DPG-based time-marching-schemes for linear parabolic and hyperbolic partial differential equations

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ABSTRACT

The Discontinuous Petrov-Galerkin (DPG) method with optimal test functions intends to approximate Partial Differential Equations (PDEs). It was introduced by Demkowicz and Gopalakrishnan in 2010 [1]. The main idea of this method is to select optimal test functions that guarantee the discrete stability of non-coercive problems. For that, they employ test functions that realize the supremum in the inf-sup condition. It has been previously applied to transient problems in the context of space-time formulations or together with finite differences in time [2, 3, 4]. In this work, we follow the approach of applying the DPG method only in the time variable in order to obtain a DPG-based time-marching scheme for linear transient PDEs [7, 8].

For parabolic problems, we first semidiscretize in space by a classical Bubnov-Galerkin method and we consider an ultraweak variational formulation of the resulting system of Ordinary Differential Equations (ODEs). Then, we calculate the optimal test functions analytically employing the adjoint norm. The optimal test functions we obtain are exponential-related functions of the stiffness matrix. Finally, we substitute the optimal test functions into the ultraweak variational formulation and we obtain the DPG-based time-marching scheme. Here, we obtain an independent formula for the trace variables and a system to locally compute the interiors of the elements. The procedure for hyperbolic problems is the same after reducing the equation to a first order system.

The equation we obtain for the trace variables is called variation-of-constants formula and it is the starting point of the so-called exponential integrators for solving systems of ODEs [6]. In these type of methods, it is necessary to approximate the exponential of the stiffness matrix and related functions, called φ-functions. For the hyperbolic case, it is also possible to express the resulting system in terms of trigonometric functions. Although the theory of exponential integrators is classical, they have recently gained popularity due to the rise of the available software and efficient algorithms to compute the action of function matrices over vectors [5]. In our work, we express the DPG-based time-marching scheme in terms of the φ-functions for computational purposes.

The main benefit of the presented method is that it fits into the DPG theory. Therefore, we can naturally apply adaptive strategies and a posteriori error estimation previously studied by the DPG community. Currently, most of the goal-oriented adaptive strategies for transient problems are based on Discontinuous Galerkin (DG) formulations in time because we need a variational formulation in time to represent the error in the quantity of interest over the whole space-time domain. Employing the variational formulation presented in this article, we will be able to design goal-oriented adaptive strategies in the future.

REFERENCES

[1] L. Demkowicz and J. Gopalakrishnan. A class of discontinuous Petrov–Galerkin methods. Part I: The transport equation. *Computer Methods in Applied Mechanics and Engineering*, 199(23-24):1558–1572, 2010.

[2] L. Demkowicz, J. Gopalakrishnan, S. Nagaraj, and P. Sepúlveda. A spacetime DPG method for the Schrödinger equation. *SIAM Journal on Numerical Analysis*, 55(4):1740–1759, 2017.

[3] T. Führer, N. Heuer, and J. S. Gupta. A time-stepping DPG scheme for the heat equation. *Computational Methods in Applied Mathematics*, 17(2):237–252, 2017.

[4] J. Gopalakrishnan and P. Sepúlveda. A space-time DPG method for the wave equation in multiple dimensions. *Space-Time Methods. Applications to Partial Differential Equations*, pages 129–154, 2017.

[5] N. J. Higham and E. Hopkins. A catalogue of software for matrix functions. version 3.0. *http://eprints.maths.manchester.ac.uk/*, 2020.

[6] M. Hochbruck and A. Ostermann. Exponential integrators. *Acta Numerica*, 19:209–286, 2010.

[7] J. Muñoz-Matute, D. Pardo, and L. Demkowicz. A DPG-based time-marching scheme for linear hyperbolic problems. *Computer Methods in Applied Mechanics and Engineering*, 373:113539, 2020.

[8] J. Muñoz-Matute, D. Pardo, and L. Demkowicz. Equivalence between the DPG method and the Exponential Integrators for linear parabolic problems. *Journal of Computational Physics*, in press, https://doi.org/10.1016/j.jcp.2020.110016, 2020.