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SSP ALGORITHM FOR LINEAR AND NON-LINEAR DYNAMIC RESPONSE SIMULATION

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SUMMARY

A direct integration algorithm to solve the spatially discretized equations of motion of a structure is proposed. This algorithm formulates the equations of motion in state space and uses their analytical solution to derive a recursive discrete-time equation. The proposed structural state procedure (SSP) can be considered as a generalization for multi-degree-of-freedom systems of the Duhamel's integral used for single-degree-of-freedom systems. It can be noted that the proposed SSP algorithm does not need a previous modal uncoupling of the equations of motion and consequently it does not require any hypothesis about damping. SSP is shown to be stable and to give accurate results with a reasonable computation time. Stability and accuracy essentially depend on the computation of the system matrix. The SSP algorithm is combined with an iterative scheme to obtain the response of structures with non-linear behaviour. Two examples of application of SSP are included: seismic response of a building structure with linear behaviour and free vibration of a non-linear system with imposed initial conditions.

INTRODUCTION

The motion of a structure spatially discretized by a linear model with n degrees of freedom is described by

$$\mathbf{M}\ddot{\mathbf{d}}(t)+\mathbf{C}\dot{\mathbf{d}}(t)+\mathbf{K}\mathbf{d}(t)=\mathbf{f}(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are, respectively, the mass, damping and stiffness matrices, $\mathbf{d}$ is the displacement vector and $\mathbf{f}$ the dynamic excitation vector.

The different methods usually applied for solving equation (1) can be included in a few families according to the kind of damping which is considered, since a classical damping allows an uncoupling of the equations of motion in modal co-ordinates, while a non-classical damping produces a coupled system of equations.

In the case of structures with non-classical damping, two direct integration solution schemes have been traditionally used. One of them is the explicit integration scheme (e.g. central differences method) and the other one is the implicit integration scheme (e.g. Newmark, Hoûbolt and θ-Wilson method).[1] All these procedures consider the hypothesis of linear variation of the response acceleration $\ddot{\mathbf{d}}$ on the temporal discretization interval to solve (1) at each instant. These procedures are very sensitive to the time increment used. Small time increments are required in order to compute stable and accurate responses. Other numerical procedures capable of dealing with non-classical damping systems use a modal uncoupling in the complex domain.[2] The solution of (1) in

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the frequency domain[2] is a procedure also used which is efficient with harmonic loading. Both of them need a considerable computation effort. In References 3 and 4 the equations of motion are formulated in state space and solved analytically, the exponential matrices involved in the solution being computed in Reference 3 by Padé approximations and in Reference 4 by Taylor series expansion.

If a classical damping is considered, the solution of the independent scalar equations can be obtained by applying any of the previously mentioned algorithms, as well as by means of the numerical solution of Duhamel's integral.[5]

This paper concerns the procedure proposed in Reference 4 which is called SSP (structural state procedure). This method does not make any hypothesis about the variation of the acceleration response between two time increments. It does not discretize equation (1) but its analytical solution in state space, leading to an accurate discrete-time model. SSP can be applied in modal co-ordinates,[6] taking into account only the modes with a significant influence on the response. Since active control problems are usually formulated in state space, SSP has been useful to simulate the dynamic behaviour of controlled structures and to test control systems.[7–9]

The purpose of this paper is to present a general formulation of SSP, to optimize the computational aspects and to perform an assessment of its potential to be used as a practical algorithm to compute the dynamic response of structures. An exhaustive analysis of the stability and the cost–accuracy ratio is performed and comparisons with other methods are included. An extension of the algorithm for the computation of the dynamic response of structures with non-linear behaviour is proposed. Numerical examples are presented to illustrate the application of SSP.

FORMULATION OF SSP ALGORITHM

The proposed algorithm starts from a state space formulation of the equation of motion (1) in the form

$$\dot{\mathbf{x}}(t)=\mathbf{F}\mathbf{x}(t)+\mathbf{v}(t) \tag{2}$$

where $\mathbf{x}$ is the $2n\times 1$ state vector, $\mathbf{F}$ is the $2n\times 2n$ *system matrix* and $\mathbf{v}$ is the $2n\times 1$ *excitation vector* defined as

$$\mathbf{x}(t)=\begin{Bmatrix}\mathbf{d}(t)\\ \dot{\mathbf{d}}(t)\end{Bmatrix} \qquad \mathbf{F}=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix} \qquad \mathbf{v}(t)=\begin{Bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\mathbf{f}(t)\end{Bmatrix} \tag{3}$$

With the initial conditions $\mathbf{x}(t_0)=\mathbf{x}_0$, the analytical solution of (2) is

$$\mathbf{x}(t)=\exp[(t-t_0)\mathbf{F}]\mathbf{x}_0+\int_{t_0}^{t}\exp[(t-\tau)\mathbf{F}]\mathbf{v}(\tau)\,\mathrm{d}\tau \tag{4}$$

By using this solution between two instants $k\Delta t$ and $(k+1)\Delta t$, the following discrete-time equation is obtained:

$$\mathbf{x}(k\Delta t+\Delta t)=\exp(\Delta t\mathbf{F})\mathbf{x}(k\Delta t)+\int_{k\Delta t}^{(k+1)\Delta t}\exp\{[(k+1)\Delta t-\tau]\mathbf{F}\}\mathbf{v}(\tau)\,\mathrm{d}\tau \tag{5}$$

To solve the integral involved in (5) the continuous-time evolution of $\mathbf{v}(\tau)$ in the discretization interval is required. If it is considered that the excitation vector is known only at discrete instants, $\mathbf{v}(\tau)$ can be defined by interpolating the discrete values. For example, the use of a linear

interpolation provides

$$\mathbf{v}(\tau)=\mathbf{v}(k\Delta t)+(\tau-k\Delta t)\frac{\mathbf{v}(k\Delta t+\Delta t)-\mathbf{v}(k\Delta t)}{\Delta t} \qquad k\Delta t\leqslant\tau\leqslant(k+1)\Delta t \tag{6}$$

By substituting (6) into (5), the following discrete-time equation is obtained:[7]

$$\mathbf{x}(k\Delta t+\Delta t)=\mathbf{A}\mathbf{x}(k\Delta t)+\mathbf{P}_1\mathbf{v}(k\Delta t+\Delta t)+\mathbf{P}_2[\mathbf{v}(k\Delta t+\Delta t)-\mathbf{v}(k\Delta t)] \tag{7}$$

where $\mathbf{A}$, $\mathbf{P}_1$ and $\mathbf{P}_2$ are $2n\times 2n$ matrices given by

$$\mathbf{A}=\exp(\Delta t\mathbf{F}) \qquad \mathbf{P}_1=\mathbf{F}^{-1}(\mathbf{A}-\mathbf{I}) \qquad \mathbf{P}_2=\mathbf{F}^{-1}\left(\frac{1}{\Delta t}\mathbf{P}_1-\mathbf{A}\right) \tag{8}$$

The system matrix $\mathbf{F}$ is always non-singular, its inverse being

$$\mathbf{F}^{-1}=\begin{bmatrix}-\mathbf{K}^{-1}\mathbf{C} & -\mathbf{K}^{-1}\mathbf{M}\\ \mathbf{I} & \mathbf{0}\end{bmatrix} \tag{9}$$

The recursive application of (7), starting from the initial conditions, allows the computation of the state vector $\mathbf{x}$ (which contains displacements and velocities) at each time instant. The accelerations can be obtained by numerical derivation of the velocities.

The numerical evaluation of the exponential matrix $\mathbf{A}=\exp(\Delta t\mathbf{F})$ is the key point of the proposed procedure. There are two main general ways to calculate the exponential matrix. One of them consists of approximating the exponential by series expansion, while the second one uses the Jordan decomposition of the matrix $\mathbf{F}$.

The proposed procedure can be also applied when the equations of motion (1) are formulated in modal co-ordinates, including the number of modes of vibration which have a significant influence on the structural response. By considering classical damping, the equations of motion are uncoupled and SSP is applied to individual differential equations, being thus a procedure similar to the use of Duhamel's integral. If the damping is non-classical, the effectiveness of the application of the procedure can be increased by using the modal co-ordinates to reduce the order of the system of equations.

COMPUTATION OF MATRIX A

(a) *Computation by series expansion*

The exponential matrix is defined by the power series

$$\exp(\Delta t\mathbf{F})=\lim_{p\to\infty}\sum_{i=0}^{p}\frac{(\Delta t\mathbf{F})^p}{p!}=\mathbf{I}+\frac{\Delta t\mathbf{F}}{1!}+\frac{(\Delta t\mathbf{F})^2}{2!}+\ \ldots\ +\frac{(\Delta t\mathbf{F})^p}{p!}+\ \ldots\ . \tag{10}$$

which can be approximated by a finite number p of terms as

$$\exp(\Delta t\mathbf{F})\simeq\mathbf{I}+\frac{\Delta t\mathbf{F}}{1!}+\frac{(\Delta t\mathbf{F})^2}{2!}+\ \ldots\ +\frac{(\Delta t\mathbf{F})^p}{p!}=T_p(\Delta t\mathbf{F}) \tag{11}$$

The error in the approximation of the series can be bounded according to[10]

$$\|T_p(\Delta t\mathbf{F})-\exp(\Delta t\mathbf{F})\|\leqslant\frac{\|\Delta t\mathbf{F}\|^{p+1}}{(p+1)!}\frac{1}{1-\|\Delta t\mathbf{F}\|/(p+2)}\leqslant\varepsilon \tag{12}$$

where $\|\ \|$ is the matrix Euclidean norm.

In some cases, $\|\Delta t\mathbf{F}\|$ can take values high enough to make this equation inefficient due to the excessive number of terms necessary to assure a reasonably small error. In such cases, the following property of the exponential function is useful:

$$\exp(\Delta t\mathbf{F}) = \exp\left(\frac{\Delta t\mathbf{F}}{m}\right)^m \tag{13}$$

Thus, Δt is replaced by $\Delta t/m$ in (10) and (11), as well as in relation (12). Moreover it is advantageous to select m as a power of two, $m = 2^q$, to turn the computation of the exponential matrix in the right side of (13) into the calculation of successive squares of a matrix in the form

$$\exp(\Delta t\mathbf{F}) = \left(\ldots \exp\left(\frac{\Delta t}{2^q}\mathbf{F}\right)^{\overbrace{2\ldots}^{q \text{ times}}} \right)^2 \tag{14}$$

Consequently, the discrete-time system matrix $\mathbf{A}$ is approximated by the following equation:

$$\mathbf{A} = \exp(\Delta t\mathbf{F}) \simeq \left[T_p\left(\frac{\Delta t}{m}\mathbf{F}\right)\right]^m \qquad m = 2^q \tag{15}$$

Choosing q in such a way that $\|(\Delta t/2^q)\mathbf{F}\| \leqslant 1$, it is possible to get a small error in (12) for $T_p(\Delta t\mathbf{F}/m)$ by using a reasonable number of terms p. By squaring q times the matrix $\exp[(\Delta t/2^q)\mathbf{F}]$ the error ε provided by equation (13) increases; however, it can be reduced as much as required by considering a number p of terms high enough.[10]

With the aim of simplifying the computer programming and of avoiding roundoff errors, it is convenient to transform the sum in (11) into the following product:

$$T_p\left(\frac{\Delta t}{2^q}\mathbf{F}\right) = \mathbf{I} + \frac{\Delta t}{2^q}\mathbf{F}\left(\mathbf{I} + \frac{\Delta t}{2^q}\frac{\mathbf{F}}{2}\left(\mathbf{I} + \frac{\Delta t}{2^q}\frac{\mathbf{F}}{3}\left(\mathbf{I} + \ldots\left(\mathbf{I} + \frac{\Delta t}{2^q}\frac{\mathbf{F}}{p-1}\left(\mathbf{I} + \frac{\Delta t}{2^q}\frac{\mathbf{F}}{p}\right)\right)\ldots\right)\right)\right) \tag{16}$$

This operation can be implemented by means of a loop in which a variable matrix $\mathbf{Y}$ is premultiplied in each step by $\mathbf{F}$. One can take advantage of the sparseness of the matrix $\mathbf{F}$ in (3) to simplify the operations and reduce the computation time. Calling $\mathbf{Y}_{ij}$ the four blocks of dimension $n \times n$ which constitute the matrix $\mathbf{Y}$, the following expression can be written:

$$\begin{aligned} \mathbf{FY} &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}\begin{bmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Y}_{21} & \mathbf{Y}_{22} \\ -\mathbf{M}^{-1}\mathbf{K}\mathbf{Y}_{11} - \mathbf{M}^{-1}\mathbf{C}\mathbf{Y}_{21} & -\mathbf{M}^{-1}\mathbf{K}\mathbf{Y}_{12} - \mathbf{M}^{-1}\mathbf{C}\mathbf{Y}_{22} \end{bmatrix} \end{aligned} \tag{17}$$

(*b*) *Computation through the Jordan form of* $\mathbf{F}$

The discrete-time system matrix $\mathbf{A}$ can be also computed starting from the Jordan form of $\mathbf{F}$, by using the following properties:

$$\mathbf{F} = \mathbf{T}\mathbf{F}_J\mathbf{T}^{-1} \qquad \mathbf{A} = \exp(\Delta t\mathbf{F}) = \mathbf{T}\exp(\Delta t\mathbf{F}_J)\mathbf{T}^{-1} \tag{18}$$

where $\mathbf{F}_J$ is the Jordan matrix in the basis defined by $\mathbf{T}$. When the damping matrix is $\mathbf{C} = \mathbf{0}$, the $2n$ eigenvalues of the system matrix $\mathbf{F}$ are $\pm \mathrm{j}\omega_i$, with $i = 1, 2, \ldots, n$, $\mathrm{j} = (-1)^{1/2}$ and ω_i the natural frequencies of the system expressed in rad/s. Consequently, for multi-degree-of-freedom systems with low damping and with distinct natural frequencies ω_i, the eigenvalues of the matrix $\mathbf{F}$ are also distinct and thus $\mathbf{F}_J$ is diagonal. In this case the matrix $\exp(\Delta t\mathbf{F})$ is also diagonal and its

computation is particularly simple since it reduces to the calculation of the complex exponential of the eigenvalues of the matrix **F** multiplied by Δt.

STABILITY OF THE ALGORITHM

The SSP procedure results in the discrete-time model (7) which simulates the response of the structure in the calculation instants. Consequently, the stability of SSP can be analysed by using concepts of stability of dynamic systems applied to (7). Two general types of stability can be considered: (a) external stability, conditioned by the excitation and (b) equilibrium stability related to the free evolution of the system starting from some initial conditions, as with Lyapunov, asymptotical and global stability. It is shown that all stability types are assured by conditions related to the modulus of the eigenvalues of matrix **A**.

External stability

The SSP algorithm is *externally stable* if, for any t_0, $\mathbf{x}(t_0)$ with $\|\mathbf{x}(t_0)\| \leqslant \delta$ and $\mathbf{v}(t)$ with $\|\mathbf{v}(t)\| \leqslant \delta$ for any $t \geqslant t_0$, there is a positive constant ε which depends only on t_0, $\mathbf{x}(t_0)$ and δ in such a way that $\|\mathbf{x}(t)\| \leqslant \varepsilon$ for any $t \geqslant t_0$. This stability condition is equivalent to that which states that to any bounded excitation corresponds a bounded response. Therefore this type of stability is important to guarantee the lack of artificial amplifications of the response. To analyse the conditions in which SSP is externally stable, it is necessary to have an expression which relates the response at an arbitrary instant k with the response at all the previous instants. A relation of this kind can be written by recurrently applying equation (7) starting from the initial instant; it leads to

$$\mathbf{x}(k\Delta t) = \mathbf{A}^k \mathbf{x}(\mathbf{t}_0) + \sum_{l=1}^{k} \mathbf{A}^{k-l} \mathbf{P}_1 \mathbf{v}(l\Delta t) + \sum_{l=1}^{k} \mathbf{A}^{k-l} \mathbf{P}_2 [\mathbf{v}(l\Delta t) - \mathbf{v}(l\Delta t - \Delta t)] \tag{19}$$

If all the eigenvalues of matrix **A** are inside the unit circle, the following inequality can be written:[11]

$$\|\mathbf{A}^l\| \leqslant C\lambda^l \tag{20}$$

where l is a positive integer, and C and λ are positive constant numbers, λ being smaller than one.

By using (20) the following inequalities can be written from (19):

$$\|\mathbf{x}(k\Delta t)\| \leqslant \|\mathbf{A}^k\| \, \|\mathbf{x}(\mathbf{t}_0)\| + \sum_{l=1}^{k} \|\mathbf{A}^{k-l}\| \, \|\mathbf{P}_1\| \, \|\mathbf{v}(l\Delta t)\| + \sum_{l=1}^{k} \|\mathbf{A}^{k-l}\| \, \|\mathbf{P}_2\| \, \|\mathbf{v}(l\Delta t) - \mathbf{v}(l\Delta t - \Delta t)\|$$

$$\leqslant \delta \left[C\lambda^k + 2(\|\mathbf{P}_1\| + \|\mathbf{P}_2\|) \sum_{l=1}^{q} C\lambda^{k-l} \right] \leqslant \varepsilon \tag{21}$$

where

$$\varepsilon = C\delta \left(1 + 2 \frac{\|\mathbf{P}_1\| + \|\mathbf{P}_2\|}{1 - \lambda} \right) \tag{22}$$

is independent of k.

Expressions (21) and (22) show that system (7) is externally stable if all the eigenvalues of matrix **A** have their modulus smaller than one.

Equilibrium stability

If there is not excitation, three types of stability can be considered in the SSP algorithm: Lyapunov, asymptotical and global.[11] The Lyapunov stability is important to assure that the

initial distortions and the round-off errors do not increase artificially in the computation process. The asymptotical and global stabilities are important in order to assure that these perturbations damp out in the computation process. If there is not excitation acting on the system, equation (19) reduces to

$$\mathbf{x}(k\Delta t)=\mathbf{A}^k\mathbf{x}(\mathbf{t}_0) \tag{23}$$

From (23) it is concluded[11] that the SSP algorithm is stable in the sense of *Lyapunov* if the moduli of the eigenvalues of matrix **A** are $\leqslant 1$ and only the eigenvalues belonging to a unit-sized Jordan block have moduli $=1$, and the SSP algorithm is *asymptotically* and *globally stable* if all the eigenvalues of **A** have moduli smaller than one.

Verification of the stability condition

Since when the moduli of **A** are inside the unit circle the system is externally and globally stable, it is useful to know the values of the parameters Δt, p and q which generate a matrix **A** in (15) with the moduli of its eigenvalues smaller than 1 to guarantee the stability of the numerical process. This study will be initially performed for a system without damping.

The equations of motion of a system in which $\mathbf{C}=\mathbf{0}$ can be always uncoupled by using its modal co-ordinates, resulting in the following n independent equations:

$$\dot{\mathbf{z}}_i=\mathbf{F}_i^*\mathbf{z}_i+\mathbf{v}_i^*(t) \quad i=1,\ldots,n \tag{24}$$

where $\mathbf{z}_i$, $\mathbf{F}_i^*$ and $\mathbf{v}_i^*$ are given by

$$\mathbf{z}_i=\begin{Bmatrix}\eta_i\\ \dot{\eta}_i\end{Bmatrix} \qquad \mathbf{F}_i^*=\begin{bmatrix}0 & 1\\ -\omega_i^2 & 0\end{bmatrix} \qquad \mathbf{v}_i^*=\begin{Bmatrix}0\\ f_i^*(t)\end{Bmatrix} \quad i=1,\ldots,n \tag{25}$$

In expressions (25) η_i is the modal co-ordinate, ω_i is the frequency and f_i^* is the force corresponding to the mode i. The eigenvalues of the matrix $\mathbf{A}_i^*=\exp(\Delta t\mathbf{F}_i^*)$ are $\pm\exp(\Delta t\omega_i)$ whose modulus is 1 and, consequently, if $\mathbf{A}_i^*$ is calculated exactly, the stability in the sense of Lyapunov is intrinsically guaranteed.

If matrix $\mathbf{A}_i^*$ is calculated closely by applying the expression (15) taking p as an odd number ($p=2s+1$), the following equations are obtained:

$$\begin{aligned}\mathbf{A}_i^*=\exp(\Delta t\mathbf{F}_i^*)&\simeq\left\{\mathbf{I}+\frac{1}{1!}\frac{\Delta t\mathbf{F}_i^*}{2^q}+\frac{1}{2!}\left(\frac{\Delta t\mathbf{F}_i^*}{2^q}\right)^2+\ldots+\frac{1}{p!}\left(\frac{\Delta t\mathbf{F}_i^*}{2^q}\right)^p\right\}^{2^q}\\ &=\left\{\left[1-\frac{1}{2!}\left(\frac{\omega_i\Delta t}{2^q}\right)^2+\frac{1}{4!}\left(\frac{\omega_i\Delta t}{2^q}\right)^4+\ldots+(-1)^s\frac{1}{(2s)!}\left(\frac{\omega_i\Delta t}{2^q}\right)^{2s}\right]\mathbf{I}\right.\\ &\quad\left.+\frac{1}{\omega_i}\left[\left(\frac{\omega_i\Delta t}{2^q}\right)-\frac{1}{3!}\left(\frac{\omega_i\Delta t}{2^q}\right)^3+\ldots+(-1)^s\frac{1}{(2s+1)!}\left(\frac{\omega_i{}^i\Delta t}{2^q}\right)^{2s+1}\right]\mathbf{F}_i^*\right\}^{2^q}\end{aligned} \tag{26}$$

This equation shows that on the moduli of the eigenvalues of the approximated matrix $\mathbf{A}_i^*$ two parameters prevail upon the others: the number of terms p and the values $\omega_i\Delta t/2^q$. It has to be emphasized that the exponent q modifies the eigenvalues of $\mathbf{A}_i^*$, but not the position of their moduli respecting the unit. Figure 1 gives curves which relate the modulus ρ of the eigenvalues of $\mathbf{A}_i^*$ (without damping) with $\Delta t/T_i 2^q$ for different values of p, where $T_i=2\pi/\omega_i$ is the period of the mode i. Figure 1 shows that, when there is not damping, for each value of the number of terms p, values of the period of discretization Δt and the exponent q can be found in such a way that the moduli of the eigenvalues of matrix $\mathbf{A}_i^*$ are equal to 1.

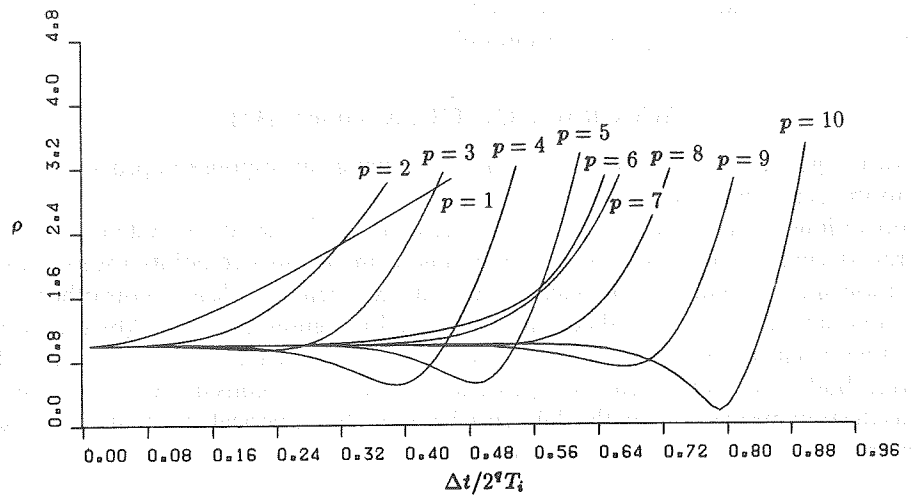

Figure 1. Maximum modulus of the eigenvalues of matrix $\mathbf{A}_i^*$

If in the mode i a damping coefficient v_i exists which is not equal to 0, the matrix $\mathbf{F}_i^*$ is now equal to

$$\mathbf{F}_i^* = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2v_i\omega_i \end{bmatrix} \tag{27}$$

The eigenvalues of matrix $\mathbf{A}_i^* = \exp(\Delta t \mathbf{F}_i^*)$ are now

$$\exp[-\omega_i(v_i \pm j\sqrt{1-v_i^2})] \tag{28}$$

their modulus being $|\exp(-\omega_i v_i)| < 1$. Consequently, damping contributes to increase the stability, and if a undamped mode is stable in the sense of Lyapunov a damped mode having the same mass and stiffness is externally and globally stable. Extending this property to the case in which matrix $\mathbf{A}_i^*$ is calculated by the approximation (15) and a general damping exists, Figure 1 allows one to analyse the stability conditions of SSP in all the applications.

In each application of the SSP algorithm, the values of Δt, p and q have to be chosen in such a way that, in the curve corresponding to p, the value $\Delta t/2^q T_n$, where n defines the higher mode of vibration, belongs to the horizontal part of the curve. If the computation process is stable for the mode n, the stability is also assured for all the modes $1, 2, \ldots, (n-1)$ according to the curves of Figure 1.

Conditional-unconditional stability

It is interesting to analyse if SSP, being a conditionally stable algorithm, behaves as an *unconditionally stable* algorithm[12] in the following sense: for any time increment Δt, there is a number p of terms for which the stability is guaranteed. With this aim, consider the maximum difference between the moduli of the eigenvalues of the matrix $\mathbf{A}_i^*$ (computed approximately by using (26)) and their exact value 1. This difference is[12]

$$2\cos(\omega_i\Delta t)\frac{(\omega_i\Delta t)^{2s+2}}{(2s+2)!} + 2\sin(\omega_i\Delta t)\frac{(\omega_i\Delta t)^{2s+3}}{(2s+3)!} + \left[\frac{(\omega_i\Delta t)^{2s+2}}{(2s+2)!}\right]^2 + \left[\frac{(\omega_i\Delta t)^{2s+3}}{(2s+3)!}\right]^2 \tag{29}$$

When p and consequently s increase indefinitely, this expression tends to vanish, taking values as small as desired, independently of the values of Δt.

ACCURACY OF THE ALGORITHM

A numerical algorithm is accurate if the computed numerical response approaches as much as desired to the exact response.

The operations required to calculate the dynamic response of a system by using SSP concentrate the numerical errors inherent to the algorithm in only two points: the interpolation of the excitation and the computation of matrix $\mathbf{A}$ as the exponential of $\Delta t\mathbf{F}$. All the other operations are algebraic and consequently theoretically exact. In Figures 2(a) and 2(b) the accuracy is compared with that of other algorithms, such as central differences, Newmark and Wilson, by considering both aforementioned error-generating points. The considered error criterion e_i is defined as the root mean square of the difference between the numerical and analytical solution for an undamped mode of vibration i for 25 sec.

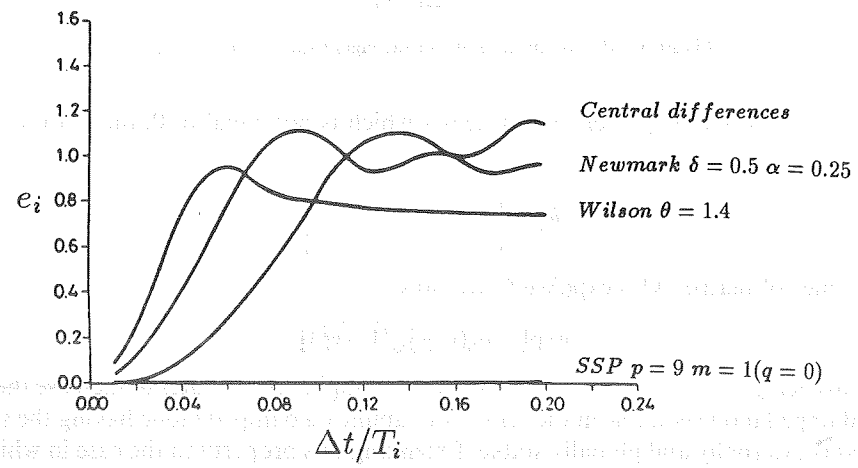

Figure 2(a). Comparison of the accuracy of different methods for free vibration

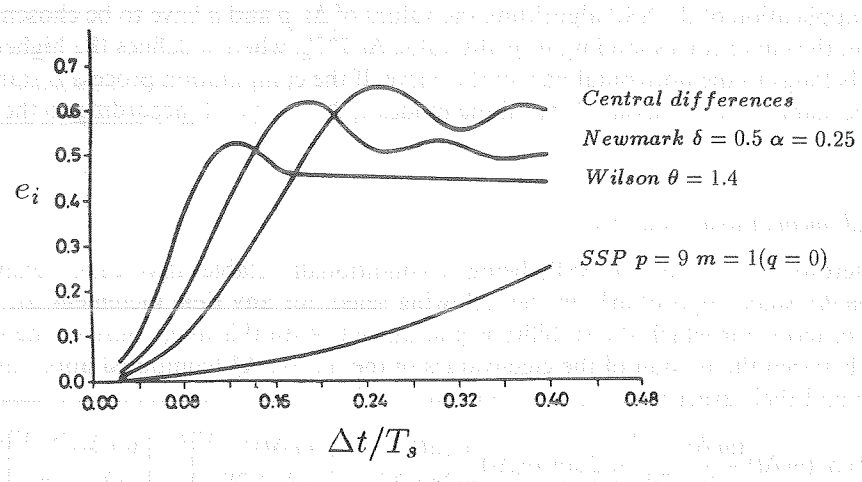

Figure 2(b). Comparison of the accuracy of different methods for sinusoidal excitation

In Figure 2(a), the free evolution of the system, starting from an unit initial displacement, has been considered. The error e_i is given as a function of the relation between the time increment Δt and the period T_i corresponding to the mode of vibration i. The matrix $\mathbf{A}_i^*$ is computed by using equation (26) with $p=9$ and $m=1$ ($q=0$). Within the considered range of values, the error corresponding to the SSP algorithm is practically null. In Figure 2(b), the undamped mode of vibration with natural frequency $\omega_i=1$ rad/sec is subjected to a sinusoidal vibration of period T_s and amplitude g. The variation of the error e_i with the coefficient $\Delta t/T_s$ is given, the accuracy of SSP depending in this case on the interpolation of the excitation performed in equation (6). The error of SSP is appreciably smaller than the error corresponding to the other algorithms. Figure 2(b) gives a measure of the error introduced in the response by the approximations due to the interpolation of the excitation within each time increment. This error does not depend on the parameters p and m, and it can be seen from Figure 2(a) that, for the considered values $p=9$ and $m=1$, the contribution to the error of the approximations in the computation of matrix $\mathbf{A}_i^*$ is negligible. Consequently, the total error of the SSP algorithm given in Figure 2(b) corresponds to the linear interpolation of the excitation.

To evaluate the approximations introduced in the response by the errors in the computation of matrix $\mathbf{A}$, a vibration mode corresponding to a free undamped vibration due to a unit initial displacement is studied. The percentage of the difference between the response predicted by SSP at the end of a complete period and its exact value 1 is represented in Figures 3(a) and 3(b) as a function of the relation between Δt and T_i for different values of p and m.

NUMBER OF OPERATIONS OF SSP ALGORITHM

The number of operations involved in the implementation of an SSP algorithm in a digital computer can be evaluated by using an unity called *flop* which is defined as the time required for a particular computer system to execute the FORTRAN statement

$$A(I, J)=A(I, J)+T*A(I, K)$$

that involves one floating point multiplication, one floating point addition, a few subscript and index calculations and a few storage references.[10]

In each application of the SSP algorithm to compute the dynamic response of a structure, it is necessary to obtain the value of matrices $\mathbf{A}$, $\mathbf{P}_1$ and $\mathbf{P}_2$ in (8) and compute the expression (7) at each discrete-time instant. The numbers of flops involved in these operations are called, respectively, N_1 and N_2 and, if matrix $\mathbf{A}$ is computed by Taylor series expansion in (15), their values are[12]

$$N_1=n^3(\tfrac{74}{3}+4p+8q)+n^2(8+4p)-n\tfrac{2}{3} \tag{30a}$$

$$N_2=(8n^2+2n)K_t \tag{30b}$$

where n is the number of degrees of freedom of the system, p and q are introduced in the computation of matrix $\mathbf{A}$ in (11) and (15) and K_t is the number of sampling instants.

The number of flops in (30a) represents the operations which must be done once for every structure, while in (30b) it represents the operations which have to be repeated for every dynamic excitation acting on that structure. Expressions (30) show that the most important computational effort is spent in (30a), that being especially true for systems with a great number of degrees of freedom.

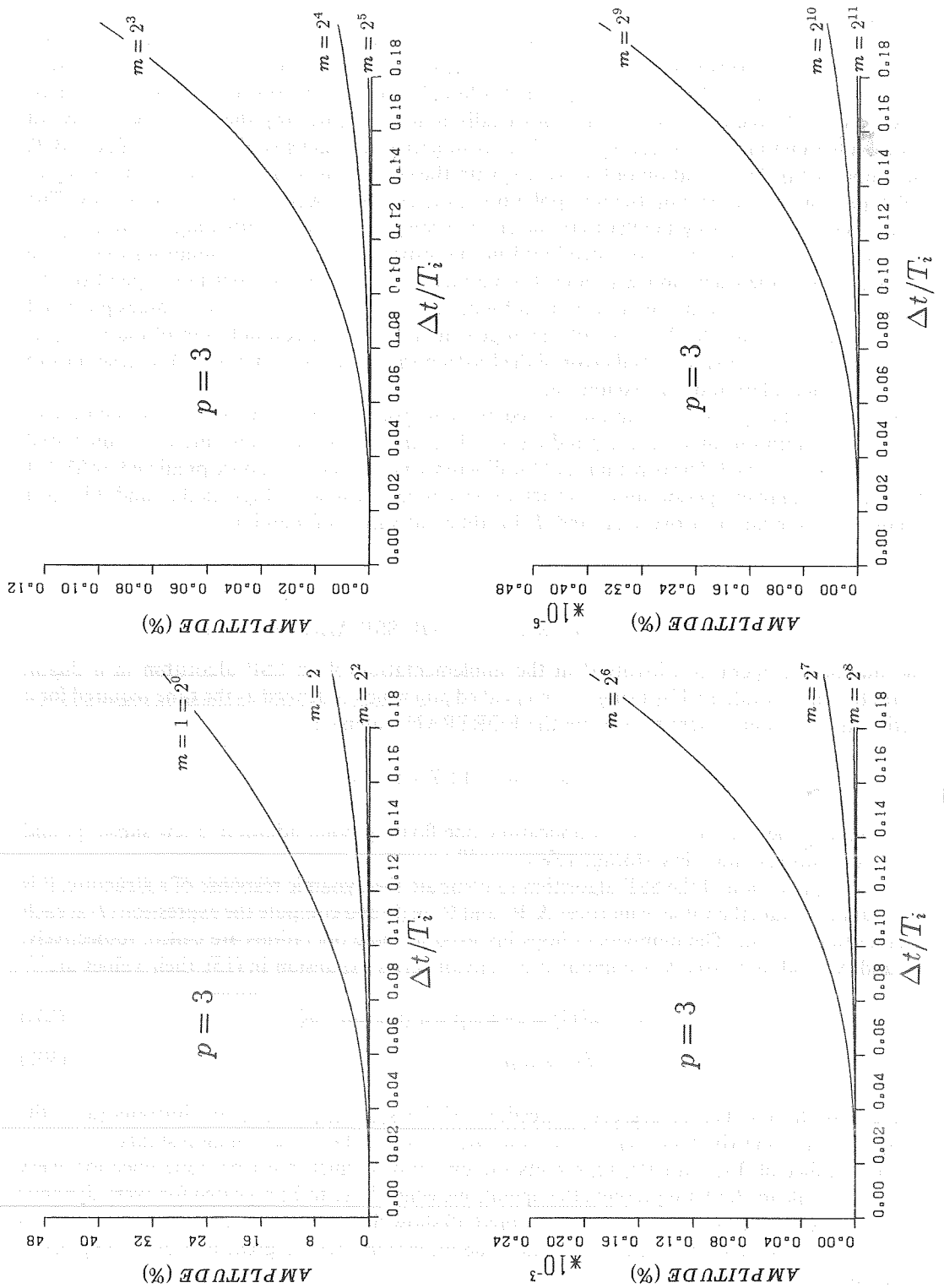

Figure 3(a). Error in the amplitude for $p=3$

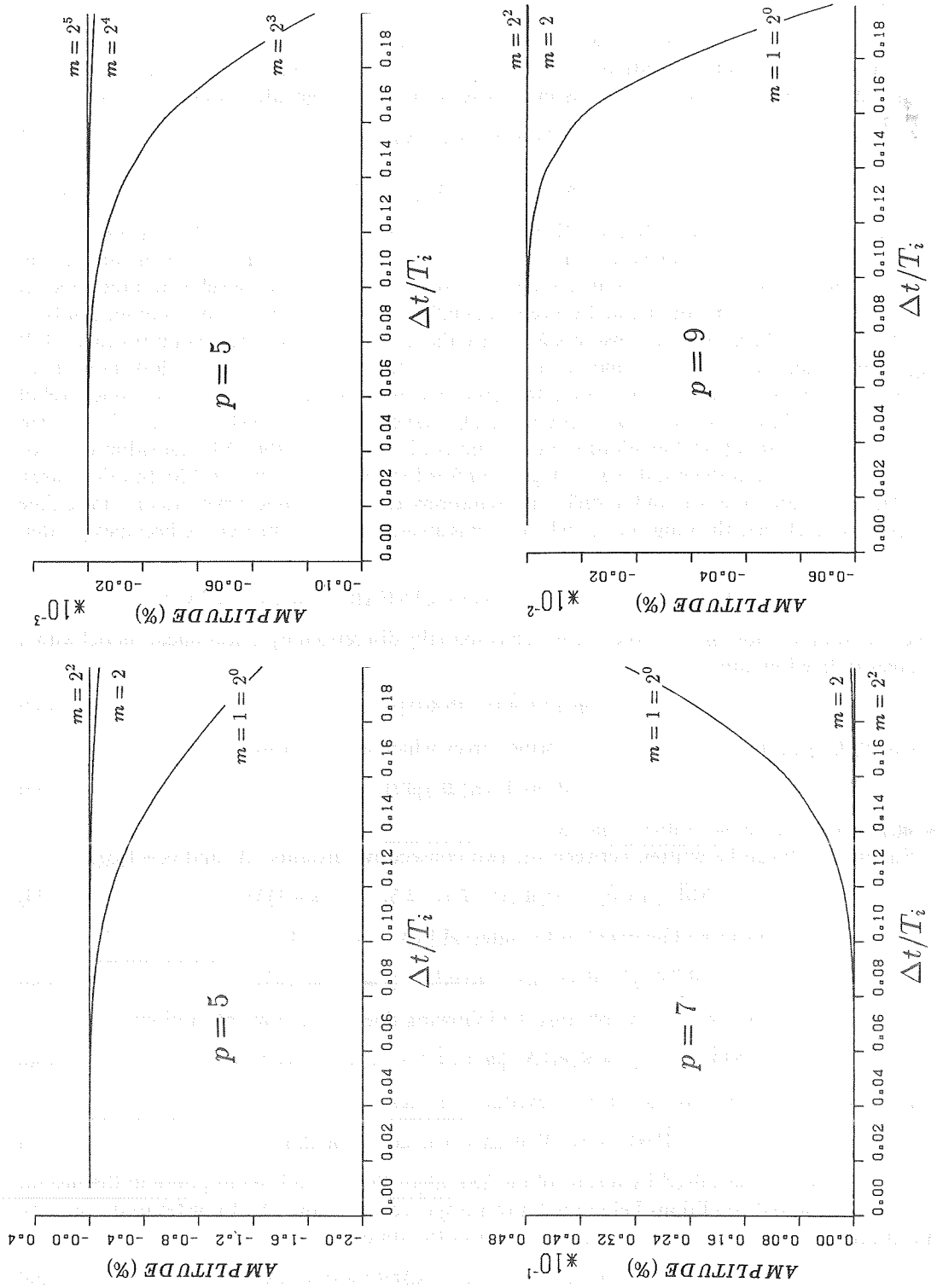

Figure 3(b). Error in the amplitude for $p = 5, 7, 9$

Comparison with other procedures

In this section SSP is compared with Newmark's procedure, which is considered as representative of direct integration methods. The operations involved in Newmark's procedure can be classified in two groups N_1 and N_2, as in the case of the SSP algorithm, whose values are[1]

$$N_1 = n^3\tfrac{1}{3} + n^2 2 - n\tfrac{1}{3} \tag{31a}$$

$$N_2 = (8n^2 + 6n)K_t \tag{31b}$$

The comparison between (30) and (31) shows that N_1 is greater in the SSP algorithm than in Newmark's method and N_2 has similar values in the two cases. Nevertheless, a real comparison must be done in the case that both procedures provide results of a similar level of accuracy and, in that case, Newmark's method usually needs a smaller time increment Δt and, consequently, a greater number of computation instants K_t. Since the more important source of error in an SSP algorithm is due to the approximate computation of matrix A in (15), Figure 2(a) shows that, taking $p=9$ and $q=0$, a ratio $\Delta t/T$ equal to 0·20 provides in the SSP algorithm the same level of accuracy that 0·02 does in Newmark's procedure. In that case, considering for instance $K_t = 100$ in (30) and $K_t = 1000$ in (31), the addition of N_1 and N_2 is smaller for the SSP algorithm than for Newmark's method if the number n of degrees of freedom of the system is $\leqslant 120$. In other cases, similar results are obtained and, if different excitations act on the same structure or a long-time analysis is developed, the range of n in which SSP is an advantageous procedure becomes greater.

APPLICATION OF SSP TO NON-LINEAR COMPUTATION

The equations of motion of a structure which is spatially discretized by a non-linear model with n degrees of freedom are

$$\mathbf{M}\ddot{\mathbf{d}}(t) + \mathbf{C}\dot{\mathbf{d}}(t) + \mathbf{P}[\mathbf{d}(t)] = \mathbf{f}(t) \tag{32}$$

where $\mathbf{P}[\mathbf{d}(t)]$ contains the internal restoring forces which are equal to

$$\mathbf{P}[\mathbf{d}(t)] = \mathbf{K}[\mathbf{d}(t)]\mathbf{d}(t) \tag{33}$$

$\mathbf{K}[\mathbf{d}(t)]$ being a variable stiffness matrix.

Equation (32) can be written between the two consecutive instants $k\Delta t$ and $(k+1)\Delta t$:

$$\mathbf{M}\ddot{\mathbf{d}}(\tau) + \mathbf{C}\dot{\mathbf{d}}(\tau) + \mathbf{P}[\mathbf{d}(\tau)] = \mathbf{f}(\tau) \quad k\Delta t \leqslant \tau \leqslant (k+1)\Delta t \tag{34}$$

The vector $\mathbf{P}[\mathbf{d}(\tau)]$ can be linearized in the interval $[k\Delta t, k\Delta t + \Delta t]$:

$$\mathbf{P}^0[\mathbf{d}(\tau)] = \mathbf{P}[\mathbf{d}(k\Delta t)] + \mathbf{K}[\mathbf{d}(k\Delta t)]\,[\mathbf{d}(\tau) - \mathbf{d}(k\Delta t)] \tag{35}$$

By substituting (35) into (34) one obtains the following linear equations of motion:

$$\mathbf{M}\ddot{\mathbf{d}}(\tau) + \mathbf{C}\dot{\mathbf{d}}(\tau) + \mathbf{K}[\mathbf{d}(k\Delta t)]\mathbf{d}(\tau) = \hat{\mathbf{f}}^0(\tau) \quad k\Delta t \leqslant \tau \leqslant (k+1)\Delta t \tag{36}$$

where vector $\hat{\mathbf{f}}^0(\tau)$ contains the effective excitation forces given by

$$\hat{\mathbf{f}}^0(\tau) = \mathbf{f}(\tau) - \mathbf{P}[\mathbf{d}(k\Delta t)] + \mathbf{K}[\mathbf{d}(k\Delta t)]\mathbf{d}(k\Delta t) \tag{37}$$

Equations (36) can be solved by means of the SSP algorithm to find the response in the instant $k\Delta t + \Delta t$, the initial conditions being the known response at instant $k\Delta t$. In order to do this, the linearized motion equations (36) can be written in the state space form:

$$\dot{\mathbf{x}}(\tau) = \mathbf{F}[\mathbf{d}(k\Delta t)]\mathbf{x}(\tau) + \hat{\mathbf{v}}^0(\tau) \quad k\Delta t \leqslant \tau \leqslant (k+1)\Delta t \tag{38}$$

where

$$\mathbf{x}(\tau)=\begin{Bmatrix}\mathbf{d}(\tau)\\ \dot{\mathbf{d}}(\tau)\end{Bmatrix} \quad \mathbf{F}[\mathbf{d}(k\Delta t)]=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K}[\mathbf{d}(k\Delta t)] & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix} \quad \hat{\mathbf{v}}^0(\tau)=\begin{Bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\hat{\mathbf{f}}^0(\tau)\end{Bmatrix} \tag{39}$$

$\mathbf{x}(\tau)$ is the state vector at the instant τ and $\mathbf{F}[\mathbf{d}(k\Delta t)]$ is the system matrix at the instant $k\Delta t$. $\hat{\mathbf{v}}^0(\tau)$ is the effective excitation vector at the instant τ.

The discrete-time solution of (38) is

$$\mathbf{x}^1(k\Delta t+\Delta t)=\mathbf{A}\mathbf{x}(k\Delta t)+\mathbf{P}_1\hat{\mathbf{v}}^0(k\Delta t+\Delta t)+\mathbf{P}_2[\hat{\mathbf{v}}^0(k\Delta t+\Delta t)-\hat{\mathbf{v}}^0(k\Delta t)] \tag{40}$$

where $\mathbf{A}$, $\mathbf{P}_1$ and $\mathbf{P}_2$ are matrices defined in (8), the system matrix $\mathbf{F}$ being now equal to $\mathbf{F}[\mathbf{d}(k\Delta t)]$. Upper index 0 refers to instant $k\Delta t$ and upper index 1 refers to instant $k\Delta t+\Delta t$.

The solution shown in (40) contains the displacement and velocity for instant $k\Delta t+\Delta t$:

$$\mathbf{x}^1(k\Delta t+\Delta t)=\begin{Bmatrix}\mathbf{d}^1(k\Delta t+\Delta t)\\ \dot{\mathbf{d}}^1(k\Delta t+\Delta t)\end{Bmatrix} \tag{41}$$

which verify the linearized equation (36). However, they do not verify the non-linear equations (34) at instant $\tau=(k+1)\Delta t$. By substituting in (34) the linear solution given in (40), it is necessary to add to the second member of equation (34) a residual force vector $\boldsymbol{\varphi}^1(k\Delta t+\Delta t)$ to verify this equation, its value being

$$\boldsymbol{\varphi}^1(k\Delta t+\Delta t)=\mathbf{f}(k\Delta t+\Delta t)-\{\mathbf{M}\ddot{\mathbf{d}}^1(k\Delta t+\Delta t)+\mathbf{C}\dot{\mathbf{d}}^1(k\Delta t+\Delta t)+\mathbf{P}[\mathbf{d}^1(k\Delta t+\Delta t)]\} \tag{42}$$

The state vector at instant $(k+1)\Delta t$ which verifies (34) for $\tau=k\Delta t+\Delta t$ is obtained by an iterative process[13] which starts from the linearized solution shown in (41). At each iteration i the SSP algorithm is applied to find a value $\mathbf{x}^i(k\Delta t+\Delta t)$ of the state vector starting from the value $\mathbf{x}^{i-1}(k\Delta t+\Delta t)$ found in the $(i-1)$th iteration. The residual forces $\boldsymbol{\varphi}^i(k\Delta t+\Delta t)$ are used only to test the convergence of the process.

The convergence criterion depends on the ratios

$$c_{r_j}=\frac{\varphi_j^i(k\Delta t+\Delta t)}{P_j[\mathbf{d}^i(k\Delta t+\Delta t)]} \quad (j=1,\ldots,n) \tag{43}$$

for the n degrees of freedom. The iterative process is stopped when the modulus of c_{r_j} is smaller than a pre-defined value ε. If a small value of ε is selected, a very accurate result is obtained.

EXAMPLES

Example 1: Seismic response of a building structure

SSP is applied to compute the seismic response of a 23-storey building modelled as a linear system with 23 degrees of freedom. The building has a prismatical form, with a 24 m square base and a height of 81 m, without underground floors. The structure is composed of five identical steel frames, with rigid nodes, spaced at 6 m and constructed with a steel having an elastic limit of 3600 kg/cm^2. Each frame has five 81 m columns and 23 continuous beams of 24 m, constituting thus a perfectly orthogonal structure. The height of each storey is of 3·5 m, excepting the first one which is 4 m. A horizontal and a vertical cross section of the building are given in Figure 4. In Table I the different cross sections of the elements used in the structural frames are defined according to the Spanish code.[14] Columns corresponding to the lower floors are reinforced with steel strips, as indicated in Figure 5.

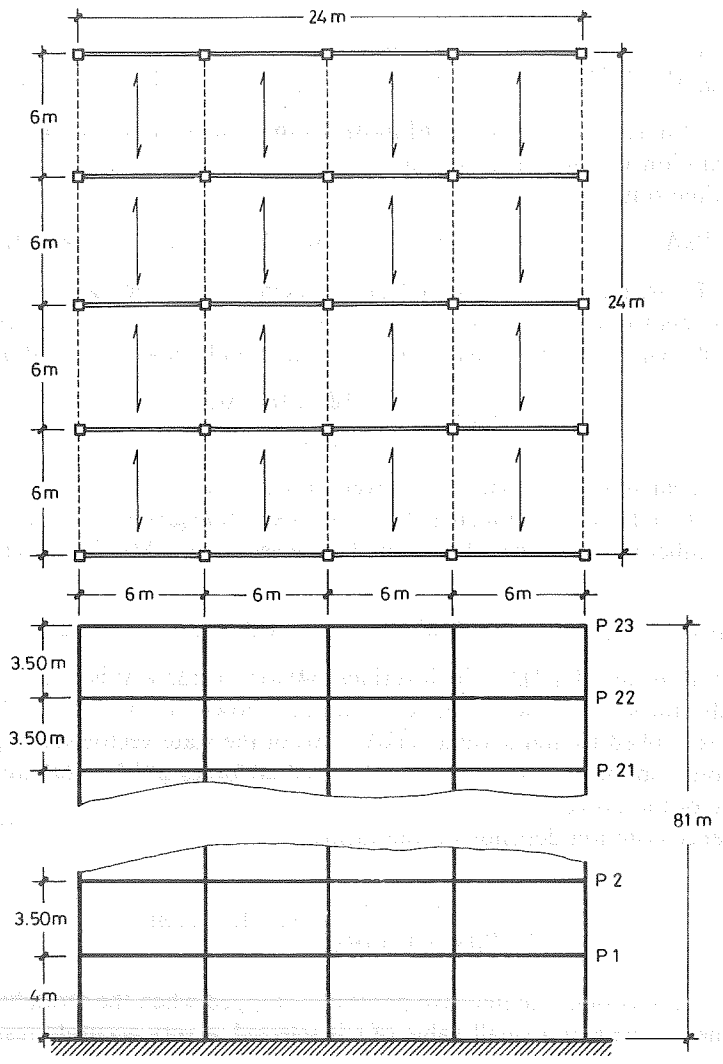

Figure 4. Structure of the building

It is considered that the ground acceleration excites the structure in the plane of the frames and consequently the mathematical model used corresponds to this assumption. A 23-degree-of-freedom linear building model has been used, according to the hypotheses which have been made. In the mass matrix **M** of the model, an uniform overload of 150 kg/m^2 has been considered on all the floors. The total mass of each floor is 374,000 kg. The stiffness matrix **K** of the model is obtained by inverting the flexibility matrix **S**. The damping matrix **C** is defined in modal co-ordinates by means of a damping ratio of 2·5. In Table II the periods of the first five modes of vibration and that of mode 23 are given.

To apply SSP, equation (1) is written in the form

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{C}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = -\mathbf{M}\mathbf{e}_n a(t) \tag{44}$$

Table I. A52 steel rolled shapes used in the columns and beams of the building structure

Floor	Column	Beam
1 and 2	HEB600 + 2□350 × 30(*)	IPN550
3 and 4	HEB600 + 2□350 × 24(*)	IPN550
5 and 6	HEB600 + 2□350 × 20(*)	IPN550
7	HEB600 + 2□350 × 15(*)	IPN550
8	HEB600 + 2□350 × 10(*)	IPN550
9 and 10	HEB600	IPN550
11	HEB550	IPN550
12	HEB500	IPN550
13	HEB450	IPN550
14	HEB400	IPN550
15	HEB360	IPN550
16	HEB340	IPN550
17	HEB320	IPN550
18	HEB300	IPN550
19	HEB280	IPN550
20	HEB260	IPN550
21	HEB240	IPN550
22	HEB220	IPN550
23	HEB200	IPN550

(*)Steel rolled section reinforced with two strips (dimensions in mm)

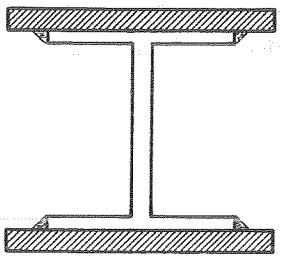

Figure 5. Cross section of the columns in the lower floors

Table II. Periods of the modes of vibration of the building model

Mode	Period (s)
1	2·833
2	1·117
3	0·666
4	0·472
5	0·368
23	0·029

where $\mathbf{e}_n$ is a column matrix having dimensions $n \times 1$ composed of unit elements, $a(t)$ is the acceleration and $\mathbf{d}$, $\dot{\mathbf{d}}$ and $\ddot{\mathbf{d}}$ are, respectively, the horizontal displacement, velocity and acceleration according to the degrees of freedom of the model.

The accelerogram is artificially generated starting from the following data:

(i) discretization interval $\Delta t_s = 0{\cdot}05$ s;
(ii) expected maximum acceleration $0{\cdot}1g$;
(iii) predominant frequency 1·667 Hz, corresponding to a period of 0·6 s.

The following values of the parameters have been selected: $\Delta t = 0{\cdot}0125$ s and $\varepsilon = 0{\cdot}01$. Starting from these data, the following values have been obtained for p and q: $p = 3$ and $q = 10$ ($m = 2^{10} = 1024$).

Previous to the application of SSP it is necessary to verify that the values of Δt, p and q guarantee a stable and accurate computation of the structural response. Figure 1 shows that the value of $\Delta t/2^q T_{23} = 0{\cdot}0004$ belongs to the horizontal part of the curve corresponding to $p = 3$ and therefore the numerical process is stable for the higher mode and for all the other modes of vibration. Taking T_s as two times the predominant period in the excitation accelerogram, it can be seen from Figure 2(b) that for the value $\Delta t/T_s = 0{\cdot}042$ the error introduced in the response by the linear interpolation of the excitation is practically negligible. The error introduced in the response by the inexact calculation of the matrix $\mathbf{A}$ in (15) is evaluated for the fifth mode assuming it is the highest mode whose influence on the response is appreciable. Taking the value of the fifth mode period T_5 from Table II, the value of the ratio $\Delta t/T_5 = 0{\cdot}034$ shows in Figure 3(a) (curve for $p = 3$ and $m = 2^{10}$) that the error in the amplitude percentage is negligible.

In Figures 6(a) and 6(b), the displacement and acceleration responses of the upper floor are given.

The response of the building has been also obtained by applying the SSP in modal co-ordinates taking into account only the five first modes. The results are very similar to those given in Figures 6, but the time of calculation has decreased ten times.

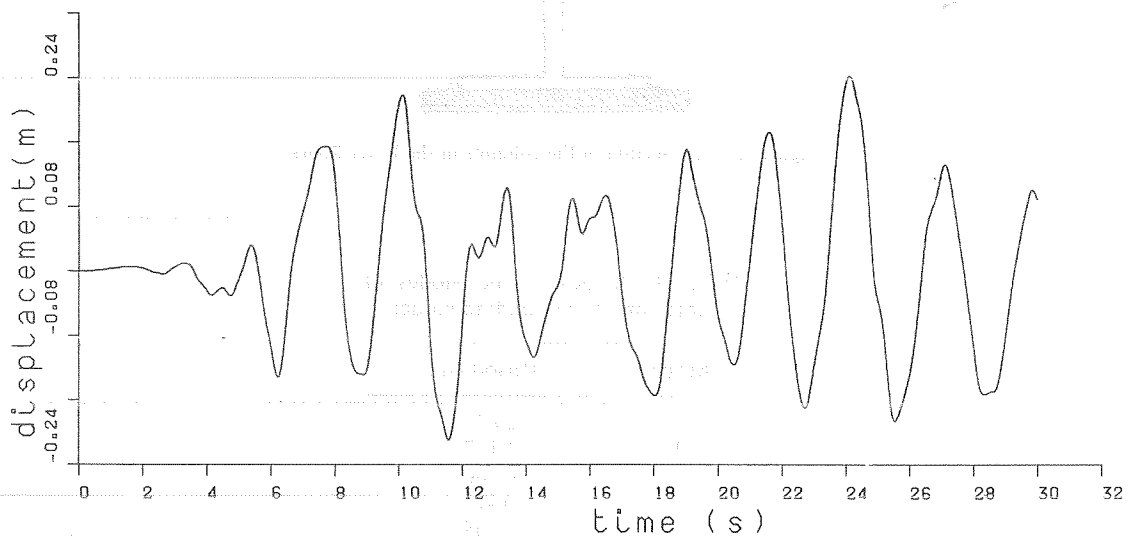

Figure 6(a). Horizontal displacement of the upper floor

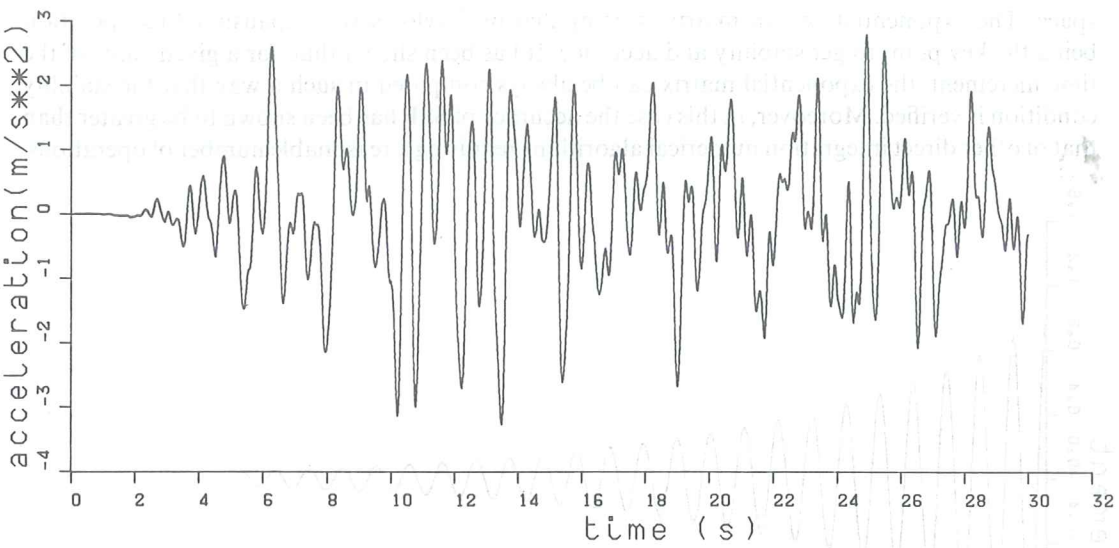

Figure 6(b). Horizontal acceleration of the upper floor

Example 2: Free evolution of a non-linear system

In this section a single-degree-of-freedom system is considered in which the relationship between the restoring forces P and the displacement d is elastic and non-linear:

$$P(d) = \pm \frac{K_e}{\beta}(1 - e^{-\beta|d|}) \tag{45}$$

where the + sign corresponds to positive values of d and the − sign to negative. K_e is the initial stiffness of the system and β is a coefficient. If $\beta = 0$, curves contained in (45) tend to the following linear equation:

$$P(d) = K_e d \tag{46}$$

If $\beta > 0$ the constitutive law (45) belongs to a more flexible material that the one whose law is shown in (46). Nevertheless, if $\beta < 0$ the law (45) corresponds to a stiffer material.

The values of mass and damping of the system are such that, for the initial sttiffness K_e, its natural frequency is 6 rad/s and its damping is 2 per cent of the critical damping.

In the application of the SSP algorithm the following values of the general parameters are considered: $\Delta t = 0{\cdot}01$ s, $p = 9$ and $q = 0$. They are enough to generate a stable and accurate calculation.

In Figure 7(a) the free evolution of the linear system ($\beta = 0$) starting from the initial conditions $d(0) = 1$ and $\dot{d}(0) = 0$ is shown. In Figures 7(b) and 7(c) values of $\beta = 4$ and -4 are considered, respectively. The comparison between Figures 7(a), 7(b) and 7(c) shows that, for great displacements near from 1, the frequency of the most flexible material in Figure 7(b) decreases, while the frequency of the stiffest material in Figure 7(c) increases.

CONCLUSIONS

A structural state procedure SSP has been formulated as a discrete-time algorithm to compute the dynamic response of structures based on the analytical solution of the equations of motion in state

space. The exponential system matrix is computed by Taylor series expansion, this operation being the key point to get stability and accuracy. It has been shown that, for a given value of the time increment, the exponential matrix can be always computed in such a way that the stability condition is verified. Moreover, in this case the accuracy of SSP has been shown to be greater than that of other direct integration numerical algorithms requiring a reasonable number of operations.

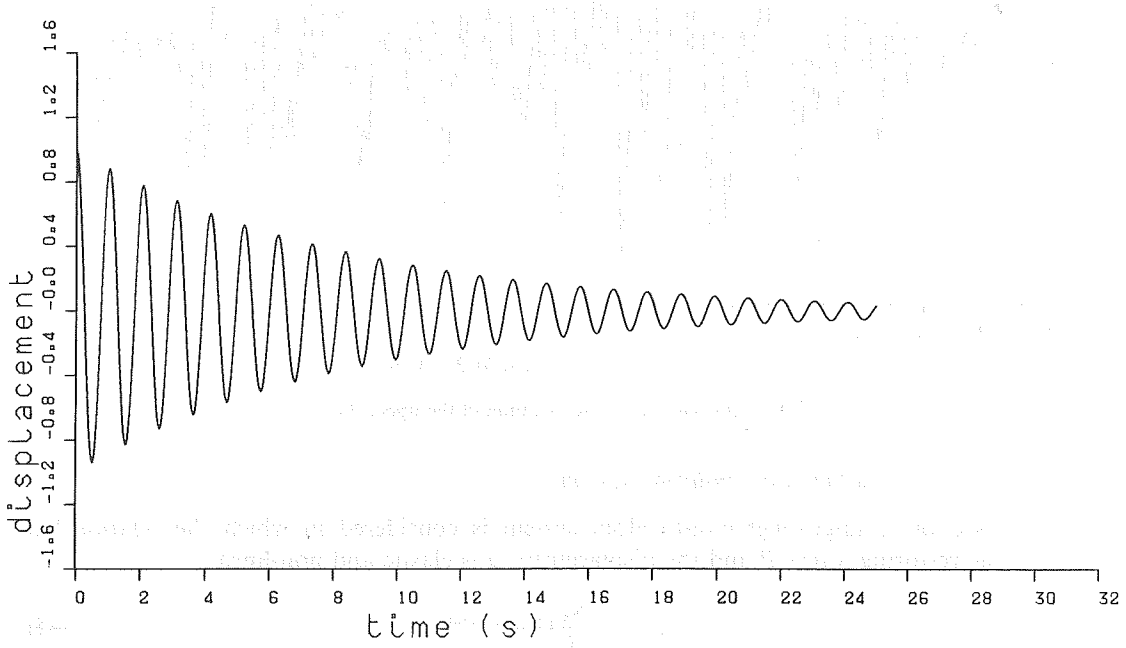

Figure 7(a). Free response of a linear system

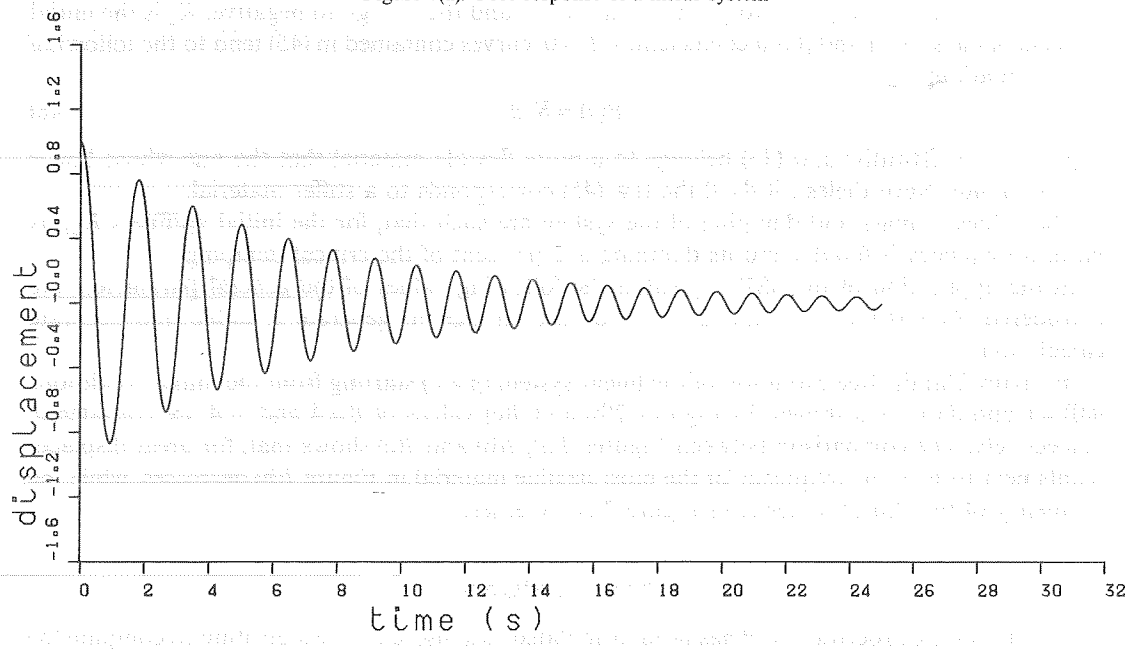

Figure 7(b). Free response of a flexible non-linear system

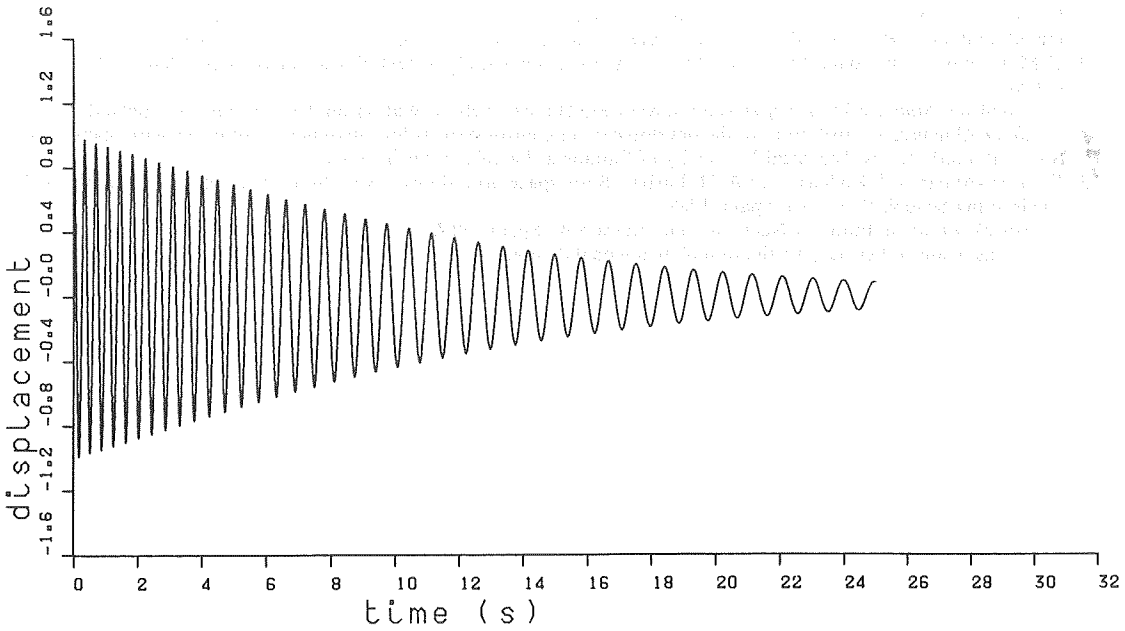

Figure 7(c). Free response of a stiff non-linear system

SSP does not need any hypothesis about damping and it appears as an extension for multi-degree-of-freedom systems of the Duhamel's integral commonly used for linear single-degree-of-freedom systems.

The combination of SSP with an iterative scheme has shown the potential of the procedure in computing the response of structures with non-linear behaviour.

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