Bayesian and Non-Bayesian Inference for Discrete Model Based on Censored Samples with Optimal Test Plan

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INFORMATION

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ABSTRACT

In reliability and lifetime studies, it is often impractical to observe the failure times of all units in a sample, particularly when the process is time-consuming or expensive. Type II censoring addresses this by terminating the experiment after a predetermined number of failures from a total sample of size. The Discrete Alpha Power Extended Inverted Weibull distribution is particularly suitable for modeling such censored discrete lifetime data. Its flexible shape parameters allow it to capture a wide range of failure behaviors, including over-dispersion, which is common in censored datasets. In this context, the likelihood function and estimation procedures (maximum likelihood and Bayesian) explicitly account for the censoring, ensuring unbiased parameter estimates and reliable predictive inferences. Consequently, the Discrete Alpha Power Extended Inverted Weibull distribution provides a practical and statistically robust framework for analyzing discrete lifetimes under type II censoring.

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1 Introduction

According to [1], modelling using continuous lifespan distributions is a prominent topic in a variety of domains, including economics, medicine, engineering science, and other closely connected sciences. The gamma and exponential distributions are substituted discretely by geometric and negative binomial distributions. These distributions, however, do not necessarily correlate to the exact facts as seen. The requirement for more reliable discrete distributions that can reproduce discrete data in a range of real-world circumstances has lately resulted in the discretization of various continuous lifespan distributions in literature.

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Discreteization, or the act of condensing probability distributions of uncertainty into a limited number of point masses, is a common task in decision analysis. Additional information is accessible in [2].

When a decision analysis model contains a high level of uncertainty, determining the values of numerous points in the distribution, let alone the true distribution, can be difficult or costly. Discretization replaces difficult and computationally expensive integrations with the evaluation of a small number of utilities in order to expedite computation and reduce the cost of computing some counterparts. Reference [3] found that discretization considerably improves a decision analyst's ability to communicate with clients. Discretization enables the investigation and evaluation of judgments that are incomprehensible to humans, even with increased processing capacity. The discretization process can be thought of as two separate processes. The first stage involves selecting the points that are relevant to discretization. These locations often indicate the original distribution's percentiles. The discretization method entails assigning a probability mass to each site in the second phase. Days can be used to symbolize the length of life for individuals with brain tumors or, in survival studies, the time it takes for remission to resume. Every second of the day, reliability engineers determine whether a system is functioning properly. Observed data shows the number of time units successfully completed prior to a breakdown. The count phenomenon can be observed in a variety of real-world circumstances, including accident frequency, ecological species types, insurance claims, and longevity data. In these cases, discrete distributions are quite beneficial for simulating lifespan data. Discretization techniques are commonly used due to their utility. The probability distribution can be made discrete by partitioning it into intervals based on values or cumulative distributions. Every time, the mean or median is assigned a percentile. Finally, each interval is allocated a probability based on its size. Choosing means and percentiles is another discretization strategy. Reference [4] provided information on the odds that correspond to the original distribution's moments.

There are many methods of optimization algorithms in engineering. One of the most popular algorithms is the Shuffled Frog Leaping Algorithm (SFLA). In 2006, Eusuff and Lansey created it. The advantages of particle swarm optimization and memetics are combined in the population-based metaheuristic algorithm known as SFLA. Because of its simplicity of usage and small number of variables, it has been applied in many fields, particularly engineering challenges. The algorithm has undergone numerous enhancements to mitigate its shortcomings, whether these were produced by alterations or hybridizations with other well-known algorithms, for more details see [5]. Also, one of the more recent metaheuristic swarm intelligence techniques is the Chimp Optimization Algorithm (ChOA). Because it has fewer parameters than other swarm intelligence techniques and doesn't require derivation knowledge for the first search, it has been widely adapted for a wide range of optimization tasks. Additionally, it is straightforward, user-friendly, adaptable, scalable, and possesses the unique capacity to achieve a favorable convergence by striking the correct balance between exploration and exploitation throughout the search. As a result, in a relatively short period of time, ChOA has attracted enormous audiences from a variety of areas and a great deal of research interest, see [6].

In reliability and lifetime studies, it is often impractical to observe the failure times of all units in a sample, particularly when the process is time-consuming or expensive. Type II censoring addresses this by terminating the experiment after a predetermined number of failures r from a total sample of size n. The Discrete Alpha Power Extended Inverted Weibull (DAPEIW) distribution is particularly suitable for modeling such censored discrete lifetime data. Its flexible shape parameters allow it to capture a wide range of failure behaviors, including over-dispersion, which is common in censored datasets. In this context, the likelihood function and estimation procedures (maximum likelihood and Bayesian) explicitly account for the censoring, ensuring unbiased parameter estimates and reliable predictive inferences. Consequently, the DAPEIW distribution provides a practical and statistically robust framework for analyzing discrete lifetimes under type II censoring, for more information on this subject, see [7].

More papers used the discretization strategy to introduce discrete distribution as follows: Discrete Poisson Quasi-XLindley distribution by [8]; discrete expansion of the Lindley distribution by [9]; bivariate exponentiated discrete Weibull distribution by [10]; overview of discrete distributions in modelling COVID-19 data by [11]; discrete binomial exponential II distribution by [12], etc.

There are various ways for discretizing continuous distributions (see [12] for further details). The most popular approach for calculating the discrete equivalent of a continuous distribution is shown below. To understand more about this subject, go to [13]. If the survival function is present in the underlying random variable $X(SF)$, $S_X(x) = P(X \geq x)$, then the probability mass function (pmf) of the random variable ($Y = [X]$, l argest integer less than or equal to X) exists.

$$p(Y = y) = p(y \leq X < y+1) = p(X \geq x) - p(X \geq y+1) = S\_X(y) - S\_X(y+1), y = 0, 1, 2, ... \quad (1)$$

The method described above generates a discrete distribution with the same functional form of the SF as its continuous counterpart. This feature has an impact on a variety of reliability criteria. This method is suitable for generating a discrete modification of the current continuous distributions. Acquiring the entire data set may be difficult since time and cost constraints can cause issues with data collection. Censored data is an example of incomplete data. The literature describes a range of censoring approaches for these data sets. For more information on censoring systems, their extensions, and analysis, see [14]. To derive relevant study conclusions and develop appropriate inferences, a precise analysis of randomly censored lifespan data is required. These data are widely used in domains such as biology, dependability research, and medical science. These data are frequently right-censored because it is impossible to observe patients or study subjects until they die, or because patients may withdraw from the study.

The DAPEIW model has several advantages over conventional one- or two-parameter distributions. Here are the primary objectives:

- Discrete distributions allow for numerous hazard rate forms, including decreasing and increasing. The proposed model's hazard rates make it suitable for modeling various data sets.
- This model can accurately model a variety of pmf forms, including positively skewed, negatively and skewed data that other models may not. The article introduces many statistical and reliability features, such as moments, probability functions, reliability indices, hazard functions, and order statistics.
- The DAPEIW distribution performed well in the real-world application, performing other discrete distribution models in literature.
- The proposed parameters are estimated using maximum likelihood and bootstrapping methods with type II censored data.
- The proposed parameters are estimated using a Bayesian method with type II censored data.
- The four methods of the optimal test plan and the sensitivity analysis of the proposed model.
- Monte Carlo simulations evaluate the performance of acquired estimators based on accurate metrics such as mean squared errors, mean absolute biases, and average interval length. Type II censoring is recommended for estimating unknown parameters, which appears reasonable.

The remainder of the essay is organized in this fashion. Section 2 explains the two-parameter DAPEIW. In Section 3, we look at several significant distributional and reliability properties. In

Section 4, we use maximum likelihood (ML) to estimate the DAPEIW distribution parameters with type II censoring. Section 5 additionally includes the appropriate test approach for the given model's censoring technique. Section 6 includes a sensitivity analysis that presents an ideal technique based on the asymptotic variances of maximum likelihood estimators. The Bayesian strategy for deriving non-classical estimators is discussed in Section 7. Section 8 discusses a simulation investigation, which supports the theoretical findings. While the concluding remarks are indicated in Section 10.

2 Discrete Alpha Power Extended Inverted Weibull

The continuous DAPEIW distribution's SF is supplied by

$$S(x) = \frac{\alpha}{\alpha - 1}\left(1 - \alpha^{e^{-x^{-\varphi}} - 1}\right), \alpha, \varphi > 0, x \geq 0. \tag{2}$$

Applying Eqs. (2) in (1) yields the probability mass function (pmf) of the DAPEIW distribution.

$$p(x) = \frac{1}{\alpha - 1}\left[\alpha^{e^{-(x+1)^{-\varphi}}} - \alpha^{e^{-x^{-\varphi}}}\right], x = 0, 1, 2, \ldots, \tag{3}$$

where $\alpha, \varphi > 0$, are the shape parameters. The accuracy of the proposed PMF, $\sum_{x=0}^{\infty} p(X = x) = 1$, is easily demonstrated.

$$F(x) = 1 - \frac{\alpha}{\alpha - 1}\left(1 - \alpha^{e^{-(x+1)^{-\varphi}} - 1}\right), \alpha, \varphi > 0, x = 0, 1, 2, \ldots. \tag{4}$$

3 The Distribution's Structural Characteristics of the DAPEIW

3.1 Shape of the pmf

VaIt is noted Fig. 1 that pmf has different types of shapes such as decreasing at $\alpha = 0.5$ and $\varphi = 0.95$, flat to slowly decreasing shape at $\alpha = 50$ and $\varphi = 1$, highly concentrated at $\alpha = 20$ and $\varphi = 50$, strong right skewed shape at $\alpha = 2$ and $\varphi = 10$, right skewed shape at $\alpha = 3$ and $\varphi = 2$ and very flat shape at $\alpha = 0.1$ and $\varphi = 0.2$.

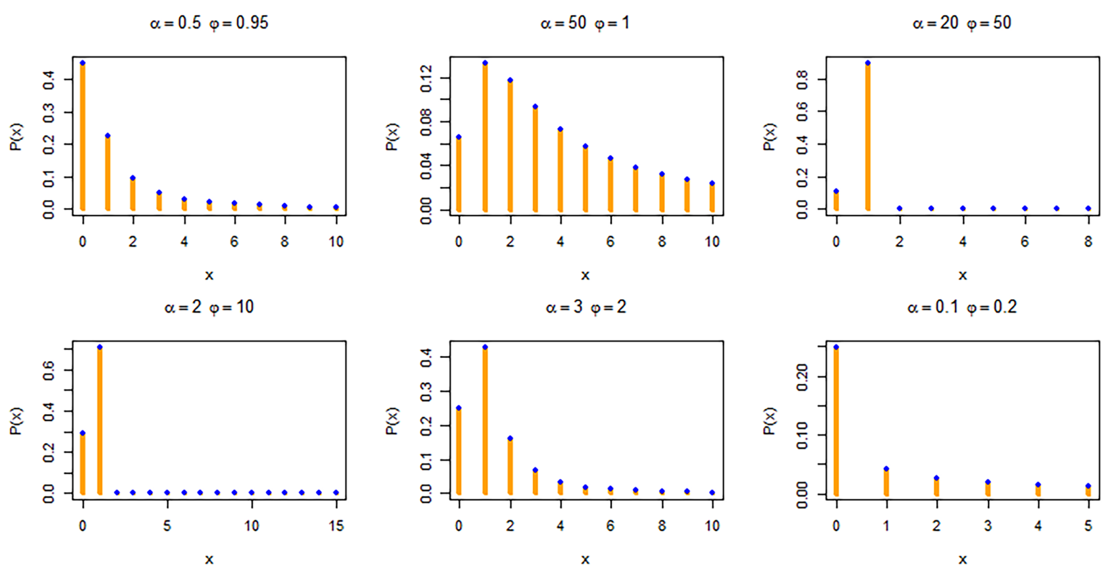

Figure 1: Pmf of the DAPEIW distribution with various

3.2 Quantiles, Random Number Generation, Skewness, and Kurtosis

If x_u is the u^{th} quantile of a discrete random variable, X, then $P(X \leq x_u) \geq u$ and $P(X \leq x_u) \geq 1-u$ are comparable; for further information, see [15]. The following follows from this outcome:

Theorem 1: *The u^{th}quantile $Qu(u)$ has the distribution DAPEIW (α, φ) given by*

$$Qu(u) = x_u = \left\{\left[-ln\left(\frac{ln(1-(1-u)(\alpha-1))}{ln\alpha}+1\right)\right]^{-1/\varphi} - 1\right\}, 0 < u < 1, \tag{5}$$

where x_u stands for the lowest integer that is greater than or equal to x_u.

Eq. (5) makes it easy to sample a uniform random number on $U(0,1)$, represented by u, from the proposed distribution. In particular, the median is given as $Qu(0.5) = x_{0.5} = \left\{\left[-ln\left(\frac{ln(1-(0.5)(\alpha-1))}{ln\alpha}+1\right)\right]^{-1/\varphi} - 1\right\}$.

The well-known quantile-based formulas for skewness and kurtosis were created by [16,17]. What sets these measures apart is their ability to be computed even for distributions lacking moments and their reduced susceptibility to outlier effects. Reference [17] provided the following representation of the skewness (Sk):

$$Sk = \frac{Qu(0.75) + Qu(0.25) - 2Qu(0.5)}{Qu(0.75) - Qu(0.25)},$$

The Moors kurtosis (MKu), as proposed by [17], can be expressed as

$$MKu = \frac{Qu(0.875) - Qu(0.625) + Qu(0.375) - Qu(0.125)}{Qu(0.75) - Qu(0.25)}.$$

The Sk and MKu for the DAPEIW distribution are easily calculated by applying Eq. (5) in the previously described formulas.

3.3 Moments

According to [18,19] A probability distribution's mean, variance, skewness, kurtosis, and other characteristics are determined using the distribution's moments. The following method can be used to find the r^{th} raw moments of the DAPEIW distribution.

$$\grave{M}_r = E(x^r) = \sum_{x=0}^{\infty}\left((x+1)^r - x^r\right)(1-F(x))$$
$$= \sum_{x=0}^{\infty}\left((x+1)^r - x^r\right)\left(1 - \frac{\alpha}{\alpha-1}\left(1-\alpha^{e^{-(x+1)^{-\varphi}}-1}\right)\right). \tag{6}$$

The first four raw moments of the DAPEIW distribution can be found using Eq. (6).

$$\grave{M}_1 = E(x) = \sum_{x=0}^{\infty}\left(1 - \frac{\alpha}{\alpha-1}\left(1-\alpha^{e^{-(x+1)^{-\varphi}}-1}\right)\right), \tag{7}$$

$$\grave{M}_2 = E\left(x^2\right) = \sum_{x=0}^{\infty}(2x+1)\left(1 - \frac{\alpha}{\alpha-1}\left(1-\alpha^{e^{-(x+1)^{-\varphi}}-1}\right)\right), \tag{8}$$

$$\grave{M}_3 = E\left(x^3\right) = \sum_{x=0}^{\infty}\left(3x^2+3x+1\right)\left(1 - \frac{\alpha}{\alpha-1}\left(1-\alpha^{e^{-(x+1)^{-\varphi}}-1}\right)\right), \tag{9}$$

$$\grave{M}_4 = E\left(x^4\right) = \sum_{x=0}^{\infty} \left(4x^3 + 6x^2 + 4x + 1\right) \left(1 - \frac{\alpha}{\alpha - 1} \left(1 - \alpha^{e^{-(x+1)^{-\varphi}} - 1}\right)\right). \tag{10}$$

The variance of the DAPEIW distribution is

$$Var\left(x\right) = \sum_{x=0}^{\infty} \left(2x + 1\right) \left(1 - \frac{\alpha}{\alpha - 1} \left(1 - \alpha^{e^{-(x+1)^{-\varphi}} - 1}\right)\right) - \left[\sum_{x=0}^{\infty} \left(1 - \frac{\alpha}{\alpha - 1} \left(1 - \alpha^{e^{-(x+1)^{-\varphi}} - 1}\right)\right)\right]^2.$$

We can rapidly find the skewness and kurtosis by using the raw moments from the following relations (Eqs. (7) and (10)):

$$sk = \frac{\grave{M}_3 - 3\grave{M}_2\grave{M}_1 + 2\grave{M}_1^3}{\left[Var\left(x\right)\right]^{3/2}} \text{ and } Mku = \frac{\grave{M}_4 - 4\grave{M}_2\grave{M}_1 + 6\grave{M}_2\grave{M}_1^2 - 3\grave{M}_1^4}{\left[Var\left(x\right)\right]^2}.$$

A method for figuring out if data is uniformly, unevenly, or too spread is the index of dispersion, or ID. $ID > 1$ indicates over-dispersion, $ID < 1$ indicates under-dispersion, and $ID = 1$ indicates equi-dispersion. The proposed model's ID is as follows:

$$ID = \frac{Var\left(x\right)}{E(x)} = \frac{\sum_{x=0}^{\infty} \left(2x + 1\right) \left(1 - \frac{\alpha}{\alpha-1} \left(1 - \alpha^{e^{-(x+1)^{-\varphi}} - 1}\right)\right) - \left[\sum_{x=0}^{\infty} \left(1 - \frac{\alpha}{\alpha-1} \left(1 - \alpha^{e^{-(x+1)^{-\varphi}} - 1}\right)\right)\right]^2}{\sum_{x=0}^{\infty} \left(1 - \frac{\alpha}{\alpha-1} \left(1 - \alpha^{e^{-(x+1)^{-\varphi}} - 1}\right)\right)}.$$

When comparing the variability of two independent samples, the coefficient of variation (CV), a relative measure of variability, is frequently utilized. Greater variability is indicated by a high CV value. One may compute the CV for the DAPEIW distribution using

$$CV = \frac{Var\left(x\right)^{0.5}}{E(x)}$$
$$= \frac{\left[\sum_{x=0}^{\infty} \left(2x + 1\right) \left(1 - \frac{\alpha}{\alpha-1} \left(1 - \alpha^{e^{-(x+1)^{-\varphi}} - 1}\right)\right) - \left[\sum_{x=0}^{\infty} \left(1 - \frac{\alpha}{\alpha-1} \left(1 - \alpha^{e^{-(x+1)^{-\varphi}} - 1}\right)\right)\right]^2\right]^{0.5}}{\sum_{x=0}^{\infty} \left(1 - \frac{\alpha}{\alpha-1} \left(1 - \alpha^{e^{-(x+1)^{-\varphi}} - 1}\right)\right)}.$$

Since the previously provided equations do not have a closed form, we use the R software to statistically illustrate these properties. The DAPEIW distribution's mean, variance, skewness, kurtosis, ID, and CV for various parameter choices are shown numerically in Table 1.

Table 1: Data summarized of DAPEIW distribution

α	φ	min	Q1	Median	Mean	SD	Q3	Max	SK	KT
	0.4	0	0	1	5149	6,6076	8	932,860	13.94467	196.2423
	0.7	0	0	1	22	187	3	2578	12.96567	176.3107
	1	0	1	1	4	19	2	244	11.04864	137.744
	1.3	0	1	1	2	6	2	69	8.980285	99.50716
0.4	1.6	0	1	1	2	3	2	31	7.266351	70.53949
	1.9	0	1	1	2	2	2	18	6.050247	52.41209
	2.2	0	1	1	1	1	1	12	5.169665	40.48558

(Continued)

Table 1 (continued)

α	φ	min	Q1	Median	Mean	SD	Q3	Max	SK	KT
	2.5	0	1	1	1	1	1	9	4.672272	33.14254
	2.8	1	1	1	1	1	1	7	4.183042	26.51089
	0.4	0	0	2	15403	197,875	20	2,793,632	13.94566	196.2616
	0.7	0	1	2	41	350	5	4825	12.97321	176.4585
	1	0	1	1	6	29	3	379	11.06404	137.9992
	1.3	0	1	1	3	8	2	96	8.979558	99.17135
0.9	1.6	0	1	1	2	4	2	41	7.295549	70.78636
	1.9	0	1	1	2	2	2	23	6.029836	52.03819
	2.2	0	1	1	2	1	2	15	5.119046	40.05089
	2.5	0	1	1	1	1	2	11	4.525434	32.26369
	2.8	1	1	1	1	1	2	8	3.745224	22.28402
	0.4	0	1	4	28,680	368,542	35	5,203,159	13.94592	196.2667
	0.7	0	1	2	59	499	7	6884	12.97522	176.4992
	1	0	1	2	8	37	4	486	11.07319	138.1682
	1.3	0	1	1	4	10	3	117	8.997849	99.44464
1.5	1.6	0	1	1	2	4	2	48	7.235074	70.21758
	1.9	0	1	1	2	2	2	26	5.975518	51.1959
	2.2	0	1	1	2	2	2	17	5.006059	38.67084
	2.5	0	1	1	2	1	2	12	4.330437	30.30449
	2.8	1	1	1	1	1	2	9	3.67214	22.72008
	0.4	0	1	7	61,236	787,006	70	11,111,152	13.94606	196.2695
	0.7	0	1	3	90	770	11	10,621	12.97611	176.5167
	1	0	1	2	11	50	5	658	11.06653	138.0055
	1.3	0	1	2	5	12	4	147	8.974529	99.05028
3	1.6	0	1	2	3	5	3	58	7.223306	69.93326
	1.9	0	1	1.5	2	3	2	30	5.733662	47.8913
	2.2	0	1	1	2	2	2	19	4.833843	36.20182
	2.5	1	1	1	2	1	2	13	4.043865	26.72298
	2.8	1	1	1	2	1	2	10	3.528242	21.35026

3.4 The Survival and Hazard Rate Functions

The SF and hrf of the DAPEIW distribution are supplied by

$$S(x) = p(X \geq x) = \frac{\alpha}{\alpha - 1}\left(1 - \alpha^{e^{-x^{-\varphi}} - 1}\right), x = 0, 1, 2, \ldots,$$

$$h(x) = p(X = x | X \geq x) = \frac{\frac{1}{\alpha - 1}\left[\alpha^{e^{-(x+1)^{-\varphi}}} - \alpha^{e^{-x^{-\varphi}}}\right]}{\frac{\alpha}{\alpha - 1}\left(1 - \alpha^{e^{-x^{-\varphi}} - 1}\right)}, x = 0, 1, 2, \ldots,$$

It is clear in Fig. 2 that hrf has different types of shapes such as decreasing at $\alpha = 0.5$ and $\varphi = 0.95$, gradually decreasing shape at $\alpha = 50$ and $\varphi = 1$, extremely peaked at $\alpha = 20$ and $\varphi = 50$, very steep peak at $\alpha = 2$ and $\varphi = 10$, right skewed shape at $\alpha = 3$ and $\varphi = 2$ and small peak shape at $\alpha = 0.1$ and $\varphi = 0.2$.

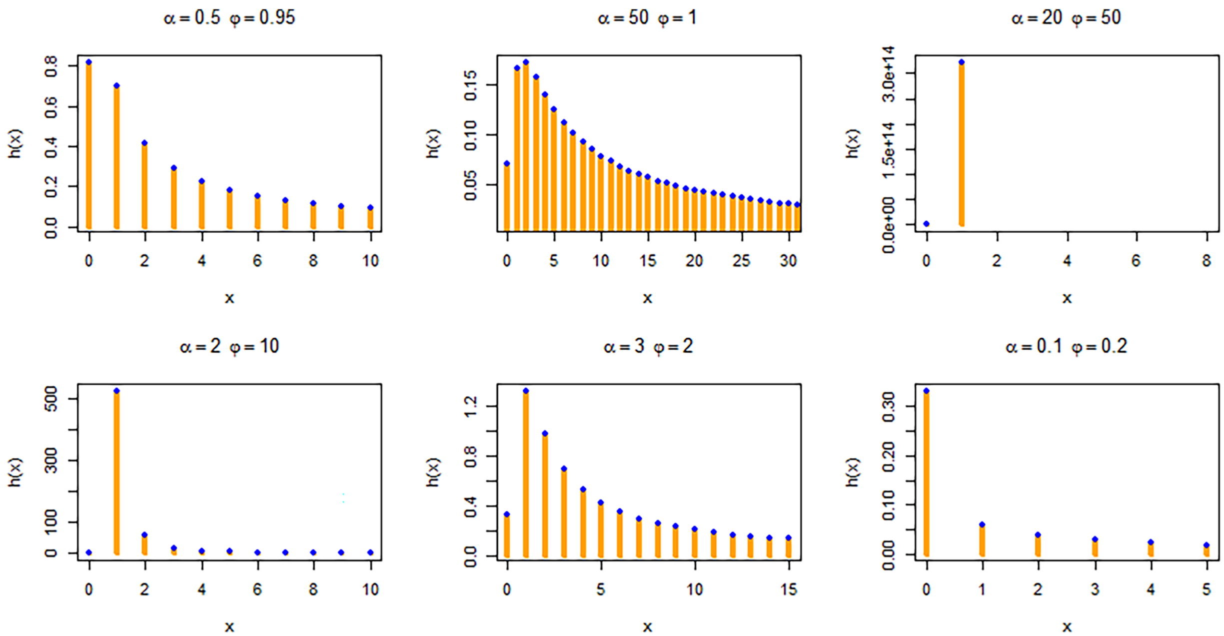

Figure 2: Hrf of DAPEIW distribution with different values

The proposed distribution is more flexible to evaluate a wide range of data than both traditional and recently published models because of the different forms of hrf. The second rate of failure (srf) and reversed hazard rate function (rhrf) of the proposed model are also analyzed.

$$rh\left(x\right) = p\left(X = x | X \leq x\right) = \frac{\frac{1}{\alpha - 1}\left[\alpha^{e^{-(x+1)^{-\varphi}}} - \alpha^{e^{-x^{-\varphi}}}\right]}{1 - \frac{\alpha}{\alpha - 1}\left(1 - \alpha^{e^{-x^{-\varphi}}-1}\right)}; x = 0, 1, 2, \ldots,$$

$$srf\left(x\right) = log\left[\frac{S\left(x\right)}{S\left(x+1\right)}\right] = log\left[\frac{\frac{\alpha}{\alpha - 1}\left(1 - \alpha^{e^{-x^{-\varphi}}-1}\right)}{\frac{\alpha}{\alpha - 1}\left(1 - \alpha^{e^{-(x+1)^{-\varphi}}-1}\right)}\right]; x = 0, 1, 2, \ldots.$$

Table 1 reports summary statistics for the DAPEIW across selected parameter values. Figs. 3 and 4 show that the central location (mean/median) is relatively stable, while dispersion and tail metrics (SD, CV, skewness, kurtosis, ID) change markedly with the parameters: smaller parameter values produce larger variance and pronounced right-skewed heavy tails, whereas larger values yield more concentrated, less skewed behavior.

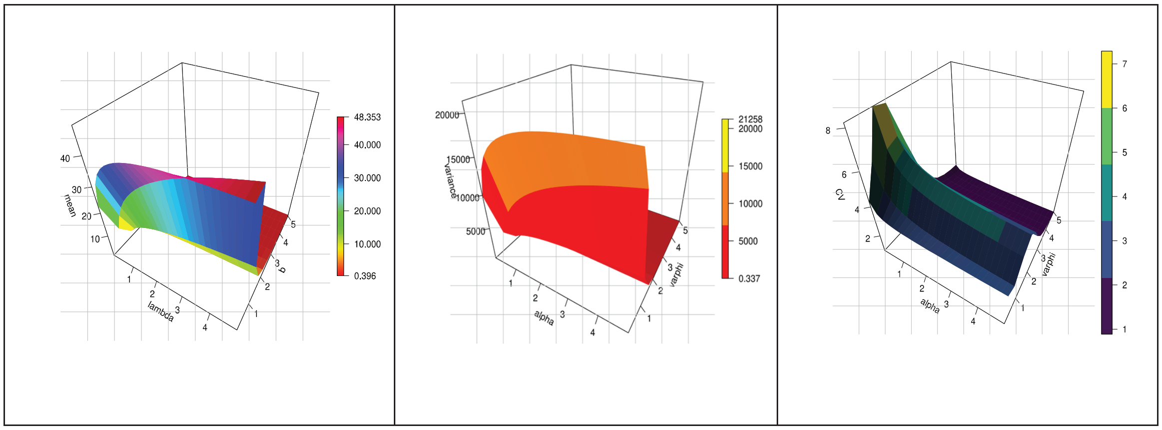

Figure 3: Data summarized (Mean, variance, and CV) of DAPEIW

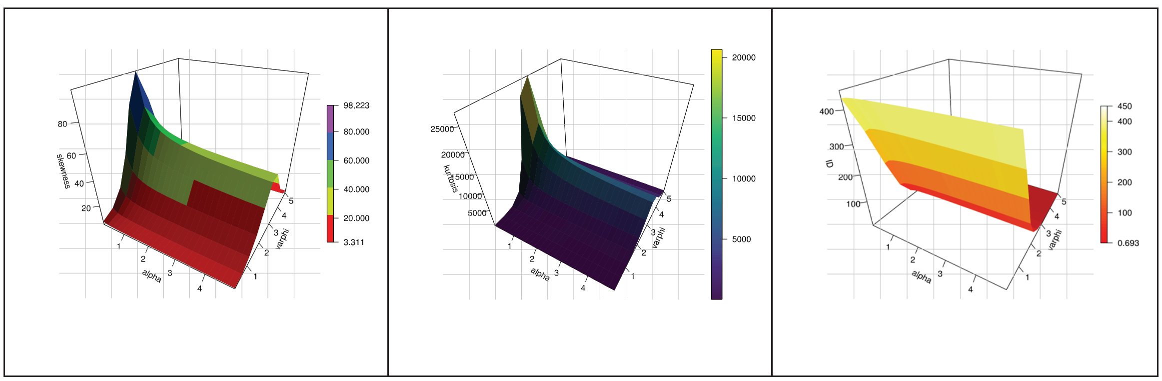

Figure 4: Data summarized (SK, KT, ID) of DAPEIW

3.5 Mean Past Life Function

The anticipated inactivity time function, also known as the mean past life function (MPL), is represented by the symbol $M^*(t)$. If the system failed before t, this function calculates the amount of time that has passed since X failed. Among its uses are forensic science, survival analysis, actuarial research, and dependability theory. In a discrete context, the MPL function is defined as

$$M^*(t) = E(t - X|X < t) = \frac{1}{F(t-1)} \sum_{j=1}^{t} F(j-1)\,; t = 1, 2, 3, \ldots$$

It just takes to obtain the MPL for the proposed model to change the cdf in Eq. (4) to $M^*(t)$ in the formula.

3.6 Parameter of Stress-Strength

The stress-strength (SS) parameter is defined as the probability $R = P[X > Z]$ and is used as a measure of component reliability. The SS model describes the life of a component with random strengths X and random stresses Z. A component fails quickly if the applied stress exceeds its strength; else, it functions as intended. The SS system paradigm is useful in many domains, such as psychology, engineering, and medical science. The analysis of SS models provided by [20] is extensive. The SS reliability for discrete independent random variables, X and Z, is defined as

$$R = p\left(X > Z\right) = \sum\nolimits_{x=0}^{\infty} p_X\left(x\right) F_Z\left(x\right).$$

When X and Z, the independent discrete random variables, are represented as $p_X\left(x\right)$ and $F_Z\left(x\right)$, respectively, by the pmf and cdf. It is necessary to consider the real values $X \sim DAPEIW\left(\alpha_1, \varphi_1\right)$ and $X \sim DAPEIW\left(\alpha_2, \varphi_2\right)$. Next, we apply Eqs. (3) and (4) as follows:

$$R = \sum\nolimits_{x=0}^{\infty} \frac{1}{\left(\alpha_1 - 1\right)} \left[\alpha_1^{e^{-(x+1)^{-\varphi_1}}} - \alpha_1^{e^{-(x)^{-\varphi_1}}}\right]. \tag{11}$$

We used the R, "base" programme to measure this feature because it was challenging to specify the SS dependability, R, in this case. Refer to Fig. 5 for additional details.

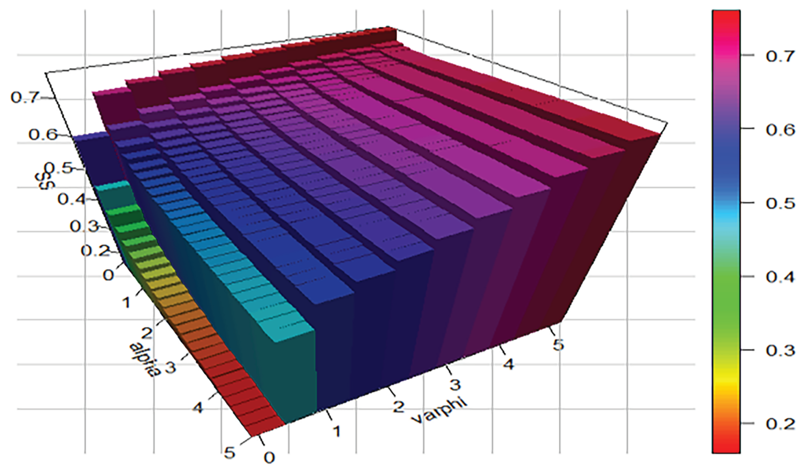

Figure 5: Stress-strength reliability of DAPEIW

3.7 Entropy

According to information theory, a random variable's entropy is the average amount of "information," "surprise," or "uncertainty" included in the various outcomes. According to [21], the Renyi entropy (RE) is an important entropy. In many domains, including physics, econometrics, statistical inference, pattern recognition in computer science, and econometrics, it is a crucial indicator of complexity and uncertainty. The RE for DAPEIW distribution can be expressed as ($\delta > 0, \delta \neq 1$).

$$RE\left(\rho\right) = \frac{1}{1-\delta} \log \sum\nolimits_{x=0}^{\infty} p_x^{\delta}\left(x\right),$$

$$= \frac{\delta}{1-\delta} \log \sum_{x=0}^{\infty} \left\{\frac{1}{\alpha - 1}\left[\alpha^{e^{-(x+1)^{-\varphi}}} - \alpha^{e^{-x^{-\varphi}}}\right]\right\}.$$

As a special example of RE, the well-known Shannon entropy (SE) can be calculated as $\delta \rightarrow 1$, where $SE = -E\left[\log p\left(x\right)\right]$.

3.8 Order Statistics

Particularly in survival analysis, order statistics are crucial for evaluating population traits and determining tolerance intervals for distributions. $X_1, X_2, \ldots, X_n$ are random sample (RS) taken from the DAPEIW (α, φ) is assumed. Assign the corresponding order statistics' representation to $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. Let us thus argue that the cdf of the r^{th} order statistic is given by $\vartheta = X_{(r)}$.

$$F_r(\vartheta) = \sum\nolimits_{i=r}^{n} \binom{n}{i} F^i(\vartheta)\left[1 - F(\vartheta)\right]^{n-i} = \sum\nolimits_{i=r}^{n} \sum\nolimits_{j=0}^{n-i} (-1)^j \binom{n}{i}\binom{n-i}{j} F^{i+k}(\vartheta)$$

$$= \sum\nolimits_{i=r}^{n} \sum\nolimits_{j=0}^{n-i} (-1)^j \binom{n}{i}\binom{n-i}{j} \left[\frac{\alpha}{\alpha-1}\left(1 - \alpha^{e^{-(x+1)^{-\varphi}}-1}\right)\right]^{i+j}. \tag{12}$$

The PMf that corresponds to the r^{th} order statistics is

$$F_r(\vartheta) = F_r(\vartheta) - F_r(\vartheta - 1) = \sum\nolimits_{i=r}^{n} \sum\nolimits_{j=0}^{n-i} (-1)^j \binom{n}{i}\binom{n-i}{j} \left[\frac{\alpha}{\alpha-1}\left(1 - \alpha^{e^{-(x+1)^{-\varphi}}-1}\right)\right]^{i+j}$$

$$-\sum\nolimits_{i=r}^{n} \sum\nolimits_{j=0}^{n-i} (-1)^k \binom{n}{i}\binom{n-i}{j} \left[\frac{\alpha}{\alpha-1}\left(1 - \alpha^{e^{-(x+1)^{-\varphi}}-1}\right)\right]^{i+j}. \tag{13}$$

By setting $r = 1$ and $r = n$ in Eq. (13), $\left(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\right)$ and max $\left(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\right)$, respectively, have pmfs that we can acquire.

4 Maximum Likelihood Method

The maximum likelihood approach is one of the most often used traditional point estimate methods. The location in the parametric space where the likelihood function is maximized is known as the maximum likelihood estimate. Its logic is so adaptable and simple that statistical inference now follows this standard process. This section provides the maximum likelihood estimators (MLEs) of the model parameters. The cdf in Eq. (4) and the related pmf in Eq. (3) are used to determine the probability function of the two parameters based on the type-II censored sample.

$$L(\alpha, \varphi) \propto \prod\nolimits_{i=1}^{r} p(x_i)\left[S(x_r)\right]^{n-r},$$

$$L(\alpha, \varphi) \propto \prod\nolimits_{i=1}^{r} \frac{1}{\alpha-1}\left[\alpha^{e^{-(x_i+1)^{-\varphi}}} - \alpha^{e^{-x_i^{-\varphi}}}\right]\left[\frac{\alpha}{\alpha-1}\left(1 - \alpha^{e^{-x_r^{-\varphi}}-1}\right)\right]^{n-r}. \tag{14}$$

The logarithm of α, and φ, or ℓ of the probability function, is as follows:

$$\ell \propto ln\prod\nolimits_{i=1}^{r} \frac{1}{\alpha-1}\left[\alpha^{e^{-(x_i+1)^{-\varphi}}} - \alpha^{e^{-x_i^{-\varphi}}}\right] + (n-r)\ln\left[\frac{\alpha}{\alpha-1}\left(1 - \alpha^{e^{-x_r^{-\varphi}}-1}\right)\right],$$

$$\ell \propto -rln(\alpha-1) + \sum\nolimits_{i=1}^{r} ln\left[\alpha^{e^{-(x_i+1)^{-\varphi}}} - \alpha^{e^{-x_i^{-\varphi}}}\right] + (n-r)\ln\left[\frac{\alpha}{\alpha-1}\left(1 - \alpha^{e^{-x_r^{-\varphi}}-1}\right)\right]. \tag{15}$$

Regarding the unknown parameters, we have the following derivatives of ℓ:

$$\frac{\partial \ell}{\partial \alpha} = -\frac{r}{(\alpha-1)} + \sum\nolimits_{i=1}^{r}\left\{\frac{e^{-(x_i+1)^{-\varphi}}\alpha^{e^{-(x_i+1)^{-\varphi}}-1} - e^{-x_i^{-\varphi}}\alpha^{e^{-x_i^{-\varphi}}-1}}{\alpha^{e^{-(x_i+1)^{-\varphi}}} - \alpha^{e^{-x_i^{-\varphi}}}}\right\} + \frac{(n-r)}{\alpha}$$

$$-\frac{(n-r)}{\alpha-1} - (n-r)\frac{\alpha^{e^{-x_r^{-\varphi}}-2}\left(e^{-x_r^{-\varphi}}-1\right)}{\left(1 - \alpha^{e^{-x_r^{-\varphi}}-1}\right)}, \tag{16}$$

$$\frac{\partial \ell}{\partial \varphi} = \sum_{i=1}^{r} \left\{ \frac{\alpha^{e^{-(x_i+1)^{-\varphi}}} e^{-(x_i+1)^{-\varphi}} (x_i+1)^{-\varphi} \log(x_i+1) - \alpha^{e^{-x_i^{-\varphi}}} e^{-(x_i)^{-\varphi}} (x_i)^{-\varphi} \, log\,(x_i)}{\alpha^{e^{-(x_i+1)^{-\varphi}}} - \alpha^{e^{-x_i^{-\varphi}}}} \right\}$$
$$+ \frac{\alpha^{e^{-x_r^{-\varphi}}} (n-r)(\alpha-1) e^{-(x_r)^{-\varphi}} (x_r)^{-\varphi} \log(x_r)}{\alpha \left(1 - \alpha^{e^{-x_r^{-\varphi}} - 1}\right)}. \tag{17}$$

We currently possess a set of nonlinear equations where the parameters α and φ are unknown. It is evident that finding the closed-form solution is challenging. As a result, an iterative technique like Newton-Raphson may be able to find a numerical solution for the nonlinear system mentioned above.

Bootstrap Confidence Interval

The bootstrap methodology is a generic resampling technique for estimating statistical distributions based on independent observations. It was first established by [22]. As an alternative to asymptotic methods, the bootstrap methodology is becoming increasingly popular due to its shown effectiveness in a range of settings. Here, we use the following methods to generate the percentile bootstrap and bootstrap-t confidence intervals for α and φ:

1) Percentile Bootstrap Confidence Interval (B-P)

i. Determine $\pi = (\alpha, \varphi)$'s MLE.

ii. To get the bootstrap estimate of α say $\hat{\alpha}^b$, and φ say $\hat{\varphi}^b$, a bootstrap sample was generated using π.

iii. To obtain $\left(\hat{\alpha}^{b(1)}, \hat{\alpha}^{b(2)}, \ldots, \hat{\alpha}^{b(B)}\right)$ and $\left(\hat{\varphi}^{b(1)}, \hat{\varphi}^{b(2)}, \ldots, \hat{\varphi}^{b(B)}\right)$, repeat step (ii) B times.

iv. As $\left(\hat{\alpha}^{b(1)}, \hat{\alpha}^{b(2)}, \ldots, \hat{\alpha}^{b(B)}\right)$ and $\left(\hat{\varphi}^{b(1)}, \hat{\varphi}^{b(2)}, \ldots, \hat{\varphi}^{b(B)}\right)$ in ascending order as $\left(\hat{\alpha}^{b(1)}, \hat{\alpha}^{b(2)}, \ldots, \hat{\alpha}^{b(B)}\right)$ and $\left(\hat{\varphi}^{b(1)}, \hat{\varphi}^{b(2)}, \ldots, \hat{\varphi}^{b(B)}\right)$, respectively.

v. For the unknown parameters α and φ, the two side percentile bootstrap confidence interval is as follows: $\left\{\hat{\alpha}^{b[B(\varnothing/2)]}, \hat{\alpha}^{b[B(1-\varnothing/2)]}\right\}$ and $\left\{\hat{\varphi}^{b[B(\varnothing/2)]}, \hat{\varphi}^{b[B(1-\varnothing/2)]}\right\}$.

2) Bootstrap-t Confidence Interval (B-T)

i. like Boot-p's steps (i–ii)

ii. After obtaining the Fisher information matrix, compute the t-statistic of $\pi = (\alpha, \varphi)$ as $T = \left(\hat{\pi}^b - \hat{\pi}\right) / \sqrt{V\left(\hat{\pi}^b\right)}$ where $V\left(\hat{\pi}^b\right)$ is the asymptotic variances of $\hat{\pi}^b$.

iii. To obtain $T^{(1)}, T^{(2)}, \ldots, T^{(B)}$, repeat step ii B several times.

iv. $T^{(1)}, T^{(2)}, \ldots, T^{(B)}$ should be arranged as $T^{[1]}, T^{[2]}, \ldots, T^{[B]}$ in ascending order. $100\,(1-\varnothing)\,\%$.

v. For the unknown parameters α and φ, the two-side $100\,(1-\varnothing)\,\%$ percentile bootstrap-t confidence interval is $\left\{\hat{\alpha} + T^{[B(\varnothing/2)]}\sqrt{V\left(\hat{\alpha}\right)}, \hat{\alpha} + T^{[B(1-\varnothing/2)]}\sqrt{V\left(\hat{\alpha}\right)}\right\}$ and $\left\{\hat{\varphi} + T^{[B(\varnothing/2)]}\sqrt{V\left(\hat{\varphi}\right)}, \hat{\varphi} + T^{[B(1-\varnothing/2)]}\sqrt{V\left(\hat{\varphi}\right)}\right\}$.

5 Optimal Test Plan

Finding the best censoring strategy has received a lot of attention lately in statistical literature; for further information, read [23]. Finding the type II censoring scheme that offers the most information about the unknown parameters among all the schemes that are available is the first step towards choosing the optimal sampling method for a given n and r. The creation of unknown parameter information measures based on type II censored data and the comparison of two different information measures based on two different type II censoring procedures are the first and second issues, respectively; for further details, refer to [24]. Some of the optimality criteria applied in this case are discussed in the next section of the study. Refer to [25] for more information. Our goal is to choose the filtering approach that yields the most insights into the unknown parameters. In this case, Table 2 offers a variety of commonly used metrics to help us choose the best censoring strategy, Op_i Technical Requirements.

Table 2: Demonstrates several useful best practices for censoring plans

Criteria	Techniques
Op_1	Maximize trace $[\boldsymbol{I}_{\mathbf{2}\times\mathbf{2}}(.)]$
Op_2	Minimize trace $[\boldsymbol{I}_{\mathbf{2}\times\mathbf{2}}(.)]^{-1}$
Op_3	Minimize det $[\boldsymbol{I}_{\mathbf{2}\times\mathbf{2}}(.)]^{-1}$
Op_4	Minimize Var $\left[\mathbf{log}\left(\hat{t}_u\right)\right], \mathbf{0} < \boldsymbol{u} < \mathbf{1}$

For Op_1, it is optimal to maximize the observed Fisher information, $\boldsymbol{I}_{\mathbf{2}\times\mathbf{2}}(.)$ values. Moreover, we like to decrease the determinant and trace of $[\boldsymbol{I}_{\mathbf{2}\times\mathbf{2}}(.)]^{-1}$ for criterion Op_2 and Op_3. Scale-invariant criteria may be used to determine the best censoring strategy for distributions with multiple parameters. Scale invariant criteria can be used to compare numerous criteria when working with single-parameter distributions, however comparing the two Fisher information matrices becomes more difficult when dealing with unknown multi-parameter distributions Op_4. The p-dependent criteria tend to reduce the variance of the logarithmic MLE of the p-th quantile, $\mathbf{log}\left(\hat{t}_u\right)$, Op_4 Consequently, at time $\hat{t}_u$, the logarithmic of the DAPEIW distribution is determined by

$$\log\left(\hat{t}_u\right) = \left\{\left[-ln\left(\frac{ln\left(1-(1-u)(\alpha-1)\right)}{ln\alpha}+1\right)\right]^{-1/\varphi} - 1\right\}, 0 < u < 1. \tag{18}$$

Using the delta technique to Eq. (4), one may determine the approximation variance for $\log\left(\hat{t}_p\right)$ of the DAPEIW distribution.

$$Var\left(\log\left(\hat{t}_u\right)\right) = \left[\nabla log\left(\hat{t}_u\right)\right]^T I_{2\times 2}^{-1}\left(\hat{\alpha}, \hat{\varphi}\right)\left[\nabla \log\left(\hat{t}_u\right)\right],$$

where

$$\left[\nabla \log\left(\hat{t}_u\right)\right]^T = \left[\frac{\partial}{\partial u}\log\left(\hat{t}_u\right), \frac{\partial}{\partial \varphi}\log\left(\hat{t}_\varphi\right)\right]_{\left(\alpha=\hat{\alpha}, \varphi=\hat{\varphi}\right)}.$$

The optimal censoring for $i = 2, 3$, and 4 is represented by the maximum value of criterion Op_{i} and the lowest value of criterion Op_1. On the other hand, the optimal censoring is represented by the largest value of the criterion Op_1 and the lowest value of the criteria Op_{i}, where $i = 2, 3, 4$.

6 The Sensitivity Analysis

The best time to change stress is shown in this section. It is based on the asymptotic variances of maximum likelihood estimators. The inverse Fisher information matrix's diagonals are used to compute these variances. Rather than depending on the sum of parameter variances which reference [26] suggested and used, we use the sum of coefficients of variations (SCV) as the best criterion. The technique presented by [26] minimizes the expected value of SCV to create an optimal plan, because the sum of variances can change based on the scale of parameter values.

Thus, $E(\phi(\mathrm{x}))$ is maximized, where

$$\phi(\mathrm{x}) = \frac{\sqrt{F_{11}^{-1}}}{\hat{\alpha}} + \frac{\sqrt{F_{22}^{-1}}}{\hat{\varphi}}.$$

On the other hand, it might not be possible to compute exact closed forms for the posterior variances of parameters. Reference [26] advised computing using the following Gibbs sampling method:

First, create samples U with the specified n and parameter values.

Step 2: Create the selected sample using the specified r size.

Step 3: Calculated parameters using Type-II cenosred samples ().

Step 4: Determine the target function $\phi(\mathrm{x})$.

Step 5: N times through Steps 1 through 4, you will obtain $\phi^1(\mathrm{x}), \phi^2(\mathrm{x}), \ldots, \phi^N(\mathrm{x})$.

Step 6: Determine the objective functions' median and write it as $\phi^m(\mathrm{x})$.

7 Bayesian Approach

Our primary interest in this case is the Bayes estimate for the unknown parameters. Prior to initiating the Bayesian analysis, the unknown parameters must be specified. For the purposes of this analysis, it is assumed that the parameters φ and α have independent gamma priors and are statistically independent. The form of the linked joint prior is as follows.

$$\pi_1(\alpha, \varphi) \propto \alpha^{\omega_1 - 1} \varphi^{\omega_2 - 1} e^{-\omega_3 \alpha - \omega_4 \varphi}, \tag{19}$$

where the pre-established hyperparameters, $j = 1, 2, 3,$ and 4 represent the prior knowledge of the unknown parameters, and $\omega_{\mathrm{j}} > 0$. The resulting unknown parameters' cumulative posterior distribution will be:

$$\pi\left(\alpha, \varphi | \underline{t}\right) \propto \pi_1(\alpha, \varphi) \prod\nolimits_{i=1}^{2} \prod\nolimits_{j=1}^{n_i} f\left(t_{ij}\right)\left(1 - F\left(t_{ij}\right)\right)^{d_i}. \tag{20}$$

It should be emphasized that inferring posteriors of the model parameter is difficult due to the structure of the joint posterior in Eq. (20). We use Markov chain Monte Carlo (MCMC) techniques to provide posterior samples of the unknown parameters required for posterior inference. The collected samples will be utilized to construct the associated credible ranges for the highest posterior density (HPD) and approximate Bayes estimates for the unknown parameters; see [27] for more information. From the current analysis, the following Bayes estimates of the unknown parameters under the symmetric squared loss function (SLF) were obtained:

$$\ell(\alpha, \tilde{\alpha}) = (\tilde{\alpha} - \alpha)^2, \ell(\varphi, \tilde{\varphi}) = (\tilde{\varphi} - \varphi)^2, \tag{21}$$

where the variables $\tilde{\alpha}$, and $\tilde{\varphi}$ display the predicted posterior means of α, and φ.

Because the joint posterior for α, and φ have no closed-form expression, it cannot be sampled directly by standard methods. Therefore, we implement a Metropolis–Hastings (M–H) sampler using a Gaussian random-walk proposal to draw samples from the posterior density. The algorithm is as follows:

Start by initializing an initial estimate, denoted as $\Theta^{(0)} \equiv (\hat{\alpha}, \hat{\varphi})$.

- Initialize the iteration counter j←1.
- Draw a proposal Θ^* from the Normal proposal distribution $q(\Theta) = N(\hat{\Theta}, var(\hat{\Theta}))$.
- Evaluate the acceptance probability for the proposed Θ^*, as

$$A\left(\alpha^{j-1}, \alpha^*\right) = \min\left[1, \frac{\pi\left(\alpha^*, \varphi^{j-1} \,|data\right) q\left(\alpha^*\right)}{\pi\left(\alpha^{j-1}, \varphi^{j-1} \,|data\right) q\left(\alpha^{j-1}\right)}\right],$$

$$A\left(\varphi^{j-1}, \varphi^*\right) = \min\left[1, \frac{\pi\left(\varphi^*, \alpha^{j-1} \,|data\right) q\left(\varphi^*\right)}{\pi\left(\varphi^{j-1}, \alpha^{j-1} \,|data\right) q\left(\varphi^{j-1}\right)}\right],$$

and repeat this probability for all parameters to obtain $A\left(\alpha^{j-1}, \alpha^*\right)$, and $A\left(\varphi^{j-1}, \varphi^*\right)$.

- Generate a sample from a uniform distribution, i.e., $u \sim U(0, 1)$.

$$\text{If} \begin{cases} u \leq A\left(\alpha^{j-1}, \alpha^*\right) & accept\, \alpha^* = \alpha^j \\ u \leq A\left(\alpha^{j-1}, \alpha^*\right) & accept\, \alpha^* = \alpha^{j-1} \end{cases}, \text{ and} \begin{cases} u \leq A\left(\varphi^{j-1}, \varphi^*\right) & accept\, \varphi^* = \varphi^j \\ u \leq A\left(\varphi^{j-1}, \varphi^*\right) & accept\, \varphi^* = \varphi^{j-1} \end{cases}$$

- Increment the iteration index: j = j + 1, and repeat steps 2–5 M times until obtaining M samples, resulting in $(\alpha^{j-1}, \varphi^{j-1})$ for t = 1, 2, . . . , M. We'll discuss simulation study next.

8 Simulation

In this section, we conduct Monte Carlo simulations (MCS) to evaluate the performance of the proposed estimators for the parameters α, and φ introduced in Sections 4–7. We will conclude by outlining the simulation setup. Subsequently, the results of the simulations will be presented for discussion.

8.1 Simulation Scenario

In this subsection, we conduct several MCS studies to evaluate the effectiveness of the obtained maximum likelihood estimates and Bayesian estimation for the parameters α, and φ of DAPEIW basde on Type-II censored samples. We propose the following steps to obtain a sample from the DAPEIW model:

- Set α, and φ to their actual values as follows:

In Table 3: $\alpha = 2, \varphi = 0.5$ and $\alpha = 2, \varphi = 2$.

In Table 4: $\alpha = 1.2, \varphi = 0.5$ and $\alpha = 3, \varphi = 0.5$.

- Determine specific values for n (total test units) as 50, 100, 200, and the censored size for r as 30, 40, 50 when n = 50, r = 70, 90, 100 when n = 100, and r = 150, 180, 200 when n = 200.
- Generate a uniform random variable within the range of 0 to 1. Utilize the quantile function described in Eq. (5) to generate a random sample from the DAPEIW distribution. Then, round the number of samples to the nearest whole number.
- Calculate the maximum likelihood estimates and $100(1-\gamma)\%$ confidence intervals using the 'optim' function in R with the Fisher information matrix (Hessian Matrix).
- Use the 'coda' package to obtain Bayesian inferences by running the MCMC sampler 12,000 times with a burn-in of 2000 iterations.

- Repeat the above steps 15,000 times.
- Compute the relative bias (RB), mean squared errors (MSE), average lower and upper bounds, and coverage probability (CP) of the parameters for each group (n or actual value of the parameter). We specifically determine these metrics to compare interval estimates, with emphasis on meeting the CP requirement in our evaluations.

All numerical analyses are conducted using the R 4.2.2 programming language. Tables 3 and 4 present the numerical findings for α, and φ, respectively, and Tables 5 and 6 indicate the optimal censoring test plan of our suggested model at various values of α and φ.

Table 3: RB, MSE, and CI for parameters of DAPEIW distribution based on Type-II censored samples $\alpha = 2$

$\alpha = 2$				MLE				Bootstrap		Bayesian		
φ	n	m		RB	MSE	LACI	CP	LBP	LBT	RB	MSE	LCCI
0.5	50	30	α	0.0168	0.1261	1.7198	94.62%	0.2010	0.2012	0.00358	0.03213	0.84138
			φ	0.0049	0.0008	0.1486	95.19%	0.0198	0.0200	0.00179	0.00054	0.13044
		40	α	0.0061	0.0383	0.7575	95.61%	0.1880	0.1927	0.00149	0.01139	0.37624
			φ	0.0017	0.0002	0.0589	95.90%	0.0144	0.0145	0.00049	0.00008	0.03575
		50	α	0.0026	0.0102	0.3109	96.10%	0.1166	0.1164	0.00059	0.00414	0.24318
			φ	0.0008	0.0001	0.0303	96.70%	0.0113	0.0112	0.00008	0.00005	0.01250
	100	70	α	0.0114	0.0554	0.9443	95.30%	0.1196	0.1195	0.00328	0.02563	0.59197
			φ	0.0032	0.0003	0.0826	94.12%	0.0106	0.0105	0.00130	0.00025	0.06469
		90	α	0.0048	0.0200	0.4272	95.40%	0.1101	0.1115	0.00126	0.00794	0.26473
			φ	0.0011	0.0001	0.0323	94.70%	0.0079	0.0078	0.00030	0.00003	0.01689
		100	α	0.0028	0.0100	0.2617	96.60%	0.1038	0.0990	0.00056	0.00415	0.19642
			φ	0.0006	0.0000	0.0211	95.30%	0.0077	0.0076	0.00007	0.00002	0.00828
	200	150	α	0.0123	0.0461	0.8818	94.58%	0.1123	0.1117	0.00315	0.02310	0.43735
			φ	0.0029	0.0002	0.0552	95.30%	0.0071	0.0070	0.00110	0.00017	0.03826
		180	α	0.0053	0.0193	0.4096	95.70%	0.1029	0.1003	0.00124	0.00746	0.22873
			φ	0.0012	0.0001	0.0261	96.20%	0.0066	0.0066	0.00030	0.00003	0.01320
		200	α	0.0019	0.0052	0.2013	96.70%	0.0750	0.0745	0.00053	0.00311	0.14561
			φ	0.0005	0.0000	0.0141	96.58%	0.0053	0.0053	0.00006	0.00002	0.00568
2	50	30	α	0.0175	0.4940	3.6677	93.62%	0.2629	0.2613	0.00715	0.13898	0.99425
			φ	0.0074	0.0025	0.3317	95.10%	0.0520	0.0519	0.00098	0.00181	0.16718
		40	α	0.0095	0.2969	2.6102	94.20%	0.2533	0.2527	0.00334	0.09397	0.50704
			φ	0.0052	0.0014	0.1010	96.10%	0.0310	0.0305	0.00036	0.00049	0.06657
		50	α	0.0062	0.1907	1.9712	96.28%	0.2344	0.2324	0.00188	0.05198	0.36310
			φ	0.0020	0.0009	0.0744	97.64%	0.0306	0.0305	0.00007	0.00041	0.03305
	100	70	α	0.0150	0.4065	3.4952	94.50%	0.6655	0.6606	0.00815	0.09411	0.97520
			φ	0.0061	0.0015	0.1278	93.67%	0.0294	0.0281	0.00083	0.00116	0.09242
		90	α	0.0087	0.2928	2.5197	95.60%	0.6518	0.6508	0.00390	0.08527	0.51712
			φ	0.0025	0.0007	0.0842	96.96%	0.0263	0.0263	0.00026	0.00025	0.04424
		100	α	0.0056	0.1694	1.3766	97.36%	0.6414	0.6408	0.00232	0.02865	0.36570
			φ	0.0011	0.0005	0.0517	97.57%	0.0239	0.0238	0.00007	0.00013	0.02233

(Continued)

Table 3 (continued)

$\alpha = 2$				MLE				Bootstrap		Bayesian		
φ	n	m		RB	MSE	LACI	CP	LBP	LBT	RB	MSE	LCCI
	200	150	α	0.0174	0.3970	3.4533	96.00%	0.9230	0.9151	0.00784	0.09129	0.96936
			φ	0.0003	0.0012	0.0620	93.20%	0.0109	0.0104	0.00067	0.00092	0.05557
		180	α	0.0087	0.2834	2.5148	96.90%	0.8013	0.7956	0.00384	0.06188	0.47600
			φ	0.0002	0.0004	0.0540	94.70%	0.0107	0.0102	0.00023	0.00022	0.03085
		200	α	0.0057	0.1236	1.0419	97.30%	0.7479	0.7207	0.00233	0.02344	0.33069
			φ	0.0002	0.0002	0.0394	96.80%	0.0097	0.0096	0.00006	0.00003	0.01729

Table 4: RB, MSE, and CI for parameters of DAPEIW distribution based on Type-II censored samples $\varphi = 0.5$

$\varphi = 0.5$				MLE				Bootstrap		Bayesian		
α	n	m		RB	MSE	LACI	CP	LBP	LBT	RB	MSE	LCCI
1.2	50	30	α	0.0223	0.0858	1.4238	94.08%	0.1803	0.1788	0.00219	0.00543	0.52402
			φ	0.0054	0.0010	0.1736	94.60%	0.0218	0.0216	0.00053	0.00066	0.07527
		40	α	0.0091	0.0297	0.6224	94.60%	0.1582	0.1597	0.00097	0.00210	0.24054
			φ	0.0019	0.0003	0.0688	95.40%	0.0169	0.0171	0.00018	0.00016	0.02637
		50	α	0.0046	0.0124	0.3609	95.40%	0.1405	0.1392	0.00061	0.00133	0.16191
			φ	0.0009	0.0001	0.0351	95.70%	0.0141	0.0139	0.00009	0.00006	0.01353
	100	70	α	0.0164	0.0421	0.8598	94.40%	0.1143	0.1150	0.00195	0.00412	0.45491
			φ	0.0041	0.0005	0.1108	95.30%	0.0146	0.0144	0.00040	0.00031	0.04444
		90	α	0.0078	0.0192	0.4124	96.10%	0.1061	0.1055	0.00088	0.00170	0.20289
			φ	0.0016	0.0001	0.0365	95.82%	0.0090	0.0089	0.00014	0.00007	0.01499
		100	α	0.0046	0.0109	0.2894	96.70%	0.1047	0.1048	0.00054	0.00108	0.13666
			φ	0.0009	0.0001	0.0247	96.90%	0.0089	0.0088	0.00008	0.00005	0.00903
	200	150	α	0.0183	0.0418	0.7319	94.26%	0.0934	0.0933	0.00189	0.00355	0.38546
			φ	0.0041	0.0004	0.0675	94.00%	0.0087	0.0087	0.00035	0.00002	0.02805
		180	α	0.0076	0.0168	0.3247	94.50%	0.0838	0.0839	0.00080	0.00130	0.16038
			φ	0.0016	0.0001	0.0300	95.30%	0.0080	0.0080	0.00013	0.00001	0.01131
		200	α	0.0041	0.0093	0.2904	97.10%	0.0711	0.0701	0.00053	0.00091	0.12064
			φ	0.0007	0.0001	0.0223	98.20%	0.0079	0.0078	0.00007	0.00001	0.00716
3	50	30	α	0.0920	2.8922	1.0748	90.00%	0.1487	0.1429	0.00529	0.06744	1.02261
			φ	−0.0015	0.0026	0.2590	89.30%	0.0327	0.0312	0.00040	0.00081	0.08820
		40	α	0.0459	1.4882	1.0152	94.87%	0.1389	0.1392	0.00249	0.03015	0.50094
			φ	−0.0004	0.0007	0.1306	96.40%	0.0323	0.0304	0.00014	0.00025	0.07172
		50	α	0.0338	1.1854	0.7863	95.38%	0.1293	0.1293	0.00146	0.01616	0.37690
			φ	0.0003	0.0007	0.1110	96.57%	0.0314	0.0300	0.00014	0.00021	0.03910

(Continued)

Table 4 (continued)

$\varphi = 0.5$				MLE				Bootstrap		Bayesian		
α	n	m		RB	MSE	LACI	CP	LBP	LBT	RB	MSE	LCCI
	100	70	α	0.0778	2.6169	1.0575	90.00%	0.8630	0.8726	0.00601	0.05847	0.95415
			φ	−0.0017	0.0025	0.2455	91.00%	0.0657	0.0652	0.00038	0.00080	0.12284
		90	α	0.0607	1.4566	0.9806	93.80%	0.3891	0.3901	0.00295	0.02941	0.50607
			φ	0.0003	0.0004	0.1038	94.20%	0.0259	0.0258	0.00009	0.00010	0.04469
		100	α	0.0415	1.0800	0.7167	95.20%	0.3891	0.3879	0.00186	0.01248	0.35552
			φ	0.0002	0.0004	0.0802	95.40%	0.0250	0.0241	0.00002	0.00009	0.02549
	200	150	α	0.0829	2.3907	1.0063	93.30%	0.2647	0.2660	0.00717	0.05118	0.93278
			φ	−0.0018	0.0021	0.1234	93.20%	0.0159	0.0157	0.00031	0.00052	0.07185
		180	α	0.0497	1.0820	0.9391	94.00%	0.2577	0.2562	0.00361	0.02600	0.46767
			φ	0.0002	0.0002	0.0815	94.40%	0.0142	0.0141	0.00010	0.00010	0.03119
		200	α	0.0374	0.9672	0.5961	96.10%	0.1954	0.1946	0.00228	0.01136	0.32369
			φ	0.0001	0.0001	0.0567	95.70%	0.0132	0.0130	0.00004	0.00005	0.01843

Table 5: Optimal censoring plan $\alpha = 2$

φ	n	r	Op_2	Op_3	Op_1	Op_4	SCV
0.5	50	30	5.7093	0.0781	75.4750	2,155,286.3797	0.1122
		40	3.5000	0.0284	132.2146	1,400,544.4415	0.0622
		50	2.1232	0.0113	182.8653	1,277,617.5400	0.0333
	100	70	2.0915	0.0097	210.9447	1,303,100.2700	0.0738
		90	1.6317	0.0050	322.6098	1,182,794.3518	0.0437
		100	1.3617	0.0036	367.9124	948,854.2386	0.0313
	200	150	1.1919	0.0025	481.4671	1,258,549.9318	0.0661
		180	1.0044	0.0016	638.8245	1,183,115.4067	0.0426
		200	0.3903	0.0005	753.4463	752,456.1509	0.0228
2	50	30	61.3673	4.3070	15.1875	10.1720	0.1842
		40	60.2662	4.2465	15.2731	4.0170	0.1439
		50	58.6610	3.7184	15.7423	3.3524	0.1160
	100	70	19.2058	0.6009	31.3285	7.3782	0.1661
		90	19.1538	0.5997	30.7211	4.0141	0.1406
		100	18.8302	0.5610	32.1371	3.2365	0.1080
	200	150	19.0962	0.3364	58.2060	4.7797	0.1634
		180	18.9151	0.3162	60.2733	3.6073	0.1370
		200	17.9342	0.2804	63.9527	3.0915	0.0905

Table 6: Optimal censoring plan $\varphi = 0.5$

α	n	r	Op_2	Op_3	Op_1	Op_4	SCV
1.2	50	30	3.8287	0.0694	55.7216	659,802.273	0.0996
		40	2.6070	0.0262	103.9510	462,005.082	0.0582
		50	1.5337	0.0105	148.2875	242,707.374	0.0371
	100	70	1.2489	0.0077	157.1100	409,492.757	0.0704
		90	1.1037	0.0043	254.5057	200,057.926	0.0454
		100	0.8409	0.0028	301.3008	181,024.348	0.0340
	200	150	0.7394	0.0020	362.1736	177,005.758	0.0675
		180	0.5645	0.0011	508.6632	120,432.550	0.0426
		200	0.4435	0.0007	606.9660	84,848.183	0.0311
3	50	30	69.7930	7.3355	8.1393	3.6355	0.4385
		40	69.4744	6.4700	8.4035	1.9975	0.3174
		50	68.6542	6.3923	8.9610	1.9670	0.2890
	100	70	29.2131	1.7858	17.9948	3.5025	0.4241
		90	27.1606	1.1074	18.2653	1.1894	0.3025
		100	27.0680	1.0912	18.5019	1.0115	0.2674
	200	150	16.1247	0.3457	35.9806	2.9437	0.4004
		180	15.8958	0.3455	36.5092	1.1212	0.2616
		200	14.5072	0.3261	36.9464	1.0039	0.2478

8.2 Simulation and Algorithms Conclusion

The primary focus of this subsection is to evaluate the performance of the proposed algorithms for both point and interval estimates based on type II censored sample. From Tables 3 and 4, we can deduce the following observations:

- The estimated values for the unknown parameters α, and φ generally demonstrate strong performance in terms of minimized Mean Squared Error (MSE), Relative Bias (RB), and narrow Confidence Intervals (CI) with high Coverage Probability (CP).
- As the sample size n increases, the MSE, RB, and CI width of α, and φ tend to decrease. This trend aligns with the consistency property of the DAPEIW distribution based on Type-II censored sample associated estimates, indicating improved performance with larger sample sizes.
- As the censored sample size r increases, the MSE, RB, and CI width of α, and φ tend to decrease.
- With an increase in the true value of α, the MSE, RB, and CI width of α, and φ generally increase.
- Conversely, as the true value of φ increases, the MSE, RB, and CI width of all unknown parameters α, and φ tend to increase within each set.
- The credible interval for Bayesian estimation typically narrows with increasing sample size.
- Across various sample sizes, Bayesian estimation using the SE loss function consistently yields minimal RB and MSE values.

- In each case, the bootstrap results were consistently superior in terms of CI length, exhibiting the shortest length of CI compared to the other methods.

9 Application

In this section, the superior performance of the DAPEIW Distribution with two parameters over traditional distributions such as discrete Marshall-Olkin inverted Topp-Leone (DMOITL) [28], discrete Buur (DB) [8], binomial (binom), and discrete alpha power IW (DAPIW) [29], are demonstrated. Tables 7 and 8 present statistics for all fitted models based on real datasets, including Kolmogorov-Smirnov discrete (KSD) with *p*-value of KS (PVKS), Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent AIC (CAIC), and Hannon and Quinn's information criterion (HQIC). The MLEs of the parameters for the examined models are also provided in these tables. Additionally, Figs. 6 and 7 display the fitted DAPEIW distribution using probability mass function (PMF), cumulative distribution function (CDF), PP-plot, and QQ-plot for the two datasets. Firstly, the dataset pertains to COVID-19 new cases in Zimbabwe from 30 June 2020 to 25 July 2020, with the following values: 7, 17, 14, 12, 8, 73, 18, 18, 53, 98, 41, 16, 40, 3, 49, 30, 25, 273, 58, 58, 133, 102, 107, 214, 90, 172.

Table 7: MLE, StEr, KSD, PVKS, AIC, BIC, HQIC, CAIC for each discrete distribution

		α	φ	μ	KSD	PVKS	AIC	CAIC	BIC	HQIC
DAPEIW	Estimates	1.16E+09	0.9543		0.1136	0.8828	278.1503	278.6721	280.6665	278.8749
	StEr	3751.4995	0.0666							
DAPIW	Estimates	29.2318	12.3966	1.1216	0.1149	0.8827	278.6395	279.7304	282.4138	279.7264
	StEr	78.9822	10.6428	0.1638						
DMOITL	Estimates	348.4498	1.9046		0.1533	0.5739	278.5726	279.0943	281.0888	278.9272
	StEr	98.4176	0.1355							
DB	Estimates	6.5684	0.9596		0.3923	0.0007	315.7284	316.2502	318.2446	316.4530
	StEr	1.1052	0.5913							
Binom	Estimates	0.2811			0.4621	0.0000	614.3214	614.4881	615.5795	614.6837
	StEr	8.40E−05								

Table 8: Bayesian estimation

	Estimates	StEr
α	1.164990e+09	3.808099e+01
φ	9.542908e−01	6.654605e−08

The dataset has been provided by the World Health Organization, accessible at https://covid19.who.int/ (accessed on 01 September 2025). Fig. 6 illustrates the trend line, showing a consistent increase over time. Additionally, Fig. 6 examines the normality check, revealing a significantly large CV, ranging from a minimum of 7 cases to a maximum of 273 cases.

Fig. 7 exhibits the Empirical CDF together with the fitted CDF, along with the Quantile-Quantile (QQ) plot and Probability-Probability (PP) plot for the DAPEIW distribution. It offers a visual comparison between the observed data and the theoretical quantiles of the distribution, facilitating the evaluation of the DAPEIW distribution's goodness-of-fit to the empirical data.

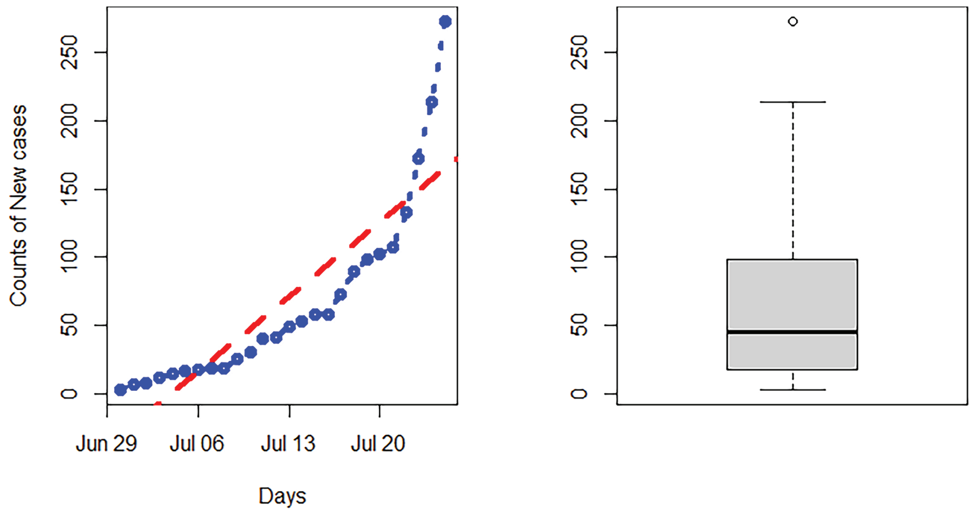

Figure 6: Trend plot and boxplot of COVID-19 data

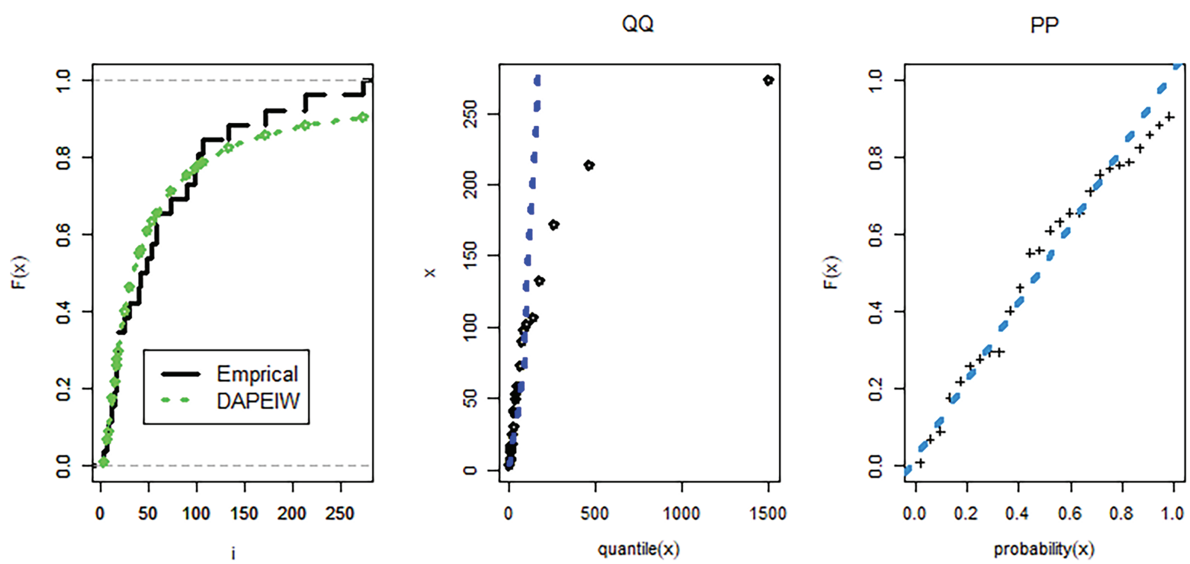

Figure 7: Emprical CDF, QQ, and PP for DAPEIW distribution

Non-parametric diagnostics were added in Figs. 6 and 7, including the empirical PMF (bars) with the fitted PMF overlaid as points; the ECDF and the fitted CDF at integer values; Q-Q and P-P plots; and a boxplot plot. The parametric tests are complemented by these diagnostics, and overall agreement between the empirical data and the fitted DAPEIW model is indicated, with minor upper-tail deviations noted.

The profile likelihood of DAPEIW distribution parameters involves creating a likelihood function where certain parameters are set to specific values and then maximizing this function for the remaining parameters in Fig. 8. This is done repeatedly for different fixed values of the parameters, generating a profile likelihood plot that illustrates how the likelihood changes as each parameter is adjusted while the others are kept constant. This method is frequently employed in statistical inference to evaluate the uncertainty linked to parameter estimates and to pinpoint areas of parameter space that align best with the observed data.

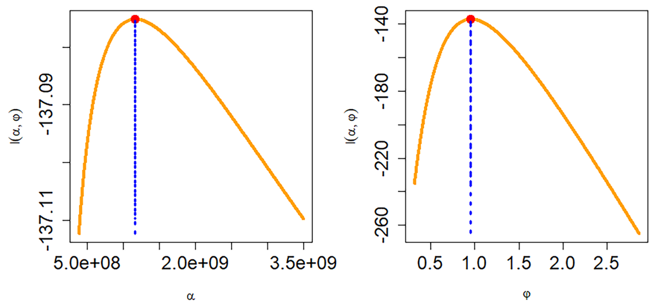

Figure 8: Likelihood profile of α, and φ for DAPEIW distribution

To verify that the reported maximum-likelihood estimates correspond to a true maximum of the log-likelihood function, we performed several diagnostics. Fig. 8 displays the profile log-likelihoods for α and φ, each showing a single, well-defined peak. The Hessian matrix of the log-likelihood evaluated at the estimates was found to be negative-definite (hence the observed Fisher information, Hessian, is positive-definite), confirming the second-order optimality conditions. In addition, the numerical optimization was repeated from multiple distinct starting values and a local grid search around the solution was carried out; all runs converged to the same estimates. Taken together, these checks provide strong evidence that the reported MLEs are not spurious local maxima.

Fig. 9 presents the trace plot and histogram of the posterior density for the MCMC results. The trace plot illustrates the successive values of the parameters sampled during the MCMC simulation, offering insight into the convergence and mixing properties of the chain. The histogram provides a visual representation of the posterior density, depicting the distribution of parameter values sampled by the MCMC algorithm. Together, these plots offer a comprehensive view of the posterior distribution and the performance of the MCMC algorithm in exploring the parameter space. Fig. 10 visually depicts the convergence lines and the Autocorrelation Function (ACF) test of the MCMC results. They provide insight into the convergence behavior and autocorrelation structure of the MCMC sampling process, aiding in the assessment of the reliability and efficiency of the MCMC algorithm in generating representative samples from the target distribution.

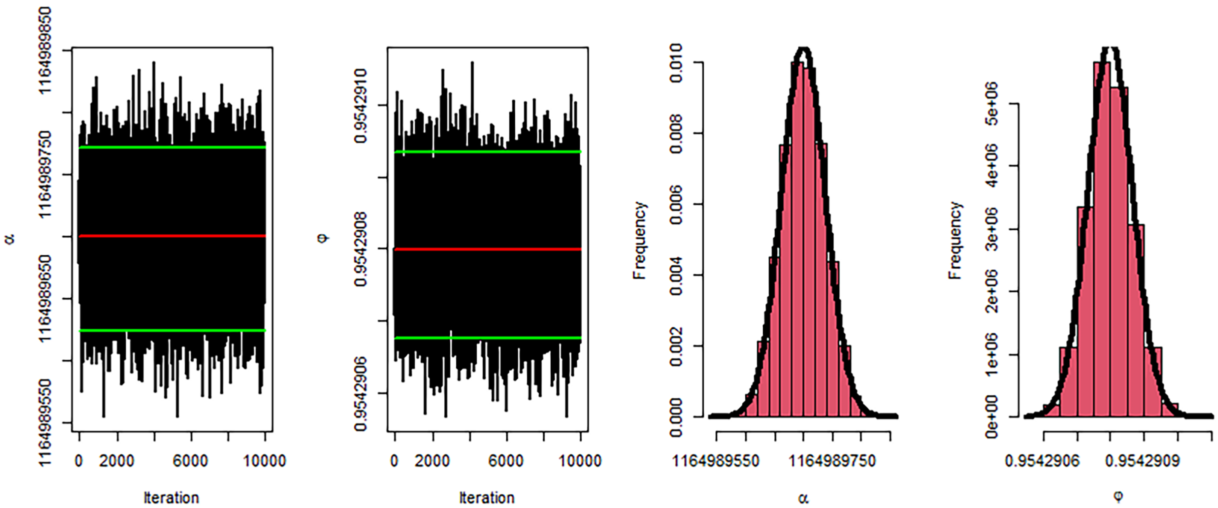

Figure 9: Trace and histogram of posterior density

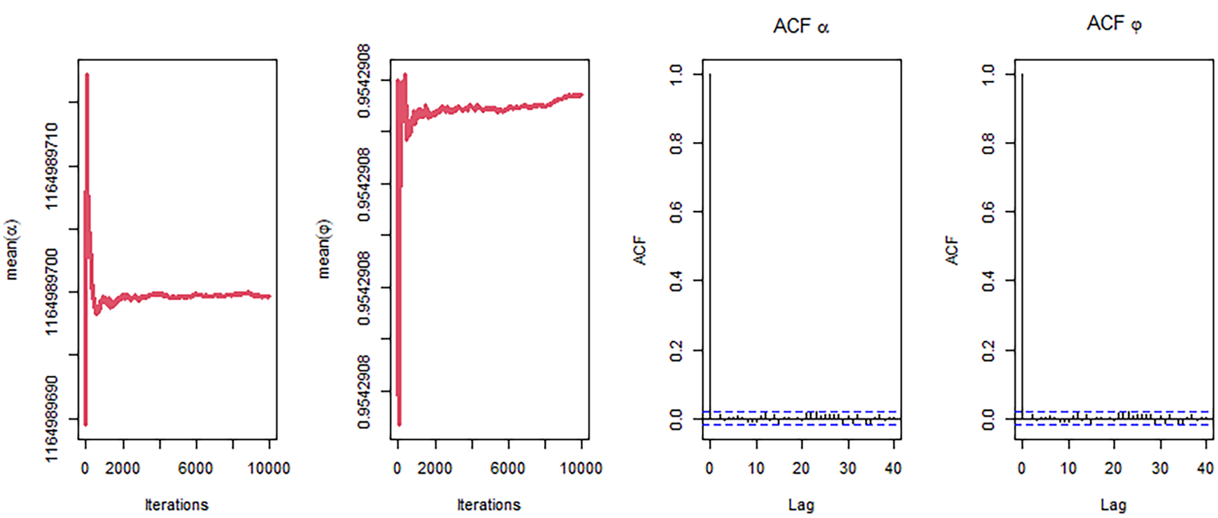

Figure 10: Convergence lines and ACF test of MCMC results

10 Concluding Remarks

In this paper, the classic optimum test approach for censored data with discontinuous alpha power and an extended inverted Weibull distribution is obtained. Specifically, the statistical properties of the model we propose are identified. These include order statistics, entropy, skewness, kurtosis, moments, quantiles, medians, and other metrics. The model parameters were computed using the maximum likelihood approach using type II censored samples. The asymptotic confidence intervals for the parameters are calculated. It is noted that as the sample size increases, the credible interval for Bayesian estimate tends to shorten. Also, Bayesian estimation with the SE loss function produces low RB and MSE values for all sample sizes. In addition, bootstrap consistently produced the shortest CI length compared to other approaches. Using a range of ideal criteria, four probable optimum test methodologies are examined. The sensitivity analysis of the best time to change stress is shown in this study. It is based on the asymptotic variances of maximum likelihood estimators. Several simulations were performed to assess the efficacy of the approaches described in this article. Finally, the medical instance is given to help understand the preferred procedure. Such as the dataset pertains to COVID-19 new cases in Zimbabwe from 30 June 2020 to 25 July 2020.

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Availability of Data and Materials: The data used to support the findings of this study are included within the article.

Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest to report regarding the present study.

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