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ABSTRACT

This study focuses on estimating reliability for the Weibull distribution within the framework of constant-stress partially accelerated life tests, incorporating the unified Type-I progressive hybrid censoring scheme that has not been previously examined. The unknown parameters of the Weibull model, including acceleration factors and key reliability measures such as the reliability function and hazard rate function under normal use conditions, are estimated. The maximum likelihood method is used to obtain point estimates and approximate confidence intervals. Additionally, a Bayesian approach based on Markov Chain Monte Carlo sampling is applied to compute point estimates and Bayesian credible intervals, assuming a squared error loss function. A simulation study is conducted to compare the performance of the estimators using criteria such as mean squared error, interval width, and coverage probability. Finally, two real-world data sets from accelerated life testing are analyzed to illustrate the practical usefulness of the proposed methods.

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1 Introduction

Nowadays, the rapid advancements in manufacturing technologies have led to the production of highly reliable products with long lifespan. One of the main challenges manufacturers face in such cases is assessing the reliability of these products, as conventional lifetime experiments require a significant amount of time to collect sufficient failure data, which is essential for reliability analysis. To address this issue, researchers commonly employ accelerated life tests (ALTs). In the ALTs, products are tested under elevated stress conditions, such as higher voltage or temperature, than those encountered under normal operating conditions. The data obtained under these stressed conditions are then used to infer the product's reliability under typical usage scenarios. See for more details, Bagdonavicius and Nikulin [1]. In practice, several approaches can be used to conduct the ALTs. The two most commonly used

methods are constant-stress ALTs (CSALTs) and step-stress ALTs (SSALTs). In CSALTs, the test sample is divided into subgroups, and each subgroup is subjected to a fixed stress level throughout the experiment. The test continues until all units have failed or a predetermined termination point is reached. Many authors considered the CSALTs, see for example the work of Yang [2], Feng et al. [3], and Wu et al. [4]. In contrast, the SSALTs involve gradually increasing the stress level at specific time intervals or after a pre-specified number of failures have been observed. This allows for faster failure data collection while still enabling inference about product reliability under normal operating conditions. For more details, see Amleh and Raqab [5], Moala and Chagas [6], and Yadav et al. [7].

To evaluate product reliability under normal operating conditions using data obtained from the ALTs, it is essential to understand the life-stress relationship, a model that describes how a product's lifetime is affected by different levels of applied stress. However, in many practical situations, this physical relationship is either unknown or difficult to define precisely. In such cases, partially ALTs (PALTs) offer a practical solution for estimating reliability under standard usage conditions. This study specifically focuses on a common type of PALTs known as constant-stress PALTs (CSPALTs), where each test unit is subjected to either a normal or an accelerated stress level throughout the entire testing period. This structure simplifies the analysis, as it eliminates the need to model the relationship between the product's lifetime and stress levels. Numerous studies in the literature have explored CSPALTs due to their analytical convenience and practical relevance, see for example, Ismail [8], Ahmad Lone et al. [9], Dey et al. [10], and Fayomi et al. [11], among others.

Another practical method for obtaining failure data more quickly is through the use of censored samples, rather than waiting for all test units to fail. This strategy is widely adopted in reliability analysis, as it helps reduce both testing time and cost. The literature offers a variety of censoring schemes, each with its own benefits and specific termination criteria, making them suitable for different experimental settings and reliability objectives. For example, in a life-testing experiment involving n test units, the Type-I censoring plan terminates the test at a pre-specified time T , regardless of the number of observed failures. In contrast, the Type-II censoring plan ends the experiment after a predetermined number of failures m have been recorded. To combine the features of both schemes, Epstein [12] proposed the Type-I hybrid censoring (HC) plan, under which the experiment stops at $\min(T, Y_{m:n})$, where $Y_{m:n}$ denotes the failure time of the m -th unit. On the other hand, Childs et al. [13] introduced the Type-II HC plan, which terminates the experiment at $\max(T, Y_{m:n})$, ensuring that at least m failures are observed while also allowing for a time constraint. However, these HC schemes come with certain limitations. The Type-I HC plan may result in too few observed failures, limiting the amount of information available, while the Type-II HC plan can lead to lengthy testing times, increasing the overall cost and duration of the experiment. Another notable HC scheme is the unified HC (UHC) plan, introduced by Balakrishnan [14]. In this design, the experimenter pre-specifies two integers r and m (with $k < m$), and two time thresholds T_1 and T_2 , such that $T_1 < T_2$. The termination time of the experiment depends on the relationship between these values and the observed failure times $Y_{k:n}$ and $Y_{m:n}$, as follows: (1) If $T_1 > Y_{k:n}$, the test ends at $\min\{\max(Y_{m:n}, T_1), T_2\}$; (2) If $T_1 < Y_{k:n} < T_2$, the test stops at $\min\{Y_{m:n}, T_2\}$; and (3) If $Y_{k:n} > T_2$, the test concludes at $Y_{k:n}$. This structure provides greater flexibility in balancing the experimental duration with the number of failures observed, making it a versatile choice for life-testing studies. See for more details about the UHC plan, Jeon and Kang [15], and Dutta et al. [16].

All the previously discussed censoring schemes, as well as several others not mentioned here, fall under the category of single-stage censoring plans. A key limitation of such plans is that they do not allow for the removal of test units before the experiment ends. One widely used alternative that addresses this limitation is the conventional progressive Type-II censoring plan, which permits the

removal of surviving units at the time of each failure according to a pre-defined removal scheme. To introduce greater flexibility in life-testing experiments, Górný and Cramer [17] proposed the unified progressive HC (UPHC) scheme. This design integrates features from both the UHC plan and the progressive Type-II censoring plan. In their work, four versions of the UPHC schemes were introduced: unified Type-I, Type-II, Type-III, and Type-IV progressive HC plans. In the present study, we concentrate on the unified Type-I progressive HC (UTIPHC) scheme. Despite the flexibility of this censoring scheme, relatively few studies have considered it in the literature. See, for example, Dutta and Kayal [18], Dutta et al. [19], Anwar et al. [20], for further details. The detailed description of the UTIPHC scheme under CSPALTs is provided in the following section.

Modeling lifetime data under either normal usage or accelerated stress conditions requires a probabilistic distribution that accurately reflects the behavior of the observed data. Among the numerous distributions available, the Weibull distribution stands out as one of the most widely used in statistical and reliability literature. Its popularity stems from its flexibility, as it can effectively model different types of failure rate shapes, including constant, increasing, and decreasing hazard rates, making it particularly suitable for a broad range of reliability and survival analysis applications. For an overview of the Weibull distribution, including its key properties and various extensions, readers are referred to the comprehensive review by Lai et al. [21], and Abdelaziz et al. [22]. For discussions on different estimation methods for the Weibull distribution, see also Teimouri et al. [23]. Under normal operating conditions, let Y be a random variable following the Weibull distribution with scale parameter $a > 0$ and shape parameter $b > 0$. The probability density function (PDF) and cumulative distribution function (CDF) are then given by

$$f_1(y; a, b) = aby^{b-1}e^{-ay^b}, y > 0, a, b > 0 \quad (1)$$

and

$$F_1(y; a, b) = 1 - e^{-ay^b}, \quad (2)$$

respectively. In addition, the reliability function (RF) and hazard rate function (HRF) are as follows

$$R_1(y; a, b) = e^{-ay^b} \quad \text{and} \quad H_1(y; a, b) = aby^{b-1}. \quad (3)$$

The Weibull distribution is widely recognized as a fundamental model in the field of ALT. The literature has extensively documented its flexibility and effectiveness under various ALT settings. For further insights into its application across different ALT methodologies, readers are referred to the works of Ismail [24], Cheng and Sheu [25], Cui et al. [26], and Feng and Tang [27], among others.

Several factors motivated the completion of this study: (1) Although the UTIPHC scheme plays a significant role in terminating life-testing experiments, to the best of our knowledge, it has not yet been explored in the context of ALTs. This omission may be attributed to its structural complexity, as the scheme involves four distinct parameters, as will be detailed in the following section. (2) The effectiveness of CSPALTs in reliability analysis is another motivation, as they allow for quicker acquisition of failure data compared to traditional tests. This advantage translates into reduced testing time and lower costs for manufacturers. (3) The Weibull distribution's flexibility as a lifetime model also underpins our motivation. Its ability to model different failure rates makes it especially useful in ALT scenarios, where models with more than two parameters are typically avoided due to practical constraints. Considering these motivating factors, the primary objective of this study is to investigate the estimation of the unknown parameters of the Weibull distribution, the acceleration factor, and two key reliability measures under the CSPALTs, when the data are obtained using the UTIPHC scheme. In this context, the RF and HRF are evaluated under normal operating conditions, which

represents the primary goal of reliability analysis when using ALTs. To achieve this, we employ two estimation approaches, maximum likelihood estimation and Bayesian estimation, to obtain both point and interval estimators. The Bayesian method is conducted under the squared error loss function, incorporating Markov Chain Monte Carlo (MCMC) sampling techniques. Given the difficulty in comparing the performance of the estimators analytically, we conduct a simulation study to evaluate and compare the accuracy of the proposed estimators based on several performance criteria for both point and interval estimation methods. Finally, to assess the practical applicability of the proposed methods, we analyze two real-world ALT data sets.

The structure of the remainder of this paper is as follows: First, we describe the UTIPHC scheme within the context of CSPALTs. [Section 3](#) introduces the maximum likelihood estimators (MLEs) and their corresponding approximate confidence intervals (ACIs) for the model parameters, the acceleration factor, and the reliability measures under normal usage conditions. In [Section 4](#), we present the Bayesian estimation approach, including both point estimates and Bayesian credible intervals (BCIs). [Section 5](#) outlines the simulation study and discusses its findings. [Section 6](#) illustrates the practical implementation of the proposed methods using two real data sets. Finally, [Section 7](#) concludes with key observations and remarks.

2 Model Description and Test Procedure

In the framework of CSPALTs, consider that a total of n test units are available. These units are randomly allocated into two groups as follows:

1. The first group comprises n_1 units, randomly selected from the total n units, and tested under normal operating conditions. It is assumed that the lifetimes of these units follow a Weibull distribution, with the PDF and CDF given by [Eqs. \(1\) and \(2\)](#), respectively.
2. The second group consists of the remaining $n_2 = n - n_1$ units, which are subjected to accelerated stress conditions. Under these conditions, the HRF of a unit is modeled as $H_2(y; \eta) = \theta H_1(y; a, b)$, $\theta > 1$, where $\eta = (a, b, \theta)^\top$ is the vector of unknown parameters. Accordingly, the RF under the accelerated condition is expressed as $R_2(y; \eta) = \exp \left\{ - \int_0^y H_2(z; \eta) dz \right\}$. Thus, both the RF and HRF under accelerated conditions are functions of the parameter vector η , as follows

$$H_2(y; \eta) = ab\theta y^{b-1}, y > 0, a, b > 0, \theta > 1 \quad (4)$$

and

$$R_2(y; \eta) = e^{-a\theta y^b}.$$

The accelerated condition is modeled through a proportional hazards structure. For the Weibull distribution, this formulation is equivalent to a time-scale acceleration model, thereby justifying the interpretation of θ as an acceleration parameter. The CDF and PDF under accelerated conditions can be expressed, respectively, as

$$F_2(y; \eta) = 1 - e^{-a\theta y^b} \quad (5)$$

and

$$f_2(y; \eta) = ab\theta y^{b-1} e^{-a\theta y^b}. \quad (6)$$

For each test group, the units are subjected to the UTIPHC censoring plan. For simplicity, let $Y_{j1}, Y_{j2}, \dots, Y_{jm_j}, j = 1, 2$, denote the ordered failure times of the test units. Let two integers k_j and m_j

be chosen such that $k_j < m_j < n_j$. Additionally, two time thresholds T_{j1} and T_{j2} , satisfying $T_{j1} < T_{j2}$, and a progressive censoring scheme $(Q_{j1}, \dots, Q_{jm_j})$, satisfying the condition $n_j = m_j + \sum_{i=1}^{m_j} Q_{ji}$, are pre-specified before the experiment begins. Under this setup, the UTIPHC censoring plan within the CSPALT framework proceeds as follows: At the time of the first failure Y_{j1} , Q_{j1} units are randomly removed from the remaining surviving units. Subsequently, at the time of the second failure Y_{j2} , an additional Q_{j2} units are randomly withdrawn, and this process continues according to the specified removal scheme. The termination time of the life-testing experiment under the UTIPHC scheme is defined as

$$T_j^* = \max \{ \min(Y_{jk_j}, T_{j2}), \min(Y_{jm_j}, T_{j1}) \}.$$

Let d_j and r_j for $j = 1, 2$ denote the number of observed failures that occur before T_{j1} and T_{j2} , respectively. As described by Górný and Cramer [17], the UTIPHC scheme can lead to six different experimental cases. However, by grouping similar scenarios, these can be reduced to four distinct cases that need to be considered when analyzing data obtained under the UTIPHC plan. The four cases that may be encountered under the UTIPHC can be summarized as follows:

- **Case 1:** If $Y_{jm_j} < T_{j1}$, terminate the test at $T_j^* = Y_{jm_j}$, with order values $Y_{j1}, \dots, Y_{jm_j}$;
- **Case 2:** If $Y_{jk_j} < T_{j1} < Y_{jm_j}$, end the test at $T_j^* = T_{j1}$, with order values $Y_{j1}, \dots, Y_{jk_j}, \dots, Y_{jd_j}$;
- **Case 3:** If $T_{j1} < Y_{jk_j} < T_{j2}$, stop the test at $T_j^* = Y_{jk_j}$, with order values $Y_{j1}, \dots, Y_{jk_j}$;
- **Case 4:** If $T_{j2} < Y_{jk_j}$, end the test at $T_j^* = T_{j2}$, with ordered values $Y_{j1}, \dots, Y_{jr_j}$.

Based on an observed sample obtained under the UTIPHC scheme, denoted by $\mathbf{y}$, the joint likelihood function corresponding to the four distinct possible cases of the observed data can be expressed, ignoring the constant term, as follows

$$L(\boldsymbol{\eta}; \mathbf{y}) = \prod_{j=1}^2 \left\{ \prod_{i=1}^{D_j} f_j(y_{ji}) [1 - F_j(y_{ji})]^{Q_{ji}} [1 - F_j(T_j^*)]^{Q_j^*} \right\}, \quad (7)$$

where D_j , T_j^* and Q_j^* are defined as follows

$$(D_j, T_j^*, Q_j^*) = \begin{cases} \left(m_j - 1, Y_{jm_j}, n_j - m_j - \sum_{i=1}^{m_j-1} Q_{ji} \right), & \text{for Case 1,} \\ \left(d_j, T_{j1}, n_j - d_j - \sum_{i=1}^{d_j} Q_{ji} \right), & \text{for Case 2,} \\ \left(k_j - 1, Y_{jk_j}, n_j - k_j - \sum_{i=1}^{k_j-1} Q_{ji} \right), & \text{for Case 3,} \\ \left(r_j, T_{j2}, n_j - r_j - \sum_{i=1}^{r_j} Q_{ji} \right), & \text{for Case 4.} \end{cases}$$

3 Likelihood Estimation

This section focuses on the classical inference for the Weibull distribution under the CSPALTs, when the test data are obtained using the UTIPHC censoring scheme. The inference involves deriving both the MLEs and ACIs for the model parameters, the acceleration factor, as well as the two reliability measures, namely the RF and the HRF.

3.1 Point Estimation

Suppose that we have an observed UTIPHC sample drawn from a Weibull population under normal operating settings, with PDF and CDF given by (1) and (2), respectively. Let another observed UTIPHC sample be obtained from a Weibull population under accelerated conditions, with CDF and

PDF specified by (5) and (6), respectively. Then, the likelihood function of the unknown parameter vector η can be expressed using (7) as follows

$$L(\eta; \mathbf{y}) = (ab)^{D_1} \theta^{D_2} \exp \left\{ b \sum_{j=1}^2 \sum_{i=1}^{D_j} \log(y_{ji}) - a \sum_{j=1}^2 \theta^{j-1} \omega_j(\mathbf{y}; b) \right\}, \quad (8)$$

where $D = D_1 + D_2$ and $\omega_j(\mathbf{y}; b) = \sum_{i=1}^{D_j} (1 + Q_{ji}) y_{ji}^b + Q_j^* T_j^{*b}$. The natural logarithm of (8) gives the log-likelihood function as follows

$$l(\eta; \mathbf{y}) = D \log(ab) + D_2 \log(\theta) + b \sum_{j=1}^2 \sum_{i=1}^{D_j} \log(y_{ji}) - a \sum_{j=1}^2 \theta^{j-1} \omega_j(\mathbf{y}; b). \quad (9)$$

The MLEs of the unknown parameters, defined by $\hat{\eta} = (\hat{a}, \hat{b}, \hat{\theta})^\top$, are computed through maximizing the log-likelihood function in (9). This can be achieved by solving the following three normal equations simultaneously

$$\frac{\partial l(\eta; \mathbf{y})}{\partial a} = \frac{D}{a} - \sum_{j=1}^2 \theta^{j-1} \omega_j(\mathbf{y}; b) = 0, \quad (10)$$

$$\frac{\partial l(\eta; \mathbf{y})}{\partial \theta} = \frac{D_2}{\theta} - a \omega_2(\mathbf{y}; b) = 0 \quad (11)$$

and

$$\frac{\partial l(\eta; \mathbf{y})}{\partial b} = \frac{D}{b} + \sum_{j=1}^2 \sum_{i=1}^{D_j} \log(y_{ji}) - a \sum_{j=1}^2 \theta^{j-1} \omega_j^*(\mathbf{y}; b) = 0, \quad (12)$$

where $\omega_j^*(\mathbf{y}; b) = \sum_{i=1}^{D_j} (1 + Q_{ji}) y_{ji}^b \log(y_{ji}) + Q_j^* T_j^{*b} \log(T_j^*)$. Using (10), the MLE of a can be acquired explicitly as a function of the shape parameter and the acceleration factor as follows

$$\hat{a}(b, \theta) = \frac{D}{\sum_{j=1}^2 \theta^{j-1} \omega_j(\mathbf{y}; b)}. \quad (13)$$

Using (13) and the normal equation in (11), we can compute the MLE of the acceleration factor θ as a function of the shape parameter as follows

$$\hat{\theta}(b) = \frac{D_2 \omega_1(\mathbf{y}; b)}{D_1 \omega_2(\mathbf{y}; b)}. \quad (14)$$

Substituting the MLE $\hat{\theta}(b)$ in (14) in (13), one can easily obtain the following simple form of the MLE of the scale parameter a as a function of the shape parameter b

$$\hat{a}(b) = \frac{D_1}{\omega_1(\mathbf{y}; b)}. \quad (15)$$

By replacing θ and a by their MLEs obtained from (14) and (15), respectively, in the normal equation in (12), the normal equation of b can be written as

$$\frac{D}{b} + \sum_{j=1}^2 \sum_{i=1}^{D_j} \log(y_{ji}) - \sum_{j=1}^2 D_j \frac{\omega_j^*(\mathbf{y}; b)}{\omega_j(\mathbf{y}; b)} = 0. \quad (16)$$

From (16), the MLE of b can be obtained by solving $\varepsilon(b) = b$ iteratively, where

$$\varepsilon(b) = \left[\sum_{j=1}^2 \frac{D_j}{D} \frac{\omega_j^*(\mathbf{y}; b)}{\omega_j(\mathbf{y}; b)} - \frac{\sum_{j=1}^2 \sum_{i=1}^{D_j} \log(y_{ji})}{D} \right]^{-1}. \quad (17)$$

The iterative procedure defined by $\varepsilon(b^{(v)}) = b^{(v+1)}$, where $b^{(v)}$ denotes the value of b at the v -th iteration, is employed to solve Eq. (17). Through the Newton–Raphson method, once the MLE $\hat{b}$ is obtained, the MLEs of θ and a can be computed using (14) and (15), yielding $\hat{\theta} = \hat{\theta}(\hat{b})$ and $\hat{a} = \hat{a}(\hat{b})$, respectively.

As mentioned earlier, the primary objective of conducting the ALTs is to estimate the reliability metrics under normal operating conditions, which are of significant interest to both manufacturers and end users. To estimate these metrics, we utilize the invariance property of the MLEs. This property states that the MLEs of the RF and HRF under normal operating conditions can be obtained by substituting the true parameter values with their corresponding MLEs. Accordingly, we can obtain the MLEs of the RF and HRF, at mission time t , as

$$\hat{R}_1(t) = e^{-\hat{a}t^{\hat{b}}} \quad \text{and} \quad \hat{H}_1(t) = \hat{a}\hat{b}t^{\hat{b}-1}$$

3.2 Interval Estimation

To construct the ACIs for the model parameters, the acceleration factor, or any function of these parameters, such as the RF or HRF under normal operating conditions, it is necessary to derive the variance-covariance matrix of the MLEs. This matrix typically requires the exact expressions of the Fisher information matrix. However, obtaining these expressions is often infeasible, particularly when dealing with complex censoring schemes like the one adopted in the current study. To address this challenge, we estimate the variance-covariance matrix by inverting the observed Fisher information matrix. Accordingly, the estimated variance-covariance matrix can be expressed as follows

$$\begin{aligned} \mathbf{I}^{-1}(\hat{\boldsymbol{\eta}}) &= \begin{pmatrix} -l_{11} & -l_{12} & -l_{13} \\ & -l_{22} & -l_{23} \\ & & -l_{33} \end{pmatrix}_{\boldsymbol{\eta}=\hat{\boldsymbol{\eta}}}^{-1} \\ &= \begin{pmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} & \hat{\sigma}_{13} \\ & \hat{\sigma}_{22} & \hat{\sigma}_{23} \\ & & \hat{\sigma}_{33} \end{pmatrix}, \end{aligned} \quad (18)$$

where

$$\begin{aligned} l_{11} &= -\frac{D}{a^2}, \quad l_{22} = -\frac{D_2}{\theta^2}, \quad l_{33} = -\frac{D}{b^2} - a \sum_{j=1}^2 \theta^{j-1} \omega_j^*(\mathbf{y}; b), \\ l_{12} &= -\omega_2(\mathbf{y}; b), \quad l_{13} = -\sum_{j=1}^2 \theta^{j-1} \omega_j^*(\mathbf{y}; b) \quad \text{and} \quad l_{23} = -a\omega_2^*(\mathbf{y}; b), \end{aligned}$$

where $\omega_j^*(\mathbf{y}; b) = \sum_{i=1}^{D_j} (1 + Q_{ji}) y_{ji}^b \log^2(y_{ji}) + Q_j^* T_j^{*b} \log^2(T_j^*)$. By utilizing the asymptotic normality of the MLEs, i.e., $\hat{\boldsymbol{\eta}} \sim N(\boldsymbol{\eta}, \mathbf{I}^{-1}(\boldsymbol{\eta}))$, where $\mathbf{I}^{-1}(\hat{\boldsymbol{\eta}})$ denotes the estimated variance-covariance matrix given in (18), we can construct the $(1 - \alpha) \times 100\%$ ACIs for the model parameters and the acceleration factor as follows

$$\hat{a} \pm z_{\alpha/2} \sqrt{\hat{\sigma}_{11}}, \quad \hat{b} \pm z_{\alpha/2} \sqrt{\hat{\sigma}_{22}}, \quad \text{and} \quad \hat{\theta} \pm z_{\alpha/2} \sqrt{\hat{\sigma}_{33}},$$

where $z_{\alpha/2}$ refers to the upper $\alpha/2$ quantile of the standard normal distribution.

To establish the ACIs for the reliability measures $R_1(t) = R_1(t; \eta)$ and $h_1(t) = h_1(t; \eta)$, it is essential to estimate the variances of their MLEs, $\hat{R}_1(t)$ and $\hat{h}_1(t)$. Direct computation of these variances poses significant analytical challenges. As an alternative, the delta method is utilized to approximate the variances. This technique involves determining the first-order partial derivatives of the reliability metrics $R_1(t)$ and hazard rate $h_1(t)$ with respect to the parameter vector η , evaluated at the MLEs $\hat{\eta}$. For the RF and HRF, we have the following

$$\hat{\Delta}_{R_1} = -t^{\hat{b}} e^{-\hat{a}t^{\hat{b}}} \begin{pmatrix} 1 \\ 0 \\ \hat{a}t^* \end{pmatrix} \quad \text{and} \quad \hat{\Delta}_{h_1} = t^{\hat{b}-1} \begin{pmatrix} \hat{b} \\ 0 \\ \hat{a}(1 + \hat{b}t^*) \end{pmatrix},$$

where $t^* = \log(t)$. Accordingly, we can use the delta method to approximate the variance of $\hat{R}_1(t)$ as follows

$$\begin{aligned} \widehat{Var}(\hat{R}_1) &\approx \hat{\Delta}_{R_1}^\top \mathbf{I}^{-1}(\hat{\eta}) \hat{\Delta}_{R_1} \\ &\approx t^{2\hat{b}} e^{-2\hat{a}t^{\hat{b}}} \begin{pmatrix} 1 & 0 & \hat{a}t^* \end{pmatrix} \begin{pmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} & \hat{\sigma}_{13} \\ & \hat{\sigma}_{22} & \hat{\sigma}_{23} \\ & & \hat{\sigma}_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \hat{a}t^* \end{pmatrix} \\ &\approx t^{2\hat{b}} e^{-2\hat{a}t^{\hat{b}}} (\hat{\sigma}_{11} + 2\hat{a}t^* \hat{\sigma}_{13} + \hat{a}^2 t^{*2} \hat{\sigma}_{33}). \end{aligned}$$

Similarly, we can approximate the variance of the MLE of the HRF at normal operating conditions as follows

$$\begin{aligned} \widehat{Var}(\hat{h}_1) &\approx \hat{\Delta}_{h_1}^\top \mathbf{I}^{-1}(\hat{\eta}) \hat{\Delta}_{h_1} \\ &\approx t^{2(\hat{b}-1)} \begin{pmatrix} \hat{b} & 0 & \hat{a}(1 + \hat{b}t^*) \end{pmatrix} \begin{pmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} & \hat{\sigma}_{13} \\ & \hat{\sigma}_{22} & \hat{\sigma}_{23} \\ & & \hat{\sigma}_{33} \end{pmatrix} \begin{pmatrix} \hat{b} \\ 0 \\ \hat{a}(1 + \hat{b}t^*) \end{pmatrix} \\ &\approx t^{2(\hat{b}-1)} (\hat{b}^2 \hat{\sigma}_{11} + 2\hat{a}\hat{b}(1 + \hat{b}t^*) \hat{\sigma}_{13} + \hat{a}^2 (1 + \hat{b}t^*)^2 \hat{\sigma}_{33}). \end{aligned}$$

Then, the $(1 - \alpha) \times 100\%$ ACI for the RF under normal operating setting can be expressed, after some simplifications, as follows

$$e^{-\hat{a}t^{\hat{b}}} \left(1 \pm z_{\alpha/2} t^{\hat{b}} \sqrt{\hat{\sigma}_{11} + 2\hat{a}t^* \hat{\sigma}_{13} + \hat{a}^2 t^{*2} \hat{\sigma}_{33}} \right). \quad (19)$$

To address the issue wherein the computed ACI for the RF may yield a lower bound that is less than zero or an upper bound that exceeds one, as is evident from (19), both of which are theoretically invalid, as the RF must reside within the interval $[0, 1]$, two corrective strategies can be implemented. First, a truncation approach can be employed; if the computed upper bound surpasses one, it is adjusted to one, and conversely, if the lower bound is negative, it is substituted with zero. Alternatively, the ACI based on the log-transformation of the MLE can be utilized. This method guarantees that the resulting confidence limits remain within the valid range $[0, 1]$. On the other hand, the $(1 - \alpha) \times 100\%$ ACI for the HRF under normal settings is given by

$$t^{\hat{b}-1} \left(\hat{a}\hat{b} \pm z_{\alpha/2} \sqrt{\hat{b}^2 \hat{\sigma}_{11} + 2\hat{a}\hat{b}(1 + \hat{b}t^*) \hat{\sigma}_{13} + \hat{a}^2 (1 + \hat{b}t^*)^2 \hat{\sigma}_{33}} \right).$$

Remark 1: To evaluate both point and interval frequentist inferences of a , b , θ , $R_1(t)$, and $H_1(t)$, we recommend implementing the `maxLik` package, proposed by Henningsen and Toomet [28], in R software that employs the ‘`maxNR()`’ function of maximization.

4 Bayesian Estimation

The Bayesian approach has attracted considerable attention in statistical inference as a strong alternative to classical estimation methods. Its power to integrate earlier knowledge into the analysis makes it particularly practical in reliability and lifetime studies, fields where data are often insufficient, particularly in the presence of censoring plans or when using ALTs. In this section, we develop Bayesian estimation procedures for both point and interval estimation of the model parameters, the acceleration factor, and two key reliability measures of the Weibull distribution. The analysis is conducted under the framework of CSPALTs, using data collected through the UTIPHC scheme, under the assumption that the three unknown parameters are independent.

4.1 Prior Knowledge and Posterior Construction

The first step in conducting the Bayesian analysis is to specify prior distributions for the unknown parameters. This step reflects any prior knowledge or structural assumptions about the parameters before observing the data. From the likelihood function in (8), it is evident that a gamma prior is conjugate for the scale parameter a . For the shape parameter b , no conjugate prior exists; therefore, we assume a gamma prior due to its flexibility and tractability. For the acceleration factor θ , we adopt a non-informative prior proportional to $1/\theta$, which corresponds to a Jeffreys-type prior commonly used for scale parameters. This choice provides scale invariance and facilitates posterior computation within the MCMC framework. Under these assumptions, the joint prior distribution of $\eta = (a, b, \theta)^\top$ is given by

$$g(\eta) \propto a^{\lambda_1-1} b^{\lambda_2-1} \theta^{-1} \exp \{-\beta_1 a - \beta_2 b\}, \quad a, b > 0, \theta > 1, \quad (20)$$

where $\lambda_v, \beta_v > 0$, $v = 1, 2$, are hyper-parameters.

By combining the likelihood function provided by (8) with the joint prior distribution given by (20), the joint posterior distribution of the unknown parameters η and the observed data can be expressed as follows

$$\mathcal{L}(\eta|\mathbf{y}) = \frac{1}{A} a^{D+\lambda_1-1} b^{D+\lambda_2-1} \theta^{D_2-1} \exp \left\{ b \left[\sum_{j=1}^2 \sum_{i=1}^{D_j} \log(y_{ji}) - \beta_2 \right] - a \left[\sum_{j=1}^2 \theta^{j-1} \omega_j(\mathbf{y}; b) + \beta_1 \right] \right\}, \quad (21)$$

where the normalized constant A is obtained as follows

$$A = \int_1^\infty \int_0^\infty \int_0^\infty g(\eta) L(\eta; \mathbf{y}) da db d\theta.$$

For any unknown parameter, including the acceleration factor or any parametric function of the unknown parameters, denoted by $\ell(\eta)$, the Bayes estimator under the squared error loss function is given by the posterior mean. Let $\tilde{\ell}(\eta)$ denote the Bayes estimate of $\ell(\eta)$. Then, $\tilde{\ell}(\eta)$ can be computed based on the posterior distribution given in (21) as follows

$$\tilde{\ell}(\eta) = \frac{\int_1^\infty \int_0^\infty \int_0^\infty \ell(\eta) g(\eta) L(\eta; \mathbf{y}) da db d\theta}{\int_1^\infty \int_0^\infty \int_0^\infty g(\eta) L(\eta; \mathbf{y}) da db d\theta}. \quad (22)$$

The Bayes estimator, as shown in Eq. (22), cannot be obtained explicitly due to the complexity involved in calculating the ratio of integrals. To overcome this challenge, we employ numerical approximation methods such as the MCMC sampling procedures. These methods allow us to generate samples from the target distribution presented in (21). We then utilize these generated samples to derive both Bayes point estimates of the various parameters and their associated BCIs.

4.2 Posterior Analysis and MCMC Procedures

Among the most commonly employed MCMC procedures, two algorithms are quite notable due to their wide-ranging applications: Gibbs sampling and the Metropolis-Hastings (MH) algorithm. To implement the MCMC sampling procedure and decide which algorithm to use, it is essential to derive the full conditional distribution for each unknown parameter. These full conditional distributions can be acquired from the joint posterior distribution shown in (21) as follows

$$\mathcal{L}(a|\eta_{-a}, \mathbf{y}) \propto a^{D+\lambda_1-1} \exp \left\{ -a \left[\sum_{j=1}^2 \theta^{j-1} \omega_j(\mathbf{y}; b) + \beta_1 \right] \right\}, \quad (23)$$

$$\mathcal{L}(b|\eta_{-b}, \mathbf{y}) \propto b^{D+\lambda_2-1} \exp \left\{ b \left[\sum_{j=1}^2 \sum_{i=1}^{D_j} \log(y_{ji}) - \beta_2 \right] - a \sum_{j=1}^2 \theta^{j-1} \omega_j(\mathbf{y}; b) \right\}, \quad (24)$$

and

$$\mathcal{L}(\theta|\eta_{-\theta}, \mathbf{y}) \propto \theta^{D_2-1} \exp \left\{ -a\theta \omega_2(\mathbf{y}; b) \right\}. \quad (25)$$

Upon checking the full conditional distributions in (23)–(25), the following observations can be made:

1. The random variable $a|\eta_{-a}, \mathbf{y}$ follows gamma distribution with shape parameter $\tau_1 = D + \lambda_1$ and scale parameter $\nu_1 = \sum_{j=1}^2 \theta^{j-1} \omega_j(\mathbf{y}; b) + \beta_1$.
2. The random variable $\theta|\eta_{-\theta}, \mathbf{y}$ follows gamma distribution truncated to the interval $(1, \infty)$ with shape parameter $\tau_2 = D_2$ and scale parameter $\nu_2 = a\omega_2(\mathbf{y}; b)$.
3. The full conditional distribution $\mathcal{L}(b|\eta_{-b}, \mathbf{y})$ cannot be reduced to any standard form.

However, in practical applications, it is possible that the stressed condition does not strictly produce acceleration. In fact, for one of the real data sets analyzed in this study, the estimated value of θ is less than one, indicating a deceleration effect. To allow the data to determine whether acceleration or deceleration occurs, we generate θ from the corresponding gamma distribution on $(0, \infty)$ during the MCMC implementation. This approach preserves the algebraic structure of the full conditional distribution while permitting greater flexibility in inference. Consequently, the generation of the MCMC samples for a and θ can be efficiently implemented employing standard gamma distribution routines. However, to generate the MCMC samples for b , we propose using the Metropolis-Hastings (MH) algorithm with a normal proposal distribution. To handle the full set of parameters a , b , and θ , we employ the MH-within-Gibbs sampling algorithm to generate the required MCMC samples. The implementation of the MH-within-Gibbs algorithm follows the steps outlined below for the sampling process.

Step 1: Set the initial values $(\hat{a}, \hat{b}, \hat{\theta})$ using the MLEs.

Step 2: Put $c = 1$.

Step 3: Generate $b^{(c)}$ employing the MH algorithm as follows:

- (1) Utilize the normal distribution with mean $b^{(c-1)}$ and variance $\hat{\sigma}_{22}$ to obtain a proposal value, denoted by b^* .
- (2) Computet the acceptance ratio as: $\delta = \min \left\{ 1, \frac{\mathcal{L}(b^* | \boldsymbol{\eta}_{-b}^{(c-1)}, \mathbf{y})}{\mathcal{L}(b^{(c-1)} | \boldsymbol{\eta}_{-b}^{(c-1)}, \mathbf{y})} \right\}$.
- (3) Simulate u from unit uniform distribution, i.e., $U(0, 1)$.
- (4) Store $b^{(c)} = b^*$ if $u \leq \delta$, otherwise set $b^{(c)} = b^{(c-1)}$.

Step 4: Use the obtained $b^{(c)}$ to generate: and $\theta^{(c)}$ as follows:

- $a^{(c)}$ using *Gamma* $\left(\tau_1, \sum_{j=1}^2 [\theta^{j-1}]^{(c-1)} \omega_j(\mathbf{y}; b^{(c)}) + \beta_1 \right)$.
- $\theta^{(c)}$ employing *Gamma* $\left(\tau_2, a^{(c)} \omega_2(\mathbf{y}; b^{(c)}) \right)$.

Step 5: Compute RF and HRF at normal operating settings at iteration c as

$$R_1^{(c)} = e^{-a^{(c)} t^{b^{(c)}}} \quad \text{and} \quad H_1^{(c)} = a^{(c)} b^{(c)} t^{b^{(c)}-1}.$$

Step 6: Put $c = c + 1$.

Step 7: Repeat Steps 3 to 7 a total of $\mathcal{M}$ times to generate the required MCMC samples as follows:

$$\{a^{(c)}, b^{(c)}, \theta^{(c)}, R_1^{(c)}, H_1^{(c)}\}, c = 1, \dots, \mathcal{M}.$$

Using the acquired MCMC samples, the Bayes estimates of the model parameter, the acceleration factor, and the two reliability metrics under the normal operating conditions can be obtained using the squared error loss function, respectively, as

$$\begin{aligned} \tilde{a} &= \frac{1}{\mathcal{M}^*} \sum_{c=\mathcal{B}+1}^{\mathcal{M}} a^{(c)}, \quad \tilde{b} = \frac{1}{\mathcal{M}^*} \sum_{c=\mathcal{B}+1}^{\mathcal{M}} b^{(c)}, \quad \tilde{\theta} = \frac{1}{\mathcal{M}^*} \sum_{c=\mathcal{B}+1}^{\mathcal{M}} \theta^{(c)} \\ \tilde{R}_1(t) &= \frac{1}{\mathcal{M}^*} \sum_{c=\mathcal{B}+1}^{\mathcal{M}} R_1^{(c)}, \quad \text{and} \quad \tilde{H}_1(t) = \frac{1}{\mathcal{M}^*} \sum_{c=\mathcal{B}+1}^{\mathcal{M}} H_1^{(c)}, \end{aligned}$$

where $\mathcal{B}$ represents the burn-in period, which is carefully selected to ensure proper convergence of the MCMC sequence. After sorting the MCMC samples as: $a^{[\mathcal{B}+1]} < \dots < a^{[\mathcal{M}]}$, $b^{[\mathcal{B}+1]} < \dots < b^{[\mathcal{M}]}$, $\theta^{[\mathcal{B}+1]} < \dots < \theta^{[\mathcal{M}]}$, $R_1^{[\mathcal{B}+1]} < \dots < R_1^{[\mathcal{M}]}$, and $H_1^{[\mathcal{B}+1]} < \dots < H_1^{[\mathcal{M}]}$. The $100(1 - \alpha)\%$ BCIs of $a, b, \theta, R_1(t)$ and $H_1(t)$, can be obtained, respectively, as

$$\begin{aligned} &\{a^{[\alpha \mathcal{M}^*/2]}, a^{[(1-\alpha/2)\mathcal{M}^*]}\}, \quad \{b^{[\alpha \mathcal{M}^*/2]}, b^{[(1-\alpha/2)\mathcal{M}^*]}\}, \quad \{\theta^{[\alpha \mathcal{M}^*/2]}, \theta^{[(1-\alpha/2)\mathcal{M}^*]}\} \\ &\{R_1^{[\alpha \mathcal{M}^*/2]}, R_1^{[(1-\alpha/2)\mathcal{M}^*]}\}, \quad \text{and} \quad \{H_1^{[\alpha \mathcal{M}^*/2]}, H_1^{[(1-\alpha/2)\mathcal{M}^*]}\}. \end{aligned}$$

where $\mathcal{M}^* = \mathcal{M} - \mathcal{B}$.

Remark 2: To approximate the Bayesian inferences of $a, b, \theta, R_1(t)$, and $H_1(t)$, in addition to their $100(1 - \alpha)\%$ BCIs, we recommend implementing the coda package, proposed by Plummer et al. [29], in R software that employs the ‘run_metropolis_MCMC()’ function of MH algorithm.

To ensure that the MCMC sampling scheme works efficiently, careful consideration is given to the choice of proposal distributions and tuning parameters. For the scale parameter a and the acceleration factor θ , the full conditional distributions given in (23) and (25) follow gamma distributions (with

θ generated on $(0, \infty)$). Consequently, these parameters are updated directly using Gibbs sampling, which allows exact sampling from their full conditional distributions and ensures computational efficiency and rapid convergence.

In contrast, the full conditional distribution of the Weibull shape parameter b , given in (24), does not correspond to any standard probability distribution. Therefore, an MH step is employed for updating b . Specifically, a normal random-walk proposal distribution of the form

$$b^* \sim \mathcal{N}(b^{(c-1)}, \hat{\sigma}_{22})$$

is adopted due to its simplicity and its widespread use in reliability and survival models involving shape parameters. The tuning parameter $\hat{\sigma}_{22}$ is selected through preliminary pilot runs in order to achieve an acceptance rate between 25% and 45%, which is commonly recommended for one-dimensional random-walk MH algorithms. This tuning strategy provides a suitable balance between efficient exploration of the posterior distribution and adequate acceptance of proposed moves.

The resulting MH-within-Gibbs sampler combines the computational efficiency of Gibbs updates with the flexibility of the MH algorithm, leading to satisfactory mixing behavior and stable convergence for all model parameters and reliability measures. The convergence properties of the generated MCMC chains are further assessed using standard diagnostic tools, as discussed in subsequent sections.

5 Monte Carlo Comparisons

To examine the accuracy, consistency, and efficiency of point and interval estimators for the Weibull parameters (a, b) , acceleration factor θ , RF $R_1(t)$, and HRF $H_1(t)$ developed in earlier sections, this section presents several Monte Carlo simulation studies aimed at evaluating their performances.

5.1 Simulation Scenarios

Using various configurations of full sample sizes n_i ($i = 1, 2$), effective sample sizes (k_i, m_i) for ($i = 1, 2$), threshold times T_{ij} ($i, j = 1, 2$), and progressive censoring patterns $(Q_{j1}, Q_{j2}, \dots, Q_{jm_j})$, $j = 1, 2$, we simulate 1000 UTIPHC samples using the CSPALT framework when the true parameter values for (a, b, θ) are taken as $(1.5, 0.5, 1.5)$. Table 1 outlines several combinations of n_i, k_i, m_i, T_{ij} , and $(Q_{j1}, Q_{j2}, \dots, Q_{jm_j})$, $j = 1, 2$, where the thresholds are fixed as $T_{11} \in (0.5, 0.8)$, $T_{12} = T_{21} \in (1.0, 1.5)$, and $T_{22} \in (1.5, 2.0)$. All offered estimators of $R_1(t)$ and $H_1(t)$ are fitted under normal operating conditions at a selected mission time point $t = 0.1$ when their plausible values are taken as 0.9843 and 0.2372, respectively. In Algorithm 1, the data generation steps for UTIPHC-Weibull-based samples are provided within the CSPALT framework.

Table 1: Various simulation setups of CSPALT from UTIPHC-based

Test	(n_1, n_2)	$\{(k_1, m_1), (k_2, m_2)\}$	$\{(Q_{11}, Q_{12}, \dots, Q_{1m_1}), (Q_{21}, Q_{22}, \dots, Q_{2m_2})\}$
A1[1]	(20, 30)	$\{(5, 10), (10, 15)\}$	$\{(2 * 5, 0 * 5), (3 * 5, 0 * 10)\}$
A1[2]			$\{(0 * 3, 2 * 5, 0 * 2), (0 * 5, 3 * 5, 0 * 5)\}$
A1[3]			$\{(0 * 5, 2 * 5), (0 * 10, 3 * 5)\}$
B1[1]		$\{(10, 15), (15, 20)\}$	$\{(5 * 1, 0 * 14), (5 * 2, 0 * 18)\}$

(Continued)

Table 1 (continued)

Test	(n_1, n_2)	$\{(k_1, m_1), (k_2, m_2)\}$	$\{(Q_{11}, Q_{12}, \dots, Q_{1m_1}), (Q_{21}, Q_{22}, \dots, Q_{2m_2})\}$
B1[2] B1[3]			$\{(0 * 7, 5 * 1, 0 * 7), (0 * 9, 5 * 2, 0 * 9)\}$ $\{(0 * 14, 5 * 1), (0 * 18, 5 * 2)\}$
A2[1] A2[2] A2[3]	(40, 50)	$\{(10, 20), (20, 30)\}$	$\{(5 * 4, 0 * 16), (5 * 4, 0 * 26)\}$ $\{(0 * 8, 5 * 4, 0 * 8), (0 * 13, 5 * 4, 0 * 13)\}$ $\{(0 * 16, 5 * 4), (0 * 26, 5 * 4)\}$
B2[1] B2[2] B2[3]		$\{(20, 30), (30, 40)\}$	$\{(5 * 2, 0 * 28), (5 * 2, 0 * 38)\}$ $\{(0 * 14, 5 * 2, 0 * 14), (0 * 19, 5 * 2, 0 * 19)\}$ $\{(0 * 28, 5 * 2), (0 * 38, 5 * 2)\}$
A3[1] A3[2] A3[3]	(60, 80)	$\{(20, 30), (30, 40)\}$	$\{(5 * 6, 0 * 24), (5 * 8, 0 * 32)\}$ $\{(0 * 12, 5 * 6, 0 * 12), (0 * 16, 5 * 8, 0 * 16)\}$ $\{(0 * 24, 5 * 6), (0 * 32, 5 * 8)\}$
B3[1] B3[2] B3[3]		$\{(30, 40), (40, 60)\}$	$\{(5 * 4, 0 * 36), (5 * 4, 0 * 56)\}$ $\{(0 * 18, 5 * 4, 0 * 18), (0 * 28, 5 * 4, 0 * 28)\}$ $\{(0 * 36, 5 * 4), (0 * 56, 5 * 4)\}$

Algorithm 1 : Generation of UTIPHC-Weibull samples using CSPALT

```

1: Input:  $a > 0, b > 0, \theta > 0$ , number of replications  $n_{\text{rep}}$ 
2: Group 1:  $n_1, k_1, m_1, T_{11}, T_{12}$ , and  $Q^{(1)} = (Q_{11}, Q_{12}, \dots, Q_{1m_1})$ 
3: Group 2:  $n_2, k_2, m_2, T_{21}, T_{22}$ , and  $Q^{(2)} = (Q_{21}, Q_{22}, \dots, Q_{2m_2})$ 
4: for  $j = 1$  to  $n_{\text{rep}}$  do
5:   (A) Generate from Group 1
6:   for  $i = 1$  to  $m_1$  do
7:     Generate  $W_i \sim \text{Uniform}(0, 1)$ 
8:     Compute  $E_i = \frac{1}{i + \sum_{j=m_1+1-i}^{m_1} Q_{1j}^{(1)}}$ 
9:     Set  $V_i = W_i^{E_i}$ 
10:  end for
11:  for  $i = 1$  to  $m_1$  do
12:    Compute  $U_i = 1 - \prod_{j=m_1+1-i}^{m_1} V_j$ 
13:  end for
14:  Obtain Weibull failure times  $Y_i^{(1)} = \left(-\frac{1}{b} \log(1 - U_i)\right)^{1/a}$ 
15:  Compute  $d_1 = \sum_{i=1}^{m_1} I(0 \leq Y_i^{(1)} < T_{11})$ ,  $r_1 = \sum_{i=1}^{m_1} I(0 \leq Y_i^{(1)} < T_{12})$ 

```

(Continued)

Algorithm 1 (continued)

```

16:   Determine
      
$$D_1 = \begin{cases} \min(r_1, m_1), & k_1 \leq d_1, \\ \min(r_1, m_1), & d_1 < k_1 \leq r_1, \\ k_1, & k_1 > r_1 \end{cases}$$

17:   Observed data:  $Y_i^{(1)} = \begin{cases} Y_i^{(1)}, & i \leq D_1 \\ 0, & i > D_1 \end{cases}$ 
18:   Determine
      
$$(D_1, T_1^*, Q_1^*) = \begin{cases} (m_1 - 1, Y_{1m_1}, n_1 - m_1 - \sum_{i=1}^{m_1-1} Q_{1i}), & \text{Case 1,} \\ (d_1, T_{11}, n_1 - d_1 - \sum_{i=1}^{d_1} Q_{1i}), & \text{Case 2,} \\ (k_1 - 1, Y_{1k_1}, n_1 - k_1 - \sum_{i=1}^{k_1-1} Q_{1i}), & \text{Case 3,} \\ (r_1, T_{12}, n_1 - r_1 - \sum_{i=1}^{r_1} Q_{1i}), & \text{Case 4.} \end{cases}$$

19:   Record observed failure times  $Y_1^{(1)}, Y_2^{(1)}, \dots, Y_{D_1}^{(1)}$ 
20:   (B) Generate from Group 2
21:   for  $i = 1$  to  $m_2$  do
22:     Redo Steps 6–13 using  $n_2, m_2$ , and  $Q^{(2)}$ 
23:   end for
24:   Obtain Weibull failure times:  $Y_i^{(2)} = \left(-\frac{1}{\theta b} \log(1 - U_i)\right)^{1/a}$ 
25:   Redo Steps 15–18 using  $k_2, d_2$ , and  $T_{2j}$  (for  $j = 1, 2$ )
26:   Record observed failure times  $Y_1^{(2)}, Y_2^{(2)}, \dots, Y_{D_2}^{(2)}$ 
27: end for
28: Output: Collect required samples  $\{Y^{(1)}, Q^{(1)}, Y^{(2)}, Q^{(2)}\}$ 

```

After generating the required 1000 UTIPHC samples, the MLEs and corresponding 95% ACIs for the parameters $a, b, \theta, R_1(t)$, and $H_1(t)$ are obtained using the Newton–Raphson method implemented utilizing the `maxLik` package, which was introduced by Henningsen and Toomet [28]. For the Bayesian analysis, two independent prior groups for the hyperparameters (λ_i, β_i) for $i = 1, 2$ are considered for each parameter of the proposed Weibull lifespan model as follows:

- Prior-1: $(\lambda_1, \lambda_2) = (7.5, 2.5)$ and $\beta_i = 5$, $i = 1, 2$;
- Prior-2: $(\lambda_1, \lambda_2) = (15, 5)$ and $\beta_i = 10$, $i = 1, 2$.

All selected values for prior parameters (λ_i, β_i) , $i = 1, 2$, are chosen such that the prior means align closely with the true parameter values of a and b , thereby incorporating informative prior knowledge into the Bayesian framework; see, for additional details, Kundu [30]. Remember that, as motivated in Section 4, we are still assuming that the acceleration operator parameter θ follows a non-informative uniform distribution. Subsequently, to assess the acquired Bayes estimator corresponding to each unknown parameter, we produce $\mathcal{M} = 12,000$ MCMC samples (with $-\mathcal{B} = 2000$ as burn-in) from the posterior distributions (23)–(25) using the MH-within-Gibbs sampling algorithm. The initial 2000 iterations are discarded as burn-in to ensure convergence. The remaining samples $\mathcal{M}^* = 10,000$ are analyzed using the `coda` package, which was developed by Plummer et al. [29], to compute Bayesian point estimates in addition to 95% BCI estimates for $a, b, \theta, R_1(t)$, and $H_1(t)$.

In this part, the MLE procedure is initialized using starting values chosen from the theoretical domains of the underlying parameters, while the Bayesian iterative algorithms are initialized at the corresponding MLEs obtained from the observed data. The process is terminated when either the relative change in successive estimates fall below a prescribed tolerance (e.g., 10^{-8}). Convergence of the MLE optimization routine was achieved within 12 iterations, and the Bayesian MCMC algorithm reached stationarity after 1500 burn-in iterations (with a total of 10,000 iterations), as verified by trace plots and stability of posterior summaries. This approach ensures stable numerical performance and facilitates reliable posterior inference. Moreover, sensitivity checks with different reasonable starting values produced negligible differences in the final estimates, indicating that the inferential results are robust to the choice of initial values. It is worth noticing here that the average computational time per replication for both the MLE and Bayesian methods is 35 s per iteration, demonstrating that while the Bayesian approach requires slightly more time, it remains computationally feasible and comparable in efficiency to the MLE procedure, which needs 25 s per iteration.

In each simulation setup, the mean point estimates (MPEs) for a , b , θ , $R_1(t)$, or $H_1(t)$ (say, Y for short) are calculated as follows:

$$\text{MPE}(\check{Y}_\varsigma) = \frac{1}{1000} \sum_{i=1}^{1000} \check{Y}_\varsigma^{[i]}, \quad \varsigma = 1, 2, \dots, 5,$$

where $\check{Y}_\varsigma^{[i]}$ denotes the estimate of Y_ς obtained from the i th simulated sample, $Y_1 = a$, $Y_2 = b$, $Y_3 = \theta$, $Y_4 = R_1(t)$, and $Y_5 = H_1(t)$.

The accuracy of the point estimates is assessed using the root mean squared-errors (RMSEs) and average relative absolute-biases (ARABs), defined respectively as

$$\text{RMSE}(\check{Y}_\varsigma) = \sqrt{\frac{1}{1000} \sum_{i=1}^{1000} \left(\check{Y}_\varsigma^{[i]} - Y_\varsigma \right)^2}, \quad \varsigma = 1, 2, \dots, 5,$$

and

$$\text{ARAB}(\check{Y}_\varsigma) = \frac{1}{1000} \sum_{i=1}^{1000} Y_\varsigma^{-1} \left| \check{Y}_\varsigma^{[i]} - Y_\varsigma \right|, \quad \varsigma = 1, 2, \dots, 5.$$

The performance of the interval estimates is evaluated using the average interval lengths (AILs) and the corresponding coverage probabilities (CPs), which are computed as

$$\text{AIL}^{95\%}(Y_\varsigma) = \frac{1}{1000} \sum_{i=1}^{1000} \left(U_{\check{Y}_\varsigma^{[i]}} - L_{\check{Y}_\varsigma^{[i]}} \right), \quad \varsigma = 1, 2, \dots, 5,$$

and

$$\text{CP}^{95\%}(Y_\varsigma) = \frac{1}{1000} \sum_{i=1}^{1000} \mathbb{J}_{(L_{\check{Y}_\varsigma^{[i]}}, U_{\check{Y}_\varsigma^{[i]}})}(Y_\varsigma), \quad \varsigma = 1, 2, \dots, 5,$$

where $\mathbb{J}_{(a,b)}(Y_\varsigma)$ is an indicator function that equals 1 if $Y_\varsigma \in (a, b)$ and 0 otherwise, and $L_{\check{Y}_\varsigma^{[i]}}$ and $U_{\check{Y}_\varsigma^{[i]}}$ denote the lower and upper bounds of the ACI (or BCI) at i -th sample.

5.2 Simulation Results and Interpretation

All simulation findings of a , b , θ , $R_1(t)$, and $H_1(t)$ are presented in Tables 2–11. Based on the goal of minimizing RMSE, ARAB, and AIL while maximizing CP, from Tables 2–11, the following observations are made:

- Overall, the calculated point and 95% interval estimates for the Weibull parameters (a, b), acceleration factor θ , RF $R_1(t)$, and HRF $H_1(t)$ demonstrate reliable and satisfactory performance across simulation setups.
- Increasing the sample sizes n_i , $i = 1, 2$ enhances estimation accuracy; a similar improvement is observed as the effective sample sizes k_i (or m_i) for $i = 1, 2$, increase.
- When the total of progressive removals in $(Q_{j1}, Q_{j2}, \dots, Q_{jm_j})$, $j = 1, 2$, is reduced, indicating higher precision and reliability of the resulting estimators.
- Bayesian estimators (alongside their associated 95% BCIs) consistently outperform classical MLEs (and their 95% ACIs), owing to the incorporation of informative gamma priors.
- As the censoring thresholds T_{ij} , $i, j = 1, 2$ increase, RMSEs, ARABs, and AILs for all parameters and functions decrease, while CPs improve accordingly.
- Bayesian evaluations (based on either point estimates from MCMC or BCI methods) under Prior-2, which has lower prior variance than Prior-1, yield superior results.
- A comparison of the proposed progressive removal strategies, listed in Table 1, reveals the following:
 - The estimation accuracy of the Weibull parameters (a, b) and HRF $H_1(t)$ exhibit superior behavior using Ai [3] and Bi [3] for $i = 1, 2, 3$ “right censoring”;
 - The estimation accuracy of the RF $R_1(t)$ exhibits superior behavior using Ai [1] and Bi [1] for $i = 1, 2, 3$ “left censoring”;
 - The estimation accuracy of the acceleration factor θ exhibits superior behavior using Ai [2] and Bi [2] for $i = 1, 2, 3$ “middle censoring.”
- In summary, the Bayesian estimation procedure implemented via the MH-within-Gibbs algorithm provides a reliable and effective approach for estimating the model parameters and associated reliability measures of the Weibull lifetime distribution under CSPALTs using UTIPHC data.

Table 2: The MPE (1st Col.), RMSE (2nd Col.), and ARAB (3rd Col.) of a

(n_1, n_2)	Test	MLE			Bayes					
					Prior-1			Prior-2		
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.5, 1.0), (1.0, 1.5)\}$										
(20, 30)	A1[1]	1.5538	0.5838	0.3083	1.4809	0.3822	0.2069	1.5680	0.3019	0.1764
	A1[2]	1.9281	0.5951	0.3177	1.6276	0.4845	0.3038	1.7168	0.3958	0.2457
	A1[3]	1.9430	0.5660	0.3062	1.6193	0.2898	0.1640	1.6926	0.2669	0.1336
	B1[1]	1.8067	0.4918	0.2861	1.4543	0.2639	0.1526	1.5400	0.1951	0.1137
	B1[2]	1.7628	0.5274	0.2963	1.4967	0.2646	0.1528	1.5635	0.2333	0.1169

(Continued)

Table 2 (continued)

(n_1, n_2)	Test	MLE			Bayes					
					Prior-1			Prior-2		
	B1[3]	1.9335	0.4312	0.2277	1.6218	0.2630	0.1502	1.7283	0.1940	0.1125
(40, 50)	A2[1]	1.6496	0.3695	0.2050	1.5610	0.1894	0.1028	1.7139	0.1778	0.0971
	A2[2]	1.6486	0.3965	0.2075	1.6749	0.1910	0.1106	1.7479	0.1885	0.1099
	A2[3]	1.9379	0.3458	0.1885	1.7927	0.1882	0.1022	1.9455	0.1741	0.0913
	B2[1]	1.7972	0.2895	0.1598	1.6653	0.1846	0.0954	1.4872	0.1669	0.0754
	B2[2]	1.7683	0.3209	0.1773	1.6726	0.1848	0.0987	1.4788	0.1736	0.0820
	B2[3]	1.6435	0.2865	0.1505	1.6636	0.1836	0.0929	1.5114	0.1517	0.0731
(60, 80)	A3[1]	1.5676	0.2756	0.1413	1.5256	0.1331	0.0723	1.3883	0.1326	0.0683
	A3[2]	1.5679	0.2843	0.1491	1.5194	0.1555	0.0869	1.3973	0.1440	0.0691
	A3[3]	1.9254	0.2693	0.1379	1.8686	0.1293	0.0691	1.6389	0.1135	0.0593
	B3[1]	1.7306	0.1843	0.0946	1.5081	0.1093	0.0529	1.6222	0.1047	0.0519
	B3[2]	1.6739	0.2368	0.1284	1.4887	0.1251	0.0585	1.5767	0.1049	0.0523
	B3[3]	1.7340	0.1821	0.0939	1.4040	0.0923	0.0492	1.4654	0.0888	0.0472
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.8, 1.5), (1.5, 2.0)\}$										
(20, 30)	A1[1]	1.4980	0.5779	0.3326	1.4333	0.3591	0.1894	1.5465	0.2630	0.1528
	A1[2]	1.9112	0.5809	0.3348	1.6212	0.4692	0.2952	1.6905	0.3787	0.2425
	A1[3]	1.9406	0.5580	0.3121	1.5947	0.2630	0.1490	1.6611	0.2333	0.1311
	B1[1]	1.7531	0.5546	0.3014	1.4438	0.2379	0.1339	1.5003	0.2097	0.1106
	B1[2]	1.6663	0.5535	0.2966	1.4537	0.2139	0.1245	1.5154	0.1960	0.0978
	B1[3]	1.9347	0.3699	0.2011	1.6218	0.1946	0.1131	1.7283	0.1849	0.0975
(40, 50)	A2[1]	1.6003	0.3470	0.1958	1.5560	0.1876	0.1096	1.6337	0.1725	0.0923
	A2[2]	1.5413	0.3612	0.2001	1.5225	0.1903	0.1103	1.6021	0.1840	0.0969
	A2[3]	1.8564	0.3138	0.1722	1.7669	0.1714	0.0926	1.9350	0.1653	0.0916
	B2[1]	1.7451	0.3052	0.1571	1.6361	0.1669	0.0856	1.5164	0.1651	0.0834
	B2[2]	1.6867	0.2658	0.1410	1.6035	0.1602	0.0833	1.4654	0.1541	0.0776
	B2[3]	1.7883	0.2467	0.1316	1.6912	0.1563	0.0787	1.5416	0.1522	0.0749
(60, 80)	A3[1]	1.5350	0.2240	0.1148	1.4863	0.1397	0.0734	1.3919	0.1338	0.0631
	A3[2]	1.4741	0.2376	0.1310	1.4403	0.1470	0.0754	1.3662	0.1375	0.0727
	A3[3]	1.9973	0.1928	0.0986	1.8638	0.1292	0.0702	1.6437	0.1231	0.0626
	B3[1]	1.6844	0.1570	0.0823	1.4954	0.1211	0.0629	1.5839	0.1050	0.0540
	B3[2]	1.6004	0.1741	0.0921	1.4365	0.1218	0.0639	1.5174	0.1104	0.0564
	B3[3]	1.7975	0.1486	0.0803	1.6094	0.0999	0.0526	1.6450	0.0996	0.0479

Table 3: The MPE (1st Col.), RMSE (2nd Col.), and ARAB (3rd Col.) of b

(n_1, n_2)	Test	MLE			Bayes					
					Prior-1			Prior-2		
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.5, 1.0), (1.0, 1.5)\}$										
(20, 30)	A1[1]	0.4975	0.3873	0.7477	0.5389	0.3795	0.7252	0.6042	0.3450	0.6836
	A1[2]	0.5608	0.4007	0.7608	0.5201	0.3934	0.7343	0.7194	0.3651	0.7277
	A1[3]	0.5164	0.3807	0.7457	0.4828	0.3680	0.7182	0.7625	0.3406	0.6697
	B1[1]	0.6711	0.3692	0.7061	0.5817	0.3534	0.6611	0.6669	0.3015	0.5971
	B1[2]	0.6977	0.3709	0.7278	0.5231	0.3562	0.6845	0.4001	0.3181	0.6303
	B1[3]	0.5427	0.3643	0.6823	0.6257	0.3485	0.6493	0.5544	0.2898	0.5685
(40, 50)	A2[1]	0.5582	0.3476	0.6719	0.6169	0.3279	0.6348	0.6097	0.2739	0.5388
	A2[2]	0.6674	0.3489	0.6778	0.6310	0.3357	0.6350	0.4312	0.2889	0.5631
	A2[3]	0.4080	0.3338	0.6420	0.5381	0.3151	0.5878	0.5283	0.2541	0.4063
	B2[1]	0.5818	0.3013	0.5537	0.5694	0.2956	0.4938	0.5264	0.2137	0.3810
	B2[2]	0.5220	0.3027	0.5751	0.6262	0.2969	0.5443	0.4122	0.2511	0.4049
	B2[3]	0.5262	0.2935	0.5453	0.5482	0.2805	0.4754	0.4562	0.2083	0.3711
(60, 80)	A3[1]	0.4582	0.2752	0.4708	0.5839	0.2599	0.4032	0.3769	0.1937	0.3182
	A3[2]	0.6647	0.2931	0.5150	0.4738	0.2667	0.4620	0.5164	0.2039	0.3435
	A3[3]	0.5399	0.2705	0.4392	0.6093	0.2234	0.3746	0.5934	0.1583	0.2508
	B3[1]	0.6131	0.2080	0.2552	0.5180	0.1972	0.2399	0.5907	0.1286	0.2015
	B3[2]	0.4928	0.2444	0.4016	0.6130	0.2108	0.2526	0.4601	0.1293	0.2132
	B3[3]	0.5091	0.2058	0.2415	0.5154	0.1487	0.1691	0.5127	0.1062	0.1540
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.8, 1.5), (1.5, 2.0)\}$										
(20, 30)	A1[1]	0.4421	0.3873	0.7472	0.3373	0.3692	0.7217	0.5447	0.3440	0.6825
	A1[2]	0.5264	0.3960	0.7593	0.5030	0.3801	0.7317	0.5856	0.3634	0.7244
	A1[3]	0.5013	0.3794	0.7424	0.4642	0.3620	0.7046	0.4729	0.3402	0.6655
	B1[1]	0.5652	0.3677	0.7100	0.5747	0.3463	0.6640	0.5275	0.3180	0.6291
	B1[2]	0.4844	0.3606	0.6954	0.6097	0.3447	0.6560	0.5174	0.3006	0.5960
	B1[3]	0.5027	0.3496	0.6821	0.5626	0.3225	0.6262	0.5544	0.2862	0.5592
(40, 50)	A2[1]	0.5231	0.3369	0.6518	0.4863	0.3189	0.5972	0.2858	0.2594	0.4884
	A2[2]	0.5147	0.3403	0.6521	0.4857	0.3191	0.6074	0.3645	0.2854	0.5458
	A2[3]	0.6155	0.3255	0.5986	0.5634	0.3027	0.5651	0.5076	0.2478	0.3760
	B2[1]	0.5877	0.3014	0.5476	0.5151	0.2958	0.5060	0.5154	0.2142	0.3522
	B2[2]	0.5774	0.2958	0.5383	0.4903	0.2933	0.4794	0.5426	0.2018	0.3476
	B2[3]	0.5054	0.2912	0.5363	0.5256	0.2599	0.4713	0.4696	0.1868	0.3142

(Continued)

Table 3 (continued)

(n_1, n_2)	Test	MLE		Bayes						
				Prior-1			Prior-2			
(60, 80)	A3[1]	0.4869	0.2664	0.4442	0.6208	0.2456	0.3581	0.6299	0.1429	0.2222
	A3[2]	0.5298	0.2828	0.4754	0.6657	0.2553	0.4442	0.4549	0.1574	0.2776
	A3[3]	0.6298	0.2465	0.3941	0.6050	0.2161	0.3046	0.5962	0.1298	0.2140
	B3[1]	0.5378	0.2080	0.2428	0.5601	0.1895	0.2099	0.4748	0.1176	0.1868
	B3[2]	0.6020	0.2375	0.3825	0.4877	0.1933	0.2322	0.5278	0.1277	0.2077
	B3[3]	0.4802	0.1873	0.2195	0.5263	0.1194	0.1614	0.4708	0.0960	0.1071

Table 4: The MPE (1st Col.), RMSE (2nd Col.), and ARAB (3rd Col.) of θ

(n_1, n_2)	Test	MLE			Bayes					
					Prior-1			Prior-2		
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.5, 1.0), (1.0, 1.5)\}$										
(20, 30)	A1[1]	1.3516	1.0675	0.7045	1.0900	0.5097	0.2858	1.1478	0.4484	0.2756
	A1[2]	1.4500	1.0042	0.5765	1.0867	0.4376	0.2742	1.1612	0.4249	0.2539
	A1[3]	1.5562	1.0399	0.6631	1.0959	0.4925	0.2824	1.1684	0.4422	0.2720
	B1[1]	1.3733	0.9609	0.5369	1.4710	0.4247	0.2638	1.7786	0.4149	0.2537
	B1[2]	1.3066	0.8399	0.4776	1.5056	0.4169	0.2548	1.7434	0.4079	0.2496
	B1[3]	1.4152	0.9080	0.5037	1.3904	0.4208	0.2596	1.6518	0.4113	0.2509
(40, 50)	A2[1]	1.3741	0.7776	0.4717	1.1335	0.4137	0.2493	1.5022	0.4072	0.2402
	A2[2]	1.5230	0.6695	0.4288	1.1205	0.4058	0.2260	1.4682	0.3676	0.2077
	A2[3]	1.3893	0.7490	0.4429	1.1225	0.4075	0.2471	1.4712	0.3939	0.2282
	B2[1]	1.3414	0.6401	0.3947	1.4422	0.3939	0.2246	1.3833	0.3582	0.2018
	B2[2]	1.1416	0.5926	0.3398	1.3451	0.3654	0.1927	1.4365	0.3093	0.1577
	B2[3]	1.0474	0.5944	0.3440	1.1235	0.3749	0.1989	1.4913	0.3160	0.1641
(60, 80)	A3[1]	1.2496	0.5888	0.3355	1.4158	0.3574	0.1845	1.1387	0.2639	0.1159
	A3[2]	1.4024	0.5466	0.3112	1.2882	0.3023	0.1582	1.1479	0.2514	0.1081
	A3[3]	1.2192	0.5607	0.3274	1.5702	0.3272	0.1826	1.2075	0.2595	0.1153
	B3[1]	1.5905	0.5449	0.3107	1.2757	0.2727	0.1297	1.4629	0.2307	0.0975
	B3[2]	1.2780	0.3965	0.2256	1.2329	0.2132	0.0975	1.3880	0.2045	0.0959
	B3[3]	1.2914	0.5308	0.2917	1.2992	0.2602	0.1160	1.5318	0.2270	0.0964
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.8, 1.5), (1.5, 2.0)\}$										
(20, 30)	A1[1]	1.6988	0.9461	0.5775	1.5748	0.4748	0.2835	1.6128	0.4279	0.2724

(Continued)

Table 4 (continued)

(n_1, n_2)	Test	MLE			Bayes					
					Prior-1			Prior-2		
	A1[2]	1.5008	0.9200	0.5017	1.4887	0.4373	0.2694	1.1651	0.4197	0.2411
	A1[3]	1.6304	0.9290	0.5738	1.1044	0.4504	0.2734	1.1759	0.4266	0.2423
	B1[1]	1.1259	0.9140	0.4561	1.3446	0.4213	0.2539	1.7296	0.4089	0.2409
	B1[2]	1.3567	0.7213	0.4424	1.4318	0.4143	0.2510	1.6463	0.4067	0.2350
	B1[3]	1.5636	0.7986	0.4514	1.3904	0.4146	0.2530	1.6518	0.4075	0.2396
(40, 50)	A2[1]	1.6955	0.6943	0.4127	1.1312	0.4121	0.2453	1.3296	0.4026	0.2336
	A2[2]	1.4676	0.6657	0.3830	1.1121	0.3908	0.2208	1.3106	0.3046	0.1521
	A2[3]	1.2281	0.6739	0.4072	1.1279	0.4069	0.2306	1.5154	0.3310	0.1791
	B2[1]	1.0785	0.6260	0.3448	1.3723	0.3774	0.2170	1.4192	0.2887	0.1451
	B2[2]	1.4496	0.5629	0.3096	1.2666	0.3542	0.1845	1.5256	0.2663	0.1267
	B2[3]	1.2438	0.5660	0.3338	1.1177	0.3715	0.1959	1.3921	0.2779	0.1352
(60, 80)	A3[1]	1.5967	0.5008	0.2872	1.3276	0.2925	0.1333	1.5194	0.2207	0.0969
	A3[2]	1.4242	0.4519	0.2476	1.2692	0.2623	0.1212	1.4397	0.2106	0.0951
	A3[3]	1.6843	0.4972	0.2857	1.5957	0.2814	0.1234	1.6027	0.2165	0.0964
	B3[1]	1.2210	0.4226	0.2367	1.3993	0.2576	0.1157	1.6269	0.2058	0.0928
	B3[2]	1.2847	0.3674	0.2048	1.4920	0.2132	0.0961	1.4797	0.1784	0.0719
	B3[3]	1.3513	0.3985	0.2057	1.6765	0.2296	0.0989	1.3713	0.1992	0.0764

Table 5: The MPE (1st Col.), RMSE (2nd Col.), and ARAB (3rd Col.) of $R_i(t)$

(n_1, n_2)	Test	MLE			Bayes					
					Prior-1			Prior-2		
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.5, 1.0), (1.0, 1.5)\}$										
(20, 30)	A1[1]	0.9835	0.2141	0.1389	0.9857	0.1895	0.1266	0.9890	0.1265	0.1241
	A1[2]	0.9911	0.2481	0.1428	0.9855	0.2008	0.1331	0.9892	0.1299	0.1282
	A1[3]	0.9921	0.2241	0.1411	0.9863	0.1964	0.1299	0.9891	0.1291	0.1269
	B1[1]	0.9967	0.1833	0.1243	0.9937	0.1633	0.1168	0.9954	0.1128	0.1109
	B1[2]	0.9951	0.1858	0.1253	0.9925	0.1811	0.1205	0.9940	0.1213	0.1180
	B1[3]	0.9912	0.1881	0.1332	0.9916	0.1833	0.1246	0.9942	0.1239	0.1214
(40, 50)	A2[1]	0.9936	0.1609	0.1124	0.9920	0.1388	0.1027	0.9955	0.1065	0.1012
	A2[2]	0.9909	0.1757	0.1202	0.9910	0.1586	0.1168	0.9939	0.1095	0.1048
	A2[3]	0.9943	0.1657	0.1145	0.9909	0.1568	0.1132	0.9948	0.1068	0.1032
	B2[1]	0.9971	0.1252	0.1064	0.9965	0.1161	0.0949	0.9930	0.1036	0.0941

(Continued)

Table 5 (continued)

(n_1, n_2)	Test	MLE			Bayes					
					Prior-1			Prior-2		
	B2[2]	0.9958	0.1421	0.1091	0.9957	0.1167	0.1004	0.9917	0.1057	0.0950
	B2[3]	0.9861	0.1474	0.1105	0.9941	0.1217	0.1009	0.9910	0.1059	0.0968
(60, 80)	A3[1]	0.9946	0.1117	0.0906	0.9948	0.1045	0.0791	0.9909	0.0837	0.0773
	A3[2]	0.9906	0.1202	0.0979	0.9918	0.1110	0.0933	0.9878	0.1035	0.0928
	A3[3]	0.9944	0.1135	0.0933	0.9959	0.1058	0.0917	0.9917	0.0993	0.0792
	B3[1]	0.9972	0.0884	0.0696	0.9936	0.0781	0.0672	0.9949	0.0665	0.0583
	B3[2]	0.9954	0.0994	0.0896	0.9922	0.0944	0.0705	0.9933	0.0801	0.0663
	B3[3]	0.9893	0.1087	0.0900	0.9790	0.0980	0.0780	0.9838	0.0830	0.0753
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.8, 1.5), (1.5, 2.0)\}$										
(20, 30)	A1[1]	0.9837	0.1978	0.1341	0.9862	0.1830	0.1256	0.9895	0.1240	0.1239
	A1[2]	0.9912	0.2029	0.1385	0.9859	0.1838	0.1274	0.9888	0.1262	0.1262
	A1[3]	0.9923	0.2363	0.1416	0.9862	0.1873	0.1312	0.9889	0.1269	0.1273
	B1[1]	0.9964	0.1796	0.1178	0.9935	0.1611	0.1127	0.9955	0.1108	0.1098
	B1[2]	0.9944	0.1864	0.1277	0.9925	0.1794	0.1231	0.9941	0.1170	0.1163
	B1[3]	0.9920	0.1851	0.1249	0.9916	0.1676	0.1185	0.9942	0.1141	0.1128
(40, 50)	A2[1]	0.9936	0.1543	0.1061	0.9930	0.1232	0.1017	0.9944	0.1047	0.0987
	A2[2]	0.9901	0.1572	0.1145	0.9909	0.1382	0.1037	0.9931	0.1054	0.1004
	A2[3]	0.9962	0.1638	0.1169	0.9910	0.1409	0.1058	0.9952	0.1075	0.1047
	B2[1]	0.9968	0.1168	0.0995	0.9967	0.1120	0.0946	0.9938	0.1025	0.0917
	B2[2]	0.9951	0.1256	0.1061	0.9953	0.1185	0.0970	0.9921	0.1034	0.0958
	B2[3]	0.9906	0.1250	0.1015	0.9942	0.1162	0.0952	0.9912	0.1028	0.0950
(60, 80)	A3[1]	0.9943	0.1110	0.0870	0.9949	0.0989	0.0776	0.9919	0.0836	0.0734
	A3[2]	0.9895	0.1123	0.0926	0.9915	0.1007	0.0844	0.9885	0.0982	0.0777
	A3[3]	0.9965	0.1140	0.0937	0.9959	0.1098	0.0929	0.9917	0.1004	0.0903
	B3[1]	0.9969	0.0806	0.0680	0.9938	0.0742	0.0627	0.9945	0.0648	0.0376
	B3[2]	0.9946	0.1007	0.0827	0.9921	0.0949	0.0766	0.9931	0.0806	0.0713
	B3[3]	0.9916	0.0988	0.0759	0.9917	0.0875	0.0702	0.9921	0.0734	0.0571

Table 6: The MPE (1st Col.), RMSE (2nd Col.), and ARAB (3rd Col.) of $H_1(t)$.

(n_1, n_2)	Test	MLE			Bayes					
					Prior-1			Prior-2		
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.5, 1.0), (1.0, 1.5)\}$										
(20, 30)	A1[1]	0.2339	0.2595	0.8437	0.2033	0.2415	0.7932	0.1607	0.1898	0.7762
	A1[2]	0.1503	0.2864	0.8492	0.2201	0.2461	0.8055	0.1681	0.1919	0.7890
	A1[3]	0.1333	0.2540	0.8125	0.2074	0.2321	0.7795	0.1673	0.1866	0.7586
	B1[1]	0.2541	0.2250	0.7533	0.2129	0.2152	0.7024	0.2070	0.1709	0.6905
	B1[2]	0.2778	0.2258	0.7760	0.3090	0.2177	0.7239	0.1922	0.1749	0.7091
	B1[3]	0.1482	0.2212	0.7433	0.1302	0.2083	0.6981	0.1894	0.1651	0.6735
(40, 50)	A2[1]	0.3100	0.2015	0.7093	0.1108	0.1855	0.6787	0.2170	0.1560	0.6116
	A2[2]	0.1426	0.2065	0.7158	0.1316	0.2016	0.6836	0.1932	0.1587	0.6566
	A2[3]	0.1823	0.1997	0.6852	0.1383	0.1812	0.6428	0.2086	0.1548	0.6012
	B2[1]	0.2490	0.1853	0.6755	0.2058	0.1702	0.6271	0.2465	0.1426	0.5386
	B2[2]	0.2710	0.1891	0.6812	0.1782	0.1743	0.6408	0.2308	0.1450	0.5903
	B2[3]	0.2125	0.1769	0.6689	0.1953	0.1662	0.5949	0.2298	0.1391	0.5217
(60, 80)	A3[1]	0.3081	0.1709	0.5622	0.2080	0.1628	0.5023	0.1245	0.1190	0.4182
	A3[2]	0.1425	0.1731	0.6177	0.1249	0.1639	0.5622	0.1681	0.1390	0.5127
	A3[3]	0.2101	0.1639	0.5261	0.1763	0.1565	0.4896	0.1330	0.1106	0.4090
	B3[1]	0.3461	0.1564	0.4673	0.1916	0.1249	0.3891	0.2076	0.1050	0.3725
	B3[2]	0.3074	0.1632	0.4887	0.3111	0.1491	0.4089	0.1975	0.1070	0.3977
	B3[3]	0.1745	0.1300	0.4378	0.2912	0.1087	0.3601	0.2363	0.0968	0.3133
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.8, 1.5), (1.5, 2.0)\}$										
(20, 30)	A1[1]	0.2255	0.2452	0.8165	0.2319	0.2287	0.7855	0.2050	0.1857	0.7510
	A1[2]	0.1947	0.2728	0.8366	0.2144	0.2403	0.7898	0.1972	0.1881	0.7593
	A1[3]	0.2031	0.2359	0.8091	0.2067	0.2219	0.7720	0.2268	0.1833	0.7338
	B1[1]	0.2057	0.2183	0.7550	0.1913	0.2103	0.7137	0.2167	0.1695	0.7009
	B1[2]	0.2186	0.2161	0.7459	0.2640	0.2091	0.7000	0.2388	0.1662	0.6870
	B1[3]	0.1373	0.2088	0.7059	0.2302	0.1963	0.6837	0.2099	0.1616	0.6678
(40, 50)	A2[1]	0.2297	0.1972	0.6874	0.2615	0.1818	0.6406	0.1978	0.1455	0.5802
	A2[2]	0.1945	0.2056	0.6931	0.2419	0.1850	0.6687	0.2098	0.1580	0.6476
	A2[3]	0.3071	0.1871	0.6730	0.2355	0.1807	0.6391	0.1813	0.1453	0.5779
	B2[1]	0.2531	0.1787	0.6651	0.2550	0.1733	0.5921	0.2090	0.1443	0.5756
	B2[2]	0.2279	0.1775	0.6578	0.2749	0.1686	0.5811	0.2287	0.1406	0.5291
	B2[3]	0.1959	0.1767	0.6065	0.2134	0.1657	0.5602	0.2129	0.1385	0.5142

(Continued)

Table 6 (continued)

(n_1, n_2)	Test	MLE		Bayes						
				Prior-1			Prior-2			
(60, 80)	A3[1]	0.2290	0.1667	0.5476	0.2376	0.1488	0.4912	0.3062	0.1083	0.4094
	A3[2]	0.2508	0.1692	0.5811	0.2219	0.1542	0.5443	0.2552	0.1377	0.5087
	A3[3]	0.2166	0.1599	0.5111	0.2076	0.1471	0.4126	0.1976	0.1080	0.3987
	B3[1]	0.2250	0.1408	0.3986	0.2186	0.1226	0.3856	0.1804	0.1023	0.3667
	B3[2]	0.2384	0.1505	0.4809	0.2174	0.1408	0.4063	0.1975	0.1059	0.3967
	B3[3]	0.2145	0.1245	0.3934	0.2097	0.1003	0.3565	0.2165	0.0901	0.1970

Table 7: The AIL (1st Col.) and CP (2nd Col.) of α .

(n_1, n_2)	Test	ACI		BCI			
				Prior-1		Prior-2	
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.5, 1.0), (1.0, 1.5)\}$							
(20, 30)	A1[1]	1.338	0.915	0.679	0.951	0.766	0.946
	A1[2]	1.340	0.913	0.722	0.949	0.958	0.935
	A1[3]	1.222	0.920	0.663	0.952	0.689	0.949
	B1[1]	1.170	0.923	0.601	0.956	0.670	0.951
	B1[2]	1.196	0.921	0.615	0.955	0.692	0.950
	B1[3]	1.092	0.927	0.584	0.956	0.663	0.952
(40, 50)	A2[1]	0.946	0.936	0.525	0.960	0.597	0.956
	A2[2]	0.958	0.935	0.576	0.957	0.605	0.955
	A2[3]	0.856	0.941	0.479	0.962	0.488	0.962
	B2[1]	0.802	0.944	0.438	0.965	0.468	0.963
	B2[2]	0.828	0.942	0.459	0.964	0.469	0.963
	B2[3]	0.791	0.945	0.435	0.965	0.466	0.963
(60, 80)	A3[1]	0.721	0.949	0.387	0.968	0.408	0.967
	A3[2]	0.751	0.947	0.423	0.966	0.424	0.966
	A3[3]	0.670	0.951	0.363	0.969	0.404	0.967
	B3[1]	0.629	0.954	0.334	0.971	0.387	0.968
	B3[2]	0.657	0.952	0.347	0.970	0.402	0.967
	B3[3]	0.605	0.955	0.296	0.973	0.320	0.972
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.8, 1.5), (1.5, 2.0)\}$							
(20, 30)	A1[1]	1.290	0.916	0.666	0.952	0.692	0.950

(Continued)

Table 7 (continued)

(n_1, n_2)	Test	ACI		BCI			
				Prior-1		Prior-2	
	A1[2]	1.292	0.916	0.680	0.951	0.877	0.940
	A1[3]	1.174	0.923	0.611	0.955	0.628	0.954
	B1[1]	1.096	0.927	0.596	0.956	0.627	0.954
	B1[2]	1.084	0.928	0.588	0.956	0.612	0.955
	B1[3]	0.986	0.933	0.552	0.958	0.599	0.956
(40, 50)	A2[1]	0.850	0.941	0.549	0.958	0.582	0.957
	A2[2]	0.944	0.936	0.551	0.958	0.588	0.956
	A2[3]	0.792	0.945	0.488	0.962	0.570	0.957
	B2[1]	0.769	0.946	0.457	0.964	0.471	0.963
	B2[2]	0.763	0.946	0.443	0.965	0.470	0.963
	B2[3]	0.751	0.947	0.432	0.965	0.446	0.964
(60, 80)	A3[1]	0.672	0.951	0.387	0.968	0.423	0.966
	A3[2]	0.717	0.949	0.396	0.967	0.424	0.966
	A3[3]	0.633	0.954	0.381	0.968	0.416	0.966
	B3[1]	0.601	0.956	0.334	0.971	0.385	0.968
	B3[2]	0.602	0.955	0.343	0.970	0.394	0.967
	B3[3]	0.551	0.958	0.298	0.973	0.381	0.968

Table 8: The AIL (1st Col.) and CP (2nd Col.) of b .

(n_1, n_2)	Test	ACI		BCI			
				Prior-1		Prior-2	
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.5, 1.0), (1.0, 1.5)\}$							
(20, 30)	A1[1]	0.835	0.930	0.694	0.941	0.595	0.949
	A1[2]	0.860	0.928	0.707	0.940	0.601	0.949
	A1[3]	0.753	0.937	0.672	0.943	0.555	0.952
	B1[1]	0.614	0.948	0.510	0.956	0.463	0.960
	B1[2]	0.670	0.943	0.641	0.946	0.530	0.954
	B1[3]	0.605	0.948	0.508	0.956	0.423	0.963
(40, 50)	A2[1]	0.575	0.951	0.482	0.958	0.366	0.968
	A2[2]	0.585	0.950	0.495	0.957	0.415	0.964
	A2[3]	0.541	0.954	0.455	0.961	0.328	0.971
	B2[1]	0.444	0.961	0.422	0.963	0.293	0.974

(Continued)

Table 8 (continued)

(n_1, n_2)	Test	ACI		BCI			
				Prior-1		Prior-2	
	B2[2]	0.485	0.958	0.449	0.961	0.316	0.972
	B2[3]	0.453	0.961	0.412	0.964	0.286	0.974
(60, 80)	A3[1]	0.448	0.962	0.400	0.965	0.195	0.981
	A3[2]	0.443	0.961	0.404	0.965	0.212	0.980
	A3[3]	0.428	0.963	0.398	0.965	0.164	0.984
	B3[1]	0.423	0.963	0.378	0.967	0.141	0.986
	B3[2]	0.428	0.963	0.391	0.966	0.145	0.985
	B3[3]	0.394	0.965	0.375	0.967	0.101	0.989
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.8, 1.5), (1.5, 2.0)\}$							
(20, 30)	A1[1]	0.702	0.941	0.665	0.944	0.577	0.951
	A1[2]	0.758	0.936	0.676	0.943	0.601	0.949
	A1[3]	0.684	0.942	0.655	0.944	0.544	0.953
	B1[1]	0.641	0.946	0.604	0.949	0.530	0.954
	B1[2]	0.565	0.952	0.512	0.956	0.434	0.962
	B1[3]	0.553	0.953	0.477	0.959	0.341	0.970
(40, 50)	A2[1]	0.512	0.956	0.461	0.960	0.297	0.973
	A2[2]	0.539	0.954	0.475	0.959	0.340	0.970
	A2[3]	0.492	0.958	0.458	0.960	0.272	0.975
	B2[1]	0.486	0.958	0.445	0.961	0.270	0.975
	B2[2]	0.445	0.961	0.431	0.962	0.259	0.976
	B2[3]	0.437	0.962	0.429	0.963	0.227	0.979
(60, 80)	A3[1]	0.428	0.963	0.398	0.965	0.189	0.982
	A3[2]	0.430	0.963	0.404	0.965	0.207	0.980
	A3[3]	0.421	0.963	0.395	0.965	0.163	0.984
	B3[1]	0.408	0.964	0.370	0.967	0.141	0.986
	B3[2]	0.416	0.964	0.380	0.967	0.145	0.985
	B3[3]	0.396	0.965	0.370	0.967	0.102	0.989

Table 9: The AIL (1st Col.) and CP (2nd Col.) of θ .

(n_1, n_2)	Test	ACI		BCI			
				Prior-1		Prior-2	
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.5, 1.0), (1.0, 1.5)\}$							
(20, 30)	A1[1]	2.921	0.893	1.428	0.937	0.991	0.950
	A1[2]	2.222	0.914	1.230	0.943	0.879	0.953
	A1[3]	2.794	0.897	1.409	0.937	0.888	0.953
	B1[1]	2.080	0.918	0.802	0.955	0.726	0.957
	B1[2]	1.527	0.934	0.769	0.956	0.696	0.958
	B1[3]	1.576	0.932	0.801	0.955	0.708	0.958
(40, 50)	A2[1]	1.506	0.935	0.727	0.957	0.671	0.959
	A2[2]	1.368	0.939	0.674	0.959	0.663	0.959
	A2[3]	1.414	0.937	0.687	0.958	0.671	0.959
	B2[1]	1.284	0.941	0.662	0.959	0.654	0.959
	B2[2]	1.176	0.944	0.637	0.960	0.630	0.960
	B2[3]	1.263	0.942	0.653	0.960	0.652	0.960
(60, 80)	A3[1]	1.156	0.945	0.630	0.960	0.615	0.961
	A3[2]	1.150	0.945	0.618	0.961	0.548	0.963
	A3[3]	1.152	0.945	0.625	0.960	0.605	0.961
	B3[1]	1.102	0.946	0.614	0.961	0.446	0.966
	B3[2]	0.579	0.962	0.548	0.963	0.398	0.967
	B3[3]	1.037	0.948	0.600	0.961	0.425	0.966
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.8, 1.5), (1.5, 2.0)\}$							
(20, 30)	A1[1]	2.960	0.892	1.382	0.938	1.021	0.949
	A1[2]	1.766	0.927	1.230	0.943	0.786	0.956
	A1[3]	2.808	0.896	1.304	0.940	0.814	0.955
	B1[1]	1.689	0.929	0.764	0.956	0.682	0.957
	B1[2]	1.466	0.936	0.728	0.957	0.667	0.959
	B1[3]	1.640	0.931	0.751	0.957	0.668	0.958
(40, 50)	A2[1]	1.377	0.938	0.716	0.958	0.657	0.959
	A2[2]	1.133	0.945	0.702	0.958	0.639	0.961
	A2[3]	1.191	0.944	0.713	0.958	0.647	0.960
	B2[1]	1.130	0.946	0.676	0.959	0.636	0.961
	B2[2]	1.029	0.948	0.632	0.960	0.628	0.962
	B2[3]	1.100	0.946	0.662	0.959	0.623	0.962

(Continued)

Table 9 (continued)

(n_1, n_2)	Test	ACI		BCI			
				Prior-1		Prior-2	
(60, 80)	A3[1]	1.007	0.949	0.627	0.960	0.615	0.963
	A3[2]	0.931	0.951	0.612	0.961	0.537	0.965
	A3[3]	0.992	0.950	0.621	0.960	0.584	0.963
	B3[1]	0.896	0.952	0.604	0.961	0.453	0.966
	B3[2]	0.510	0.964	0.443	0.966	0.396	0.967
	B3[3]	0.554	0.962	0.475	0.965	0.432	0.966

Table 10: The AIL (1st Col.) and CP (2nd Col.) of $R_1(t)$

(n_1, n_2)	Test	ACI		BCI			
				Prior-1		Prior-2	
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.5, 1.0), (1.0, 1.5)\}$							
(20, 30)	A1[1]	0.500	0.947	0.418	0.955	0.247	0.972
	A1[2]	0.554	0.942	0.457	0.951	0.452	0.952
	A1[3]	0.521	0.945	0.446	0.952	0.257	0.971
	B1[1]	0.461	0.951	0.325	0.964	0.173	0.979
	B1[2]	0.476	0.950	0.334	0.963	0.217	0.974
	B1[3]	0.496	0.948	0.350	0.962	0.238	0.972
(40, 50)	A2[1]	0.259	0.970	0.257	0.971	0.144	0.981
	A2[2]	0.303	0.966	0.287	0.968	0.168	0.979
	A2[3]	0.286	0.968	0.268	0.969	0.146	0.981
	B2[1]	0.229	0.973	0.162	0.980	0.103	0.985
	B2[2]	0.235	0.973	0.182	0.978	0.122	0.984
	B2[3]	0.245	0.972	0.243	0.972	0.125	0.983
(60, 80)	A3[1]	0.210	0.975	0.081	0.986	0.071	0.988
	A3[2]	0.216	0.975	0.154	0.981	0.100	0.986
	A3[3]	0.212	0.975	0.110	0.985	0.093	0.986
	B3[1]	0.055	0.990	0.049	0.991	0.045	0.991
	B3[2]	0.071	0.988	0.061	0.989	0.054	0.990
	B3[3]	0.108	0.984	0.071	0.988	0.062	0.989
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.8, 1.5), (1.5, 2.0)\}$							

(Continued)

Table 10 (continued)

(n_1, n_2)	Test	ACI		BCI			
				Prior-1		Prior-2	
(20, 30)	A1[1]	0.497	0.947	0.412	0.956	0.220	0.974
	A1[2]	0.506	0.947	0.442	0.953	0.242	0.972
	A1[3]	0.525	0.945	0.448	0.952	0.433	0.954
	B1[1]	0.415	0.955	0.374	0.959	0.176	0.978
	B1[2]	0.446	0.952	0.395	0.957	0.210	0.975
	B1[3]	0.438	0.953	0.376	0.959	0.181	0.978
(40, 50)	A2[1]	0.322	0.964	0.311	0.965	0.136	0.982
	A2[2]	0.342	0.962	0.322	0.964	0.136	0.982
	A2[3]	0.397	0.957	0.349	0.962	0.154	0.981
	B2[1]	0.253	0.971	0.188	0.977	0.096	0.986
	B2[2]	0.287	0.968	0.245	0.972	0.132	0.983
	B2[3]	0.253	0.971	0.197	0.976	0.101	0.986
(60, 80)	A3[1]	0.222	0.974	0.087	0.987	0.077	0.988
	A3[2]	0.241	0.972	0.113	0.984	0.082	0.987
	A3[3]	0.245	0.972	0.176	0.978	0.087	0.987
	B3[1]	0.051	0.990	0.051	0.990	0.045	0.991
	B3[2]	0.082	0.987	0.076	0.988	0.063	0.989
	B3[3]	0.068	0.989	0.064	0.989	0.063	0.989

Table 11: The AIL (1st Col.) and CP (2nd Col.) of $H_1(t)$

(n_1, n_2)	Test	ACI		BCI			
				Prior-1		Prior-2	
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.5, 1.0), (1.0, 1.5)\}$							
(20, 30)	A1[1]	0.613	0.938	0.509	0.947	0.346	0.962
	A1[2]	0.641	0.935	0.530	0.945	0.418	0.955
	A1[3]	0.589	0.940	0.468	0.951	0.334	0.963
	B1[1]	0.534	0.945	0.419	0.955	0.305	0.965
	B1[2]	0.564	0.942	0.427	0.954	0.319	0.964
	B1[3]	0.516	0.946	0.383	0.958	0.225	0.973
(40, 50)	A2[1]	0.341	0.962	0.339	0.962	0.182	0.976
	A2[2]	0.359	0.960	0.352	0.961	0.219	0.973
	A2[3]	0.337	0.962	0.337	0.962	0.182	0.976

(Continued)

Table 11 (continued)

(n_1, n_2)	Test	ACI		BCI			
				Prior-1		Prior-2	
	B2[1]	0.300	0.966	0.253	0.970	0.166	0.978
	B2[2]	0.320	0.964	0.300	0.966	0.177	0.977
	B2[3]	0.299	0.966	0.239	0.971	0.151	0.979
(60, 80)	A3[1]	0.277	0.968	0.165	0.978	0.115	0.983
	A3[2]	0.292	0.967	0.231	0.972	0.124	0.982
	A3[3]	0.265	0.969	0.121	0.982	0.109	0.983
	B3[1]	0.095	0.984	0.094	0.984	0.084	0.985
	B3[2]	0.110	0.983	0.098	0.984	0.093	0.984
	B3[3]	0.086	0.985	0.072	0.986	0.067	0.987
$\{(T_{11}, T_{12}), (T_{21}, T_{22})\} = \{(0.8, 1.5), (1.5, 2.0)\}$							
(20, 30)	A1[1]	0.596	0.939	0.505	0.947	0.337	0.962
	A1[2]	0.604	0.938	0.513	0.947	0.393	0.958
	A1[3]	0.585	0.940	0.457	0.952	0.313	0.965
	B1[1]	0.499	0.948	0.441	0.953	0.297	0.966
	B1[2]	0.495	0.948	0.426	0.954	0.253	0.970
	B1[3]	0.463	0.951	0.419	0.955	0.211	0.974
(40, 50)	A2[1]	0.388	0.958	0.358	0.961	0.177	0.977
	A2[2]	0.438	0.953	0.400	0.957	0.194	0.975
	A2[3]	0.382	0.958	0.357	0.961	0.169	0.978
	B2[1]	0.380	0.959	0.352	0.961	0.157	0.980
	B2[2]	0.320	0.964	0.285	0.967	0.126	0.982
	B2[3]	0.312	0.965	0.260	0.969	0.123	0.982
(60, 80)	A3[1]	0.298	0.966	0.162	0.978	0.113	0.983
	A3[2]	0.299	0.966	0.255	0.970	0.121	0.982
	A3[3]	0.288	0.967	0.141	0.980	0.098	0.984
	B3[1]	0.118	0.982	0.086	0.985	0.083	0.985
	B3[2]	0.127	0.981	0.110	0.983	0.096	0.984
	B3[3]	0.084	0.985	0.078	0.986	0.066	0.987

6 Accelerated Data Applications

This section presents two real accelerated datasets from engineering field to empirically evaluate the performance of the proposed estimation methods of the Weibull parameters (a, b) , acceleration factor θ , RF $R_1(t)$, and HRF $H_1(t)$. The first dataset involves the lifetimes of micro-droplets under

ambient environmental conditions, while the second pertains to the failure times of light-emitting diodes.

6.1 Micro-Droplets

Micro-droplets play a critical role in the transmission of infectious agents, particularly in the spread of respiratory diseases. When an individual sneezes, they expel aerosolized particles, typically around 10 μm in diameter, that can remain airborne and suspended for significant durations. These particles may carry viral pathogens, posing a substantial risk when inhaled by others. To mitigate such risks, individuals are advised to avoid physical contact with unfamiliar surfaces, maintain physical distancing, and minimize exposure to potentially contaminated environments. Importantly, airflow velocity is a key factor influencing the airborne lifetime of micro-droplets: increased airflow tends to reduce their persistence, while reduced airflow allows them to remain suspended longer (see Aliabadi et al. [31]). In this application, we analyze droplet behavior under different airflow velocities. Following the methodology of Asadi et al. [32], we treat measurements taken at an airflow speed of 0.35 m/s (with sample size $n_1 = 15$) as representing normal-use conditions, and those recorded at 0.2 m/s (with sample size $n_2 = 19$) as representing accelerated test conditions; see Table 12. Recently, this dataset has been reanalyzed under the inverted Weibull model by Nassar and Elshahhat [33].

Table 12: Times of micro-droplets in air

Normal Use Condition (0.35 m/s)									
0.47	0.54	0.55	0.80	0.96	1.14	1.24	1.30	1.38	1.50
1.61	1.61	1.70	1.79	1.80					
Accelerated Stress Condition (0.20 m/s)									
0.8	0.9	1.15	1.15	1.25	1.3	1.4	0.97	1.01	1.09
1.15	1.18	1.22	1.27	1.29	1.31	1.34	1.36	1.37	

To assess the adequacy of the Weibull distribution in modeling the micro-droplets failure time data, the Kolmogorov–Smirnov (KS) test statistic and its corresponding p -value (at a $\alpha = 5\%$ significance level) are computed. Specifically, the MLEs of the Weibull parameters a and b , along with their standard errors (Std.Es) and associated 95% ACIs and interval widths (IW), are reported in Table 13. The computed p -values corresponding to the estimated KS statistics indicate that the Weibull distribution provides a satisfactory fit to the micro-droplet data. To further support this conclusion, several goodness-of-fit visualization techniques are employed using both the normal-use and accelerated-stress datasets. Fig. 1 displays the estimated and empirical RF, estimated and empirical probability–probability (PP) plots, estimated and empirical quantile–quantile (QQ) plots, and scaled total-time-on-test (TTT) plots. From Fig. 1a–c, it is evident that the Weibull distribution provides a good fit to both micro-droplet datasets. Moreover, the scaled-TTT plot in Fig. 1d indicates an increasing failure rate, which aligns with the theoretical behavior of the Weibull hazard function. To confirm the existence and uniqueness of the fitted values $\hat{a}$ and $\hat{b}$ (as reported in Table 13), contour

plots of the log-likelihood function over a grid of (a, b) values are shown in Fig. 2 for both micro-droplet datasets. These plots illustrate a well-defined peak, supporting the stability of the MLE procedure. For subsequent modeling steps involving the micro-droplets dataset, we recommend using the estimated values $\hat{a}$ and $\hat{b}$ from the normal-use and accelerated-stress conditions as initial guesses.

Table 13: Summary fit for the Weibull lifespan model from LED datasets

Stress	Par.	MLE (Std.E)	95% ACI			KS (p -Value)
			Low.	Upp.	IL	
Normal	a	3.1635 (0.6955)	1.8002	4.5267	2.7264	0.1463 (0.9051)
	b	0.3656 (0.1451)	0.0813	0.6500	0.5687	
Accelerated	a	9.5117 (1.8168)	5.9508	13.073	7.1217	0.1034 (0.9872)
	b	0.1178 (0.0627)	0.0015	0.2417	0.2402	

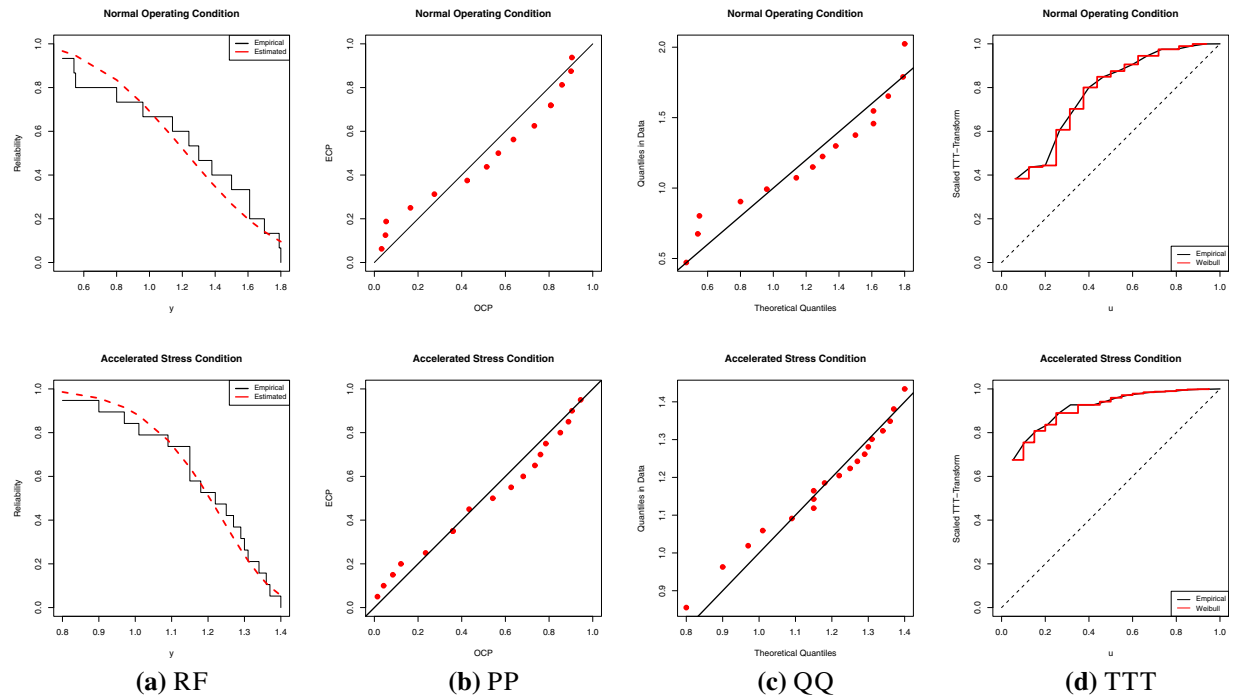


Figure 1: Fitting diagrams of the Weibull model using micro-droplets data

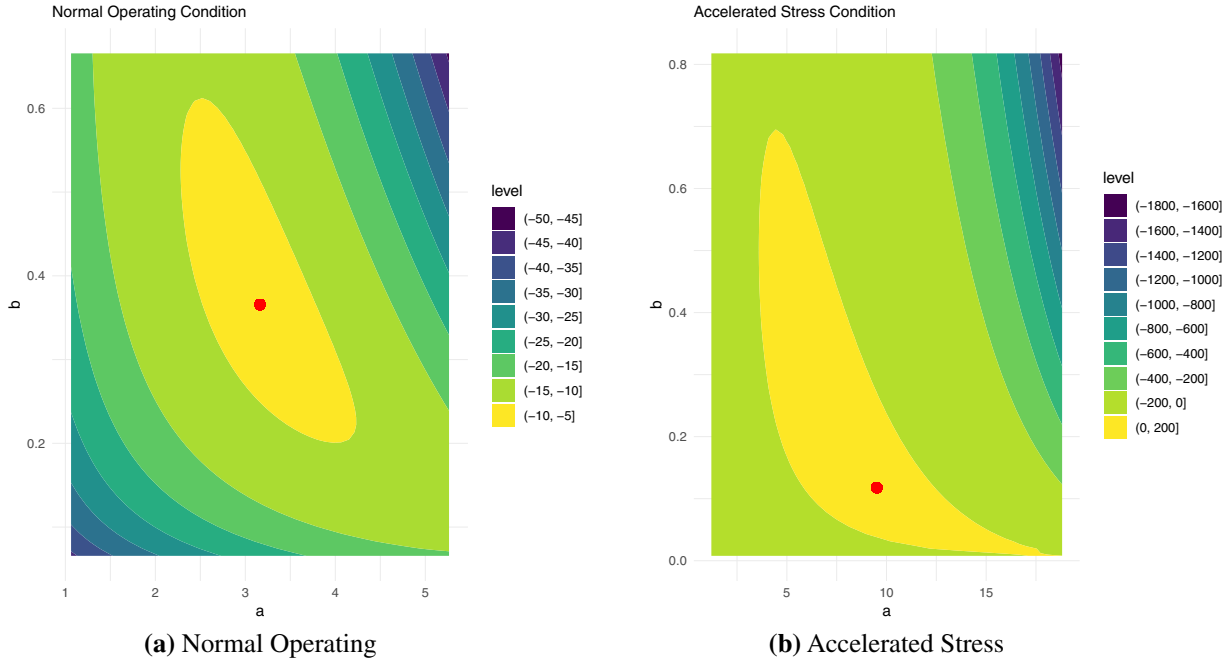


Figure 2: Contours of the Weibull parameters using micro-droplets data

Now, by setting $k_i = 8$ and $m_i = 11$ for $i = 1, 2$, three UTIPHC samples under the CSPALT are generated using different progressive censoring patterns $(Q_{j1}, Q_{j2}, \dots, Q_{jm_j})$, $j = 1, 2$ and different threshold values T_{ij} , $i, j = 1, 2$; see Table 14. For each sample $\mathbb{S}[i]$, $i = 1, 2, 3$, the MLEs alongside their associated 95% ACIs are obtained for $a, b, \theta, R_1(t)$, and $H_1(t)$ evaluated at $t = 1$. Given the absence of prior information regarding the Weibull parameters a and b for the micro-droplet datasets, the hyperparameters λ_i and β_i ($i = 1, 2$) are specified as improper, with each set to 0.001. This choice yields nearly non-informative gamma prior densities. By setting $\mathcal{M} = 50,000$ and $\mathcal{B} = 10,000$ in the MCMC procedures described in Section 4, the Bayesian MCMC point estimates and corresponding 95% BCI estimates for the same unknown subjects are computed. The initial values of a and b used to initialize the MCMC sampler are taken as their respective frequentist estimates. Table 15 reports both the classical and Bayesian point estimates (along with their Std.Es) as well as the ACI/BCI estimates (with their IWs). Based on the observed patterns of smaller Std.Es and narrower IWs, Table 15 demonstrates that the Bayesian point and interval estimates generally outperform their frequentist counterparts. Moreover, the findings summarized in Table 15 align closely with the conclusions drawn earlier in Section 5. Now, several descriptive measures, such as posterior mean, mode, quartiles (Q_i , $i = 1, 2, 3$), standard deviation (SD), and skewness (Skew.), are computed from the 40,000 posterior samples of $a, b, \theta, R_1(t)$, and $H_1(t)$; see Table 16. To assess the existence and uniqueness of the likelihood estimates of a, b , and θ based on Samples $\mathbb{S}_i$, $i = 1, 2, 3$, the corresponding contour plots of the log-likelihood surfaces are presented in Fig. 3. These plots confirm that the fitted MLEs for a, b , and θ , as reported in Table 14, are well-defined and unique. They also validate the numerical results summarized in Table 15. Accordingly, we recommend using these estimates as initial values in subsequent MCMC computations. Fig. 4 displays the Gaussian kernel density estimates and trace plots for the 40,000 MCMC samples of $a, b, \theta, R_1(t)$, and $H_1(t)$, using Sample $\mathbb{S}_1$ as a representative example. The trace plots exhibit good mixing behavior of the MCMC chains. The posterior samples of a, θ , and $R_1(t)$ appear approximately symmetric, while those of μ and $H_1(t)$ exhibit positive skewness. These visual

patterns suggest that the chosen burn-in period is sufficient to remove the influence of initial values and achieve proper convergence of the chains.

Table 14: Three UTIPHC samples using CSPALT from micro-droplets data

Sample	$(Q_{11}, \dots, Q_{1m_1})$ $(Q_{21}, \dots, Q_{2m_2})$	$(T_{11}(d_1), T_{12}(r_1))$ $(T_{21}(d_2), T_{22}(r_2))$	(Q_1^*, Q_2^*)	(T_1^*, T_2^*)	Data
S[1]	$(1 * 4, 0 * 7)$	$\{(1.80 (11), 1.85 (11))\}$	$(0, 1)$	$(1.79, 1.40)$	0.47, 0.55, 0.80, 0.96, 1.24, 1.30, 1.38, 1.50, 1.61, 1.70, 1.79
	$(2 * 4, 0 * 7)$	$\{(1.40 (10), 1.45 (10))\}$			0.80, 0.97, 1.01, 1.15, 1.18, 1.22, 1.25, 1.29, 1.34, 1.36
S[2]	$(0*4, 1*4, 0*3)$	$\{(1.75 (10), 1.80 (10))\}$	$(1, 3)$	$(1.75, 1.31)$	0.47, 0.54, 0.55, 0.80, 0.96, 1.14, 1.24, 1.38, 1.61, 1.70
	$(0*4, 2*4, 0*3)$	$\{(1.05 (4), 1.35 (8))\}$			0.80, 0.90, 0.97, 1.01, 1.09, 1.18, 1.27, 1.31
S[3]	$(0 * 7, 1 * 4)$	$\{(1.10 (5), 1.35 (8))\}$	$(7, 13)$	$(1.30, 1.25)$	0.47, 0.54, 0.55, 0.80, 0.96, 1.14, 1.24, 1.30
	$(0 * 7, 2 * 4)$	$\{(0.95 (2), 1.25 (6))\}$			0.80, 0.90, 0.97, 1.01, 1.09, 1.15

Table 15: Estimates of a , b , θ , $R_1(t)$, and $H_1(t)$ from micro-droplets data

Sample	Par.	MLE MCMC		95% ACI 95% BCI		IW
		Est.	Std.E	Low.	Upp.	
S[1]	a	3.2707	2.4150	0.0163	8.0040	7.9877
		3.2688	0.1458	2.9842	3.5543	0.5701
	b	0.3513	0.0512	0.2509	0.4516	0.2007
		0.3618	0.0728	0.2326	0.5145	0.2819
	θ	1.0041	0.4705	0.0818	1.9263	1.8445
		1.0028	0.1393	0.7338	1.2820	0.5482
	$R_1(t)$	0.7038	0.0360	0.6332	0.7744	0.1413

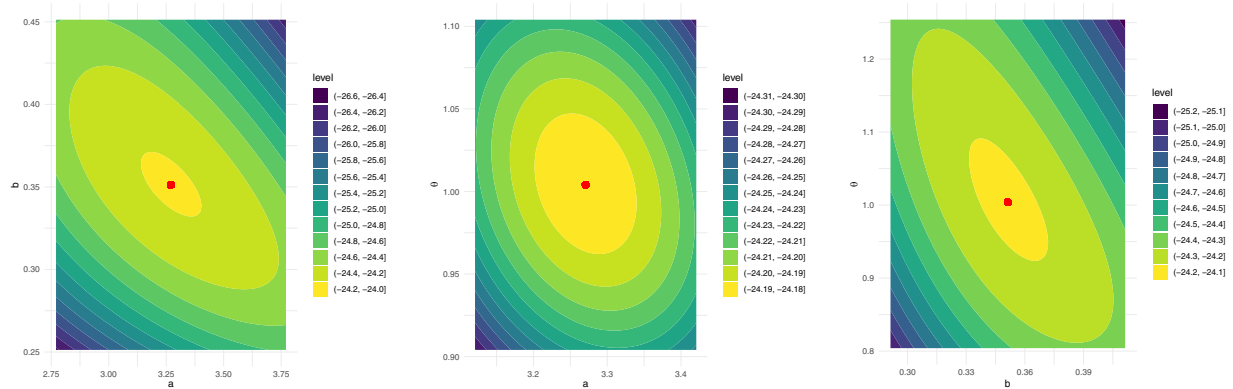
(Continued)

Table 15 (continued)

Sample	Par.	MLE MCMC		95% ACI 95% BCI		
		Est.	Std.E	Low.	Upp.	IW
S[2]	$H_1(t)$	0.6982	0.0500	0.5978	0.7925	0.1947
		1.1489	0.7433	0.0079	2.6058	2.5979
		1.1809	0.2345	0.7622	1.6719	0.9096
	a	2.2013	1.2936	0.0341	4.7368	4.7027
		2.2008	0.1447	1.9195	2.4852	0.5657
	b	0.4426	0.0706	0.3042	0.5809	0.2767
		0.4547	0.0914	0.2908	0.6423	0.3515
	θ	0.6632	0.2106	0.2505	1.0760	0.8255
		0.6665	0.1319	0.4160	0.9298	0.5138
	$R_1(t)$	0.6424	0.0453	0.5535	0.7313	0.1777
		0.6372	0.0573	0.5260	0.7476	0.2216
S[3]	$H_1(t)$	0.9742	0.5163	0.0378	1.9862	1.9484
		0.9992	0.2036	0.6371	1.4256	0.7885
	a	1.8904	1.1629	0.0039	4.1696	4.1657
		1.8907	0.1458	1.6056	2.1759	0.5703
	b	0.3996	0.0592	0.2836	0.5156	0.2320
		0.4110	0.0910	0.2486	0.6003	0.3517
	θ	0.5833	0.1837	0.2232	0.9434	0.7202
		0.5880	0.1301	0.3422	0.8526	0.5105
	$R_1(t)$	0.6706	0.0397	0.5928	0.7484	0.1556
		0.6657	0.0595	0.5486	0.7799	0.2312
	$H_1(t)$	0.7555	0.4442	0.0151	1.6260	1.6109
		0.7761	0.1780	0.4622	1.1539	0.6917

Table 16: Statistics of a , b , θ , $R_1(t)$, and $H_1(t)$ from micro-droplets data

Sample	Par.	Mean	Q_1	Q_2	Q_3	Mode	SD	Skew.
S[1]	a	3.2688	3.3094	3.1698	3.2681	3.3667	0.1458	0.0209
	b	0.3618	0.3425	0.3116	0.3570	0.4077	0.0720	0.3524
	θ	1.003	1.020	0.909	0.999	1.097	0.1393	0.0679
	$R_1(t)$	0.6982	0.7100	0.6652	0.6998	0.7323	0.0497	-0.1342
	$H_1(t)$	1.1809	1.1333	1.0189	1.1687	1.3284	0.2323	0.3352
S[2]	a	2.2008	2.2122	2.1037	2.2003	2.2987	0.1447	-0.0059
	b	0.4547	0.4184	0.3903	0.4503	0.5147	0.0905	0.2814
	θ	0.6665	0.7236	0.5763	0.6634	0.7550	0.1318	0.1404
	$R_1(t)$	0.6372	0.6581	0.5977	0.6374	0.6769	0.0571	-0.0278
	$H_1(t)$	0.9992	0.9257	0.8553	0.9886	1.1303	0.2021	0.3216
S[3]	a	1.8907	1.7966	1.7918	1.8903	1.9888	0.1458	0.0131
	b	0.4110	0.3463	0.3480	0.4060	0.4691	0.0903	0.3297
	θ	0.5880	0.6459	0.4985	0.5834	0.6749	0.1300	0.1528
	$R_1(t)$	0.6657	0.6793	0.6256	0.6663	0.7061	0.0593	-0.0610
	$H_1(t)$	0.7761	0.6375	0.6525	0.7645	0.8881	0.1768	0.3846



(a) Sample S[1]

Figure 3: (Continued)

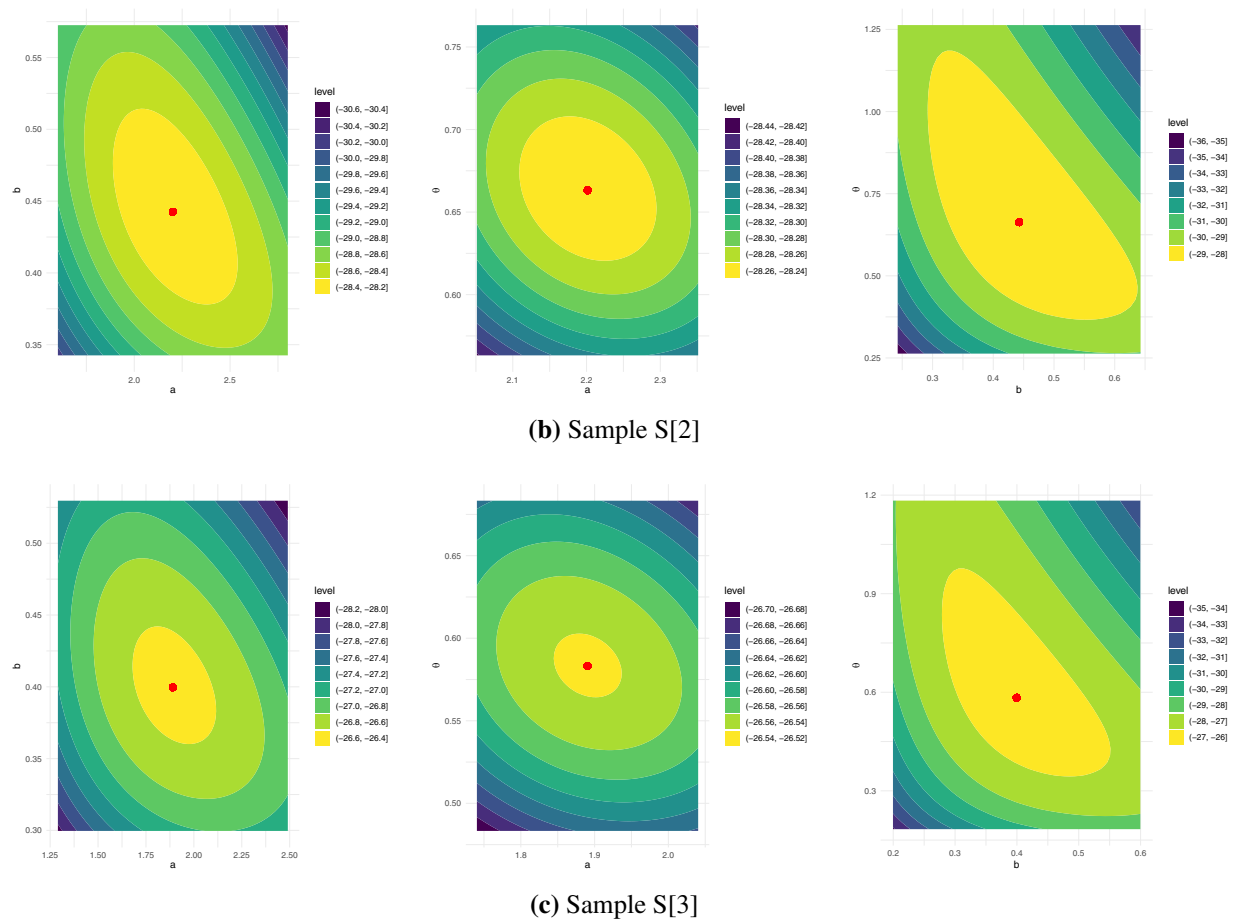


Figure 3: Contours of a (left), b (middle), and θ (right) from micro-droplets data

6.2 Light-Emitting Diode

Light-emitting diodes (LEDs) represent a widely utilized category of semiconductor devices, frequently integrated into contemporary technologies such as televisions, multicolor digital displays, and indicator systems. Each LED comprises a thin, engineered semiconductor layer designed to emit light of a specific wavelength when an electric current flows in the forward direction. The emission characteristics are governed by the type and treatment of the materials used. Table 17 presents failure time data (in thousands of hours) for LEDs subjected to both normal-use and accelerated-stress environments, each containing 58 observed failure instances. This dataset was originally analyzed by Cheng and Wang [34] and was later rediscussed by Nassar and Elshahhat [33].

To evaluate the suitability of the Weibull distribution as a potential model for the lifetime LED data, the KS statistic, along with its associated p -value, is employed. From Table 17, the MLEs (with their Std.Es) and the 95% ACIs (with their IWs) of the Weibull parameters a and b for the normal-use and stress conditions data are obtained; see Table 18. Furthermore, the KS statistics and their corresponding p -values for the complete LED datasets under each condition are also computed and provided in Table 18. The estimated p -values of the LED devices under both operational conditions

are higher than the specified significance level of 5%; thus, Table 18 supports the appropriateness of the Weibull model to analyze the LED datasets.

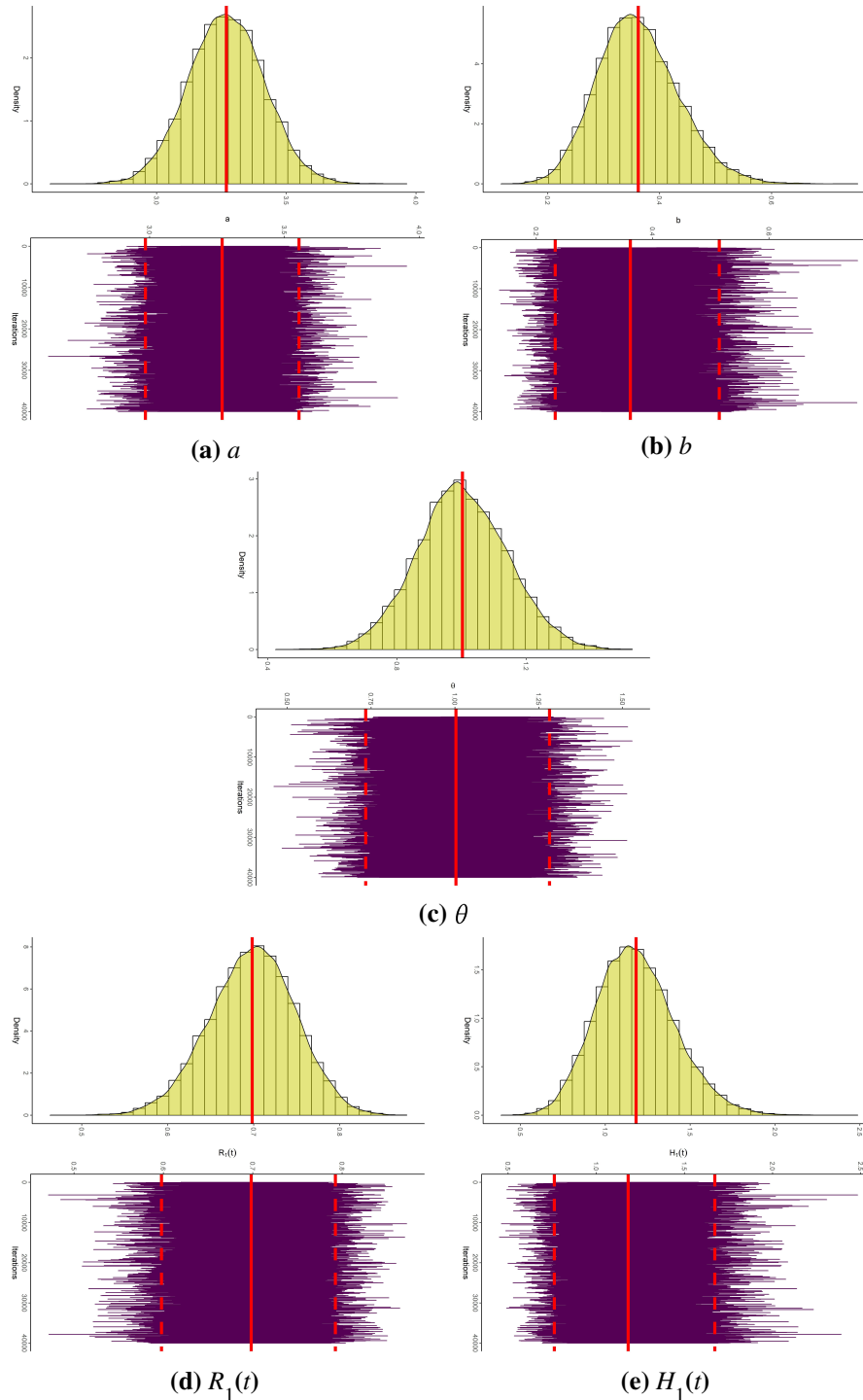


Figure 4: Density (left) and Trace (right) of a , b , θ , $R_1(t)$, and $H_1(t)$ from micro-droplets data

Table 17: Failure times of 58 LED units

Normal-Use Data

0.18	0.19	0.19	0.34	0.36	0.40	0.44	0.44	0.45	0.46	0.47	0.53	0.57	0.57	0.63
0.65	0.70	0.71	0.71	0.75	0.76	0.76	0.79	0.80	0.85	0.98	1.01	1.07	1.12	1.14
1.15	1.17	1.20	1.23	1.24	1.25	1.26	1.32	1.33	1.33	1.39	1.42	1.50	1.55	1.58
1.59	1.62	1.68	1.70	1.79	2.00	2.01	2.04	2.54	3.61	3.76	4.65	8.97		

Accelerated Stress Data

0.13	0.16	0.20	0.20	0.21	0.25	0.26	0.28	0.28	0.30	0.31	0.33	0.35	0.35	0.35
0.39	0.50	0.52	0.58	0.60	0.60	0.62	0.63	0.67	0.71	0.73	0.75	0.75	0.78	0.80
0.80	0.86	0.90	0.91	0.93	0.93	0.94	0.98	0.99	1.01	1.03	1.06	1.06	1.10	1.22
1.22	1.24	1.28	1.39	1.39	1.46	1.48	1.52	1.74	1.95	2.46	3.02	5.167		

Table 18: Summary fit for the Weibull lifespan model from LED datasets

Stress	Par.	MLE (Std.E)	95% ACI			KS (<i>p</i> -Value)
			Low.	Upp.	IL	
Normal	<i>a</i>	1.2544 (0.1119)	1.0351	1.4737	0.4386	0.1369 (0.2269)
	<i>b</i>	0.6325 (0.0999)	0.4367	0.8283	0.3916	
Accelerated	<i>a</i>	1.3516 (0.1248)	1.1070	1.5963	0.4894	0.0927 (0.7002)
	<i>b</i>	0.9769 (0.1367)	0.7090	1.2448	0.5358	

To further clarification, several visualization techniques are applied to both the normal-use and accelerated-stress LED datasets. As shown in Fig. 5a–c, the Weibull distribution demonstrates a strong visual fit to the observed data under both conditions. The TTT plot in Fig. 5d suggests a possible unimodal hazard pattern. Since the Weibull hazard function is monotonic, this may indicate some degree of model misspecification. However, due to the limited real data sets and for the purpose of illustrating the proposed inferential procedures under the UTIPHC scheme, the Weibull model is retained as a parsimonious baseline model. To verify the existence and uniqueness of the maximum likelihood estimates $\hat{a}$ and $\hat{b}$ (as presented in Table 18), contour plots of the log-likelihood surface over a grid of (*a*, *b*) values are depicted in Fig. 6 for both LED datasets. These plots show a distinct and well-defined maximum, confirming the numerical stability and identifiability of the MLEs. Based on these results, the estimated values $\hat{a}$ and $\hat{b}$ under normal-use and accelerated-stress conditions are recommended as reliable initial values for subsequent modeling procedures involving the LED failure time data.

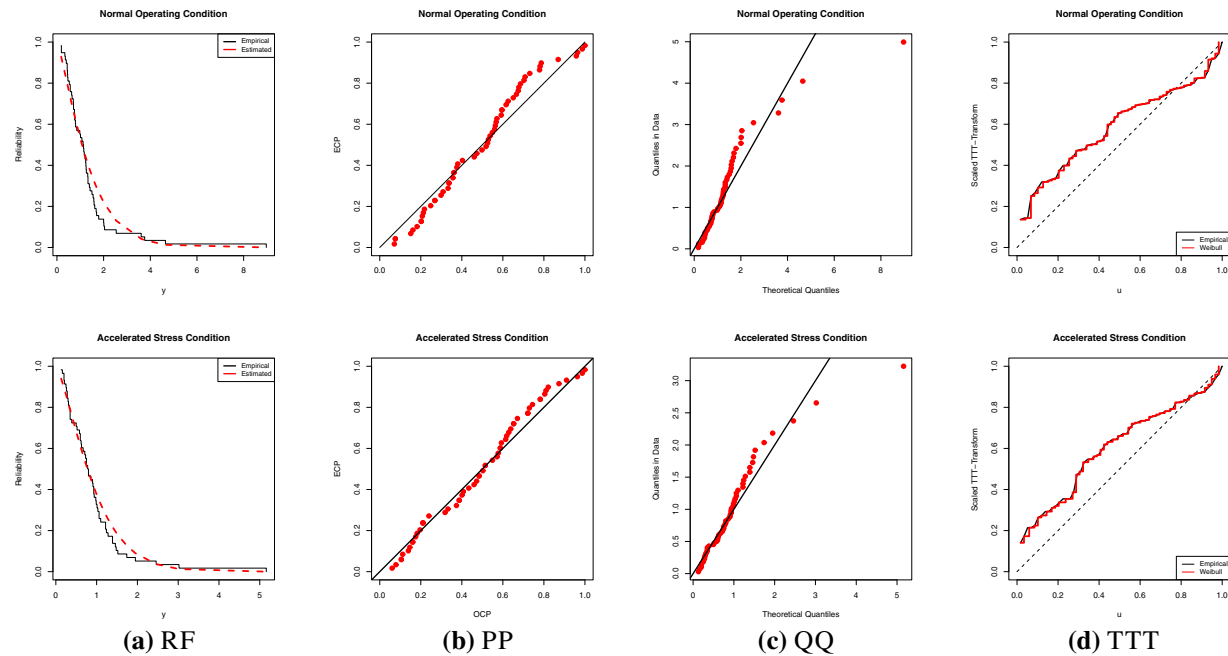


Figure 5: Fitting diagrams of the Weibull model using LED data

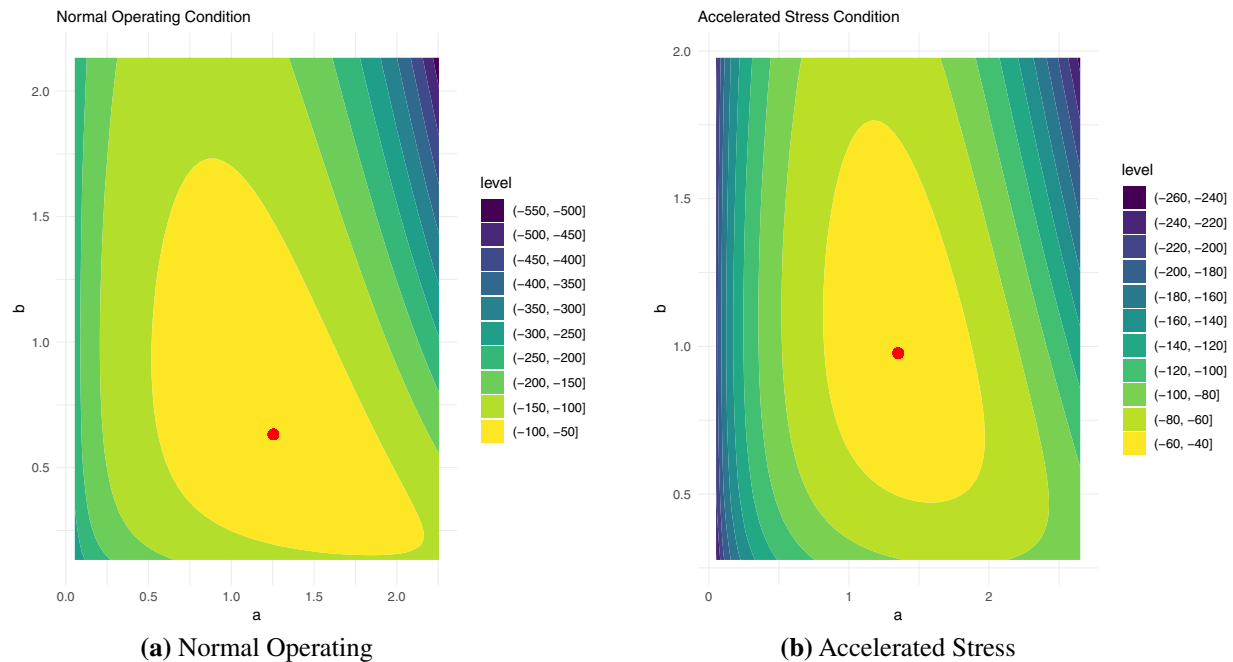


Figure 6: Contours of the Weibull parameters using LED data.

By setting $k_i = 20$ and $m_i = 331$ for $i = 1, 2$, three UTIPHC samples under the CSPALT scheme are generated using various options of $(Q_{j1}, Q_{j2}, \dots, Q_{jm_j})$, $j = 1, 2$, and T_{ij} , $i, j = 1, 2$; see Table 19.

For each sample $\mathbb{S}[i]$, $i = 1, 2, 3$, the MLEs and associated 95% ACIs are obtained for the parameters a , b , θ , and the reliability and hazard functions $R_i(t)$ and $H_i(t)$ evaluated at $t = 0.5$.

Table 19: Three UTIPHC samples using CSPALT from LED data

Sample	$(Q_{11}, \dots, Q_{1m_1})$ $(Q_{21}, \dots, Q_{2m_2})$	$(T_{11}(d_1), T_{12}(r_1))$ $(T_{21}(d_2), T_{22}(r_2))$	(Q_1^*, Q_2^*)	(T_1^*, T_2^*)	Data
$\mathbb{S}[1]$	$(5 * 5, 0 * 28)$	$\{(2.05 (33), 2.15 (33))\}$	$(0, 8)$	$(2.00, 1.50)$	0.18, 0.19, 0.34, 0.40, 0.44, 0.45, 0.46, 0.47, 0.53, 0.57, 0.57, 0.63, 0.65, 0.71, 0.71, 0.75, 0.76, 0.79, 0.85, 0.98, 1.01, 1.12, 1.14, 1.17, 1.23, 1.24, 1.25, 1.32, 1.42, 1.55, 1.58, 1.70, 2.00
	$(5 * 5, 0 * 28)$	$\{(1.50 (25), 1.60 (25))\}$			0.13, 0.20, 0.21, 0.26, 0.28, 0.30, 0.31, 0.35, 0.35, 0.50, 0.60, 0.63, 0.67, 0.71, 0.75, 0.78, 0.80, 0.91, 0.93, 0.94, 0.98, 1.01, 1.06, 1.28, 1.46
$\mathbb{S}[2]$	$(0 * 14, 5 * 5, 0 * 14)$	$\{(1.80 (25), 1.90 (25))\}$	$(8, 13)$	$(1.80, 0.91)$	0.18, 0.19, 0.19, 0.34, 0.36, 0.40, 0.44, 0.44, 0.45, 0.46, 0.47, 0.53, 0.57, 0.57, 0.63, 0.76, 0.85, 1.07, 1.17, 1.25, 1.33, 1.50, 1.62, 1.70, 1.79
	$(0 * 14, 5 * 5, 0 * 14)$	$\{(0.40 (15), 0.95 (20))\}$			0.13, 0.16, 0.20, 0.20, 0.21, 0.25, 0.26, 0.28, 0.28, 0.30, 0.31, 0.33, 0.35, 0.35, 0.35, 0.50, 0.62, 0.71, 0.80, 0.91
$\mathbb{S}[3]$	$(0 * 28, 5 * 5)$	$\{(0.55 (12), 0.85 (20))\}$	$(38, 43)$	$(0.75, 0.40)$	0.18, 0.19, 0.19, 0.34, 0.36, 0.40, 0.44, 0.44, 0.45, 0.46, 0.47, 0.53, 0.57, 0.57, 0.63, 0.65, 0.70, 0.71, 0.71, 0.75
	$(0 * 28, 5 * 5)$	$\{(0.40 (15), 0.45 (15))\}$			0.13, 0.16, 0.20, 0.20, 0.21, 0.25, 0.26, 0.28, 0.28, 0.30, 0.31, 0.33, 0.35, 0.35, 0.35

Given the absence of prior information on the Weibull parameters a and b for the micro-droplet dataset, the hyperparameters λ_i and β_i ($i = 1, 2$) are specified as improper, each set to 0.001, resulting in nearly non-informative gamma prior densities. Using $\mathcal{M} = 50,000$ iterations and a burn-in of $\mathcal{B} = 10,000$, the Bayesian MCMC point estimates and corresponding 95% BCI estimates for the

same parameters are computed using the procedures outlined in [Section 4](#). Here, the fitted frequentist estimates of a and b are recommended as initial values for MCMC-based computations.

[Table 20](#) summarizes both classical and Bayesian point estimates (with their Std.Es) and the ACI/BCI intervals (with corresponding IWs). Based on the smaller Std.Es and narrower IWs, the results in [Table 20](#) demonstrate that the Bayesian estimators generally outperform their frequentist counterparts. These findings are consistent with the conclusions drawn in [Section 5](#). Further descriptive statistics of a , b , θ , $R_1(t)$, and $H_1(t)$ are computed from the 40,000 posterior samples; see [Table 21](#). To assess the existence and uniqueness of the MLEs for a , b , and θ from Samples $\mathbb{S}_i$, $i = 1, 2, 3$, the corresponding log-likelihood contour plots are depicted in [Fig. 7](#). These plots confirm that the likelihood surfaces have well-defined global maxima, supporting the stability and uniqueness of the fitted estimates reported in [Table 19](#).

Table 20: Estimates of a , b , θ , $R_1(t)$, and $H_1(t)$ from LED data

Sample	Par.	MLE MCMC		95% ACI 95% BCI		
		Est.	Std.E	Low.	Upp.	IW
S[1]	a	1.4841	0.2515	0.9911	1.9771	0.9860
		1.4903	0.1390	1.2264	1.7662	0.5398
	b	0.9536	0.1585	0.6430	1.2643	0.6212
		0.9591	0.1289	0.7266	1.2300	0.5034
	θ	0.8188	0.1778	0.4704	1.1672	0.6969
		0.8317	0.1577	0.5540	1.1647	0.6106
	$R_1(t)$	0.7111	0.0571	0.5993	0.8230	0.2236
		0.7109	0.0392	0.6303	0.7839	0.1537
	$H_1(t)$	1.0119	0.1680	0.6826	1.3411	0.6586
		1.0131	0.1363	0.7673	1.2997	0.5324
S[2]	a	0.9118	0.1452	0.6271	1.1965	0.5693
		0.9273	0.1333	0.6793	1.1912	0.5119
	b	0.4493	0.0405	0.3700	0.5286	0.1586
		0.4583	0.0726	0.3292	0.6116	0.2824
	θ	1.2516	0.4875	0.2961	2.2071	1.9110
		1.2580	0.2090	0.8636	1.6783	0.8147
	$R_1(t)$	0.7875	0.0262	0.7363	0.8388	0.1025

(Continued)

Table 20 (continued)

Sample	Par.	MLE MCMC		95% ACI 95% BCI		
		Est.	Std.E	Low.	Upp.	IW
S[3]	$H_1(t)$	0.7859	0.0338	0.7153	0.8462	0.1308
		0.4355	0.0455	0.3463	0.5247	0.1784
		0.4432	0.0748	0.3109	0.6020	0.2912
	a	1.3631	0.3935	0.5919	2.1342	1.5424
		1.3840	0.1767	1.0422	1.7291	0.6869
	b	0.5984	0.0884	0.4251	0.7718	0.3467
		0.6180	0.1142	0.4148	0.8549	0.4400
	θ	1.6917	1.1019	0.1468	3.8513	3.7046
		1.6942	0.2279	1.2555	2.1457	0.8903
	$R_1(t)$	0.7924	0.0459	0.7024	0.8824	0.1800
		0.7903	0.0311	0.7268	0.8472	0.1205
	$H_1(t)$	0.6342	0.0984	0.4413	0.8271	0.3859
		0.6505	0.1213	0.4319	0.9015	0.4696

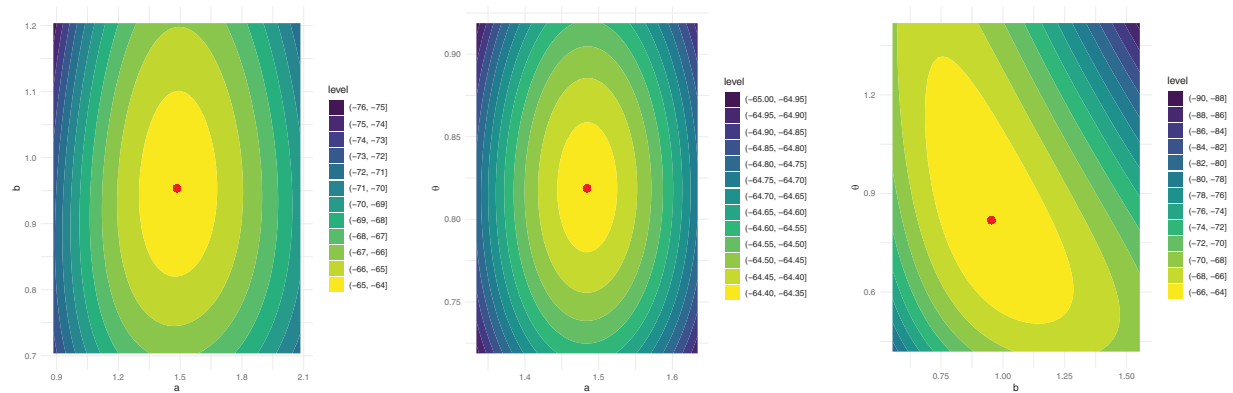
Table 21: Statistics of a , b , θ , $R_1(t)$, and $H_1(t)$ from LED data

Sample	Par.	Mean	Q_1	Q_2	Q_3	Mode	SD	Skew.
S[1]	a	1.4903	1.5065	1.3952	1.4897	1.5819	0.1389	0.1035
	b	0.9591	0.9364	0.8676	0.9539	1.0439	0.1288	0.2658
	θ	0.832	0.809	0.721	0.823	0.933	0.1572	0.3271
	$R_1(t)$	0.7109	0.7192	0.6856	0.7116	0.7383	0.0392	-0.1747
	$H_1(t)$	1.0131	0.9930	0.9166	1.0070	1.1036	0.1363	0.2662
S[2]	a	0.9273	1.0219	0.8338	0.9241	1.0173	0.1324	0.1404
	b	0.4583	0.4933	0.4072	0.4543	0.5042	0.0720	0.3602
	θ	1.2580	1.1303	1.1150	1.2451	1.3956	0.2089	0.1481
	$R_1(t)$	0.7859	0.7843	0.7637	0.7877	0.8096	0.0338	-0.3281
	$H_1(t)$	0.4432	0.4966	0.3908	0.4386	0.4909	0.0744	0.3450
S[3]	a	1.3840	1.2514	1.2634	1.3826	1.5015	0.1755	0.0532
	b	0.6180	0.5904	0.5390	0.6108	0.6896	0.1125	0.3394

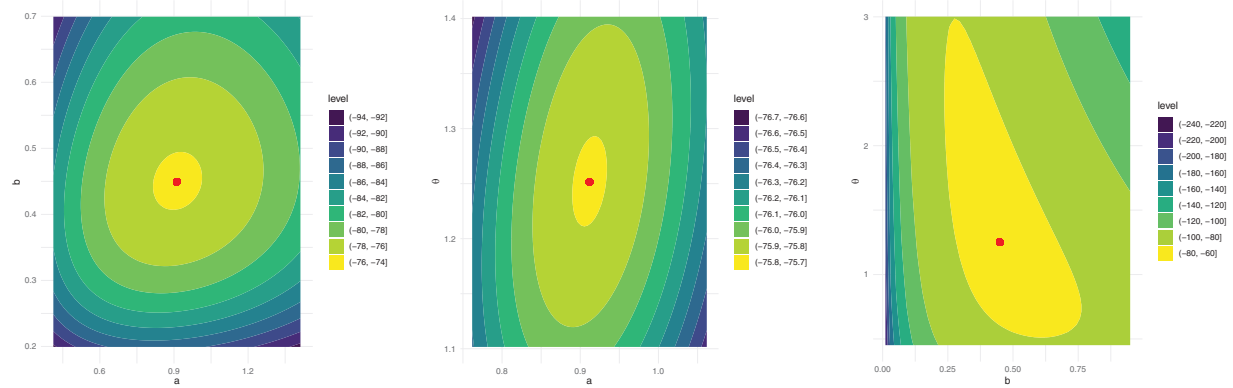
(Continued)

Table 21 (continued)

Sample	Par.	Mean	Q_1	Q_2	Q_3	Mode	SD	Skew.
	θ	1.6942	1.6042	1.5390	1.6873	1.8478	0.2279	0.0565
	$R_1(t)$	0.7903	0.7605	0.7697	0.7913	0.8116	0.0310	-0.2054
	$H_1(t)$	0.6505	0.6266	0.5666	0.6437	0.7274	0.1202	0.2911



(a) Sample S[1]



(b) Sample S[2]

Figure 7: (Continued)

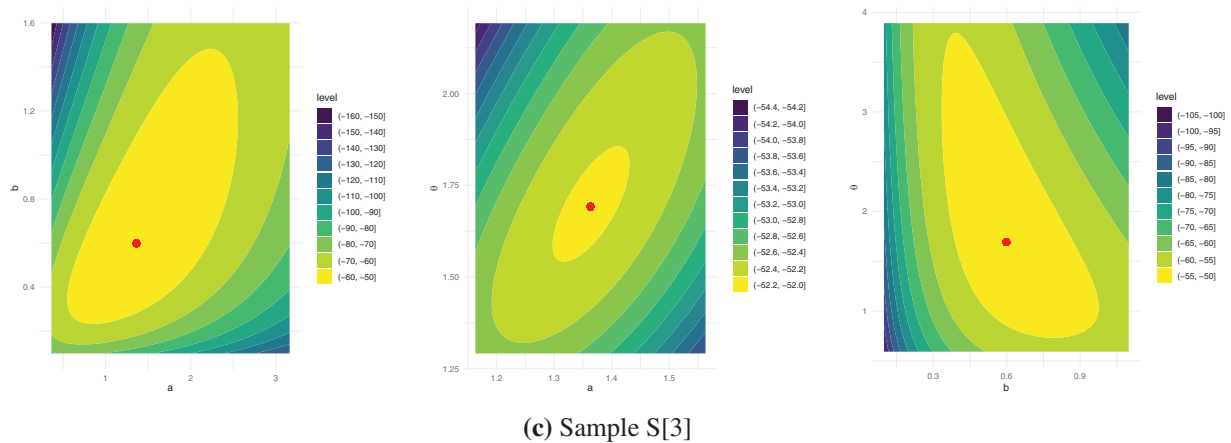


Figure 7: Contours of a (left), b (middle), and θ (right) from LED data

Finally, Fig. 8 displays the Gaussian kernel density estimates and trace plots for the 40,000 MCMC samples of a , b , θ , $R_1(t)$, and $H_1(t)$, using Sample S_1 as a representative example. The trace plots exhibit good mixing, indicating proper convergence. The posterior samples of a , θ , and $R_1(t)$ appear approximately symmetric, while those of b and $H_1(t)$ display positive skewness. These visual diagnostics confirm the same facts shown in Fig. 4.

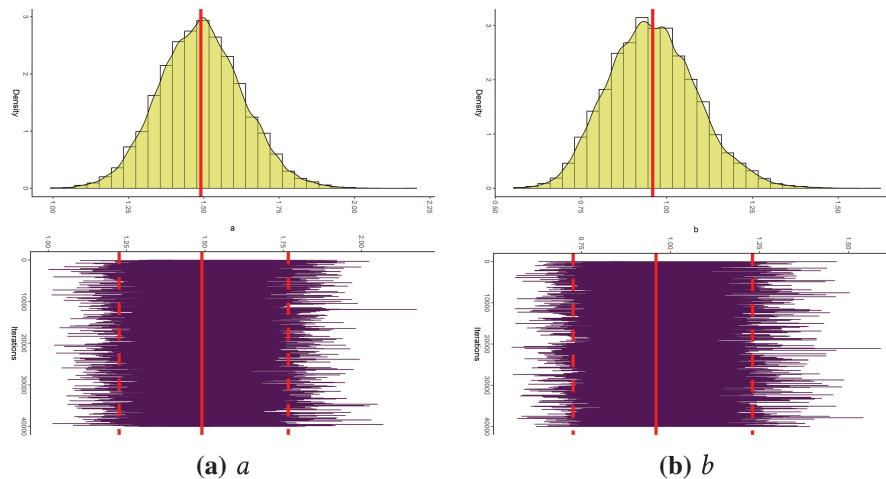


Figure 8: (Continued)

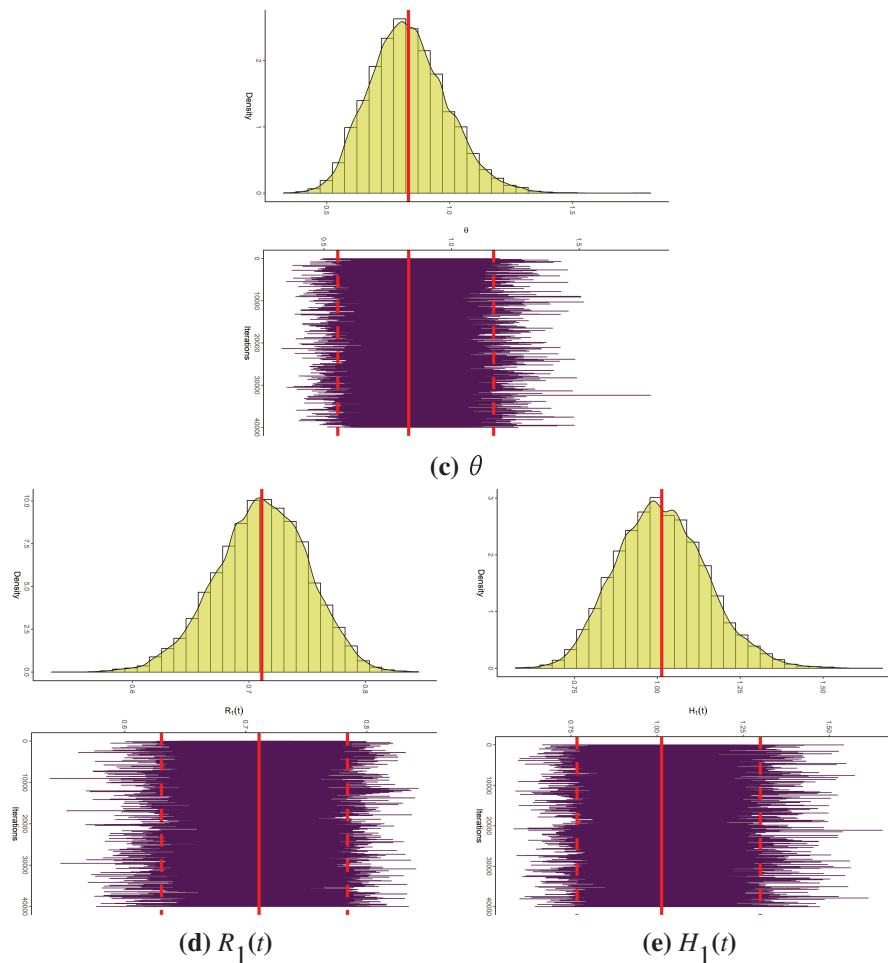


Figure 8: Density (left) and Trace (right) of a , b , θ , $R_1(t)$, and $H_1(t)$ from LED data

7 Concluding Remarks

In this study, we explored the estimation of parameters and reliability measures for the Weibull distribution under constant-stress partially accelerated life tests, utilizing the unified Type-I progressive hybrid censoring scheme that had not been previously investigated in accelerated life testing. Both model parameters and the acceleration factor, as well as the reliability and hazard rate functions under normal conditions, were estimated using two approaches: the frequentist method of maximum likelihood estimation and the Bayesian method based on Markov Chain Monte Carlo sampling. From a classical perspective, we found that closed-form expressions for the scale parameter and acceleration factor could be derived as functions of the shape parameter. However, the shape parameter itself had to be estimated iteratively, as no closed-form solution existed. In addition to point estimation, the study provided approximate confidence intervals and Bayesian credible intervals. A simulation study under various censoring scenarios was carried out to evaluate the performance of the estimators based on different accuracy metrics. Analysis of two real data sets, one based on the lifetimes of suspended microdroplets under ambient conditions and the other representing the times to failure of light-emitting diodes, confirmed the practical effectiveness of the proposed methodology, demonstrating

its ability to extract meaningful information from accelerated life tests while preserving estimation accuracy.

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