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Error estimates for an operator-splitting method for incompressible flows

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Abstract

In this paper we provide an error analysis of a fractional-step method for the numerical solution of the incompressible Navier–Stokes equations. Under mild regularity assumptions on the continuous solution, we obtain first order error estimates in the time step size, both for the intermediate and the end-of-step velocities of the method; we also give some error estimates for the pressure solution.

Keywords: Incompressible viscous flow; Navier–Stokes equations; Fractional-step methods; Finite elements; Error analysis

1. Introduction

The numerical solution of the unsteady, incompressible Navier–Stokes equations has received much attention in the last decades, and many numerical schemes are now available for that purpose. The difficulties encountered in this problem are mainly of three different kinds: the mixed type of the equations, which is due to the coupling of the momentum equation with the incompressibility condition, and, subsequently, the treatment of the pressure; the advective–diffusive character of the equations, which have a viscous and a convective term; and finally, the nonlinearity of the problem.

Fractional-step methods are becoming widely used in this context. By splitting the time advancement into a number of (generally two) substeps, they allow to separate the effects of the different operators

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appearing in the equations (see, for instance, [3]). They have been used together with different space discretizations: finite difference [4,14,15], finite element [8,17] and spectral element methods [26]. However, semidiscrete presentations of these methods, in which the space variables are not discretized, seem more appropriate to study the effect of the time discretization itself.

The origin of this category of methods is generally credited to the work of Chorin [4] and Temam [22]. They developed the well-known *projection* method, which is a two step method in which the second step consists of the projection of an intermediate velocity field onto the space of solenoidal vector fields, thus enforcing incompressibility. The incompatibility of the projection boundary conditions with those of the original problem may introduce a numerical boundary layer of size $O(\sqrt{\nu\,\delta t}\,)$ in these methods [18,25], where ν is the kinematic viscosity and δt is the time step size. However, convergence of this method to a continuous solution as δt tends to zero was proved in [23], for the semidiscrete method, and [5], for a fully discrete method with periodic boundary conditions. The end-of-step velocities of the projection method do not converge in the space $\mathbf{H}_0^1(\Omega)$, since they do not satisfy the correct boundary conditions.

More recently, analytical studies of fractional step methods have turned into obtaining error estimates in the time step size, so as to establish their order of accuracy. Thus, Shen proved in [20] that the projection method, both with and without pressure correction, is first order accurate in a certain norm. A more recent analysis given in [12] for a fully discrete, finite element implementation of the incremental fractional step projection method yielded error estimates of first order in the time step size and optimal order in the mesh size, assuming a finite element interpolation satisfying the discrete inf–sup condition. First order error estimates were also obtained by Ying (see [16] and the references therein) for another fractional step method, called *viscosity splitting* method, in which the viscosity is not fully uncoupled from incompressibility. In this sense, a fully discrete version of the so-called *θ-scheme* [11], in which viscosity and incompressibility are also coupled, was proved to converge to a continuous solution in [9] (see also [6] for a convergence analysis of a related parallel scheme). In [19] another fractional step method that keeps part of the viscous term in the second step is derived from an inexact factorization of the fully discrete original problem; this method is referred to as Yosida scheme in this reference.

In this paper we provide some error estimates for an *operator splitting*, fractional step method which was introduced and studied in [1]. It is a two step scheme in which the nonlinearity and the incompressibility of the problem are split into different steps. It allows to enforce the original boundary conditions of the problem in all substeps of the scheme, which leads to convergence of both the intermediate and the end-of-step velocities of the method to a continuous solution in the spaces $\mathbf{L}^2(\Omega)$ and $\mathbf{H}_0^1(\Omega)$ (see [1]). Here we prove that these velocities are first order accurate in the time step size.

Moreover, the study of this method was originally motivated by the consideration of a well-known predictor–multicorrector algorithm (see [2]), as detailed in [1]; this fact provides a theoretical explanation of why the original boundary conditions of the problem can be prescribed in this algorithm, and in what sense it can be understood as a fractional step method.

The paper is organized as follows: in Section 2 we introduce the notation we use and some generalities about the incompressible Navier–Stokes equations, such as the regularity assumed for their solutions. In Section 3 we recall the fractional step method of [1] and introduce a finite element spatial approximation, while in Section 4 we give an error analysis of this method; we first obtain some error estimates for both the intermediate and the end-of-step velocities and then analyze the pressure solution.

2. Preliminaries

The evolution of viscous, incompressible fluid flow in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is governed, in the primitive variable formulation, by the unsteady, incompressible Navier–Stokes equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega \times (0, T), \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega \times (0, T), \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega \times (0, T), \tag{3}$$

$$\boldsymbol{u} = \boldsymbol{u}^0 \quad \text{in } \Omega \times \{0\}, \tag{4}$$

where $\boldsymbol{u}(\boldsymbol{x}, t) \in \mathbb{R}^d$ is the fluid velocity at position $\boldsymbol{x} \in \Omega$ and time $t \in (0, T)$ (with $T > 0$ given), $p(\boldsymbol{x}, t) \in \mathbb{R}$ is the fluid kinematic pressure, $\nu > 0$ is the kinematic viscosity (which is assumed constant), $\boldsymbol{f}(\boldsymbol{x}, t)$ is an external force term, ∇ is the gradient operator, $\nabla\cdot$ is the divergence operator and Δ is the Laplacian operator (here, and in what follows, boldface characters denote vector quantities). We consider only the homogeneous Dirichlet type boundary condition (3) for the sake of simplicity.

In order to study approximation schemes for this problem, we first introduce some notation. We denote by $(\cdot, \cdot)$ the scalar product in $L^2(\Omega)$, and by $\|u\|_0 = (u, u)^{1/2}$ its norm; the quotient space $L^2_0(\Omega) = L^2(\Omega)/\mathbb{R}$ is needed in the case of Dirichlet type boundary conditions only, since the pressure is then determined only up to an additive constant; moreover, given $m \in \mathbb{N}$, the scalar product and norm in $H^m(\Omega)$ are denoted by $(u, v)_m$ and $\|u\|_m$, respectively. If $\mathcal{D}(\Omega)$ denotes the space of $\mathcal{C}^\infty$ functions with compact support in Ω, then $H^1_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$; the Poincaré–Friedrich inequality ensures that $\|\nabla u\|_0 = (\nabla u, \nabla u)^{1/2}$ is a norm on $H^1_0(\Omega)$, equivalent to the norm induced by $H^1(\Omega)$. The dual space of $H^1_0(\Omega)$ is denoted by $H^{-1}(\Omega)$ with norm $\|\cdot\|_{-1}$, the duality pairing between these spaces being denoted by $\langle\, , \rangle$. All these definitions carry over to d-dimensional vector valued function spaces.

Due to the incompressibility condition (2), closed subspaces of solenoidal vector fields of these Hilbert spaces are also needed. Thus, we define:

$$H = \left\{\boldsymbol{u} \in \mathbf{L}^2(\Omega)/\nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{n} \cdot \boldsymbol{u}_{|\partial\Omega} = 0\right\},$$

$$V = \left\{\boldsymbol{u} \in \mathbf{H}^1_0(\Omega)/\nabla \cdot \boldsymbol{u} = 0\right\}.$$

In this notation, assuming $\boldsymbol{f} \in L^2(0, T; \mathbf{H}^{-1}(\Omega))$ and $\boldsymbol{u}_0 \in H$ problem (1)–(4) has at least one solution $(\boldsymbol{u}, p)$ which satisfies $\boldsymbol{u} \in L^\infty(0, T; H) \cap L^2(0, T; V)$ (see [24]). Uniqueness and more regularity of the solution can also be proved by strengthening the assumptions on the data. In particular, we will assume that $\boldsymbol{u}$ and p satisfy:

(R1) $\boldsymbol{u} \in \mathcal{C}^0(0, T; H) \cap L^\infty(0, T; \mathbf{H}^2(\Omega))$, $\nabla p \in L^\infty(0, T; \mathbf{L}^2(\Omega))$,
(R2a) $\boldsymbol{u}_t \in L^2(0, T; \mathbf{L}^2(\Omega))$,
(R2b) $\boldsymbol{u}_t \in L^2(0, T; \mathbf{H}^1_0(\Omega))$,
(R3) $\sqrt{t}\, \boldsymbol{u}_{tt} \in L^2(0, T; \mathbf{H}^{-1}(\Omega))$,

so that we will assume either (R2a) of (R2b) depending on the context (the subindex t is employed hereafter for $\partial/\partial t$ and V' stands for the dual space of V). Conditions (R1) and (R2b) can be proved, for instance, assuming that:

$$\boldsymbol{u}_0 \in \mathbf{H}^2(\Omega) \cap V, \qquad \boldsymbol{f} \in L^\infty(0,T;H), \qquad \boldsymbol{f}_t \in L^1(0,T;H)$$

and, if $d=3$, a condition relating ν, $\boldsymbol{f}$, $\boldsymbol{u}_0$ and T (namely, [24, 3.115, p. 304]), when Ω is of class $\mathcal{C}^2$ or is a convex polygon or polyhedron (see [24, Theorem 3.7, p. 303 and Theorem 3.8, p. 306]); under these assumptions, (R3) follows from [13, Theorem 2.3, pp. 284–285]. These results also hold when Ω is a convex polygon, since some of them rely on the additional regularity of solutions of the Stokes problem in Ω with right side in $\mathbf{L}^2(\Omega)$, and are also generally believed to hold on a convex polyhedron (see [13] and the references therein). Furthermore, we will also assume (see [20,21]) that:

(R4) $\boldsymbol{u}_{tt} \in L^2(0,T;V')$,

a condition which requires some nonlocal compatibility conditions.

Error analysis of time integration schemes for time-dependent partial differential equations are usually given in terms of the following norms: given a Banach space W with norm $\|\cdot\|_W$, a continuous function $u:[0,T]\to W$ and two real numbers $p>0$ and $\alpha>0$, for each time step size $\delta t>0$ let $t_n = n\delta t$ for $n=0,\dots,M=[T/\delta t]$; a family of finite sequences $\{u^n\}_{n=1,\dots,M}$ is said to be an order α approximation of u in $l^p(W)$ if there exists a constant C such that, for all δt:

$$\left(\delta t \sum_{n=1}^{M} \left\| u(t_n) - u^n \right\|_W^p\right)^{1/p} < C\,\delta t^\alpha.$$

Moreover, $\{u^n\}_{n=1,\dots,M}$ is an order α approximation of u in $l^\infty(W)$ if:

$$\left\| u(t_n) - u^n \right\|_W < C\,\delta t^\alpha, \quad \forall n = 1,\dots,M.$$

Here, and in what follows, C denotes a generic constant, possibly different at different occurrences, which may depend on the data $\boldsymbol{f}$, $\boldsymbol{u}_0$, T and ν, the domain Ω and the continuous solution $\boldsymbol{u}$, but is independent of the time step δt and the mesh size h.

For the treatment of the convective term in the momentum equation (1), the following trilinear form is usually considered:

$$c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \big((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}\big), \quad \forall \boldsymbol{u}\in\mathbf{H}^1(\Omega),\ \boldsymbol{v}\in\mathbf{H}^1(\Omega),\ \boldsymbol{w}\in\mathbf{H}_0^1(\Omega).$$

This form is well defined and continuous on these spaces (see [24]), and it is skew-symmetric in its last two arguments if $\boldsymbol{u}\in H$, that is, if $\nabla\cdot\boldsymbol{u}=0$ and $\boldsymbol{n}\cdot\boldsymbol{u}=0$:

$$c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0, \quad \forall \boldsymbol{u}\in H,\ \boldsymbol{v}\in\mathbf{H}_0^1(\Omega). \tag{5}$$

Moreover, c has some continuity properties which hold when Ω is regular enough (see [7]) and which we will use in our proofs, such as:

$$c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \leqslant C \begin{cases} \|\boldsymbol{u}\|_0\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|_1, \\ \|\boldsymbol{u}\|_0\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_2, \\ \|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_0, \\ \|\boldsymbol{u}\|_0^{1/2}\|\boldsymbol{u}\|_1^{1/2}\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1, \\ \|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_0^{1/2}\|\boldsymbol{w}\|_1^{1/2}, \\ \|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_{L^3(\Omega)}, \\ \|\boldsymbol{u}\|_{L^8(\Omega)}\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_{L^{8/3}(\Omega)}. \end{cases}$$

Although this form is suitable for our analysis of the semidiscrete method, we will use the skew-symmetric part of c in the fully discrete problem, since incompressibility is only enforced weakly in the discrete setting; thus, we define:

$$\tilde{c}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = (1/2)\big(c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) - c(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})\big), \quad \forall \boldsymbol{u} \in \mathbf{H}^1(\Omega),\ \boldsymbol{v} \in \mathbf{H}_0^1(\Omega),\ \boldsymbol{w} \in \mathbf{H}_0^1(\Omega).$$

Obviously, this form retains the continuity properties of the original form c (but for the fifth one), and is skew-symmetric in its last two arguments for any $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$.

In some of our proofs we will also make use of the operator A^{-1}, defined as the inverse of the Stokes operator $A = -P_H\Delta$, P_H being the projection onto H. The latter is defined for $\boldsymbol{v} \in D(A) = V \cap \mathbf{H}^2(\Omega)$, and is an unbounded, positive, self-adjoint closed operator onto H. Given $\boldsymbol{u} \in H$, by definition of A, $\boldsymbol{v} = A^{-1}\boldsymbol{u}$ is the solution of the following Stokes problem:

$$\begin{aligned} &-\Delta\boldsymbol{v} + \nabla r = \boldsymbol{u} && \text{in } \Omega, \\ &\nabla\cdot\boldsymbol{v} = 0 && \text{in } \Omega, \\ &\boldsymbol{v} = 0 && \text{on } \partial\Omega. \end{aligned} \tag{6}$$

When Ω is of class $\mathcal{C}^2$, or is a convex polygon or polyhedron (see [13]), there exists a constant $C_1 > 0$ such that:

$$\left\|A^{-1}\boldsymbol{u}\right\|_s \leqslant C_1\|\boldsymbol{u}\|_{s-2}, \quad \text{for } s = 1, 2. \tag{7}$$

Furthermore, from (6) one gets $(A^{-1}\boldsymbol{u},\boldsymbol{u}) = \|A^{-1}\boldsymbol{u}\|_1$, and then it is easily seen that:

$$\|\boldsymbol{u}\|_{V'}^2 = \big(A^{-1}\boldsymbol{u},\boldsymbol{u}\big), \tag{8}$$

for all $\boldsymbol{u} \in H$. We will use these results in what follows.

3. Fractional-step method

The fractional-step method we analyze here was introduced in [1], where stability and convergence, both in the spaces $L^\infty(0,T;\mathbf{L}^2(\Omega))$ and $L^2(0,T;\mathbf{H}_0^1(\Omega))$ and of both the intermediate and the end-of-step velocities, where proved. Given $\boldsymbol{u}^n \in V$, approximation of $\boldsymbol{u}$ at $t = t_n$, the time advancement to t_{n+1} is split into two steps:

First step. The first step of the method, which includes viscous and convective effects, consists of finding an intermediate velocity $\boldsymbol{u}^{n+1/2}$ such that:

$$\frac{\boldsymbol{u}^{n+1/2} - \boldsymbol{u}^n}{\delta t} - \nu \Delta \boldsymbol{u}^{n+1/2} + \big(\boldsymbol{u}^n \cdot \nabla\big)\boldsymbol{u}^{n+1/2} = \boldsymbol{f}(t_{n+1}), \tag{9}$$

$$\boldsymbol{u}^{n+1/2}|_{\partial\Omega} = 0. \tag{10}$$

Second step. Given $\boldsymbol{u}^{n+1/2}$ from (9), (10), the second step of the method consists of finding $\boldsymbol{u}^{n+1}$ and p^{n+1} such that:

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+1/2}}{\delta t} - \nu \Delta\big(\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+1/2}\big) + \nabla p^{n+1} = 0, \tag{11}$$

$$\nabla \cdot \boldsymbol{u}^{n+1} = 0, \tag{12}$$

$$\boldsymbol{u}^{n+1}|_{\partial\Omega} = 0. \tag{13}$$

As can be observed in (11), the main difference between this method and the standard projection method is the introduction of a viscous term in the incompressibility step, which allows the imposition of the original boundary condition (13) on the end-of-step velocity $\boldsymbol{u}^{n+1}$. Similar ideas can be found in the θ-*method* of Glowinski and others (see [11], for instance) and in several other methods such as those of [6,16,17] or [26], all of which involve an incompressible step with part of the viscous term. It can be observed in (9), (10) and (11)–(13) how in this method convection is split from incompressibility, which are the two main difficulties of the problem. We have adopted here a linearized, first order form of the convective term, although there are obviously other possibilities.

The motivations that led us to the study of this fractional-step method are mainly twofold. First, it can be used to explain theoretically a class of predictor–multicorrector algorithms widely used in practice (see [1] for a more detailed explanation). These methods are based on an iterative scheme consisting of two steps per iteration with the same structure as the two steps above. Second, the imposition of the original boundary conditions on the end-of-step velocity. It is common practice among some users of the classical projection method to enforce all the boundary conditions on this field, although this is in principle not allowed if the viscous term in Eq. (11) is dropped. The present scheme, however, is not subject to this controversy; moreover, the fact that $\boldsymbol{u}^{n+1}$ satisfies the correct boundary conditions led to improved convergence results in [1] with respect to those known for that variable in the standard projection method, and will allow us to obtain improved error estimates here too.

The computational efficiency of the scheme (9)–(13) was studied in [1]. The first step of the method, which is a linear, elliptic problem, can be seen as a linearized Burger's problem; on the other hand, the second step has the structure of a Stokes (mixed) problem, the discretization of which leads to a symmetric system of linear equations. Based on ideas taken from the predictor–multicorrector algorithm used in [2], we developed in [1] an iterative technique for the solution of these two problems, in which each iteration consists of the solution of two linear systems with a diagonal matrix and a system with a symmetric, positive (semi)definite matrix, which is the same for all iterations and time steps (and thus needs being computed and factorized only once at the beginning of the calculations); this iteration showed good convergence results in several test cases, which makes the present fractional-step method feasible from a practical viewpoint. One drawback of this method is the need for the spatial discretization used to satisfy the discrete inf–sup compatibility condition, something which is nowadays known to apply to most versions of the standard projection method too (see [12]).

4. Error analysis

We present here an error analysis of the fractional-step method introduced in the previous section. Although we consider the first order, linearized form of the convective term $(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^{n+1/2}$, similar error estimates can be obtained for other approaches, such as the fully nonlinear form $(\boldsymbol{u}^{n+1/2}\cdot\nabla)\boldsymbol{u}^{n+1/2}$; likewise, other approximations of the viscous term than the backward Euler method used here, such as the trapezoidal rule, could also be studied.

4.1. Error estimates for the semidiscrete velocities

Let us define the semidiscrete velocity errors as:

$$\boldsymbol{e}_c^{n+1}=\boldsymbol{u}(t_{n+1})-\boldsymbol{u}^{n+1},$$
$$\boldsymbol{e}_c^{n+1/2}=\boldsymbol{u}(t_{n+1})-\boldsymbol{u}^{n+1/2},$$

where the subscript c refers to the fact that the space variables remain 'continuous'. We give a first estimate for $\boldsymbol{e}_c^{n+1}$ and $\boldsymbol{e}_c^{n+1/2}$ which shows that both $\boldsymbol{u}^{n+1}$ and $\boldsymbol{u}^{n+1/2}$ are order $1/2$ approximations to $\boldsymbol{u}$ in $l^\infty(\mathbf{L}^2(\Omega))$ and in $l^2(\mathbf{H}_0^1(\Omega))$:

Lemma 1. *Assume that* (R1), (R2a) *and* (R3) *hold*; *then for* $N=0,\ldots,[T/\delta t]-1$, *and for all* $\delta t>0$:

$$\begin{aligned}&\|\boldsymbol{e}_c^{N+1}\|_0^2+\|\boldsymbol{e}_c^{N+1/2}\|_0^2+\sum_{n=0}^{N}\left\{\|\boldsymbol{e}_c^{n+1}-\boldsymbol{e}_c^{n+1/2}\|_0^2+\|\boldsymbol{e}_c^{n+1/2}-\boldsymbol{e}_c^{n}\|_0^2\right\}\\&\qquad+\delta t\nu\sum_{n=0}^{N}\left\{\|\boldsymbol{e}_c^{n+1}\|_1^2+\|\boldsymbol{e}_c^{n+1/2}\|_1^2+\|\boldsymbol{e}_c^{n+1}-\boldsymbol{e}_c^{n+1/2}\|_1^2\right\}\leqslant C\delta t.\end{aligned}\tag{14}$$

Proof. The first part of the proof is similar to that of [20]. We call $\mathbf{R}^n$ the truncation error defined by:

$$\frac{1}{\delta t}\big(\boldsymbol{u}(t_{n+1})-\boldsymbol{u}(t_n)\big)-\nu\Delta\big(\boldsymbol{u}(t_{n+1})\big)+\big(\boldsymbol{u}(t_{n+1})\cdot\nabla\big)\boldsymbol{u}(t_{n+1})+\nabla p(t_{n+1})=\boldsymbol{f}(t_{n+1})+\mathbf{R}^n,\tag{15}$$

so that

$$\mathbf{R}^n=\frac{1}{\delta t}\int_{t_n}^{t_{n+1}}(t-t_n)\boldsymbol{u}_{tt}(t)\,\mathrm{d}t.$$

Subtracting (9) from (15), we get:

$$\frac{1}{\delta t}\big(\boldsymbol{e}_c^{n+1/2}-\boldsymbol{e}_c^{n}\big)-\nu\Delta\big(\boldsymbol{e}_c^{n+1/2}\big)=\big(\boldsymbol{u}^n\cdot\nabla\big)\boldsymbol{u}^{n+1/2}-\big(\boldsymbol{u}(t_{n+1})\cdot\nabla\big)\boldsymbol{u}(t_{n+1})+\mathbf{R}^n-\nabla p(t_{n+1}).\tag{16}$$

We take the inner product of (16) with $2\delta t\boldsymbol{e}_c^{n+1/2}$, use the identity $(a-b,2a)=|a|^2-|b|^2+|a-b|^2$ and split the nonlinear terms on the right side of (16) as:

$$\begin{aligned}&\big(\boldsymbol{u}^n\cdot\nabla\big)\boldsymbol{u}^{n+1/2}-\big(\boldsymbol{u}(t_{n+1})\cdot\nabla\big)\boldsymbol{u}(t_{n+1})\\&\quad=-\big(\boldsymbol{e}_c^n\cdot\nabla\big)\boldsymbol{u}^{n+1/2}+\big(\big(\boldsymbol{u}(t_n)-\boldsymbol{u}(t_{n+1})\big)\cdot\nabla\big)\boldsymbol{u}^{n+1/2}-\big(\boldsymbol{u}(t_{n+1})\cdot\nabla\big)\boldsymbol{e}_c^{n+1/2},\end{aligned}\tag{17}$$

to obtain:

$$\begin{aligned}
&\|\boldsymbol{e}_c^{n+1/2}\|_0^2 - \|\boldsymbol{e}_c^n\|_0^2 + \|\boldsymbol{e}_c^{n+1/2} - \boldsymbol{e}_c^n\|_0^2 + 2\delta t\nu\|\boldsymbol{e}_c^{n+1/2}\|_1^2 \\
&\quad = 2\delta t\langle \mathbf{R}^n, \boldsymbol{e}_c^{n+1/2}\rangle - 2\delta t\big(\nabla p(t_{n+1}), \boldsymbol{e}_c^{n+1/2}\big) - 2\delta t c\big(\boldsymbol{e}_c^n, \boldsymbol{u}^{n+1/2}, \boldsymbol{e}_c^{n+1/2}\big) \\
&\qquad + 2\delta t c\big(\boldsymbol{u}(t_n) - \boldsymbol{u}(t_{n+1}), \boldsymbol{u}^{n+1/2}, \boldsymbol{e}_c^{n+1/2}\big) - 2\delta t c\big(\boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1/2}, \boldsymbol{e}_c^{n+1/2}\big).
\end{aligned} \tag{18}$$

We bound each term in the RHS of (18) independently:

- Taylor residual term:

$$\begin{aligned}
2\delta t\langle \mathbf{R}^n, \boldsymbol{e}_c^{n+1/2}\rangle &\leqslant \frac{\delta t\nu}{3}\|\boldsymbol{e}_c^{n+1/2}\|_1^2 + \frac{C}{\delta t}\left\| \int_{t_n}^{t_{n+1}} (t - t_n)\boldsymbol{u}_{tt}\,\mathrm{d}t \right\|_{-1}^2 \\
&\leqslant \frac{\delta t\nu}{3}\|\boldsymbol{e}_c^{n+1/2}\|_1^2 + C\delta t \int_{t_n}^{t_{n+1}} t\|\boldsymbol{u}_{tt}\|_{-1}^2\,\mathrm{d}t.
\end{aligned}$$

- Pressure gradient term:

$$\begin{aligned}
-2\delta t\big(\nabla p(t_{n+1}), \boldsymbol{e}_c^{n+1/2}\big) &= -2\delta t\big(\nabla p(t_{n+1}), \boldsymbol{e}_c^{n+1/2} - \boldsymbol{e}_c^n\big) \\
&\leqslant \frac{1}{2}\|\boldsymbol{e}_c^{n+1/2} - \boldsymbol{e}_c^n\|_0^2 + 2\delta t^2\|\nabla p(t_{n+1})\|_0^2,
\end{aligned}$$

  since $\nabla \cdot \boldsymbol{e}_c^n = 0$.
- Nonlinear terms:

$$\begin{aligned}
&-2\delta t c\big(\boldsymbol{e}_c^n, \boldsymbol{u}^{n+1/2}, \boldsymbol{e}_c^{n+1/2}\big) = -2\delta t c\big(\boldsymbol{e}_c^n, \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1/2}\big) \\
&\quad \leqslant C\delta t\|\boldsymbol{e}_c^n\|_0\|\boldsymbol{u}(t_{n+1})\|_2\|\boldsymbol{e}_c^{n+1/2}\|_1 \leqslant \frac{\delta t\nu}{3}\|\boldsymbol{e}_c^{n+1/2}\|_1^2 + C\delta t\|\boldsymbol{e}_c^n\|_0^2, \\
&2\delta t c\big(\boldsymbol{u}(t_n) - \boldsymbol{u}(t_{n+1}), \boldsymbol{u}^{n+1/2}, \boldsymbol{e}_c^{n+1/2}\big) \\
&\quad = 2\delta t c\big(\boldsymbol{u}(t_n) - \boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1/2}\big) \\
&\quad \leqslant C\delta t\|\boldsymbol{u}(t_n) - \boldsymbol{u}(t_{n+1})\|_0\|\boldsymbol{u}(t_{n+1})\|_2\|\boldsymbol{e}_c^{n+1/2}\|_1 \\
&\quad \leqslant \frac{\delta t\nu}{3}\|\boldsymbol{e}_c^{n+1/2}\|_1^2 + C\delta t^2 \int_{t_n}^{t_{n+1}} \|\boldsymbol{u}_t\|_0^2\,\mathrm{d}t - 2\delta t c\big(\boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1/2}, \boldsymbol{e}_c^{n+1/2}\big) = 0,
\end{aligned}$$

where we have used (R1) and the continuity and skew-symmetry properties of the trilinear form c. From all these inequalities we deduce:

$$\begin{aligned}
&\|\boldsymbol{e}_c^{n+1/2}\|_0^2 - \|\boldsymbol{e}_c^n\|_0^2 + \frac{1}{2}\|\boldsymbol{e}_c^{n+1/2} - \boldsymbol{e}_c^n\|_0^2 + \delta t\nu\|\boldsymbol{e}_c^{n+1/2}\|_1^2 \\
&\quad \leqslant C\delta t \int_{t_n}^{t_{n+1}} t\|\boldsymbol{u}_{tt}\|_{-1}^2\,\mathrm{d}t + C\delta t^2 \int_{t_n}^{t_{n+1}} \|\boldsymbol{u}_t\|_0^2\,\mathrm{d}t + 2\delta t^2\|\nabla p(t_{n+1})\|_0^2 + C\delta t\|\boldsymbol{e}_c^n\|_0^2.
\end{aligned} \tag{19}$$

The proof is now different from that of [20]. We rewrite (11) as:

$$\frac{\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n+1/2}}{\delta t} - \nu \Delta\big(\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n+1/2}\big) - \nabla p^{n+1} = 0. \tag{20}$$

Taking the inner product of (20) with $2\delta t \boldsymbol{e}_c^{n+1}$, we get, given that $\nabla \cdot \boldsymbol{e}_c^{n+1} = 0$:

$$\begin{aligned} &\big\|\boldsymbol{e}_c^{n+1}\big\|_0^2 - \big\|\boldsymbol{e}_c^{n+1/2}\big\|_0^2 + \big\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n+1/2}\big\|_0^2 \\ &\quad + \delta t \nu \big(\big\|\boldsymbol{e}_c^{n+1}\big\|_1^2 - \big\|\boldsymbol{e}_c^{n+1/2}\big\|_1^2 + \big\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n+1/2}\big\|_1^2\big) = 0. \end{aligned} \tag{21}$$

Adding up (19) and (21) for $n = 0, \ldots, N$, we find:

$$\begin{aligned} &\big\|\boldsymbol{e}_c^{N+1}\big\|_0^2 + \sum_{n=0}^{N} \Big\{ \big\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n+1/2}\big\|_0^2 + \frac{1}{2}\big\|\boldsymbol{e}_c^{n+1/2} - \boldsymbol{e}_c^{n}\big\|_0^2 \Big\} \\ &\quad + \delta t \nu \sum_{n=0}^{N} \big\{ \big\|\boldsymbol{e}_c^{n+1}\big\|_1^2 + \big\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n+1/2}\big\|_1^2 \big\} \\ &\quad \leqslant C\delta t \left(\int_0^T t \|\boldsymbol{u}_{tt}\|_{-1}^2 \,\mathrm{d}t + \delta t \int_0^T \|\boldsymbol{u}_t\|_0^2 \,\mathrm{d}t + \sup_{t \in [0,T]} \big\|\nabla p(t)\big\|_0^2 \right) + C\delta t \sum_{n=0}^{N} \big\|\boldsymbol{e}_c^n\big\|_0^2. \end{aligned}$$

Applying the discrete Gronwall lemma to the last inequality and using the regularity properties (R1), (R2a) and (R3) of the continuous solution, we obtain:

$$\begin{aligned} &\big\|\boldsymbol{e}_c^{N+1}\big\|_0^2 + \sum_{n=0}^{N} \big\{ \big\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n+1/2}\big\|_0^2 + \big\|\boldsymbol{e}_c^{n+1/2} - \boldsymbol{e}_c^{n}\big\|_0^2 \big\} \\ &\quad + \delta t \nu \sum_{n=0}^{N} \big\{ \big\|\boldsymbol{e}_c^{n+1}\big\|_1^2 + \big\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n+1/2}\big\|_1^2 \big\} \leqslant C\delta t. \end{aligned} \tag{22}$$

Finally, the bounds for $\boldsymbol{u}^{n+1/2}$ follow from (22) and the triangle inequality, so that (14) is proved. □

Remark 2. Lemma 1 shows, in particular, that the method provides uniformly stable velocities in $\mathbf{H}_0^1(\Omega)$, that is to say, that there exists a constant $C > 0$ independent of the time step δt such that for all $n = 0, \ldots, [T/\delta t] - 1$:

$$\big\|\boldsymbol{u}^{n+1}\big\|_1 \leqslant C, \qquad \big\|\boldsymbol{u}^{n+1/2}\big\|_1 \leqslant C, \tag{23}$$

since $\|\boldsymbol{e}_c^{n+1}\|_1 \leqslant C$, $\|\boldsymbol{e}_c^{n+1/2}\|_1 \leqslant C$ and $\boldsymbol{u} \in L^\infty(0, T; \mathbf{H}_0^1(\Omega))$. Moreover, we also have:

$$\big\|\boldsymbol{e}_c^{n+1}\big\|_0 \leqslant C\delta t^{1/2}, \qquad \big\|\boldsymbol{e}_c^{n+1/2}\big\|_0 \leqslant C\delta t^{1/2}. \tag{24}$$

We will use these bounds later on.

Next we give a first order error estimate for both $\boldsymbol{u}^{n+1/2}$ and $\boldsymbol{u}^{n+1}$ in the norm of $l^2(\mathbf{L}^2(\Omega))$, which is what was proved for the standard projection method in [20] when applied to the (linear) Stokes problem, that is, when dropping the convective term in (1):

Theorem 3. *Assume* (R1), (R2a), (R3) *and* (R4) *hold*; *then, for* $N = 0, \dots, [T/\delta t] - 1$ *and for small enough* δt:

$$\left\| \boldsymbol{e}_c^{N+1} \right\|_{V'}^2 + \delta t \sum_{n=0}^{N} \left(\left\| \boldsymbol{e}_c^{n+1} \right\|_0^2 + \left\| \boldsymbol{e}_c^{n+1/2} \right\|_0^2 \right) \leqslant C \delta t^2, \tag{25}$$

that is, $\boldsymbol{u}^{n+1}$ *converges to* $\boldsymbol{u}(t_{n+1})$ *in* $l^\infty(V') \cap l^2(\mathbf{L}^2(\Omega))$ *and* $\boldsymbol{u}^{n+1/2}$ *in* $l^2(\mathbf{L}^2(\Omega))$ *with order* δt.

Proof. By adding (9) and (11), we get:

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\delta t} - \nu \Delta \boldsymbol{u}^{n+1} + \left(\boldsymbol{u}^n \cdot \nabla \right) \boldsymbol{u}^{n+1/2} + \nabla p^{n+1} = \boldsymbol{f}(t_{n+1}). \tag{26}$$

Calling $r_c^{n+1} = p(t_{n+1}) - p^{n+1}$ the pressure error and subtracting (26) from (15), we have:

$$\frac{1}{\delta t} \left(\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n \right) - \nu \Delta \left(\boldsymbol{e}_c^{n+1} \right) + \nabla r_c^{n+1} = \left(\boldsymbol{u}^n \cdot \nabla \right) \boldsymbol{u}^{n+1/2} - \left(\boldsymbol{u}(t_{n+1}) \cdot \nabla \right) \boldsymbol{u}(t_{n+1}) + \mathbf{R}^n. \tag{27}$$

We take the inner product of (27) with $2\delta t A^{-1} \boldsymbol{e}_c^{n+1}$, as in [20], and use the self-adjointness of A^{-1} to get:

$$\begin{aligned} &\left(\boldsymbol{e}_c^{n+1}, A^{-1} \boldsymbol{e}_c^{n+1} \right) - \left(\boldsymbol{e}_c^n, A^{-1} \boldsymbol{e}_c^n \right) + \left(\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n, A^{-1} \left(\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n \right) \right) - 2\delta t \nu \left(\Delta \boldsymbol{e}_c^{n+1}, A^{-1} \boldsymbol{e}_c^{n+1} \right) \\ &\quad = 2\delta t c \left(\boldsymbol{u}^n, \boldsymbol{u}^{n+1/2}, A^{-1} \boldsymbol{e}_c^{n+1} \right) \\ &\qquad - 2\delta t c \left(\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), A^{-1} \boldsymbol{e}_c^{n+1} \right) + 2\delta t \left\langle \mathbf{R}^n, A^{-1} \boldsymbol{e}_c^{n+1} \right\rangle. \end{aligned} \tag{28}$$

Taking now $\boldsymbol{u} = \boldsymbol{e}_c^{n+1}$ in (6), we get:

$$\begin{aligned} &-2\delta t \nu \left(\Delta \boldsymbol{e}_c^{n+1}, A^{-1} \boldsymbol{e}_c^{n+1} \right) = 2\delta t \nu \left(\boldsymbol{e}_c^{n+1}, -\Delta \left(A^{-1} \boldsymbol{e}_c^{n+1} \right) \right) \\ &\quad = 2\delta t \nu \left(\boldsymbol{e}_c^{n+1}, \boldsymbol{e}_c^{n+1} - \nabla r \right) = 2\delta t \nu \left\| \boldsymbol{e}_c^{n+1} \right\|_0^2, \end{aligned}$$

since $\nabla \cdot \boldsymbol{e}_c^{n+1} = 0$. The RHS terms in (28) are bounded as follows. For the Taylor residual term, we have:

$$\begin{aligned} &2\delta t \left\langle \mathbf{R}^n, A^{-1} \boldsymbol{e}_c^{n+1} \right\rangle \leqslant 2\delta t \left\| \mathbf{R}^n \right\|_{V'} \left\| A^{-1} \boldsymbol{e}_c^{n+1} \right\|_1 \leqslant 2\delta t \left\| \mathbf{R}^n \right\|_{V'} \left\| \boldsymbol{e}_c^{n+1} \right\|_{V'} \\ &\quad \leqslant \delta t \left\| \boldsymbol{e}_c^{n+1} \right\|_{V'}^2 + C \delta t \left\| \mathbf{R}^n \right\|_{V'}^2 \leqslant \delta t \left\| \boldsymbol{e}_c^{n+1} \right\|_{V'}^2 + C \delta t^2 \int_{t_n}^{t_{n+1}} \| \boldsymbol{u}_{tt} \|_{V'}^2 \, \mathrm{d}t. \end{aligned}$$

For the nonlinear terms, we use the splitting (17) again and bound the corresponding three terms as follows:

$$\begin{aligned} &-2\delta t c \left(\boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1/2}, A^{-1} \boldsymbol{e}_c^{n+1} \right) \leqslant C \delta t \left\| \boldsymbol{u}(t_{n+1}) \right\|_2 \left\| A^{-1} \boldsymbol{e}_c^{n+1} \right\|_1 \left\| \boldsymbol{e}_c^{n+1/2} \right\|_0 \\ &\quad \leqslant C \delta t \left\| \boldsymbol{e}_c^{n+1} \right\|_{V'}^2 + \frac{\delta t \nu}{4} \left\| \boldsymbol{e}_c^{n+1/2} \right\|_0^2 \\ &\quad = C \delta t \left\| \boldsymbol{e}_c^{n+1} \right\|_{V'}^2 + \frac{\delta t \nu}{4} \Big\{ \left\| \boldsymbol{e}_c^{n+1} \right\|_0^2 + \left\| \boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n+1/2} \right\|_0^2 \\ &\qquad\qquad + \delta t \nu \left\| \boldsymbol{e}_c^{n+1} \right\|_1^2 + \delta t \nu \left\| \boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n+1/2} \right\|_1^2 - \delta t \nu \left\| \boldsymbol{e}_c^{n+1/2} \right\|_1^2 \Big\}, \end{aligned}$$

where we have used (5) and (21);

$$
\begin{aligned}
&2\delta t c\big(\boldsymbol{u}(t_n)-\boldsymbol{u}(t_{n+1}),\boldsymbol{u}^{n+1/2},A^{-1}\boldsymbol{e}_c^{n+1}\big)\\
&\quad\leqslant C\delta t\big\|\boldsymbol{u}(t_n)-\boldsymbol{u}(t_{n+1})\big\|_0\big\|\boldsymbol{u}^{n+1/2}\big\|_1\big\|A^{-1}\boldsymbol{e}_c^{n+1}\big\|_2\\
&\quad\leqslant C\delta t\left\|\int_{t_n}^{t_{n+1}}\boldsymbol{u}_t\,\mathrm{d}t\right\|_0\big\|\boldsymbol{e}_c^{n+1}\big\|_0\leqslant C\delta t^2\int_{t_n}^{t_{n+1}}\|\boldsymbol{u}_t\|_0^2\,\mathrm{d}t+\frac{\delta t\nu}{4}\big\|\boldsymbol{e}_c^{n+1}\big\|_0^2,
\end{aligned}
$$

where we have used (23); and:

$$
\begin{aligned}
&-2\delta t c\big(\boldsymbol{e}_c^{n},\boldsymbol{u}^{n+1/2},A^{-1}\boldsymbol{e}_c^{n+1}\big)\\
&\quad=2\delta t c\big(\boldsymbol{e}_c^{n},A^{-1}\boldsymbol{e}_c^{n+1},\boldsymbol{u}(t_{n+1})\big)-2\delta t c\big(\boldsymbol{e}_c^{n},A^{-1}\boldsymbol{e}_c^{n+1},\boldsymbol{e}_c^{n+1/2}\big)=\mathrm{T}_1+\mathrm{T}_2,
\end{aligned}
$$

so that:

$$
\begin{aligned}
\mathrm{T}_1&\leqslant C\delta t\big\|\boldsymbol{e}_c^{n}\big\|_0\big\|A^{-1}\boldsymbol{e}_c^{n+1}\big\|_1\big\|\boldsymbol{u}(t_{n+1})\big\|_2\\
&\leqslant C\delta t\big\|\boldsymbol{e}_c^{n}\big\|_0\big\|\boldsymbol{e}_c^{n+1}\big\|_{V'}\leqslant C\delta t\big(\big\|\boldsymbol{e}_c^{n+1}\big\|_0+\big\|\boldsymbol{e}_c^{n+1}-\boldsymbol{e}_c^{n+1/2}\big\|_0+\big\|\boldsymbol{e}_c^{n+1/2}-\boldsymbol{e}_c^{n}\big\|_0\big)\big\|\boldsymbol{e}_c^{n+1}\big\|_{V'}\\
&\leqslant\frac{\delta t\nu}{4}\big\|\boldsymbol{e}_c^{n+1}\big\|_0^2+C\delta t\big(\big\|\boldsymbol{e}_c^{n+1}-\boldsymbol{e}_c^{n+1/2}\big\|_0^2+\big\|\boldsymbol{e}_c^{n+1/2}-\boldsymbol{e}_c^{n}\big\|_0^2\big)+C\delta t\big\|\boldsymbol{e}_c^{n+1}\big\|_{V'}^2,
\end{aligned}
$$

due to (R1); and finally:

$$
\begin{aligned}
\mathrm{T}_2&\leqslant C\delta t\big\|\boldsymbol{e}_c^{n}\big\|_0\big\|A^{-1}\boldsymbol{e}_c^{n+1}\big\|_2\big\|\boldsymbol{e}_c^{n+1/2}\big\|_1\leqslant C\delta t\big\|\boldsymbol{e}_c^{n}\big\|_0\big\|\boldsymbol{e}_c^{n+1}\big\|_0\big\|\boldsymbol{e}_c^{n+1/2}\big\|_1\\
&\leqslant C\delta t^{3/2}\big\|\boldsymbol{e}_c^{n+1}\big\|_0\big\|\boldsymbol{e}_c^{n+1/2}\big\|_1\leqslant\frac{\delta t\nu}{4}\big\|\boldsymbol{e}_c^{n+1}\big\|_0^2+C\delta t^2\big\|\boldsymbol{e}_c^{n+1/2}\big\|_1^2,
\end{aligned}
$$

where we have used (24). Adding up (28) for $n=0,\dots,N$, and using all these inequalities, we get:

$$
\begin{aligned}
&\big(\boldsymbol{e}_c^{N+1},A^{-1}\boldsymbol{e}_c^{N+1}\big)+\sum_{n=0}^{N}\big(\boldsymbol{e}_c^{n+1}-\boldsymbol{e}_c^{n},A^{-1}\big(\boldsymbol{e}_c^{n+1}-\boldsymbol{e}_c^{n}\big)\big)+\delta t\nu\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1}\big\|_0^2\\
&\quad\leqslant C\delta t^2\int_0^T\|\boldsymbol{u}_{tt}\|_{V'}^2\,\mathrm{d}t+C\delta t^2\int_0^T\|\boldsymbol{u}_t\|_0^2\,\mathrm{d}t+C\delta t\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1}\big\|_{V'}^2+C\delta t^2\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1}\big\|_1^2\\
&\qquad+C\delta t\sum_{n=0}^{N}\big\{\big\|\boldsymbol{e}_c^{n+1}-\boldsymbol{e}_c^{n+1/2}\big\|_0^2+\big\|\boldsymbol{e}_c^{n+1/2}-\boldsymbol{e}_c^{n}\big\|_0^2\big\}+C\delta t^2\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1}-\boldsymbol{e}_c^{n+1/2}\big\|_1^2\\
&\qquad+C\delta t^2\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1/2}\big\|_1^2.
\end{aligned}
$$

Using now (8), the regularity properties (R2a) and (R4) of the continuous solution and the estimates of Lemma 1, we get:

$$
\big\|\boldsymbol{e}_c^{N+1}\big\|_{V'}^2+\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1}-\boldsymbol{e}_c^{n}\big\|_{V'}^2+\delta t\nu\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1}\big\|_0^2\leqslant C\delta t^2+C\delta t\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1}\big\|_{V'}^2.
$$

For sufficiently small δt, we can apply the discrete Gronwall lemma to the last inequality, and we get:

$$
\big\|\boldsymbol{e}_c^{N+1}\big\|_{V'}^2+\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1}-\boldsymbol{e}_c^{n}\big\|_{V'}^2+\delta t\nu\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1}\big\|_0^2\leqslant C\delta t^2,\tag{29}
$$

and the estimate for $\boldsymbol{u}^{n+1}$ is proved. For $\boldsymbol{u}^{n+1/2}$, we have:

$$\delta t\nu \sum_{n=0}^{N} \|\boldsymbol{e}_c^{n+1/2}\|_0^2 \leqslant 2\delta t\nu \sum_{n=0}^{N} \left(\|\boldsymbol{e}_c^{n+1}\|_0^2 + \|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n+1/2}\|_0^2\right) \leqslant C\delta t^2,$$

due to (29) and Lemma 1, so that (25) is proved. □

The estimates of Theorem 3 allow us to obtain now enhanced stability properties of the semidiscrete solution.

Theorem 4. *Assume that* (R1), (R2a), (R3) *and* (R4) *hold*; *assume also that* $\boldsymbol{f} \in L^\infty(0,T;H)$ *and that the domain* Ω *is of class* $\mathcal{C}^2$ (*or is a convex polygon or polyhedron*); *then, for* $N = 0, \ldots, [T/\delta t] - 1$, *and for small enough* δt:

$$\delta t \sum_{n=0}^{N} \left\{\|\boldsymbol{u}^{n+1}\|_2^2 + \|\boldsymbol{u}^{n+1/2}\|_2^2\right\} \leqslant C, \qquad \delta t \sum_{n=0}^{N} \|p^{n+1}\|_1^2 \leqslant C,$$

that is, $\boldsymbol{u}^{n+1}$ *and* $\boldsymbol{u}^{n+1/2}$ *are uniformly bounded in* $l^2(\mathbf{H}^2(\Omega))$ *and* p^{n+1} *is uniformly bounded in* $l^2(\mathbf{H}^1(\Omega))$.

Proof. We use a similar argument to that of [24, Theorem III.3.8]. We rewrite (9) as:

$$-\nu\Delta\boldsymbol{u}^{n+1/2} = \boldsymbol{f}(t_{n+1}) - \frac{1}{\delta t}\left(\boldsymbol{u}^{n+1/2} - \boldsymbol{u}^n\right) - \left(\boldsymbol{u}^n \cdot \nabla\right)\boldsymbol{u}^{n+1/2}. \tag{30}$$

Then:

$$\begin{aligned}
&\delta t \sum_{n=0}^{N} \left\| \frac{1}{\delta t}\left(\boldsymbol{u}^{n+1/2} - \boldsymbol{u}^n\right)\right\|_0^2 \\
&\quad\leqslant \frac{C}{\delta t} \sum_{n=0}^{N} \left\{\|\boldsymbol{u}^{n+1/2} - \boldsymbol{u}(t_{n+1})\|_0^2 + \|\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)\|_0^2 + \|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|_0^2\right\} \\
&\quad\leqslant \frac{C}{\delta t} \sum_{n=0}^{N} \left\{\|\boldsymbol{e}_c^{n+1/2}\|_0^2 + \delta t \int_{t_n}^{t_{n+1}} \|\boldsymbol{u}_t\|_0^2 \,\mathrm{d}t + \|\boldsymbol{e}_c^n\|_0^2\right\} \leqslant C,
\end{aligned}$$

due to Theorem 3 and assumption (R2a). Moreover:

$$\left\|\left(\boldsymbol{u}^n \cdot \nabla\right)\boldsymbol{u}^{n+1/2}\right\|_{\mathbf{L}^{3/2}(\Omega)} = \sup_{\boldsymbol{w}\in\mathbf{L}^3(\Omega)} \frac{((\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^{n+1/2}, \boldsymbol{w})}{\|\boldsymbol{w}\|_{\mathbf{L}^3(\Omega)}} \leqslant C\|\boldsymbol{u}^n\|_1\|\boldsymbol{u}^{n+1/2}\|_1 \leqslant C,$$

due to the continuity properties of the trilinear form c and Remark 2; from (30), we can now deduce that $\Delta\boldsymbol{u}^{n+1/2}$ is bounded in $l^2(\mathbf{L}^{3/2}(\Omega))$. Next, we rewrite (11) as:

$$\begin{aligned}
&-\nu\Delta\boldsymbol{u}^{n+1} + \nabla p^{n+1} = -\nu\Delta\boldsymbol{u}^{n+1/2} - \frac{1}{\delta t}\left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+1/2}\right), \\
&\nabla \cdot \boldsymbol{u}^{n+1} = 0, \\
&\boldsymbol{u}^{n+1}|_{\partial\Omega} = 0.
\end{aligned} \tag{31}$$

The term $\frac{1}{\delta t}(\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+1/2})$ can be easily bounded in $l^2(\mathbf{L}^2(\Omega))$ as before, so that, using the regularity of solutions of the Stokes problem (31) on regular domains, we can assure that $\boldsymbol{u}^{n+1}$ is bounded in $l^2(\mathbf{W}^{2,3/2}(\Omega))$ and p^{n+1} is bounded in $l^2(W^{1,3/2}(\Omega))$. Due to Sobolev's compactness theorem, we then have that $\boldsymbol{u}^{n+1}$ is bounded in $l^2(\mathbf{L}^8(\Omega))$ both when $d = 2$ and 3. Furthermore:

$$\left\|(\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^{n+1/2}\right\|_{\mathbf{L}^{8/5}(\Omega)} = \sup_{\boldsymbol{w}\in\mathbf{L}^{8/3}(\Omega)} \frac{((\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^{n+1/2}, \boldsymbol{w})}{\|\boldsymbol{w}\|_{\mathbf{L}^{8/3}(\Omega)}} \leqslant C\left\|\boldsymbol{u}^n\right\|_{\mathbf{L}^8(\Omega)}\left\|\boldsymbol{u}^{n+1/2}\right\|_1,$$

according to the last property of the form c on page 5, which ensures that $(\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^{n+1/2}$ is bounded in $l^2(\mathbf{L}^{8/5}(\Omega))$. Returning to (30), we improve the regularity of $\Delta\boldsymbol{u}^{n+1/2}$ to $l^2(\mathbf{L}^{8/5}(\Omega))$, and then that of $\boldsymbol{u}^{n+1}$ to $l^2(\mathbf{W}^{2,8/5}(\Omega))$ and p^{n+1} to $l^2(W^{1,8/5}(\Omega))$, as solutions of the Stokes problem (31). Sobolev's theorem ensures now that $\boldsymbol{u}^{n+1}$ is bounded in $l^2(\mathbf{L}^\infty(\Omega))$. This fact, together with Remark 2, implies that $(\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^{n+1/2}$ is bounded in $l^2(\mathbf{L}^2(\Omega))$, which, returning to (30) once more, provides a bound for $\Delta\boldsymbol{u}^{n+1/2}$ also in $l^2(\mathbf{L}^2(\Omega))$, which is sufficient to bound $\boldsymbol{u}^{n+1/2}$ in $l^2(\mathbf{H}^2(\Omega))$ when Ω is regular enough (see [10]). Finally, the bounds for $\boldsymbol{u}^{n+1}$ and p^{n+1} follow again from the regularity of the Stokes problem. □

The error estimates of Theorem 3 can be improved to first order in the norms of $l^\infty(\mathbf{L}^2(\Omega))$ and $l^2(\mathbf{H}_0^1(\Omega))$ for the end-of-step velocities $\boldsymbol{u}^{n+1}$ assuming some slightly stronger regularity on the continuous solution, namely, (R2b) rather than (R2a). Estimates in these norms were also obtained in [12] for the intermediate velocities of a fully discrete, incremental version of the fractional step projection method, assuming a finite element spatial discretization satisfying the discrete inf–sup condition and under much stronger regularity assumptions on the continuous solution:

Theorem 5. *Assume that* (R1), (R2b), (R3) *and* (R4) *hold; then, for* $N = 0, \ldots, [T/\delta t] - 1$, *and for small enough* δt:

$$\left\|\boldsymbol{e}_c^{N+1}\right\|_0^2 + \delta t \nu \sum_{n=0}^{N} \left\|\boldsymbol{e}_c^{n+1}\right\|_1^2 \leqslant C\delta t^2, \tag{32}$$

that is, $\boldsymbol{u}^{n+1}$ *converges to* $\boldsymbol{u}(t_{n+1})$ *in* $l^\infty(\mathbf{L}^2(\Omega)) \cap l^2(\mathbf{H}_0^1(\Omega))$ *with order* δt.

Proof. Unlike in the standard projection method, we can take the inner product of (27) with $2\delta t \boldsymbol{e}_c^{n+1}$, since in our case $\boldsymbol{e}_c^{n+1} \in V$, to get:

$$\begin{aligned} &\left\|\boldsymbol{e}_c^{n+1}\right\|_0^2 - \left\|\boldsymbol{e}_c^{n}\right\|_0^2 + \left\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n}\right\|_0^2 + 2\delta t \nu \left\|\boldsymbol{e}_c^{n+1}\right\|_1^2 \\ &\quad = 2\delta t c\left(\boldsymbol{u}^n, \boldsymbol{u}^{n+1/2}, \boldsymbol{e}_c^{n+1}\right) - 2\delta t c\left(\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1}\right) + 2\delta t\left\langle \mathbf{R}^n, \boldsymbol{e}_c^{n+1}\right\rangle. \end{aligned} \tag{33}$$

The RHS terms in (33) are bounded as follows. For the Taylor residual term, we have:

$$2\delta t\left\langle \mathbf{R}^n, \boldsymbol{e}_c^{n+1}\right\rangle \leqslant 2\delta t\left\|\mathbf{R}^n\right\|_{V'}\left\|\boldsymbol{e}_c^{n+1}\right\|_1 \leqslant \frac{\delta t \nu}{5}\left\|\boldsymbol{e}_c^{n+1}\right\|_1^2 + C\delta t^2 \int_{t_n}^{t_{n+1}} \|\boldsymbol{u}_{tt}\|_{V'}^2 \,\mathrm{d}t.$$

For the nonlinear terms, we use again the splitting (17) and bound the corresponding terms as:

$$
\begin{aligned}
&-2\delta t c\big(\boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1/2}, \boldsymbol{e}_c^{n+1}\big) \\
&\quad \leqslant C\delta t \big\|\boldsymbol{u}(t_{n+1})\big\|_2 \big\|\boldsymbol{e}_c^{n+1}\big\|_1 \big\|\boldsymbol{e}_c^{n+1/2}\big\|_0 \leqslant \frac{\delta t \nu}{5} \big\|\boldsymbol{e}_c^{n+1}\big\|_1^2 + C\delta t \big\|\boldsymbol{e}_c^{n+1/2}\big\|_0^2, \\
&2\delta t c\big(\boldsymbol{u}(t_n) - \boldsymbol{u}(t_{n+1}), \boldsymbol{u}^{n+1/2}, \boldsymbol{e}_c^{n+1}\big) \\
&\quad \leqslant C\delta t \big\|\boldsymbol{u}(t_n) - \boldsymbol{u}(t_{n+1})\big\|_1 \big\|\boldsymbol{u}^{n+1/2}\big\|_1 \big\|\boldsymbol{e}_c^{n+1}\big\|_1 \leqslant C\delta t^2 \int_{t_n}^{t_{n+1}} \|\boldsymbol{u}_t\|_1^2 \,\mathrm{d}t + \frac{\delta t \nu}{5} \big\|\boldsymbol{e}_c^{n+1}\big\|_1^2, \\
&-2\delta t c\big(\boldsymbol{e}_c^{n}, \boldsymbol{u}^{n+1/2}, \boldsymbol{e}_c^{n+1}\big) = 2\delta t c\big(\boldsymbol{e}_c^{n}, \boldsymbol{e}_c^{n+1/2}, \boldsymbol{e}_c^{n+1}\big) - 2\delta t c\big(\boldsymbol{e}_c^{n}, \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1}\big) = \mathrm{T}_1 + \mathrm{T}_2,
\end{aligned}
$$

so that:

$$
\begin{aligned}
\mathrm{T}_1 &\leqslant C\delta t \big\|\boldsymbol{e}_c^{n}\big\|_1 \big\|\boldsymbol{e}_c^{n+1}\big\|_1 \big\|\boldsymbol{e}_c^{n+1/2}\big\|_0^{1/2} \big\|\boldsymbol{e}_c^{n+1/2}\big\|_1^{1/2} \\
&\leqslant C\delta t \big\|\boldsymbol{e}_c^{n}\big\|_1 \big\|\boldsymbol{e}_c^{n+1}\big\|_1 \big\|\boldsymbol{e}_c^{n+1/2}\big\|_0^{1/2} \leqslant C\delta t^{5/4} \big\|\boldsymbol{e}_c^{n}\big\|_1 \big\|\boldsymbol{e}_c^{n+1}\big\|_1 \leqslant C\delta t^{3/2}\nu \big\|\boldsymbol{e}_c^{n}\big\|_1^2 + \frac{\delta t \nu}{5} \big\|\boldsymbol{e}_c^{n+1}\big\|_1^2, \\
\mathrm{T}_2 &\leqslant C\delta t \big\|\boldsymbol{e}_c^{n}\big\|_0 \big\|\boldsymbol{u}(t_{n+1})\big\|_2 \big\|\boldsymbol{e}_c^{n+1}\big\|_1 \leqslant C\delta t \big\|\boldsymbol{e}_c^{n}\big\|_0^2 + \frac{\delta t \nu}{5} \big\|\boldsymbol{e}_c^{n+1}\big\|_1^2,
\end{aligned}
$$

where we have used (24) and the continuity properties of the trilinear form c. Adding up (33) for $n = 0, \ldots, N$ and taking into account (21) and the previous inequalities, we get:

$$
\begin{aligned}
&\big\|\boldsymbol{e}_c^{N+1}\big\|_0^2 + \sum_{n=0}^{N} \big\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n}\big\|_0^2 + \delta t \nu \sum_{n=0}^{N} \big\|\boldsymbol{e}_c^{n+1}\big\|_1^2 + C\delta t^2 \nu \sum_{n=0}^{N} \big\|\boldsymbol{e}_c^{n+1/2}\big\|_1^2 \\
&\quad \leqslant C\delta t^2 \int_0^T \|\boldsymbol{u}_{tt}\|_{V'}^2 \,\mathrm{d}t + C\delta t^2 \int_0^T \|\boldsymbol{u}_t\|_1^2 \,\mathrm{d}t + C\delta t \sum_{n=0}^{N} \big\|\boldsymbol{e}_c^{n+1}\big\|_0^2 + C\delta t \sum_{n=0}^{N} \big\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n+1/2}\big\|_0^2 \\
&\qquad + C\delta t^2 \sum_{n=0}^{N} \big\{ \big\|\boldsymbol{e}_c^{n+1}\big\|_1^2 + \big\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n+1/2}\big\|_1^2 \big\} + C\delta t^{3/2}\nu \sum_{n=0}^{N} \big\|\boldsymbol{e}_c^{n}\big\|_1^2.
\end{aligned}
$$

Using the regularity properties of the solution (R2b) and (R4) and the estimates of Lemma 1, we get:

$$
\begin{aligned}
&\big\|\boldsymbol{e}_c^{N+1}\big\|_0^2 + \sum_{n=0}^{N} \big\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n}\big\|_0^2 + \delta t \nu \sum_{n=0}^{N} \big\|\boldsymbol{e}_c^{n+1}\big\|_1^2 + C\delta t^2 \nu \sum_{n=0}^{N} \big\|\boldsymbol{e}_c^{n+1/2}\big\|_1^2 \\
&\quad \leqslant C\delta t^2 + C\delta t \sum_{n=0}^{N} \big\|\boldsymbol{e}_c^{n+1}\big\|_0^2 + C\delta t^{3/2}\nu \sum_{n=0}^{N} \big\|\boldsymbol{e}_c^{n}\big\|_1^2.
\end{aligned}
$$

For sufficiently small δt, we can apply the discrete Gronwall lemma to the last inequality and take the last term to the left side, to get:

$$
\big\|\boldsymbol{e}_c^{N+1}\big\|_0^2 + \sum_{n=0}^{N} \big\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^{n}\big\|_0^2 + \delta t \nu \sum_{n=0}^{N} \big\|\boldsymbol{e}_c^{n+1}\big\|_1^2 \leqslant C\delta t^2,
$$

and (25) is proved. □

4.2. Error estimates for the semidiscrete pressure

As a side product of the estimates of Theorem 5, we obtain order $1/2$ error estimates for the pressure approximation in $l^2(L^2_0(\Omega))$, which is what one can expect for the present scheme. We first recall a technical result, similar to that of [21, Lemma A1]. In Theorem 5 we have proved, in particular, that:

$$\sum_{n=0}^{N} \left\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n\right\|_0^2 \leqslant C\delta t^2.$$

This implies that:

$$\sum_{n=0}^{N} \left\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n\right\|_{-1}^2 \leqslant C\delta t^2, \tag{34}$$

since for all $\boldsymbol{v} \in \mathbf{L}^2(\Omega)$, $\|\boldsymbol{v}\|_{-1} \leqslant \|\boldsymbol{v}\|_0$. This is what we actually use to prove the following error estimate for the pressure:

Theorem 6. *Assume that* (R1), (R2b), (R3) *and* (R4) *hold*; *then, for* $N = 0, \ldots, [T/\delta t] - 1$ *and for small enough* δt:

$$\delta t \sum_{n=0}^{N} \left\|p(t_{n+1}) - p^{n+1}\right\|_{L^2_0(\Omega)}^2 \leqslant C\delta t, \tag{35}$$

that is, p^{n+1} *converges to* $p(t_{n+1})$ *in* $l^2(L^2_0(\Omega))$ *with order* $\delta t^{1/2}$.

Proof. We rewrite (27) as:

$$-\nabla r_c^{n+1} = \frac{1}{\delta t}\left(\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n\right) - \nu\Delta\left(\boldsymbol{e}_c^{n+1}\right) - \mathbf{R}^n - \left(\boldsymbol{u}^n \cdot \nabla\right)\boldsymbol{u}^{n+1/2} + \left(\boldsymbol{u}(t_{n+1}) \cdot \nabla\right)\boldsymbol{u}(t_{n+1}). \tag{36}$$

Using the continuous LBB condition:

$$\left\|r_c^{n+1}\right\|_{L^2_0(\Omega)} \leqslant C \sup_{\boldsymbol{v}\in\mathbf{H}^1_0(\Omega)} \frac{(\nabla r_c^{n+1}, \boldsymbol{v})}{\|\boldsymbol{v}\|_1}, \tag{37}$$

we need to bound the products of the RHS of (36) with an arbitrary $\boldsymbol{v} \in \mathbf{H}^1_0(\Omega)$. We have:

$$\frac{1}{\delta t}\left(\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n, \boldsymbol{v}\right) \leqslant \frac{1}{\delta t}\left\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n\right\|_{-1}\|\boldsymbol{v}\|_1, \qquad \left\langle -\nu\Delta\left(\boldsymbol{e}_c^{n+1}\right), \boldsymbol{v}\right\rangle = \nu\left\|\boldsymbol{e}_c^{n+1}\right\|_1\|\boldsymbol{v}\|_1,$$

$$\left\langle -\mathbf{R}^n, \boldsymbol{v}\right\rangle \leqslant \left\|\mathbf{R}^n\right\|_{-1}\|\boldsymbol{v}\|_1 \leqslant C\left(\int_{t_n}^{t_{n+1}} t\|\boldsymbol{u}_{tt}\|_{-1}^2\,\mathrm{d}t\right)^{1/2}\|\boldsymbol{v}\|_1.$$

For the nonlinear terms, we use the following splitting:

$$\begin{aligned} &-\left(\boldsymbol{u}^n \cdot \nabla\right)\boldsymbol{u}^{n+1/2} + \left(\boldsymbol{u}(t_{n+1}) \cdot \nabla\right)\boldsymbol{u}(t_{n+1}) \\ &\quad = \left(\left(\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)\right) \cdot \nabla\right)\boldsymbol{u}(t_{n+1}) + \left(\boldsymbol{e}_c^n \cdot \nabla\right)\boldsymbol{u}(t_{n+1}) + \left(\boldsymbol{u}^n \cdot \nabla\right)\boldsymbol{e}_c^{n+1/2}. \end{aligned} \tag{38}$$

Calling I, II and III the three terms obtained after testing (38) with $\boldsymbol{v}$, we have:

$$\mathrm{I} \leqslant C \left\| \boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n) \right\|_0 \left\| \boldsymbol{u}(t_{n+1}) \right\|_2 \|\boldsymbol{v}\|_1 \leqslant C \left(\delta t \int_{t_n}^{t_{n+1}} \|\boldsymbol{u}_t\|_0^2 \, \mathrm{d}t \right)^{1/2} \|\boldsymbol{v}\|_1,$$

$$\mathrm{II} \leqslant C \left\| \boldsymbol{e}_c^n \right\|_1 \left\| \boldsymbol{u}(t_{n+1}) \right\|_1 \|\boldsymbol{v}\|_1 \leqslant C \left\| \boldsymbol{e}_c^n \right\|_1 \|\boldsymbol{v}\|_1,$$

$$\mathrm{III} \leqslant C \left\| \boldsymbol{u}^n \right\|_1 \left\| \boldsymbol{e}_c^{n+1/2} \right\|_1 \|\boldsymbol{v}\|_1 \leqslant C \left\| \boldsymbol{e}_c^{n+1/2} \right\|_1 \|\boldsymbol{v}\|_1,$$

where we have used (R1) and (23). Thus, we obtain:

$$\left\| r_c^{n+1} \right\|_{L_0^2(\Omega)} \leqslant \frac{C}{\delta t} \left\| \boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n \right\|_{-1} + C \Bigg\{ \left\| \boldsymbol{e}_c^{n+1} \right\|_1 + \left\| \boldsymbol{e}_c^n \right\|_1 + \left\| \boldsymbol{e}_c^{n+1/2} \right\|_1 + \left(\int_{t_n}^{t_{n+1}} t \|\boldsymbol{u}_{tt}\|_{-1}^2 \, \mathrm{d}t \right)^{1/2} + \left(\delta t \int_{t_n}^{t_{n+1}} \|\boldsymbol{u}_t\|_0^2 \, \mathrm{d}t \right)^{1/2} \Bigg\},$$

which yields:

$$\left\| r_c^{n+1} \right\|_{L_0^2(\Omega)}^2 \leqslant \frac{C}{\delta t^2} \left\| \boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n \right\|_{-1}^2 + C \Bigg\{ \left\| \boldsymbol{e}_c^{n+1} \right\|_1^2 + \left\| \boldsymbol{e}_c^n \right\|_1^2 + \left\| \boldsymbol{e}_c^{n+1/2} \right\|_1^2 + \int_{t_n}^{t_{n+1}} t \|\boldsymbol{u}_{tt}\|_{-1}^2 \, \mathrm{d}t + \delta t \int_{t_n}^{t_{n+1}} \|\boldsymbol{u}_t\|_0^2 \, \mathrm{d}t \Bigg\},$$

and (35) results from (34), the regularity properties (R3) and (R2a) (which is implied by (R2b)) of the continuous solution $\boldsymbol{u}$, and the estimates of Lemma 1. □

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