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COMPUTATION OF EIGENVALUES IN LINEAR ELASTICITY WITH LEAST-SQUARES FINITE ELEMENTS: DEALING WITH THE MIXED SYSTEM

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Abstract. In this paper we discuss some aspects related to the practical implementation of a method that has been introduced recently for the approximation of the eigenvalues of the linear elasticity problem. The scheme, based on a least-squares finite element formulation, gives rise to a non-symmetric discrete formulation that may have complex eigenvalues. Moreover the algebraic eigenvalue problem to be solved is singular, so that the theoretical estimates about the convergence of the scheme should be carefully interpreted.

1 INTRODUCTION

This paper continues the research started in [3] about a least-squares finite element formulation for the approximation of the eigenvalue problem associated with linear elasticity. The formulation was introduced in [1] for the source problem. It is a two field formulation where, using as unknowns the displacement vector and the stress tensor, the second order elasticity equation is decomposed into a first order system.

In this note we recall the main results of [3] and discuss more specifically the definition of convergence of the numerical scheme. Indeed, the algebraic system that is obtained after the finite element discretization has the following structure which resembles an eigenvalue problem in mixed form

$$\begin{pmatrix} A & B^\top \\ B & C \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\sigma}} \\ \hat{\mathbf{u}} \end{pmatrix} = \omega \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\sigma}} \\ \hat{\mathbf{u}} \end{pmatrix}$$

At least two issues are associated with the analysis of such a problem: first of all the matrix on the right hand side is clearly singular so that the definition itself of the solution should be carefully stated. Then, it is apparent that the system cannot be reduced to a symmetric form due to the fact that the matrix D is different from $B^\top$. If the two matrices were related to each other, then a similar strategy as the one performed in [2] for the Laplace operator would lead to a symmetric problem.

A typical technique to deal with mixed problem consists in the introduction of appropriate Schur complements. In this case we present two possible Schur complement factorizations and we show how they can be used to solve the system. Analogies and differences are described.

Finally, we recall the definition of convergence and we discuss how to apply this to the case when complex eigenvalues are present in the discrete system.

2 PROBLEM SETTING

Let Ω be an open, bounded, and connected subset of $\mathbb{R}^2$ with its boundary $\partial\Omega$ being Lipschitz continuous. The boundary of the domain Ω is divided into two sets, Γ_D and Γ_N. By considering the first-order system for stress/displacement of linear elasticity with its boundary conditions in [1], we have the following: find a displacement vector field $\mathbf{u}$ and a symmetric 2×2 tensor $\underline{\boldsymbol{\sigma}}$ such that

$$\begin{cases} \mathcal{A}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\epsilon}}(\mathbf{u}) = 0 & \text{in} \quad \Omega \\ \mathbf{div}\underline{\boldsymbol{\sigma}} = -\mathbf{f} & \text{in} \quad \Omega \\ \mathbf{u} = 0 & \text{on} \quad \Gamma_D \\ \underline{\boldsymbol{\sigma}}\mathbf{n} = 0 & \text{on} \quad \Gamma_N \end{cases} \tag{1}$$

where $\mathbf{n}$ denotes the outward unit vector normal to the boundary, $\mathcal{A}$ and $\underline{\boldsymbol{\epsilon}}(\boldsymbol{v})$ are the compliance tensor and the symmetric gradient, respectively considered in [1], where some least-squares formulations for linear elasticity were presented. This was done by applying the L^2 norm least-square principle to (1) and then minimizing the functional

$$\mathcal{F}(\underline{\boldsymbol{\tau}}, \mathbf{v}; \mathbf{f}) = ||\mathcal{A}\underline{\boldsymbol{\tau}} - \underline{\boldsymbol{\epsilon}}(\mathbf{v})||_0^2 + ||\mathbf{div}\underline{\boldsymbol{\tau}} + \mathbf{f}||_0^2 \tag{2}$$

in $\underline{\boldsymbol{X}}_N \times H^1_{0,D}(\Omega)^2$, with

$$\underline{\boldsymbol{X}} = \begin{cases} \boldsymbol{H}(\mathbf{div};\Omega)^2 & \text{if} \quad \Gamma_N \neq \emptyset \\ \{\underline{\boldsymbol{\tau}} \in \boldsymbol{H}(\mathbf{div};\Omega)^2 : \int_\Omega \operatorname{tr}(\underline{\boldsymbol{\tau}})\mathrm{d}\mathbf{x} = 0\} & \text{if} \quad \Gamma_N = \emptyset \end{cases}$$

where $\underline{\boldsymbol{X}}_N$ is the subset of $\underline{\boldsymbol{X}}$ corresponding to the boundary condition $\underline{\boldsymbol{\tau}}\mathbf{n} = 0$ on Γ_N.

2.1 Eigenvalue problem for linear elasticity

We are interested in studying the properties of the spectrum of the operator arising from the least-squares approximation of (1). This is done by replacing the source term $\mathbf{f}$ in (1) with the expression $\omega\mathbf{u}$ and by writing the corresponding least-squares formulation. The problem reads:

find $\omega \in \mathbb{R}$ such that for non vanishing $\mathbf{u}$ and for some $\underline{\boldsymbol{\sigma}}$ we have

$$\begin{cases} \mathcal{A}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\epsilon}}(\mathbf{u}) = 0 & \text{in} \quad \Omega \\ \mathbf{div}\underline{\boldsymbol{\sigma}} = -\omega \mathbf{u} & \text{in} \quad \Omega \\ \mathbf{u} = 0 & \text{on} \quad \Gamma_D \\ \underline{\boldsymbol{\sigma}}\boldsymbol{n} = 0 & \text{on} \quad \Gamma_N \end{cases} \tag{3}$$

Due to the symmetry of linear elasticity problems, we seek for eigenvalues which are real ($\omega \in \mathbb{R}$). This idea has been exploited originally in [2] in the case of the Laplace operator and extended to non symmetric problems associated with linear elasticity in [3]. This was done by studying the spectrum of the operators associated with the least-squares *source* formulation, which deduced the so called **two-field** formulation.

2.2 Two-field formulation and the eigenvalue problem

The two-field formulation of the source problem considered in [1] consists in minimizing the functional $\mathcal{F}(\underline{\boldsymbol{\tau}}, \mathbf{v}; \mathbf{f})$. This gives rise to the variational formulation which reads: find $\underline{\boldsymbol{\sigma}} \in \underline{\boldsymbol{X}}_N$ and $\mathbf{u} \in H^1_{0,D}(\Omega)^2$ such that

$$\begin{cases} (\mathcal{A}\underline{\boldsymbol{\sigma}}, \mathcal{A}\underline{\boldsymbol{\tau}}) + (\mathbf{div}\underline{\boldsymbol{\sigma}}, \mathbf{div}\underline{\boldsymbol{\tau}}) - (\mathcal{A}\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\epsilon}}(\mathbf{u})) = -(\mathbf{f}, \mathbf{div}\underline{\boldsymbol{\tau}}) & \forall \underline{\boldsymbol{\tau}} \in \underline{\boldsymbol{X}}_N \\ -(\mathcal{A}\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\epsilon}}(\mathbf{v})) + (\underline{\boldsymbol{\epsilon}}(\mathbf{u}), \underline{\boldsymbol{\epsilon}}(\mathbf{v})) = 0 & \forall \mathbf{v} \in H^1_{0,D}(\Omega)^2 \end{cases} \tag{4}$$

The eigenvalue variational formulation associated with the two-field is obtained by replacing the source $\mathbf{f}$ with $\omega\mathbf{u}$. This leads to: find $(\omega, \mathbf{u}) \in \mathbb{C} \times H^1_{0,D}(\Omega)^2$ with $\mathbf{u} \neq 0$ such that for some $\underline{\boldsymbol{\sigma}} \in \underline{\boldsymbol{X}}_N$ we have

$$\begin{cases} (\mathcal{A}\underline{\boldsymbol{\sigma}}, \mathcal{A}\underline{\boldsymbol{\tau}}) + (\mathbf{div}\underline{\boldsymbol{\sigma}}, \mathbf{div}\underline{\boldsymbol{\tau}}) - (\mathcal{A}\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\epsilon}}(\mathbf{u})) = -\omega(\mathbf{u}, \mathbf{div}\underline{\boldsymbol{\tau}}) & \forall \underline{\boldsymbol{\tau}} \in \underline{\boldsymbol{X}}_N \\ -(\mathcal{A}\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\epsilon}}(\mathbf{v})) + (\underline{\boldsymbol{\epsilon}}(\mathbf{u}), \underline{\boldsymbol{\epsilon}}(\mathbf{v})) = 0 & \forall \mathbf{v} \in H^1_{0,D}(\Omega)^2 \end{cases} \tag{5}$$

Let us consider the bilinear forms $(\mathcal{A}\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\epsilon}}(\mathbf{u}))$ and $(\mathbf{u}, \mathbf{div}\underline{\boldsymbol{\tau}})$ in the first equation of (5). We perceive that the left hand side has the symmetric gradient $\underline{\boldsymbol{\epsilon}}(\mathbf{u})$ while the right hand side has $\mathbf{div}\underline{\boldsymbol{\tau}}$ in the bilinear forms. As a consequence, the problem is non-symmetric, hence, some of the computed eigenvalues may be in the complex plane even if we are approximating a symmetric problem. This is a natural consequence of the least-squares method.

3 NUMERICAL APPROXIMATION

The Galerkin discretization of eigenvalue problem introduced in (5) is as follows. Let $\Sigma_h \subset \underline{\boldsymbol{X}}_N$ and $U_h \subset H^1_{0,D}(\Omega)^2$ be conforming finite element spaces. Then the discrete formulation reads: find the pair $(\omega_h, \mathbf{u}_h) \in \mathbb{C} \times U_h$ with $\mathbf{u}_h \neq 0$ such that for some $\underline{\boldsymbol{\sigma}}_h \in \Sigma_h$ we have

$$\begin{cases} (\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \mathcal{A}\underline{\boldsymbol{\tau}}) + (\mathbf{div}\underline{\boldsymbol{\sigma}}_h, \mathbf{div}\underline{\boldsymbol{\tau}}) - (\mathcal{A}\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\epsilon}}(\mathbf{u}_h)) = -\omega_h(\mathbf{u}_h, \mathbf{div}\underline{\boldsymbol{\tau}}) & \forall \underline{\boldsymbol{\tau}} \in \Sigma_h \\ -(\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\epsilon}}(\mathbf{v})) + (\underline{\boldsymbol{\epsilon}}(\mathbf{u}_h), \underline{\boldsymbol{\epsilon}}(\mathbf{v})) = 0 & \forall \mathbf{v} \in U_h \end{cases} \tag{6}$$

The eigenvalue problem given in (6) has the typical structure of a mixed form given by

$$\begin{pmatrix} A & B^T \\ B & C \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\sigma}} \\ \hat{\mathbf{u}} \end{pmatrix} = \omega \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\sigma}} \\ \hat{\mathbf{u}} \end{pmatrix} \tag{7}$$

where the various operators correspond to the bilinear forms of our formulation as follows

$$\begin{cases} A \leftrightarrow (\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \mathcal{A}\underline{\boldsymbol{\tau}}) + (\mathbf{div}\underline{\boldsymbol{\sigma}}_h, \mathbf{div}\underline{\boldsymbol{\tau}}) \\ B \leftrightarrow -(\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\epsilon}}(\mathbf{v})) \\ C \leftrightarrow (\underline{\boldsymbol{\epsilon}}(\mathbf{u}_h), \underline{\boldsymbol{\epsilon}}(\mathbf{v})) \\ D \leftrightarrow -(\mathbf{u}_h, \mathbf{div}\underline{\boldsymbol{\tau}}) \end{cases} \tag{8}$$

As explained in Section 2.2, the operators B^T and D are different which leads to a non symmetric problem. The numerical approximation in (7) gives rise to an eigenvalue problem of the form

$$\mathbb{A}x = \omega \mathbb{B}x \tag{9}$$

The matrix $\mathbb{B}$ in our framework is clearly singular, however, $\mathbb{A}$ is not. For this reason, it may be appropriate to swap the roles of $\mathbb{A}, \mathbb{B}$ by considering the problem

$$\mathbb{B}x = \eta \mathbb{A}x \tag{10}$$

where $\eta = \frac{1}{\omega}$.

Since we are seeking for the pair (ω, x) in the original system (9), it is clear that if $\eta = 0$ in (10) then (η, x) corresponds to the pair (∞, x) where $\omega = \infty$.

Remark 1. *The discrete problem in* (9) *is a generalized eigenvalue problem; some specific properties of its solution are presented in [2, 3].*

3.1 Solving the system: two Schur complements

The discretization of linear elasticity problem studied in this paper gives rise to the system of block matrices considered in (7). This algebraic structure can be rewritten in the form:

$$\begin{cases} A\hat{\boldsymbol{\sigma}}_h + B^T\hat{\mathbf{u}}_h &= \omega_h D\hat{\mathbf{u}}_h \\ B\hat{\boldsymbol{\sigma}}_h + C\hat{\mathbf{u}}_h &= 0 \end{cases} \tag{11}$$

We are seeking for the discrete eigenvalues ω_h in the complex plane with a non-vanishing $\hat{\mathbf{u}}_h$. Moreover, as it has been explained at the end of the previous section, the solution of the algebraic eigenvalue problems associated with the variational formulation given in (6), will also include eigenvalues $\omega_h = \infty$. By Proposition 1 in [3], the multiplicity of such eigenvalues is equal to $dim(\Sigma_h) + dim(ker(D))$ where ker is the kernel of the matrix. In addition, the number of finite eigenvalues (counted with their multiplicity) is equal to the rank of the matrix D.

The Schur complement technique is one of the strategies to deal with mixed problems. At this stage we approach the problem by finding two Schur complements associated with the system in (11), one related to the variable $\hat{\boldsymbol{\sigma}}_h$ and the other to $\hat{\mathbf{u}}_h$.

1. The first Schur complement considered, denoted by $Schur_\sigma$ and the eigenpair by $(\omega_h^\sigma, \hat{\boldsymbol{\sigma}}_h)$, is obtained after a straightforward calculation when solving (11), and is given by

$$(A - B^T C^{-1} B)\hat{\boldsymbol{\sigma}}_h = -\omega_h^\sigma D C^{-1} B \hat{\boldsymbol{\sigma}}_h \tag{12}$$

After solving (12), we can recover the other component of the solution by

$$\hat{\mathbf{u}}_h = -C^{-1}B\hat{\boldsymbol{\sigma}}_h$$

Note that by this approach, we first seek for eigenvalues associated with the $\hat{\boldsymbol{\sigma}}_h$ variable and then look for the pair of our interest. In other words, we first find the eigenpair $(\omega_h^\sigma, \hat{\boldsymbol{\sigma}}_h)$ and then $(\omega_h^\sigma, \hat{\mathbf{u}}_h)$ for $\hat{\mathbf{u}}_h \neq 0$.

Remark 2. *This procedure might introduce some solutions which do not correspond to the ones we are interested in. Indeed, in common applications, the dimension of the space Σ_h is larger than the dimension of U_h so that more eigenvalues than the ones we are interested in may be computed by this Schur complement approach.*

2. The second Schur complement is associated with the variable $\hat{\mathbf{u}}_h$ and the pair is denoted by $(\omega_h^u, \hat{\mathbf{u}}_h)$. By using (11) and with A being invertible we have

$$\hat{\boldsymbol{\sigma}}_h = (\omega_h A^{-1}D - A^{-1}B^T)\hat{\mathbf{u}}_h$$

and, after some manipulation, we arrive at the following algebraic system

$$(BA^{-1}B^T - C)\hat{\mathbf{u}}_h = \omega_h^u BA^{-1}D\hat{\mathbf{u}}_h \tag{13}$$

Since this system involves only the variable $\hat{\mathbf{u}}_h$, we refer to it as $Schur_u$ complement approach.

When comparing both approaches and with Remark 2 in mind, we clearly see that the second approach guarantees the solution in a direct manner. Thus, in what follows we use $Schur_u$ to solve System (11) and find the pair $(\omega_h^u, \hat{\mathbf{u}}_h)$.

4 CONVERGENCE ANALYSIS OF EIGENVALUES

Looking at the convergence analysis, we will only discuss the uniform convergence of the eigenvalues of the problem considered. Due to the regularity properties of the solution of (1), the continuous eigenvalue problem in (3) is compact and therefore its eigenvalues form an increasing sequence

$$0 < \omega_1 \leq \omega_2 \leq \omega_3 \leq \cdots \leq \omega_i \leq \cdots \quad \text{with} \quad \lim_{i\to\infty} \omega_i = \infty \tag{14}$$

where the eigenvalues are real and positive and the eigenspaces are finite dimensional. As always, the same eigenvalue can be repeated several times according to its multiplicity so that we have one dimensional eigenspace for each eigenvalue (see [4, 5] for details).

Moving to the discrete problem (6), the discrete eigenvalues are complex and ordered according to their magnitude as follows

$$0 < |\omega_{1,h}^u| \leq |\omega_{2,h}^u| \leq |\omega_{3,h}^u| \leq \cdots \tag{15}$$

Thus, convergence of eigenvalues means that if we take a number of eigenvalues, counted with their multiplicity, within a circle of radius $R > 0$, for h small enough there is exactly the same number of discrete eigenvalues approximating them.

This follows from the general theory (see [4], Theorem 9.1) related to the approximation of eigenvalues of compact operators.

Theorem 4.1. *For all compact sets K in the complex plane, that is $K \subset \mathbb{C}$, that does not contain any eigenvalue of the continuous problem, there exits h_0 such that $\forall h < h_0$ no eigenvalue of the discrete problem belongs to K.*

This notion of convergence can be made more precise as follows.

$\forall R > 0$ let $\omega_1, \dots, \omega_N$ be the N eigenvalues, counted with their multiplicity, satisfying $|\omega_i| < R$. Then, $\forall \epsilon > 0, \exists h_0$ such that for $h < h_0$ exactly N discrete eigenvalues, counted with their multiplicity, satisfy $|\omega^u_{i,h}| < R$. Moreover, the N discrete eigenvalues can be sorted such that the following inequality is satisfied

$$|\omega_i - \omega^u_{i,h}| < \epsilon \qquad i = 1, \dots, N \tag{16}$$

We have performed a preliminary set of numerical experiments that confirm the theory. Some tests are reported in [3] and more detailed experiments will be presented in a forthcoming paper. The numerical evidence shows that discrete eigenvalues can be present everywhere in the complex plane, including negative values, eigenvalues with negative real part, and, as expected, infinite eigenvalues. However, in all cases it is possible to find a circle centered at zero such that inside the circle only relevant eigenvalues are present. Moreover, as h goes to zero, the radius of the circle can be taken larger and larger. This is in perfect agreement with the convergence theory that has been described above.

5 CONCLUSIONS

In this paper we investigate further the theory of least-squares linear elasticity problem developed in [3]. We succeed in finding two Schur complements associated with the problem (6), that are $Schur_\sigma$ and $Schur_u$. We then look into the convergence of the eigenvalues. Further study should account for the convergence order of eigenvalues and eigenvectors applied to the above theory presented. Moreover, numerical results for solving the system and the application of both Schur shall be analyzed. Another interesting proposition is to look into the mesh being considered in such problems. Since the problem itself is non symmetric, a valid question would be what are the eigenvalues and eigenvectors if a symmetric/non symmetric mesh is applied to such a problem.

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