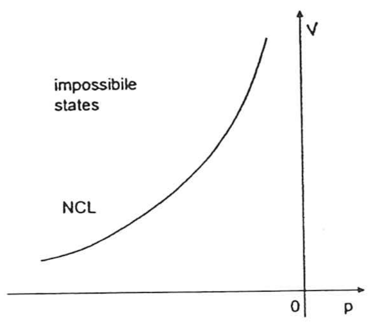
1. The (effective) stress σ and the specific volume v are the *State Variables* of soil, that is: the mechanical soil response $\delta\epsilon$ for a given $\delta\sigma$, and viceversa, is *uniquely* determined only by the current values of (σ, v), regardless the previous stress history.
2. Isotropic mechanical behavior.
3. Soils, subjected to cycles of loading and unloading, present accumulation of irreversible deformations.
4. There exists a single-value function
$$v = v_{max}(p, q, \theta) \tag{11}$$
which bounds always the value of the specific volume v of a soil at a stress state (p, q, θ), that is
$$v \leq v_{max}(p, q, \theta)$$
For any given θ, Eq. 11 is the equation of a surface, in the (p, q, v) space, called *Limit State Boundary Surface*, LSS.



a) LSS in the p, q, v space.



b) LSS projected on the q vs. p plane.



c) LLS intersection with the v vs. p plane.

Figure 1: The Limit State Surface, LSS.

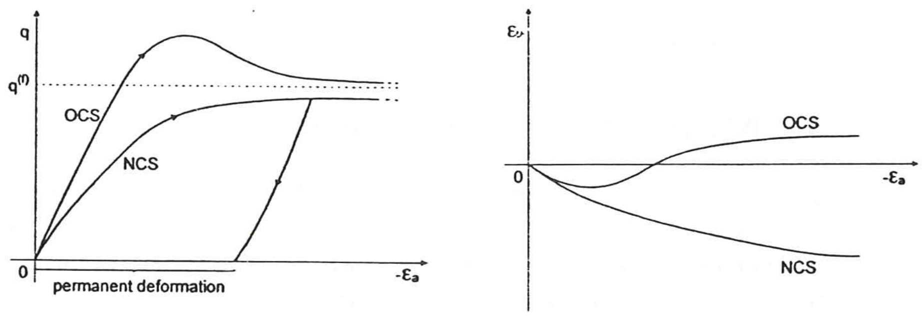


Figure 2: A drained triaxial test on NCS and OCS

5. There exists a *Critical State condition* where indeterminate shear strain ϵ_s occurs with no change in the stress σ and in the specific volume v. The locus in the σ space of all the stress levels corresponding to critical state conditions is called *Critical State Surface*, CSS. For isotropic soils, this CSS is represented by a function of the type

$$q = q_c(p, \theta)$$

At this critical state condition:

- the value of v is the maximum compatible with the stress state, that is

$$v = v_{max}(p, q, \theta)$$

- for any given θ, the locus in the (p, q, v) space of the soil states under critical state conditions is a line known as the *Critical State Line*, CSL.

The first requirement is the minimum assumption on which it is possible to describe the typical soil behavior. In fact we observe in Fig. 2 that

1. the soil response at the same initial stress value is different for NCS and OCS, hence it depends on the specific volume v;

5. The space region bounded by F is a subspace of $\overline{F}$, that is

$$F(p,q,\theta,\overline{p}_y,p_y) \subseteq \overline{F}(p,q,\theta,\overline{p}_y) \tag{20}$$

If the current stress point σ lies on $\overline{F}$, then F and $\overline{F}$ must coincide, that is

$$F(p,q,\theta,\overline{p}_y,p_y) \equiv \overline{F}(p,q,\theta,\overline{p}_y) \tag{21}$$

and

$$p_y = \overline{p}_y \tag{22}$$

$$\frac{\partial p_y}{\partial v^{(p)}} = \frac{d\overline{p}_y}{dv^{(p)}} \tag{23}$$

$$\frac{\partial p_y}{\partial h_j} = 0 \tag{24}$$

for all $j = 1, 2, \ldots, m$.

6. There exists an *Elastic Surface*

$$\widehat{F} = \widehat{F}(p,q,\theta,\overline{p}_y) \tag{25}$$

defined as

$$\widehat{F}(p,q,\theta,\overline{p}_y) \equiv F(p,q,\theta,\overline{p}_y,p_y = \widehat{p}_y)$$

where

$$\widehat{p}_y = \widehat{p}_y(\overline{p}_y)$$

Note that the above definition implies that, if the current stress point σ lies on $\widehat{F}$, then F coincides with $\widehat{F}$.

7. In general, for $p_y \rightarrow \widehat{p}_y$,

$$\frac{\partial p_y}{\partial v^{(p)}} \rightarrow \infty$$

$$\frac{\partial p_y}{\partial \epsilon_s^{(p)}} \rightarrow \infty$$

$$A \rightarrow +\infty$$

The definition of the *Plastic Modulus* A is postponed to item 9. If $\widehat{F}$ and $\overline{F}$ always coincide, the above assumptions do not apply.

8. There exists a *Potential Function* for plastic deformations, of the form

$$G(p, q, \theta, \overline{p}_y, p_y)$$

9. Analogously to the standard incremental theory of plasticity, it is assumed that an infinitesimal strain increment $\delta\epsilon$ can be expressed as

$$\delta\epsilon = \delta\epsilon^{(e)} + \delta\epsilon^{(p)} \tag{26}$$

where:

- $\delta\epsilon^{(e)}$ represents the elastic (fully recoverable) component which may be calculated according to the generalized Hooke's law

$$\delta\epsilon^{(e)} = \left(\mathbf{C}^{(e)}\right)^{-1} \delta\sigma \tag{27}$$

where $\mathbf{C}^{(e)}$, the tangential elastic stiffness matrix, may be function of the current σ and ϵ.

- $\delta\epsilon^{(p)}$ is the plastic (irreversible) component, defined as

$$\delta\epsilon^{(p)} = \delta\lambda\mathbf{b} \tag{28}$$

where

$$\delta\lambda \quad \begin{cases} \geq 0, & \text{if elasto-plastic response occurs.} \\ = 0, & \text{if elastic response occurs.} \end{cases}$$

$$\begin{aligned} \mathbf{b} &= \frac{\partial G}{\partial \sigma} = \\ &= \left\{ \frac{\partial G}{\partial \sigma_{11}}, \frac{\partial G}{\partial \sigma_{22}}, \frac{\partial G}{\partial \sigma_{33}}, \frac{\partial G}{\partial \sigma_{12}}, \frac{\partial G}{\partial \sigma_{13}}, \frac{\partial G}{\partial \sigma_{21}}, \frac{\partial G}{\partial \sigma_{23}}, \frac{\partial G}{\partial \sigma_{31}}, \frac{\partial G}{\partial \sigma_{32}} \right\}^T \end{aligned}$$

The value of the plastic multiplier $\delta\lambda$ can be calculated as, (De Crescenzo & Fusco, 1993),

- If $\delta\sigma$ is assigned,

$$\delta\lambda = \begin{cases} \dfrac{\mathbf{a}^T \delta\sigma}{A}, & \text{if } A \neq 0. \\ \text{indeterminate}, & \text{if } A = 0. \end{cases} \tag{29}$$

- if $\delta\epsilon$ is assigned,

$$\delta\lambda = \begin{cases} \dfrac{\mathbf{a}^T \delta\sigma^{(e)}}{A + \mathbf{a}^T \mathbf{c}^{(G)}}, & \text{if } A \neq -\mathbf{a}^T\mathbf{c}^{(G)}. \\ \text{indeterminate}, & \text{if } A = -\mathbf{a}^T\mathbf{c}^{(G)}. \end{cases} \tag{30}$$

where

$$\delta\sigma^{(e)} = \mathbf{C}^{(e)}\delta\epsilon \tag{31}$$

$$\mathbf{c}^{(G)} = \mathbf{C}^{(e)}\mathbf{b} \tag{32}$$

and the *Plastic Modulus* A is defined as

$$\begin{aligned} A &= -\frac{1}{\delta\lambda}\left[\frac{\partial F}{\partial \overline{p}_y}\delta\overline{p}_y + \frac{\partial F}{\partial p_y}\delta p_y\right] \\ &= -\left[v\left(\frac{\partial F}{\partial \overline{p}_y}\frac{d\overline{p}_y}{dv^{(p)}} + \frac{\partial F}{\partial p_y}\frac{\partial p_y}{\partial v^{(p)}}\right)\frac{\partial G}{\partial p} + \frac{\partial F}{\partial p_y}\frac{\partial p_y}{\partial h_j}c_j\right] \end{aligned} \tag{33}$$

for $j = 1, 2, \ldots, m$.

The type of mechanical response is established according to the criterion in item 10.

10. By definition, if

$$\widehat{F}(p, q, \theta, \overline{p}_y) < 0 \tag{34}$$

the material response is always elastic, regardless the applied stress increment $\delta\sigma$ or strain increment $\delta\epsilon$. Instead, if

$$\widehat{F}(p, q, \theta, \overline{p}_y) \geq 0 \tag{35}$$

it is *assumed* that the type of material response is established as follows:

- *Stress Based Criterion.* Let $\delta\sigma$ be a stress increment applied on any material state $(p, q, \theta, \overline{p}_y)$, then:

  (a) Elasto-plastic response occurs if

$$\begin{aligned} A > 0 \quad &; \quad \mathbf{a}^T\delta\sigma \geq 0 \\ A = 0 \quad &; \quad \mathbf{a}^T\delta\sigma = 0 \\ A < 0 \quad &; \quad \mathbf{a}^T\delta\sigma = 0 \end{aligned}$$

(b) Elastic response occurs if

$$A \geq 0 \quad ; \quad \mathbf{a}^T \delta\sigma < 0$$

(c) Either elastic or elasto-plastic response may occur if

$$A < 0 \quad ; \quad \mathbf{a}^T \delta\sigma < 0$$

This is the only ambiguous situation which this model does not solve by itself.

(d) Stress increments by which

$$A \leq 0 \quad ; \quad \mathbf{a}^T \delta\sigma > 0$$

are not admissible; that is, according to the theory, the material cannot sustain such type of stress increment.

- *Strain Based Criterion.* Let $\delta\epsilon$ be a strain increment applied on any material state $(p, q, \theta, \overline{p}_y)$, then:

(a) Elasto-plastic response occurs if

$$\begin{aligned} A &> -\mathbf{a}^T \mathbf{c}^{(F)} \quad ; \quad \mathbf{a}^T \delta\sigma^{(e)} \geq 0 \\ A &= -\mathbf{a}^T \mathbf{c}^{(F)} \quad ; \quad \mathbf{a}^T \delta\sigma^{(e)} = 0 \\ A &< -\mathbf{a}^T \mathbf{c}^{(F)} \quad ; \quad \mathbf{a}^T \delta\sigma^{(e)} = 0 \end{aligned}$$

(b) Elastic response occurs if

$$A \geq -\mathbf{a}^T \mathbf{c}^{(F)} \quad ; \quad \mathbf{a}^T \delta\sigma^{(e)} < 0$$

(c) Either elastic or elasto-plastic response may occur if

$$A < -\mathbf{a}^T \mathbf{c}^{(F)} \quad ; \quad \mathbf{a}^T \delta\sigma^{(e)} < 0$$

This is the only ambiguous situation which this model does not solve by itself.

(d) Strain increments by which

$$A \leq -\mathbf{a}^T \mathbf{c}^{(F)} \quad ; \quad \mathbf{a}^T \delta\sigma^{(e)} > 0$$

are not admissible; that is, according to the theory, the material cannot sustain such type of strain increment.

6 Additional Mathematical Assumptions

In addition to the main assumptions of the general mathematical framework, presented in Section 5, SUOLO makes the following own assumptions:

1. The bounding surface of equation

$$\overline{F}(p, q, \theta, \overline{p}_y) = 0$$

is assumed to have the shape in Fig. 3. This type of surface has the following mathematical characteristics:

- It is a closed and simply connected surface in the $\sigma_1, \sigma_2, \sigma_3$ space.
- It crosses the spatial diagonal $\widehat{\mathbf{m}}$ at two points $\overline{Y}$ and $\overline{D}$.
- The hardening parameter $\overline{p}_y$ is the value of p at $\overline{Y}$, namely, $\overline{p}_y$ is the intersection of smaller value of $\overline{F}$ with the p-axis.
- The value of p at $\overline{D}$ is said $\overline{p}_d$ and it is function of $\overline{p}_y$ only, namely

$$\overline{p}_d = \overline{p}_d(\overline{p}_y)$$

- The value of $\overline{F}$, calculated in any stress point (p, q, θ) is a continuous monotonically increasing function of $\overline{p}_y$, that is:

$$\frac{\partial \overline{F}}{\partial \overline{p}_y} > 0 \tag{36}$$

for any (p, q, θ). This implies that the bounding surface expands or contracts itself as $\overline{p}_y$ decreases or increases, Fig. 3c.
In fact, consider a bounding surface of equation

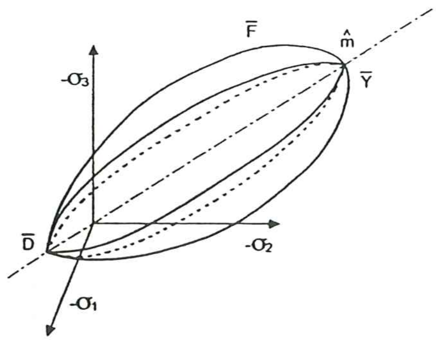
$$\overline{F}^{(o)}(\sigma) = \overline{F}(\sigma, \overline{p}_y = \overline{p}_y^{(o)}) \tag{37}$$

Suppose that the hardening parameter is varied by an infinitesimal amount $d\overline{p}_y$, so that the new location of the bounding surface has equation

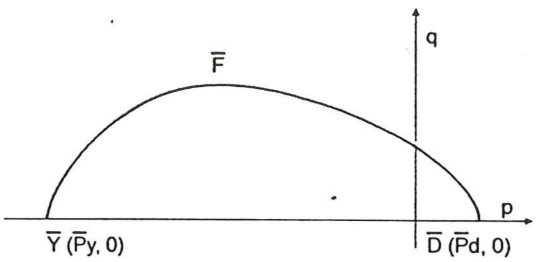
$$\overline{F}^{(f)}(\sigma) = \overline{F}(\sigma, \overline{p}_y = \overline{p}_y^{(f)} = \overline{p}_y^{(o)} + d\overline{p}_y)$$

Hence, the value that $\overline{F}^{(f)}$ takes for any stress point σ^* lying on $\overline{F}^{(o)}$, for which

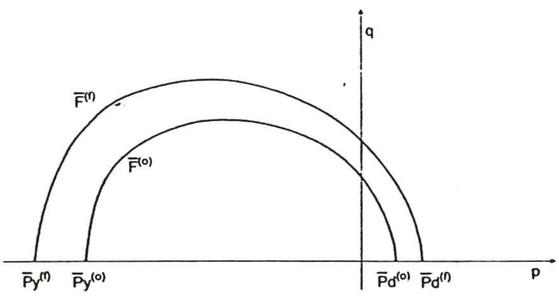
$$\overline{F}^{(o)}(\sigma^*) = \overline{F}(\sigma = \sigma^*, \overline{p}_y = \overline{p}_y^{(o)}) = 0$$



a) $\overline{F}$ in the $\sigma_1, \sigma_2, \sigma_3$ space.



b) Mapping of $\overline{F}$ in the q vs p plane.



c) Expansion of $\overline{F}$ in the q vs p plane.

Figure 3: The Bounding Surface, $\overline{F}$

may be *exactly* evaluated as

$$\overline{F}^{(f)}(\sigma^*) = \overline{F}^{(o)}(\sigma^*) + \left(\frac{\partial \overline{F}^{(o)}}{\partial \overline{p}_y}\right)_{\sigma=\sigma^*} d\overline{p}_y = \left(\frac{\partial \overline{F}^{(o)}}{\partial \overline{p}_y}\right)_{\sigma=\sigma^*} d\overline{p}_y$$

The mathematical condition in Eq. 36 assures that

$$\overline{F}^{(f)}(\sigma^*) = \left(\frac{\partial \overline{F}^{(o)}}{\partial \overline{p}_y}\right)_{\sigma=\sigma^*} d\overline{p}_y = \begin{cases} < 0 & \text{for } d\overline{p}_y < 0 \\ > 0 & \text{for } d\overline{p}_y > 0 \end{cases} \tag{38}$$

Since σ^* is any arbitrary point on $\overline{F}^{(o)}$, this implies that

$$\overline{F}(p, q, \theta, \overline{p}_y^{(f)}) \subseteq \overline{F}^{(o)}(p, q, \theta, \overline{p}_y^{(o)})$$

for any $\overline{p}_y^{(f)} \geq \overline{p}_y^{(o)}$.

2. The hardening parameter $\overline{p}_y$ is a continuous monotonically increasing function of $v^{(p)}$, that is

$$\frac{d\overline{p}_y}{dv^{(p)}} > 0 \tag{39}$$

This implies that the relationship $\overline{p}_y = \overline{p}_y(v^{(p)})$ in item 2 of Section 6 has a unique inverse relationship

$$v^{(p)} = \overline{v}_y^{(p)}(\overline{p}_y) \tag{40}$$

which is a continuous function.

3. The partial derivatives $\partial p_y / \partial v^{(p)}$ and $\partial p_y / \partial h_j$ as well as the coefficients c_j are functions of $(p, q, \theta, \overline{p}_y, p_y, v)$ only, that is

$$\begin{aligned} \frac{\partial p_y}{\partial v^{(p)}} &= \frac{\partial p_y}{\partial v^{(p)}}(p, q, \theta, \overline{p}_y, p_y, v) \\ \frac{\partial p_y}{\partial h_j} &= \frac{\partial p_y}{\partial h_j}(p, q, \theta, \overline{p}_y, p_y, v) \\ c_j &= c_j(p, q, \theta, \overline{p}_y, p_y, v) \end{aligned}$$

for $j = 1, 2, \ldots, m$.

4. The system of equations

$$\begin{cases} \overline{F}(p,q,\theta,\overline{p}_y) & = 0 \\ \\ \dfrac{\partial G}{\partial p}(p,q,\theta,\overline{p}_y,p_y) & = 0 \end{cases}$$

admits a solution (p,q,θ). For any given θ this solution is unique and satisfies the critical state surface equation, defined in item 5 Section 5,

$$q = q_c(p,\theta) \tag{41}$$

5. For any soil state where

$$\begin{cases} \overline{F}(p,q,\theta,\overline{p}_y) & < 0 \\ \\ \dfrac{\partial G}{\partial p}(p,q,\theta,\overline{p}_y,p_y) & = 0 \end{cases}$$

the plastic modulus A, defined in Eq. 33, can not be equal to zero, namely

$$A \neq 0$$

This condition is clearly equivalent to require that for any soil state as above

$$\frac{\partial F}{\partial p_y}\frac{\partial p_y}{\partial h_j}c_j \neq 0$$

sum on j, for $j = 1,2,\ldots,m$. Notice that this implies $m \geq 1$, that is p_y must depend on two variables at least: the plastic specific volume $v^{(p)}$ and another one, h_1.

6. The elastic stiffness matrix $\mathbf{C}^{(e)}$ has the following form

$$\mathbf{C}^{(e)} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & . & 0 \\ \nu & (1-\nu) & \nu & 0 & . & 0 \\ \nu & \nu & (1-\nu) & 0 & . & 0 \\ 0 & 0 & 0 & (1-2\nu) & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & (1-2\nu) \end{bmatrix}_{(9\times 9)} \tag{42}$$

The *Young Modulus* E and the *Poisson Coefficient* ν are function of (p,q,θ,v) only, that is

$$\begin{aligned} E &= E(p,q,\theta,v) \\ \nu &= \nu(p,q,\theta,v) \end{aligned}$$

and the elements of $\mathbf{v}$ are given by

$$v_{ij} = w_{ij} - \frac{\delta_{ij}}{3} w_{kk} \tag{50}$$

where

$$[w_{ij}] = \left[\frac{\partial J_3}{\partial s_{ij}}\right] = \begin{bmatrix} (s_{22}s_{33} - s_{23}^2) & (s_{13}s_{23} - s_{33}s_{12}) & (s_{12}s_{23} - s_{22}s_{13}) \\ & (s_{11}s_{33} - s_{13}^2) & (s_{12}s_{13} - s_{11}s_{23}) \\ \text{Symmetric} & & (s_{11}s_{22} - s_{12}^2) \end{bmatrix}$$

$$w_{kk} = \frac{\partial J_3}{\partial s_{kk}} = -J_2 = -\frac{q^2}{3}$$

3. From the previous item, it follows that the plastic strain, its deviatoric components and the volumetric strain result to be defined as

$$\delta\epsilon^{(p)} = \delta\lambda(c_1\widehat{\mathbf{m}} + c_2\mathbf{s} + c_3\mathbf{v}) \tag{51}$$

$$\delta\mathbf{e}^{(p)} = \delta\lambda\nabla_s G = \delta\lambda(c_2\mathbf{s} + c_3\mathbf{v}) \tag{52}$$

$$\delta\epsilon_v^{(p)} = \delta\lambda\frac{\partial G}{\partial p} \tag{53}$$

4. The yield surface F, item 3 Section 5, is function of the stress invariants and of the parameter $\overline{p}_y$ and p_y only. Consequently, also its gradient $\mathbf{a}$ can be decomposed as $\mathbf{b}$, Eq. 47. Moreover, it can be simply verified that

$$\mathbf{a}^T\delta\sigma = \frac{\partial F}{\partial p}\delta p + \frac{\partial F}{\partial q}\delta q + \frac{\partial F}{\partial \theta}\delta\theta \tag{54}$$

5. The requirement on $\overline{F}$ in Eq. 36 has two important consequences:

   - Being $\overline{F}$ a continuous monotonically increasing function of $\overline{p}_y$, the equation
   $$\overline{F}(p, q, \theta, \overline{p}_y) = 0$$
   admits only one solution $\overline{p}_y$, for any admissible stress state (p, q, θ).
   - The value of the current $\overline{p}_y$ of a soil at an admissible stress state (p, q, θ) is bounded as
   $$\overline{p}_y \leq \overline{p}_{ymax}(p, q, \theta) \tag{55}$$

where $\overline{p}_{ymax}$ is the unique solution of the equation

$$\overline{F}(p, q, \theta, \overline{p}_y) = 0$$

in terms of $\overline{p}_y$. In fact, by definition,

$$\overline{F}(p, q, \theta, \overline{p}_{ymax}) = \overline{F}^{(1)} = 0$$

while, according to the requirement in item 1 in Section 5,

$$\overline{F}(p, q, \theta, \overline{p}_y) = \overline{F}^{(2)} \leq 0$$

Thus, being $\overline{F}$ a continuous monotonically increasing function of $\overline{p}_y$ and $\overline{F}^{(2)} \leq \overline{F}^{(1)}$, it results

$$\overline{p}_y \leq \overline{p}_{ymax}$$

6. From the particular form of the elastic stiffness matrix in Eq. 42, it follows that the elastic strain, its deviatoric components and the volumetric strain result to be defined as

$$\delta\epsilon^{(e)} = \frac{\delta p}{3B^{(e)}}\widehat{\mathrm{m}} + \frac{1}{2G^{(e)}}\delta\mathrm{s} \tag{56}$$

$$\delta\mathrm{e}^{(e)} = \frac{\delta\mathrm{s}}{2G^{(e)}} \tag{57}$$

$$\delta\epsilon_v^{(e)} = \frac{\delta p}{B^{(e)}} \tag{58}$$

where $B^{(e)}$ and $G^{(e)}$ are defined in Eqs. 43 and 44.

Notice that, from the assumption on $B^{(e)}$ in Eq. 45, it follows that

$$\delta v^{(e)} = v\delta\epsilon_v^{(e)} = \frac{\delta p}{\mathcal{B}^{(e)}(p)} \tag{59}$$

8 Specific Volume Prediction

From the complete mathematical formulation of the present constitutive model, Sections 5 and 6, it is possible to derive the following important relationships for the specific volume v:

3. From the above items 1 and 2, the following remarks immediately follow:

   - the limit state boundary surface, LSS, defined in Section 4 for any given θ, has equation

   $$v = v_{max}(p, q, \theta)$$

   where $v_{max}(p, q, \theta)$ is calculated in Eq. 68;
   - the normal consolidation line, NCL, defined as the intersection of the LSS with the v, p plane, Section 3, has equation

   $$\begin{aligned} v &= v_{max}(p, q, \theta) \\ q &= 0 \\ \theta &= \text{indeterminate} \end{aligned}$$

   - if, and only if, $v = v_{max}(p, q, \theta)$, NCS, the current stress point lies on the bounding surface $\overline{F}$,

   $$\overline{F}(p, q, \theta, \overline{p}_y) = 0$$

   Instead, if, and only if, $v < v_{max}(p, q, \theta)$, OCS, the current stress point is inside the bounding surface $\overline{F}$,

   $$\overline{F}(p, q, \theta, \overline{p}_y) < 0$$

4. A possible soil state, to be used in the relationships in Eqs. 60, 61 and 62, is

$$\begin{aligned} v_o &= v_\lambda \\ p_o &= p_\lambda \\ q_o &= 0 \\ \theta_o &= \text{indeterminate} \\ \overline{p}_{yo} &= p_\lambda \end{aligned} \tag{70}$$

where (p_λ, v_λ) is an arbitrary point lying on the NCL. In this case, Eq. 63 becomes

$$v = v_\lambda + \int_{p_\lambda}^{p} \frac{1}{\mathcal{B}^{(e)}} dp + \overline{v}_y^{(p)}(\overline{p}_y) - \overline{v}_y^{(p)}(p_\lambda)$$

In fact, on the basis of item 1 Section 6, one can verify that the unique solution of

$$\overline{F}(p = p_o, q = q_o, \theta = \theta_o, \overline{p}_y) = 0$$

in terms of $\overline{p}_y$ is $\overline{p}_y = p_o$. Hence, from item 2, to the stress point with

$$\begin{aligned} p_o &= p_\lambda && (71) \\ q_o &= 0 && (72) \\ \theta_o &= \text{indeterminate} && (73) \\ \overline{p}_{yo} &= p_\lambda && (74) \end{aligned}$$

it corresponds the specific volume

$$v_o = v_{max}(p_o, q_o, \theta_o) \quad (75)$$

Eqs. 72, 73 and 75 prove that the soil state defined in Eq. 70 lies on the NCL; then it is admissible.

9 Plastic Strain Prediction

According to SUOLO, the plastic strain increment $\delta\epsilon^{(p)}$, caused by a given stress increment $\delta\boldsymbol{\sigma}$, eventually results to be function of $(\boldsymbol{\sigma}, v)$ and $\delta\boldsymbol{\sigma}$ only.

In fact, Eqs. 28 and 29,

$$\delta\epsilon^{(p)} = \begin{cases} \frac{\mathbf{a}^T \delta\boldsymbol{\sigma}}{A}\mathbf{b} & \text{for } A \neq 0 \\ \text{indeterminate} & \text{for } A = 0 \end{cases}$$

where:

- Being F and G function of $(p, q, \theta, \overline{p}_y, p_y)$ only, it results, items 2 and 4 of Section 7,

$$\begin{aligned} \mathbf{a} &= \mathbf{a}(p, q, \theta, \boldsymbol{\sigma}, \overline{p}_y, p_y) \\ \mathbf{b} &= \mathbf{b}(p, q, \theta, \boldsymbol{\sigma}, \overline{p}_y, p_y) \end{aligned}$$

- Taking into account the assumptions in item 3 of Section 6, the plastic modulus expression in Eq. 33 may be eventually expressed as

$$A = A(p, q, \theta, \overline{p}_y, p_y, v) \quad (76)$$

On the other hand, the hardening parameters $\overline{p}_y$ and p_y are given by scalar functions of the type, Eqs. 62 and 18,

$$\begin{aligned} \overline{p}_y &= \overline{p}_y(v, p) \\ p_y &= p_y(p, q, \theta, \overline{p}_y) = p_y(p, q, \theta, v) \end{aligned}$$

Being the stress invariants (p, q, θ) function of $\boldsymbol{\sigma}$ only, it immediately follows that all the above relationships are eventually function of $(\boldsymbol{\sigma}, v)$ only.

Finally, we remind that in the stress based criterion, item 10 Section 5, only the quantities $\mathbf{a}, \delta\boldsymbol{\sigma}$ and A appear; then, this criterion eventually depends on $(\boldsymbol{\sigma}, v)$ and $\delta\boldsymbol{\sigma}$ only.

10 Critical State Conditions

One of the physical hypotheses states the existence of a critical state line, CSL, in the (p, q, v) space, described, for any given θ, by the system of equations, item 5 in Section 4,

$$\begin{cases} q = q_c(p, \theta) \\ v = v_{max}(p, q, \theta) \end{cases} \tag{77}$$

On this line, and only on this line, indeterminate shear strain occurs with no change in the stress and specific volume value:

$$\begin{cases} \boldsymbol{\sigma} & = \text{constant} \\ v & = \text{constant} \\ \delta\epsilon_s & = \text{indeterminate} \end{cases} \tag{78}$$

The mathematical model SUOLO, described in Sections 5 and 6, satisfies this physical requirement.

In fact, the conditions in Eq. 78 are mathematically verified if, and only if, for a stress variation $\delta\boldsymbol{\sigma} = \mathbf{o}$, the material responds with

$$\delta\mathbf{e} = \delta\mathbf{e}^{(e)} + \delta\mathbf{e}^{(p)} \qquad \text{indeterminate} \tag{79}$$

$$\delta v = \delta v^{(e)} + \delta v^{(p)} \qquad 0 \tag{80}$$

Taking into account Eqs. 43, 44, 57 and 58, it immediately result $\delta\mathbf{e}^{(e)} = \mathbf{o}$ and $\delta\epsilon_v^{(e)} = 0$. Consequently, the conditions in Eqs. 79 and 80 are satisfied if, and only if,

$$\delta\mathbf{e}^{(p)} \qquad \text{indeterminate}$$

$$\delta v^{(p)} \qquad 0$$

that is, for Eqs. 52 and 53, if, and only if,

$$\begin{aligned} \delta\lambda & \quad \text{indeterminate} \\ \frac{\partial G}{\partial p} &= 0 \end{aligned}$$

Since, in correspondence of $\delta\boldsymbol{\sigma} = \mathbf{o}$, $\delta\lambda$ is indeterminate only for $A = 0$, Eq. 29, the conditions in Eq. 78 are *equivalent* to

$$\begin{cases} A = 0 \\ \dfrac{\partial G}{\partial p} = 0 \end{cases} \tag{81}$$

In a NCS we have, by definition,

$$v = v_{max}(p, q, \theta) \tag{82}$$

Moreover, item 3 in Section 9 and item 1 in Section 8,

$$\overline{F}(p, q, \theta, \overline{p}_y) = 0 \tag{83}$$

$$A = -v \frac{\partial \overline{F}}{\partial \overline{p}_y} \frac{d\overline{p}_y}{dv^{(p)}} \frac{\partial G}{\partial p} \tag{84}$$

From Eqs. 83 and 84 we recognize that in a NCS the conditions in Eq. 81 are equivalent to

$$\begin{cases} \overline{F}(p, q, \theta, \overline{p}_y) = 0 \\ \dfrac{\partial G}{\partial p} = 0 \end{cases}$$

This system, because of the assumption in item 4 Section 6, admits a unique solution, which satisfies the CSS equation

$$q = q_c(p, \theta) \tag{85}$$

Hence, in a NCS the critical conditions occur only on the CSL, given by the equations in Eqs. 82 and 85.

Instead, in a OCS we have, item 3 in Section 9,

$$\overline{F}(p, q, \theta, \overline{p}_y) < 0 \tag{86}$$

1. The yielding surface, Fig. 4b, has equation

$$\overline{F} = \overline{F}(p, q, \overline{p}_y) = p^2 - p\overline{p}_y + \frac{q^2}{M^2} \tag{87}$$

where M is a positive experimental constant and the hardening parameter $\overline{p}_y$ is a negative scalar value, function of $v^{(p)}$ only:

$$\overline{p}_y = p_\lambda \exp\left[-\frac{v^{(p)} - v_\lambda}{\lambda - \chi}\right] \tag{88}$$

and

- (v_λ, p_λ) is any point belonging to the NCL, with $p_\lambda < 0$.
- λ and χ are the slope of the NCL and of the *swelling* line, SWL, in a plane v vs. $\ln|p|$, Fig. 4a. Moreover:

$$\lambda > \chi$$

2. The potential surface G coincides with the yielding surface:

$$G \equiv \overline{F}(p, q, \overline{p}_y) = p^2 - p\overline{p}_y + \frac{q^2}{M^2}$$

3. The stiffness elastic matrix $\mathbf{C}^{(e)}$ has the isotropic form in Eq. 42, with

$$\begin{aligned} E &= -\frac{3(1-2\nu)}{\chi} vp \\ \nu &= \text{constant experimental parameter} \end{aligned} \tag{89}$$

According to the classical incremental theory of plasticity, the *Plastic Modulus A* is defined as

$$A = -\frac{1}{\delta\lambda}\frac{\partial \overline{F}}{\partial \overline{p}_y}\delta\overline{p}_y = -v\frac{\partial \overline{F}}{\partial \overline{p}_y}\frac{d\overline{p}_y}{dv^{(p)}}\frac{\partial G}{\partial p} = -v\frac{p\overline{p}_y}{\lambda - \chi}\left(2p - \overline{p}_y\right)$$

Moreover, it can be proved, (see for example De Crescenzo & Fusco, 1993), that:

- The plastic strain is related to the stress (or strain) increment by relationships identical to those in item 9, Section 5.
- For stress states lying on the yielding surface $\overline{F}$ the type of mechanical response is established according to the same criteria in item 10 of Section 5.

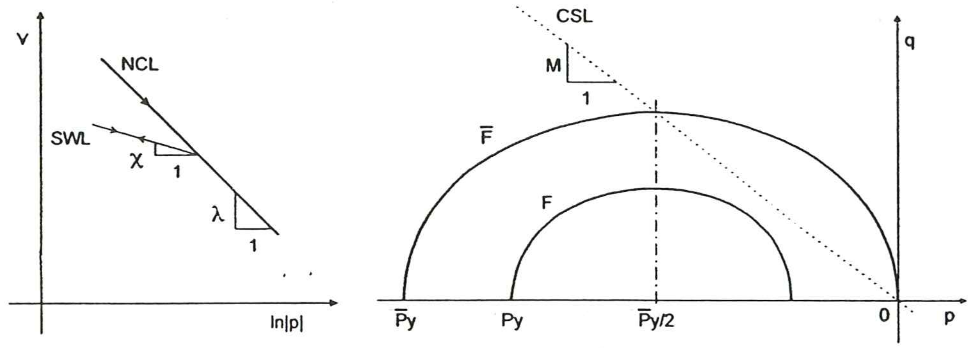


Figure 4: a) The NCL and the SWL. b) Mapping of F in the q vs. p plane for the Cam-Clay model.

12.2 A new formulation of Cam Clay

Let consider the following particular case of SUOLO:

1. The bounding surface $\overline{F}$, Fig. 4b, has the same equation of the yield surface in Eq. 87. Then, according to item 1 in Section 5, if $(p, q, \overline{p}_y)$ represents the current material state, the only admissible alternative conditions are

$$\begin{cases} \overline{F}(p, q, \overline{p}_y) = p^2 - p\overline{p}_y + \dfrac{q^2}{M^2} = 0; & \text{i.e. } \sigma \text{ lies on } \overline{F}. \\ \overline{F}(p, q, \overline{p}_y) = p^2 - p\overline{p}_y + \dfrac{q^2}{M^2} < 0; & \text{i.e } \sigma \text{ lies inside } \overline{F} \end{cases}$$

while

$$\overline{F}(p, q, \overline{p}_y) = p^2 - p\overline{p}_y + \frac{q^2}{M^2} > 0$$

is not admissible.

2. The hardening parameter $\overline{p}_y$ is function of $v^{(p)}$ only, and is given by Eq. 88. Moreover, it results

$$\frac{d\overline{p}_y}{dv^{(p)}} = -\frac{\overline{p}_y}{\lambda - \chi} > 0$$

3. The new yield surface F, Fig. 4b, has equation

$$F = F(p, q, \overline{p}_y, p_y) = p^2 - p\overline{p}_y + \frac{q^2}{M^2} - p_y^2 + p_y\overline{p}_y = 0$$

   where p_y is a negative scalar respecting the inequalities

$$\overline{p}_y \leq p_y \leq \frac{\overline{p}_y}{2}$$

4. The functional relationships for the hardening parameter p_y are given by:

   (a) There exists a scalar function for p_y, derived from the above yield function condition,

$$p_y = \frac{\overline{p}_y}{2} - \left[\left(p - \frac{\overline{p}_y}{2}\right)^2 + \frac{q^2}{M^2}\right]^{\frac{1}{2}}$$

   (b) If plasticity occurs, the dependence of p_y from $v^{(p)}$ and the internal variables h_j is arbitrary; nevertheless it must respects the limitations

$$\begin{aligned} \frac{\partial p_y}{\partial v^{(p)}} &= \frac{d\overline{p}_y}{dv^{(p)}} \\ \frac{\partial p_y}{\partial h_j} &= 0 \qquad \text{for all } j = 1, 2, \ldots, m \end{aligned}$$

   for $p_y = \overline{p}_y$.

5. The requirements in item 5 in Section 5 are trivially verified.

6. The elastic surface coincides with the bounding surface, $\widehat{F} \equiv \overline{F}$.

7. The requirements in item 7 in Section 5 have not to be satisfied, being $\widehat{F} \equiv \overline{F}$.

8. The potential surface coincides with the yielding surface, $G \equiv F$.

9. The elastic and plastic deformations are calculated in accordance with the rules in item 9 of Section 5. The stiffness matrix $\mathbf{C}^{(e)}$ is of the type in Eq. 42 and the Young modulus E is given in Eq. 89.

   According to the remark 1 in Section 7, the plastic modulus for a stress state lying on $\overline{F}$, NCS, is given by

$$A = -v\frac{\partial F}{\partial \overline{p}_y}\frac{d\overline{p}_y}{dv^{(p)}}\frac{\partial G}{\partial p} = -v\frac{p\overline{p}_y}{\lambda - \chi}\left(2p - \overline{p}_y\right)$$

   which coincides with the expression of A in Cam Clay.

10. According to item 10 in Section 5, taking into account that

$$\widehat{F} \equiv \overline{F}$$

   it results:

   - If
   $$\overline{F}(p, q, \overline{p}_y) < 0$$
   the material response is purely elastic.
   - If
   $$\overline{F}(p, q, \overline{p}_y) = 0$$
   the material skeleton is established on the basis of the stress and strain criteria in item 10 in Section 5.

It is easy to verify that

- Stress points with $p > 0$, or with $p = 0$ and $q \neq 0$, are not admissible.
- The stress point $(p = 0, q = 0)$ is a particular point in which the Young modulus E is zero, so that the elastic strain is unbounded.
- For $p < 0$ this formulation respects all the mathematical requirements listed in Sections 6 and 7.
- The mechanical behavior of the above presented model is identical to that of the classical Cam Clay model.

13 Conclusions

The most significant aspect of the class of models presented in this paper is to be based on simple and clear physical hypotheses, Section 5. In particular, the current effective stress state and the specific volume, $(\boldsymbol{\sigma}, v)$, are the state variables of soils. This allows a possible experimental refutation of all the models belonging to this class.

In Section 13 we have proved that Cam Clay belongs to SUOLO. It is widely accepted that it well describes the behavior of NCS under monotonic loading. On the contrary, it fails in describing the behavior of OCS under cyclic loading.

The mathematical framework presented in Sections 6 and 7 gives a recipe to extend Cam Clay type models to a better description of the behavior of OCS under cyclic loading.

The identification of all the functional relationships and of the internal variables h_j of the model require experimental investigations, which will be the object of successive research works.

NOTATION

$\mathbf{a}$	gradient vector of F
A	plastic modulus
$\mathbf{b}$	gradient vector of G
$B^{(e)}$	elastic bulk modulus
$\mathbf{C}^{(e)}$	elastic stiffness matrix
CSL	critical state line
CSS	critical state surface
$\mathbf{e}$	deviatoric strain
E	Young modulus
F	yielding surface
$\overline{F}$	bounding surface
$\widehat{F}$	elastic surface
G	potential surface
$G^{(e)}$	elastic shear modulus
LSS	limit state boundary surface
NCL	normal consolidation line

p	mean effective pressure
p_λ	reference pressure on the NCL
p_y	hardening parameter for F
$\overline{p}_y$	hardening parameter for $\overline{F}$
$\widehat{p}_y$	hardening parameter for $\widehat{F}$
q	equivalent shear stress
$\mathbf{s}$	deviatoric stress
v	specific volume
v_λ	value of v on the NCL, at $p = p_\lambda$
SWL	swelling line
χ	slope of a SWL in a semi-log graph
$\delta\lambda$	plastic multiplier
$\boldsymbol{\epsilon}$	total strain
$\boldsymbol{\epsilon}^{(e)}$	elastic strain
$\boldsymbol{\epsilon}^{(p)}$	plastic strain
ϵ_s	equivalent shear strain
ϵ_v	volumetric strain
λ	slope of the NCL in a semi-log graph
ν	Poisson modulus
$\boldsymbol{\sigma}$	effective stress
θ	angular invariant

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