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INFORMATION

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ABSTRACT

This paper proposes a new framework for the design of functional $\mathcal{H}_\infty$ filters tailored for nonlinear descriptor systems affected by disturbances. Earlier methods have some significant drawbacks: they rely on the restrictive assumption of system regularity, employ implicit descriptor-form filters that complicate implementation, and emphasize full-order filtering, which is often unnecessary and computationally expensive. To overcome these drawbacks, the proposed filter is developed in an explicit state-space form that allows simple implementation with arbitrary initial conditions. Moreover, its order is minimized by matching it to the dimension of the functional vector, which reduces computational complexity compared to conventional filters. A new set of sufficient conditions is presented for the existence of a functional $\mathcal{H}_\infty$ filter, expressed through a rank condition and a linear matrix inequality (LMI) formulation. These conditions guarantee the stability of the estimation error dynamics while ensuring that the $\mathcal{L}_2$ gain from disturbances to errors remains below a specified bound. A numerical example based on a simple constrained mechanical system is presented to illustrate the effectiveness of the proposed method.

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1 Introduction

Mathematical models play a crucial role in understanding and analyzing control systems. Control theory often deals with physical systems represented by ordinary differential equations (ODEs) and certain algebraic constraints. Such systems are called descriptor systems, which are also known as generalized state-space systems, implicit systems, singular systems, or systems described by differential algebraic equations (DAEs). In some cases, it is feasible to solve the algebraic constraints explicitly by transforming the system model into a well-known state space model, which is essentially a set of ODEs. Nevertheless, these transformations necessitate manual adjustments or the elimination of system variables. As a result, the transformed state space models may fail to reflect key properties of the underlying physical phenomena. For example, state space models can not effectively describe impulses in electrical circuits [1]. Consequently, it becomes practically essential to study the properties of descriptor systems. For a more in-depth exploration of descriptor systems and their applications in engineering, readers are encouraged to see the books [2–4] and the literature cited therein.

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The challenge of estimating the system's states is a fundamental problem in control systems theory. The first solution to this problem for linear time-invariant (LTI) state space systems were introduced in the early 1960s through the works of Luenberger [5] and Kalman [6]. The Luenberger approach gave rise to the well-known observer design problem, which later inspired extensive research on observer design problem for linear descriptor systems. For a detailed discussion on the existence conditions, readers are referred to the work by Jaiswal et al. [7]. Moreover, extensive research has focused on state estimation for linear descriptor systems with corrupted input and output data using Kalman filtering techniques [8–10]. However, it is notable that the Kalman filtering method assumes that the plant system has dynamics described by external noises with precisely known statistical characteristics [11,12]. To address these challenges, an alternative approach, $\mathcal{H}_\infty$ filtering, was first proposed by Xu et al. [13] for square and regular linear descriptor systems. This method does not rely on assuming any statistical characteristics for the exogenous noises, but only requires bounded (within the $\mathcal{L}_2$-norm) noises [14–16]. Moreover, $\mathcal{H}_\infty$ filtering proves to be more robust than Kalman filtering, especially when additional uncertainties are present in descriptor system models [17], and has also demonstrated effectiveness in various engineering applications [18,19].

The nonlinear extension of descriptor systems has also drawn considerable attention, with its foundations laid by the work of Luenberger [20]. Since then, numerous studies have contributed to analysis and solution methods for nonlinear descriptor systems. For comprehensive discussions, readers may consult the articles [21,22] and the books [23,24]. It is essential that the research on the problem of state estimation for nonlinear descriptor systems often focuses on systems with structured nonlinearities. In particular,observers for descriptor systems with Lipschitz nonlinearities have attracted considerable research interest over the past three decades, see [25–27]. Roughly speaking, a Lipschitz function is characterized by the condition that the magnitude of its slope is bounded above by a Lipschitz constant. On the other hand, monotone nonlinearities encompass functions that exhibit lower bounds on their slope and frequently appear in physical systems, such as stiffening springs in constrained mechanical systems and capacitors in electric circuits [28,29]. Building on these concepts, several observer designs have been proposed for nonlinear descriptor systems. A full-order observer for descriptor systems satisfying a generalized monotonicity condition was proposed by Yang et al. [30], though their approach is limited to square systems. Meanwhile, Gupta et al. [31] introduced a reduced-order observer for nonlinear descriptor systems with generalized monotone nonlinearities. Subsequently, in 2020, Berger et al. [32] developed an observer design methodology for descriptor systems characterized by Lipschitz or monotone nonlinearities. However, the observer structures in [30,32] are expressed in descriptor form, which is less desirable for simulations due to its implicit nature and the requirement for consistent initial conditions during simulations. Moysis et al. [33] expanded upon the work in [31], focusing on nonlinearities that adhere to incremental quadratic constraints in output equations. The $\mathcal{H}_\infty$ filters for descriptor systems were developed in the early 2010s for systems with Lipschitz nonlinearities [34] and monotone-type nonlinearities [35,36]. Recently, Liu et al. [37] developed the design of state and adaptive disturbance $\mathcal{H}_\infty$ observers for nonlinear descriptor systems, where the disturbances are generated by an unknown exogenous system.

The aforementioned papers primarily focus on observers and filters to estimate the entire state vector of a given control system. Observers/filters that target only the required states, rather than the full state vector, are referred to as functional (or partial) observers/filters. Such observer/filters are particularly valuable in applications including feedback control, fault detection, and disturbance estimation [38]. With the use of functional observers, $\mathcal{H}_\infty$ controller design challenges have been explored for both linear descriptor systems [17] and nonlinear descriptor systems [39]. Notably, functional observers can often be constructed under less restrictive conditions than full-state observers;

for recent advancements in functional observer design for linear descriptor systems, the interested readers are referred to [40–42] and the references therein. Nevertheless, research on functional observers or filters for nonlinear descriptor systems remains limited, and to the best of our knowledge, no functional $\mathcal{H}_\infty$ observers or filters have been developed for monotone nonlinear descriptor systems.

This paper introduces a novel approach to design the functional $\mathcal{H}_\infty$ filter for nonlinear descriptor systems with generalized monotone nonlinearities affected by external disturbances. Existing $\mathcal{H}_\infty$ filtering methods mainly address full- or reduced-order designs for square and regular descriptor systems and rely on implicit formulations that complicate implementation and demand consistent initial conditions; the full-order filter also inflates filter order and computational cost. In contrast, the proposed approach is constructed in an explicit state-space form, which facilitates practical implementation and allows arbitrary initial conditions. The filter is designed with an order equal to the dimension of the functional vector, thereby avoiding the high computational burden of full- or reduced-order filters. The proposed definition of the functional $\mathcal{H}_\infty$ filter (cf. Definition 2) is grounded in the behavioral solution theory for descriptor systems. Moreover, a new set of sufficient conditions, based on a rank condition on the system coefficient matrices and an LMI formulation, ensures the stability of the estimation error dynamics and keeps the induced $\mathcal{L}_2$ gain from disturbances within a prescribed bound. Consequently, the filter design procedure is presented as a systematic algorithm suitable for implementing standard LMI solvers, such as those available in MATLAB.

The paper is organized as follows: Section 2 presents the problem statement along with the necessary preliminaries for the forthcoming analysis. In Section 3, we establish the primary findings of the paper. Initially, a method is developed for determining coefficient matrices for the proposed filter, followed by developing error system stability using the Lyapunov theory. In Section 4, the computational complexity of the proposed filter is compared in detail with full- or reduced-order filters using flop counts. Section 5 presents a numerical example to demonstrate the applicability of the theoretical results. Lastly, Section 6 concludes the paper.

Notations

$\mathbb{N}, \mathbb{N}_0$	$\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$;
$0, I$	The zero and the identity matrices of suitable dimensions, respectively;
$\mathbb{C}, \overline{\mathbb{C}}^+$	$\mathbb{C}$ stands for the set of complex numbers, and $\overline{\mathbb{C}}^+ := \{\alpha \mid \alpha \in \mathbb{C},\ Re(\alpha) \geq 0\}$;
$A \in \mathbb{R}^{m\times n}$	Matrix A belongs to the space of real-valued $m \times n$ matrices;
$A^+, A^\top$	Moore-Penrose inverse (MP-inverse) and Transpose of $A \in \mathbb{R}^{m\times n}$, respectively;
$A > 0$	A is symmetric and positive definite matrix;
$\mathscr{C}^k(X \to Y)$	Set of functions $f : X \to Y$, that are k-times continuously differentiable, $k \in \mathbb{N}_0$;
$\lVert x(t)\rVert$	$\sqrt{x^\top(t)x(t)}$, the Euclidean norm of $x(t) \in \mathbb{R}^n$;
$\mathcal{L}_2([0, t_f], \mathbb{R}^n)$	$\{f : \int_0^{t_f} \lVert f(t)\rVert^2 \mathrm{d}t < \infty$ for a fixed $t_f > 0$ and $\lVert f\rVert_2^2 = \int_0^{t_f} \lVert f(t)\rVert^2 \mathrm{d}t\}$.

2 Problem Description and Preliminaries

Consider the following nonlinear descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t) + Dv(t) + Fg(HKx, u), \tag{1a}$$

$$y(t) = Cx(t) + Gv(t), \tag{1b}$$

$$z(t) = Kx(t), \tag{1c}$$

where $x(t) \in \mathbb{R}^n$ is the semistate vector, $u(t) \in \mathbb{R}^k$ is the (known) control input vector, $y(t) \in \mathbb{R}^r$ is the (measured) output vector, $z(t) \in \mathbb{R}^p$ is the (unmeasured) output functional vector, and $v(t) \in \mathbb{R}^q$ is the unknown disturbance vector. The coefficients $E, A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times k}$, $C \in \mathbb{R}^{r\times n}$, $D \in \mathbb{R}^{m\times q}$, $F \in \mathbb{R}^{m\times l}$, $G \in \mathbb{R}^{r\times q}$, $H \in \mathbb{R}^{l\times p}$ and $K \in \mathbb{R}^{p\times n}$ are known constant matrices. Moreover, for some open sets $\mathscr{X}_1 \subseteq \mathbb{R}^p$, $\mathscr{U} \subseteq \mathbb{R}^k$ and $\mathscr{X}_2 = H\mathscr{X}_1 \subseteq \mathbb{R}^l$, the nonlinear function $g \in \mathscr{C}^1(\mathscr{X}_2 \times \mathscr{U} \to \mathbb{R}^l)$ satisfies a generalized monotone condition [32]

$$\forall\ x_1, x_2 \in \mathscr{X}_2 : (x_1 - x_2)^\top \big(g(x_1, u) - g(x_2, u)\big) \geq \frac{1}{2}\rho\|x_1 - x_2\|^2, \quad \text{where } \rho \in \mathbb{R}. \tag{2}$$

Clearly, if we define $\Delta g = g(x_1, u) - g(x_2, u)$ and $\Delta x = x_1 - x_2$, then (2) reduces to

$$\Delta x^\top \Delta g + \Delta g^\top \Delta x \geq \rho\|\Delta x\|^2. \tag{3}$$

The property (2) (equivalently (3)) is also called the multivariable sector property, and it can be proved easily that (2) holds if the function g satisfies the slope bound property (Lemma 1 in [43]):

$$\forall\ s \in \mathbb{R}^l : \frac{\partial g(s)}{\partial s} + \left(\frac{\partial g(s)}{\partial s}\right)^\top \geq \rho I_l. \tag{4}$$

Notably, if $\rho = 0$, characterization (4) is a natural extension of the monotonic property of single variables to multivariable functions. In addition, a nonlinear function that satisfies (4) may not necessarily fulfil the globally Lipschitz condition, for example, take $g(x) = x^3$. Notably, monotone nonlinearities arise in modeling many physical systems like stiffening springs in constrained mechanical systems and capacitors in circuit systems [28,29]. Throughout this work, we assume that the system designer has defined the system variables in such a way that system (1) possesses at least one solution. More precisely, there exists an admissible pair of $Ex(0^-)$ and input $u(t)$ such that system (1) admits at least one solution. We now recall the behavioral framework introduced in [26] to define the solution of system (1).

Definition 1: *Let $\mathcal{I} \subseteq \mathbb{R}$ be an open interval. A trajectory $(x, u, v, y, z) \in \mathscr{C}(\mathcal{I}; \mathbb{R}^{n+k+q+r+p})$ is said to be a solution of system (1), if $x \in \mathscr{C}^1(\mathcal{I}; \mathbb{R}^n)$ and (1) holds for all $t \in \mathcal{I}$. The collection of all such solution trajectories*

$$\mathscr{B} := \{(x, u, v, y, z) \in \mathscr{C}(\mathcal{I}; \mathbb{R}^{n+k+q+r+p}) \mid (x, u, v, y, z) \text{ is a solution of (1)}\}$$

is referred to as the behavior of the system (1).

To estimate the functional state $z(t)$ from (1), we introduce the following functional filter structure

$$\dot{w}(t) = Nw(t) + TBu(t) + Ly(t) + TFg(H\hat{z}, u), \tag{5a}$$

$$\hat{z}(t) = w(t) + My(t), \tag{5b}$$

where $w(t) \in \mathbb{R}^p$ is the internal state of the filter, and $\hat{z}(t) \in \mathbb{R}^p$ represents the estimate of the functional output $z(t)$ in (1). The matrices $N \in \mathbb{R}^{p\times p}$, $T \in \mathbb{R}^{p\times m}$, $L \in \mathbb{R}^{p\times r}$, and $M \in \mathbb{R}^{p\times r}$ are to be determined. Before introducing the functional $\mathcal{H}_\infty$ filter design problem, define the estimation error $e(t) = z(t) - \hat{z}(t)$ as the difference between the actual and estimated functional vectors and the auxiliary error $e_1(t) = TEx(t) - w(t)$.

Definition 2: *System (5) is called a functional $\mathcal{H}_\infty$ filter for (1), if for every trajectory $(x, u, v, y, z) \in \mathscr{B}$, there exist functions $w \in \mathscr{C}^1(\mathcal{I}; \mathbb{R}^p)$ and $\hat{z} \in \mathscr{C}(\mathcal{I}; \mathbb{R}^p)$ such that the tuple $(w, u, y, \hat{z})$ satisfy (5) for all $t \in \mathcal{I}$, and for all such w and $\hat{z}$ the following conditions hold:*

1. *If $v \equiv 0$, then $e(t) \to 0$ for $t \to \infty$.*
2. *If v is not identically zero, then*

   $\|e\|_2^2 < \gamma^2 \|v\|_2^2 + \beta^2 \|e_1(0)\|^2,$

   where $\gamma > 0$ and $\beta > 0$ are two real numbers.

In this paper, we focus on designing the matrices N, T, L, and M such that (5) becomes a functional $\mathcal{H}_\infty$ filter for a given system of the form (1). We conclude this section by recalling a key result for the analysis presented in the following section.

Lemma 1: *[44] The linear system $\mathcal{X}\mathcal{Y} = \mathcal{Z}$ has a solution for $\mathcal{X}$ if and only if*

$$\text{rank}\begin{bmatrix} \mathcal{Y} \\ \mathcal{Z} \end{bmatrix} = \text{rank } \mathcal{Y}.$$

Moreover,

$$\mathcal{X} = \mathcal{Z}\mathcal{Y}^+ - \mathcal{V}(\mathcal{I} - \mathcal{Y}\mathcal{Y}^+),$$

where $\mathcal{Y}^+$ is the MP-inverse of $\mathcal{Y}$ and $\mathcal{V}$ is any matrix of appropriate dimension.

3 Functional $\mathcal{H}_\infty$ Filter Design

We begin by formulating some linear matrix equations and solving them to determine the coefficient matrices of the functional $\mathcal{H}_\infty$ filter (5). Subsequently, we employ an LMI approach to demonstrate that the designed filter satisfies both the properties in Definition 2. Throughout the section, we assume that (1) satisfies a rank condition:

$$\text{rank}\begin{bmatrix} E & A \\ C & 0 \\ 0 & C \\ 0 & K \\ K & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} E & A \\ C & 0 \\ 0 & C \\ 0 & K \end{bmatrix}. \tag{6}$$

Error dynamics and design of coefficient matrices for functional $\mathcal{H}_\infty$ filter (5): Using the system (1) and (5), we obtain

$$\begin{aligned} e(t) &= Kx(t) - (w(t) + My(t)), \\ &= e_1(t) - (TE + MC - K)x(t) - MGv(t), \end{aligned} \tag{7}$$

and

$$\begin{aligned} \dot{e}_1(t) &= TE\dot{x}(t) - \dot{w}(t), \\ &= Ne_1(t) + (TD - LG)v(t) + (TA - LC - NTE)\,x(t) + TF\big(g(Hz, u) - g(H\hat{z}, u)\big), \\ &= Ne_1(t) + (TD - LG)v(t) + (TA - LC - NTE)\,x(t) + TF\Delta g, \end{aligned} \tag{8}$$

where $\Delta g = g(Hz, u) - g(H\hat{z}, u)$. Therefore, if the design matrices satisfy the following algebraic constraints

$$TA - LC - NTE = 0, \tag{9a}$$

$$TE + MC - K = 0, \tag{9b}$$

then Eqs. (7) and (8) simplify to the following error dynamics

$$\dot{e}_1(t) = Ne_1(t) + (TD - LG)v(t) + TF\Delta g, \tag{10a}$$

$$e(t) = e_1(t) - MGv(t). \tag{10b}$$

It is important to note that condition (9a) is nonlinear due to the presence of unknown matrices. To simplify, we substitute $TE = K - MC$ from (9b) into (9a), and set

$$P = NM - L. \tag{11}$$

With these substitutions, (9a) transforms into a linear matrix equation

$$TA + PC - NK = 0. \tag{12}$$

Thus, we can rewrite Eqs. (9b) and (12) as

$$\begin{bmatrix} T & M & P & N \end{bmatrix} \Psi = \Theta, \tag{13}$$

where $\Psi = \begin{bmatrix} E & A \\ C & 0 \\ 0 & C \\ 0 & -K \end{bmatrix}$ and $\Theta = \begin{bmatrix} K & 0 \end{bmatrix}$.The solvability of the matrix Eq. (13) is addressed in Lemma 1, which states that a solution for the unknowns T, M, P, and N exists if and only if the rank condition (6) holds. Moreover, the general solution to (13) is given by

$$\begin{bmatrix} T & M & P & N \end{bmatrix} = \Theta\Psi^+ - Z(I - \Psi\Psi^+), \tag{14}$$

where Z is an arbitrary matrix of suitable size. Thus, we get

$$T = T_1 - ZT_2, \tag{15a}$$

$$M = M_1 - ZM_2, \tag{15b}$$

$$P = P_1 - ZP_2, \tag{15c}$$

$$N = N_1 - ZN_2, \tag{15d}$$

where

$$T_1 = \Theta\Psi^+ \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}^\top, \quad T_2 = (I - \Psi\Psi^+) \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}^\top, \tag{16a}$$

$$M_1 = \Theta\Psi^+ \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix}^\top, \quad M_2 = (I - \Psi\Psi^+) \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix}^\top, \tag{16b}$$

$$P_1 = \Theta\Psi^+ \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix}^\top, \quad P_2 = (I - \Psi\Psi^+) \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix}^\top, \tag{16c}$$

$$N_1 = \Theta\Psi^+ \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix}^\top, \quad N_2 = (I - \Psi\Psi^+) \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix}^\top. \tag{16d}$$

Hence, the remaining task is to determine a suitable choice of the matrix Z such that the resulting filter (5), with matrices T, M, P, and N defined as above, meets all the conditions stated in Definition 2.

Before stating the main theorem, we assume that Z_1 is an arbitrary matrix of the same dimension as Z and set the following notations.

$$\bar{M}_2 = M_2 G, \tag{17a}$$

$$Z = Z_1(I - \bar{M}_2\bar{M}_2^{+}), \tag{17b}$$

$$\mathcal{T}_2 = (I - \bar{M}_2\bar{M}_2^{+})T_2, \tag{17c}$$

$$\mathcal{M}_2 = (I - \bar{M}_2\bar{M}_2^{+})M_2, \tag{17d}$$

$$\mathcal{P}_2 = (I - \bar{M}_2\bar{M}_2^{+})P_2, \tag{17e}$$

$$\mathcal{N}_2 = (I - \bar{M}_2\bar{M}_2^{+})N_2, \tag{17f}$$

$$B_1 = T_1 D - N_1 M_1 G + P_1 G, \tag{17g}$$

$$B_2 = \mathcal{T}_2 D - \mathcal{N}_2 M_1 G + \mathcal{P}_2 G, \tag{17h}$$

$$\tilde{H} = M_1 G, \tag{17i}$$

$$\mathscr{H} = H^\top H. \tag{17j}$$

Therefore, due to the property of MP-inverse that $\bar{M}_2\bar{M}_2^{+}\bar{M}_2 = \bar{M}_2$, we obtain

$$\mathcal{M}_2 G = (I - \bar{M}_2\bar{M}_2^{+})\bar{M}_2 = 0. \tag{18}$$

We proceed to formulate the main theorem.

Theorem 1: *Consider the nonlinear system (1) that satisfies the rank condition (6) and the nonlinear function g satisfies (3). Then, for a given $\gamma > 0$, the filter defined in (5), with system coefficient matrices (15) and error dynamics (10), is a functional $\mathcal{H}_\infty$ filter, if there exists a symmetric positive definite matrix $Q = Q^\top > 0$, and a matrix Y such that the following LMI holds.*

$$\tilde{\Pi} = \begin{bmatrix} \mathcal{A}_{11} + I & \mathcal{A}_{12} & \mathcal{A}_{13} - \tilde{H} \\ \mathcal{A}_{12}^\top & 0 & \mathcal{A}_{23} \\ (\mathcal{A}_{13} - \tilde{H})^\top & \mathcal{A}_{23}^\top & \mathcal{A}_{33} + \tilde{H}^\top\tilde{H} - \gamma^2 I \end{bmatrix} \leq 0, \tag{19}$$

where

$$\mathcal{A}_{11} = N_1^\top Q + QN_1 - \mathcal{N}_2^\top Y^\top - Y\mathcal{N}_2 - \rho\mathscr{H}, \tag{20a}$$

$$\mathcal{A}_{12} = (QT_1 - Y\mathcal{T}_2)F + H^\top, \tag{20b}$$

$$\mathcal{A}_{13} = QB_1 - YB_2 + \rho\mathscr{H}\tilde{H}, \tag{20c}$$

$$\mathcal{A}_{23} = -H\tilde{H}, \tag{20d}$$

$$\mathcal{A}_{33} = -\rho\tilde{H}^\top\mathscr{H}\tilde{H}, \tag{20e}$$

$$Y = QZ_1. \tag{20f}$$

Proof. The proof is divided into three steps.

Step 1: In this step, we transform the error dynamics (10) into a convenient form so that the solution (Q, Y) of LMI (19) provides $Z_1 = Q^{-1}Y$ (cf. (20f)) in such a way that (5) becomes a functional $\mathcal{H}_\infty$ filter of (1) with the coefficient matrices (15). Clearly, (17a)–(17i) infer that (15) can be rewrite as

$$T = T_1 - Z_1\mathcal{T}_2, \tag{21a}$$

$$M = M_1 - Z_1\mathcal{M}_2, \tag{21b}$$

$$P = P_1 - Z_1\mathcal{P}_2, \tag{21c}$$

$$N = N_1 - Z_1\mathcal{N}_2. \tag{21d}$$

Moreover, if we denote

$$\mathcal{B} = B_1 - Z_1B_2, \tag{22}$$

then, by using (18) and (22), we obtain

$$\begin{aligned} TD - LG &= TD - (NM - P)G, \\ &= (T_1 - Z_1\mathcal{T}_2)D - [(N_1 - Z_1\mathcal{N}_2)(M_1 - Z_1\mathcal{M}_2) - (P_1 - Z_1\mathcal{P}_2)]G, \\ &= B_1 - Z_1B_2 = \mathcal{B}, \end{aligned} \tag{23a}$$

and $$MG = (M_1 - Z_1\mathcal{M}_2)G = M_1G. \tag{23b}$$

Thus, we can rewrite the error dynamics (10) as

$$\dot{e}_1(t) = Ne_1(t) + TF\Delta g + \mathcal{B}v(t), \tag{24a}$$

$$e(t) = e_1(t) - \tilde{H}v(t), \tag{24b}$$

where N, T, $\mathcal{B}$ and $\tilde{H}$ are defined in (21d), (21a), (22) and (17i), respectively.

Since g satisfies the generalized monotone property, using (24b), Eq. (3) reduces to

$$\begin{aligned} \Delta x^\top \Delta g + \Delta g^\top \Delta x \geq \rho\|\Delta x\|^2, \\ &= \rho\|He(t)\|^2, \\ &= \rho\|H\big(e_1(t) - \tilde{H}v(t)\big)\|^2, \\ &= \rho\big(e_1^\top(t)\mathscr{H}e_1(t) - e_1^\top(t)\mathscr{H}\tilde{H}v(t) - v^\top(t)\tilde{H}^\top\mathscr{H}e_1(t) \\ &\quad + v^\top(t)\tilde{H}^\top\mathscr{H}\tilde{H}v(t)\big). \end{aligned} \tag{25}$$

Step 2: In this step, we show that if LMI (19) holds, then the error dynamics (24) satisfy condition 2 in Definition 2. Consider the Lyapunov function V such that

$$V : \mathbb{R}^p \to \mathbb{R},\ e_1(t) \to e_1^\top(t)Qe_1(t), \tag{26}$$

where $Q = Q^\top > 0$. Thus, by using (24), we obtain

$$\begin{aligned}\dot{V}(e_1(t)) &= \dot{e}_1^\top(t)Qe_1(t) + e_1^\top(t)Q\dot{e}_1(t),\\ &= (Ne_1(t) + TF\Delta g + \mathcal{B}v(t))^\top Qe_1(t) + e_1^\top(t)Q(Ne_1(t) + TF\Delta g + \mathcal{B}v(t)),\\ &= e_1^\top(t)\big((N_1 - Z_1\mathcal{N}_2)^\top Q + Q(N_1 - Z_1\mathcal{N}_2)\big)e_1(t) + e_1^\top(t)\big(Q(T_1 - Z_1\mathcal{T}_2)\big)F\Delta g\\ &+ \Delta g^\top\big(Q\,(T_1 - Z_1\mathcal{T}_2)\,F\big)^\top e_1(t) + e_1^\top(t)Q(B_1 - Z_1B_2)v(t)\\ &+ v^\top(t)\big(Q(B_1 - Z_1B_2)\big)^\top e_1(t).\end{aligned}$$

Therefore, (25) reveals that

$$\begin{aligned}\dot{V}(e_1(t)) \leq{}& e_1^\top(t)\big((N_1 - Z_1\mathcal{N}_2)^\top Q + Q(N_1 - Z_1\mathcal{N}_2)\big)e_1(t) + e_1^\top(t)\big(Q\,(T_1 - Z_1\mathcal{T}_2)\,\big)F\Delta g\\ &+ \Delta g^\top\big(Q(T_1 - Z_1\mathcal{T}_2)F\big)^\top e_1(t) + e_1^\top(t)Q(B_1 - Z_1B_2)v(t)\\ &+ v^\top(t)\big(Q(B_1 - Z_1B_2)\big)^\top e_1(t) + \big(H(e_1(t) - \tilde{H}v(t))\big)^\top\Delta g\\ &+ \Delta g^\top\big(H(e_1(t) - \tilde{H}v(t))\big) - \rho e_1^\top(t)\mathscr{H}e_1(t) + \rho e_1^\top(t)\mathscr{H}\tilde{H}v(t)\\ &+ \rho v^\top(t)\tilde{H}^\top\mathscr{H}e_1(t) - \rho v^\top(t)\tilde{H}^\top\mathscr{H}\tilde{H}v(t),\\ ={}& e_1^\top(t)\big((N_1 - Z_1\mathcal{N}_2)^\top Q + Q(N_1 - Z_1\mathcal{N}_2) - \rho\mathscr{H}\big)e_1(t)\\ &+ e_1^\top(t)\big(Q(T_1 - Z_1\mathcal{T}_2)F + H^\top\big)\Delta g + \Delta g^\top\big(\big(Q(T_1 - Z_1\mathcal{T}_2)F\big)^\top + H\big)e_1(t)\\ &+ e_1^\top(t)\big(Q(T - Z_1B_2) + \rho\mathscr{H}\tilde{H}\big)v(t) + v^\top(t)\big(\big(Q(B_1 - Z_1B_2)\big)^\top + \rho\tilde{H}^\top\mathscr{H}\big)e_1(t)\\ &- v^\top(t)\rho\tilde{H}^\top\mathscr{H}\tilde{H}v(t) - v^\top(t)\tilde{H}^\top H^\top\Delta g - \Delta g^\top H\tilde{H}v(t),\end{aligned} \tag{27}$$

which is equivalent to the fact that

$$\dot{V}(e_1(t)) \leq \zeta^\top(t)\begin{bmatrix}\mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13}\\ \mathcal{A}_{12}^\top & 0 & \mathcal{A}_{23}\\ \mathcal{A}_{13}^\top & \mathcal{A}_{23}^\top & \mathcal{A}_{33}\end{bmatrix}\zeta(t), \quad \text{where} \quad \zeta(t) = \begin{bmatrix}e_1(t)\\ \Delta g\\ v(t)\end{bmatrix}. \tag{28}$$

Using (24b) and (28), we derive the following inequality:

$$\dot{V}(e_1(t)) + e^\top(t)e(t) - \gamma^2 v^\top(t)v(t) \leq \zeta^\top(t)\tilde{\Pi}\zeta(t),$$

where $\tilde{\Pi}$ is the same as in (19).

Thus, if $\tilde{\Pi} \leq 0$, then the above inequality gives that

$$\dot{V}(e_1(t)) \leq \gamma^2 v^\top(t)v(t) - e^\top(t)e(t).$$

Integrating both sides of the above inequality from 0 to t_f yields

$$\int_0^{t_f}\dot{V}(\tau)\mathrm{d}\tau \leq \int_0^{t_f}\gamma^2 v^\top(\tau)v(\tau)\mathrm{d}\tau - \int_0^{t_f}e^\top(\tau)e(\tau)\mathrm{d}\tau,$$

or equivalently

$$V(e_1(t_f)) - V(e_1(0)) \leq \int_0^{t_f}\gamma^2 v^\top(\tau)v(\tau)\mathrm{d}\tau - \int_0^{t_f}e^\top(\tau)e(\tau)\mathrm{d}\tau.$$

Since $V(e_1(t_f)) \geq 0$, we conclude that

$$\int_0^{t_f} e^\top(\tau)e(\tau)\mathrm{d}\tau < \gamma^2 \int_0^{t_f} v^\top(\tau)v(\tau)\mathrm{d}\tau + V(e_1(0)). \tag{29}$$

Now, using the Lyapunov function (26), we obtain

$$V(e_1(0)) = e_1^\top(0)Qe_1(0) \leq \beta^2\|e_1(0)\|^2, \tag{30}$$

where $\beta^2 = \lambda_{\max}(Q)$ and $\lambda_{\max}(Q)$ denotes the largest eigenvalue of matrix Q. Finally, by substituting $V(e_1(0))$ from (30) in (29), we obtain

$$\|e\|_2^2 < \gamma^2\|v\|_2^2 + \beta^2\|e_1(0)\|^2, \tag{31}$$

which infers that the condition *2* in Definition 2 is satisfied if (19) holds.

Step 3: This step proves that, if $v \equiv 0$, the error dynamics (24) also satisfy condition *1* in Definition 2. Take $v \equiv 0$, then the error dynamics (24) reveals that $e(t) = e_1(t)$ and

$$\dot{e}_1(t) = (N_1 - Z_1\mathcal{N}_2)e_1(t) + (T_1 - Z_1\mathcal{T}_2)F\Delta g, \tag{32}$$

and Eq. (27) reduces to

$$\begin{aligned}\dot{V} &\leq e_1{}^\top(t)\big((N_1 - Z_1\mathcal{N}_2)^\top Q + Q(N_1 - Z_1\mathcal{N}_2) - \rho\mathscr{H}\big)e_1(t) + e_1^\top(t)\big(Q\,(T_1 - Z_1\mathcal{T}_2)\,F \\ &\quad + H^\top\big)\Delta g + \Delta g^\top\big((Q\,(T_1 - Z_1\mathcal{T}_2)\,F)^\top + H\big)e_1(t), \\ &= \begin{bmatrix} e_1(t) \\ \Delta g \end{bmatrix}^\top \begin{bmatrix} N_1^\top Q + QN_1 - \mathcal{N}_2^\top Y^\top - Y\mathcal{N}_2 - \rho\mathscr{H} & (QT_1 - Y\mathcal{T}_2)F + H^\top \\ \big((QT_1 - Y\mathcal{T}_2)F\big)^\top + H & 0 \end{bmatrix} \begin{bmatrix} e_1(t) \\ \Delta g \end{bmatrix}.\end{aligned}$$

Thus, $\dot{V} < 0$ if

$$\Omega = \begin{bmatrix} N_1^\top Q + QN_1 - \mathcal{N}_2^\top Y^\top - Y\mathcal{N}_2 - \rho\mathscr{H} + I & (QT_1 - Y\mathcal{T}_2)F + H^\top \\ \big((QT_1 - Y\mathcal{T}_2)F\big)^\top + H & 0 \end{bmatrix} \leq 0 \tag{33}$$

which is a principle submatrix of $\tilde{\Pi}$. Thus, if (19) holds, the error dynamics (32) is asymptotically stable ([45], Thm. 5.14) and hence $e(t) = e_1(t) \to 0$ as $t \to \infty$. □

Based on Theorem 1, we present the step-by-step design procedure of a functional $\mathcal{H}_\infty$ filter in Algorithm 1 below.

Algorithm 1: Computational procedure to design a functional $\mathcal{H}_\infty$ filter (5) for the system described in (1).

Step 1: Check whether (3) and (6) are satisfied. If yes, go to the next step.
Step 2: Find T_i, M_i, P_i and N_i for $i = 1, 2$ using (16a)–(16d).
Step 3: Determine $\mathcal{T}_2$, $\mathcal{M}_2$, $\mathcal{P}_2$ and $\mathcal{N}_2$ from (17c) to (17f).
Step 4: Compute B_1, B_2 and $\tilde{H}$ from (17g)–(17i), respectively.
Step 5: Solve the LMI condition (19) for Q and Y.
Step 6: Compute $Z_1 = Q^{-1}Y$ and $Z = Z_1(I - \bar{M}_2\bar{M}_2{}^+)$.
Step 7: Obtain the matrices T, M, P and N from (21a) to (21d).
Step 8: Compute the matrix L from (11).

4 Computational Complexity

The computational complexity of the proposed functional $\mathcal{H}_\infty$ filtering algorithm is quantified in terms of floating-point operations (flops), where each flop represents an elementary arithmetic operation (*i.e.*, addition, subtraction, multiplication, or division). The main computational steps involve constructing the matrices T, M, P, N, and solving the LMI (19), which includes the matrices $Q = Q^T > 0$, and Y.

The flop counts for basic matrix operations are as follows: addition of a $m \times n$ matrix requires mn flops, multiplication of $m \times n$ and $n \times r$ matrices requires $mr(2n - 1)$ flops, and inversion of an $n \times n$ matrix or its Cholesky decomposition requires $O(n^3)$ and $O(\frac{n^3}{3} + 2n^2)$ flops, respectively. The LMI (19) is characterized by $\mathcal{N}_r = p + l + q$ constraints and $\mathcal{N}_d = \frac{3}{2}p^2 + p(m + 2r) + \frac{p}{2}$ scalar decision variables. The computational cost of solving this LMI via interior-point methods scales as $O(\mathcal{N}_d^3 \mathcal{N}_r)$ [46]. Table 1 summarizes the flop counts required for computing each filter parameter of the Algorithm 1, where $s_1 = 2n$, $s_2 = m + 2r + p$ and $s_3 = \min(s_1, s_2)$.

Table 1: Computational complexity of filter parameters

Eqs.	Parameters	Flop counts
(20f)	Z_1	$2p^2m + 4rp^2 - p^2 - pm - 2rp + O(p^3)$
(17b)	Z	$2s_2^2s_1 + s_2(2s_2 - 1)(r + p + q) + 2s_2rq + O(s_3^3) + O(q^3)$
(21a)	T	$2s_2(s_1s_2 + ps_1 + ms_2 + 2pm) - s_2(p + m) - pm + 2O(s_3^3)$
(21b)	M	$2s_2(s_1s_2 + ps_1 + rs_2 + 2pr) - s_2(p + r) - pr + 2O(s_3^3)$
(21c)	P	$2s_2(s_1s_2 + ps_1 + rs_2 + 2pr) - s_2(p + r) - pr + 2O(s_3^3)$
(21d)	N	$2s_2(s_1s_2 + ps_1 + ps_2 + 2p^2) - 2ps_2 - p^2 + 2O(s_3^3)$
(11)	L	$2p^2r$

For a fair comparison of computational burden, the proposed approach is compared with conventional filters, where the full-order case corresponds to $p = n$ and the reduced-order case to $p = n - r$ in the expressions of Table 1. Fig. 1 illustrates the growth of total flop counts for the three filter designs as the system order n increases. The results clearly indicate that the full-order filter imposes the highest computational burden, with complexity scaling rapidly with n. The reduced-order filter provides some reduction in cost by decreasing the order to $n - r$, but the computational demand is still significant. In contrast, the proposed functional filter consistently yields a substantial reduction in complexity across all tested values of n. These findings demonstrate that the functional filter is a computationally efficient for high-dimensional descriptor systems.

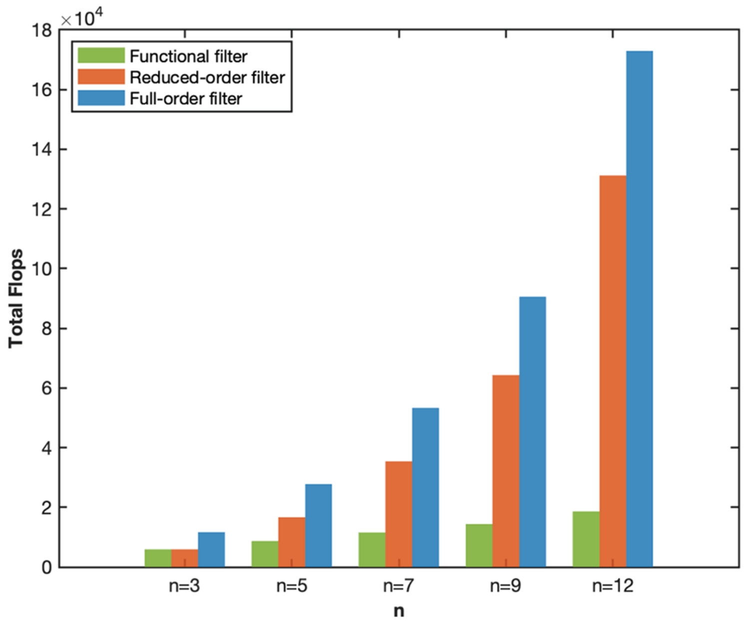


Figure 1: Computational complexity of different filters for $m = 5, r = 2, q = 1$, and $p = 2$

5 Illustrative Example

This section demonstrates the application of the developed functional $\mathcal{H}_\infty$ filter design to a mechanical setup consisting of a disc with mass m that rolls without slipping on a flat surface. The disc is connected to a fixed wall by a nonlinear spring characterized by stiffness parameters $k_1 > 0, k_2 > 0$, along with a linear damper having damping constant $b > 0$, as shown in Fig. 2 ([47], p. 78). The centre of mass position, linear velocity, and angular velocity are denoted by x_1, x_2, and x_3, respectively. A control torque u is applied at the centre, while the opposing friction force λ is treated as an unknown, bounded disturbance.

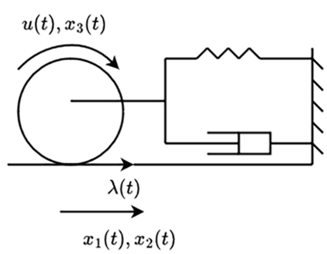


Figure 2: Rolling disc

The system dynamics, along with the kinematic constraints on motion, are determined as [47]

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\frac{k_1}{m}x_1 - \frac{k_2}{m}x_1^3 - \frac{b}{m}x_2 + \frac{\lambda}{m}, \\
0 &= x_2 - rx_3,
\end{aligned}
\tag{34}
$$

$$0 = -\frac{k_2}{m}x_1^3 - \frac{k_1}{m}x_1 - \frac{b}{m}x_2 + \left(\frac{1}{m} + \frac{r^2}{J}\right)\lambda - \frac{r}{J}u,$$

where the disc has a moment of inertia J about its centre and a radius r. The dynamics described by system (34) can be reformulated into the structure of system (1) by identifying the corresponding coefficient matrices as follows

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 \\ -k_1/m & -b/m & 0 \\ -k_1/m & 0 & -rb/m \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ -r/J \end{bmatrix}, D = \begin{bmatrix} 0 \\ 1/m \\ 1/m + r^2/J \end{bmatrix},$$

$F = \begin{bmatrix} 0 \\ -k_2/m \\ -k_2/m \end{bmatrix}$, $H = 1$ and $g(Hz, u) = x_1^3$. Let the output vector and functional vector of the system be taken as

$$y(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} \lambda(t) \text{ and } z(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t).$$

Clearly, the nonlinear function $g = x_1^3$ satisfies (2) with $\rho = 0$. For simulation purposes, we take $\frac{k_1}{m} = \frac{k_2}{m} = 1$, $\frac{b}{m} = 2$, $r = 2$, $m = 1$ and $J = 4$. It can be readily verified that the system satisfies the rank condition (6). Now, for $\gamma = 0.45$, we solve the LMI (19) by using *feasp* function in MATLAB and obtain the parameter matrix Z as

$$Z = \begin{bmatrix} 0.7761 & 0.3208 & 1.1722 & 1.5911 & 0.1890 & 0.6306 & 1.7492 & -0.4537 \end{bmatrix}$$

Using the above matrix Z, we compute the filter coefficient matrices by using Algorithm 1 as

$$T = \begin{bmatrix} -0.5067 & -0.4933 & -0.4126 \end{bmatrix}, \quad L = \begin{bmatrix} -0.7965 & 1.6504 \end{bmatrix}, \quad M = \begin{bmatrix} -0.5067 & 0.0000 \end{bmatrix},$$

$$N = -1.4126 \text{ and } P = \begin{bmatrix} 0.0807 & -1.6504 \end{bmatrix}.$$

To substantiate robustness, we have scaled the disturbance vector $\lambda(t)$ by different amplitudes $\mu \in \{0.5, 1.0, 2.0\}$, *i.e.*, $\lambda_\mu(t) = \mu\lambda(t)$ as shown in Fig. 3. The actual and estimated values of $z(t)$ are shown in Fig. 4, with the initial conditions set as $x(0) = [0.9 \quad 0.2 \quad 0.15]^\top$, $w(0) = 0.3$ and the control input $u(t) = 0.2\sin(\pi t)$. As shown in Fig. 4a,b, the filter closely follows the true state for all disturbance amplitudes, and the resulting error remains well within the specified $\mathcal{H}_\infty$ performance bound even when the disturbance magnitude is doubled.

We computed the error energy in MATLAB and, as summarized in Table 2, for disturbances scaled by factors of 0.5, 1.0, and 2.0, the resulting error energy $\|e\|_2^2$ remains well below the prescribed $\mathcal{H}_\infty$ bound, demonstrating the robustness of the filter.

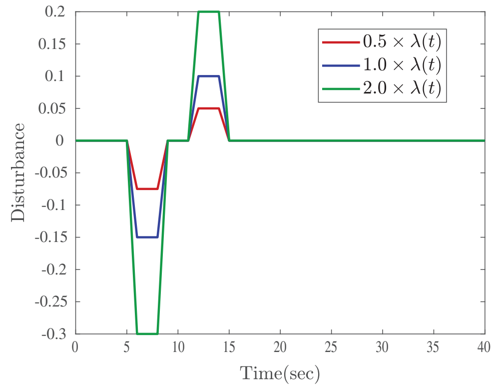
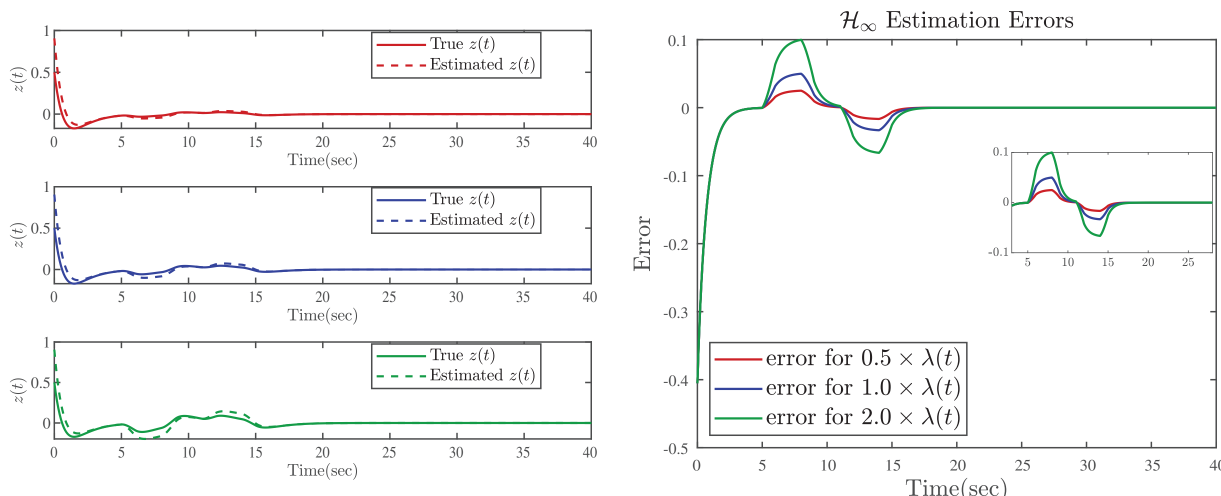


Figure 3: Disturbance $\lambda(t)$ scaled by 0.5 (red), 1.0 (blue) and 2.0 (green)



(a) Time responses of true and estimated state $z(t)$

(b) Time Responses of error $e(t) = z(t) - \hat{z}(t)$

Figure 4: Robustness of the proposed functional $\mathcal{H}_\infty$ filter under scaled disturbances

Table 2: Error energy under varying disturbance amplitudes

Amplitude (μ)	$\|e\|_2^2$	$\gamma^2\|v\|_2^2 + \beta^2\|e_1(0)\|^2$
0.5	0.1125	1.3671
1.0	0.2450	1.5433
2.0	0.9121	2.7820

6 Conclusion

This paper has contributed to the development of functional $\mathcal{H}_\infty$ filtering for nonlinear descriptor systems with generalized monotone nonlinearities. The existence conditions for the functional $\mathcal{H}_\infty$ filter have been established through a rank condition on system matrices and an LMI. Notably, the proposed method for the functional filter does not impose specific assumptions such as system regularity, requiring only the essential condition of system well-posedness. Furthermore, the designed filter exhibits robustness against unknown disturbances, as demonstrated through a numerical example. Finally, this paper opens up future opportunities for advancing the theory and application of functional $\mathcal{H}_\infty$ filtering in nonlinear descriptor systems. For instance, future research could explore the development of conditions and algorithms for constructing the smallest possible order functional filter. Additionally, investigating the incorporation of other types of nonlinearities in the system and output presents a fascinating avenue for further exploration.

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References

1. Belur MN, Shankar S. The persistence of impulse controllability. Math Control Signals Syst. 2019;31(4):487–501. doi:10.1007/s00498-019-00250-x.
2. Eich Soellner E, Führer C. Numerical methods in multibody dynamics. Vol. 45. Stuttgart, Germany: Springer; 1998.
3. Riaza R. Differential-algebraic systems: analytical aspects and circuit applications. Basel, Switzerland: World Scientific Publishing; 2008.
4. Kumar A, Daoutidis P. Control of nonlinear differential algebraic equation systems with applications to chemical processes. Vol. 397. New York, NY, USA: CRC Press; 1999.
5. Luenberger DG. Observing the state of a linear system. IEEE Trans Mil Electron. 1964;8(2):74–80. doi:10.1109/tme.1964.4323124.
6. Kalman RE. A new approach to linear filtering and prediction problems. J Basic Eng. 1960;82(1):35–45. doi:10.1115/1.3662552.
7. Jaiswal J, Gupta MK, Tomar NK. Necessary and sufficient conditions for ODE observer design of descriptor systems. Syst Control Lett. 2021;151:104916. doi:10.1016/j.sysconle.2021.104916.
8. Dai L. Singular control systems. Vol. 118. Berlin, Germany: Springer; 1989.

9. Ishihara JY, Terra MH, Campos JC. Optimal recursive estimation for discrete-time descriptor systems. Int J Syst Sci. 2005;36(10):605–15. doi:10.1080/00207720500219302.
10. Goel A, Bhaumik S, Tomar NK. Kalman filtering for linear singular systems subject to round-robin protocol. Control Theory Technol. 2024;22(4):543–51. doi:10.1007/s11768-024-00205-x.
11. Brown RG, Hwang PY. Introduction to random signals and applied Kalman filtering with MATLAB exercises. New York, NY, USA: John Wiley and Sons; 1997.
12. Meng S, Meng F, Chi H, Chen H, Pang A. A robust observer based on the nonlinear descriptor systems application to estimate the state of charge of lithium-ion batteries. J Franklin Instit. 2023;360(16):11397–413. doi:10.1016/j.jfranklin.2023.08.037.
13. Xu S, Lam J, Zou Y. H_∞ filtering for singular systems. IEEE Trans Autom Control. 2003;48(12):2217–22. doi:10.1109/tac.2003.820149.
14. Xu S, Lam J. Reduced-order H_∞ filtering for singular systems. Syst Control Letters. 2007;56(1):48–57.
15. Gao F, Chen W, Wu M, She J. Robust H_∞ control of uncertain singular systems based on equivalent-input-disturbance approach. Asian J Control. 2020;22(5):2071–9. doi:10.1002/asjc.2138.
16. Tunga PK, Tomar NK. H_∞ filter based functional observers for descriptor systems. In: 2023 62nd IEEE Conference on Decision and Control (CDC). Piscataway, NJ, USA: IEEE; 2023. p. 7681–6.
17. Xu S, Lam J. Robust control and filtering of singular systems. Vol. 332. Berlin, Germany: Springer; 2006.
18. Saravanakumar R, Kazemy A, Cao Y. Robust reliable H_∞ control for offshore steel jacket platforms via memory sampled-data strategy. Math Methods Appl Sci. 2025;48(7):6987–99. doi:10.1002/mma.8390.
19. Meng S, Li S, Chi H, Meng F, Pang A. H_∞ observer based on descriptor systems applied to estimate the state of charge. Entropy. 2022;24(3):420. doi:10.3390/e24030420.
20. Luenberger DG. Non-linear descriptor systems. J Econ Dyn Control. 1979;1(3):219–42. doi:10.1016/s0165-1889(79)90122-2.
21. Petzold L, Lötstedt P. Numerical solution of nonlinear differential equations with algebraic constraints II: practical implications. SIAM J Sci Stat Comput. 1986;7(3):720–33. doi:10.1137/0907049.
22. Liberzon D, Trenn S. Switched nonlinear differential algebraic equations: solution theory, Lyapunov functions, and stability. Automatica. 2012;48(5):954–63. doi:10.1016/j.automatica.2012.02.041.
23. Kunkel P. Differential algebraic equations: analysis and numerical solution. Vol. 2. Berlin/Heidelberg, Germany: European Mathematical Society; 2006.
24. Lamour R, März R, Tischendorf C. Differential-algebraic equations: a projector based analysis. Berlin/Heidelberg, Germany: Springer; 2013.
25. Lu G, Ho DW. Full-order and reduced-order observers for Lipschitz descriptor systems: the unified LMI approach. IEEE Trans Circuits Syst II Express Briefs. 2006;53(7):563–7. doi:10.1109/tcsii.2006.875332.
26. Berger T. On observers for nonlinear differential-algebraic systems. IEEE Trans Autom Control. 2018;64(5):2150–7. doi:10.1109/tac.2018.2866438.
27. Tripathi M, Moysis L, Gupta MK, Volos C. Reduced-order adaptive observer for rectangular descriptor systems with incremental quadratic constraints. Int J Syst Sci. 2025. doi:10.1080/00207721.2025.2514590.
28. Fan X, Arcak M. Observer design for systems with multivariable monotone nonlinearities. Syst Control Letters. 2003;50(4):319–30.
29. Campbell S. Singular systems of differential equations. New York, NY, USA: Pitman; 1982.
30. Yang C, Zhang Q, Chai T. Nonlinear observers for a class of nonlinear descriptor systems. Optim Control Appl Methods. 2013;34(3):348–63. doi:10.1002/oca.2028.
31. Gupta MK, Tomar NK, Darouach M. Unknown inputs observer design for descriptor systems with monotone nonlinearities. Int J Robust Nonlinear Control. 2018;28(17):5481–94. doi:10.1002/rnc.4331.
32. Berger T, Lanza L. Observers for differential-algebraic systems with Lipschitz or monotone nonlinearities. In: Progress in differential-algebraic equations II. Cham, Switzerland: Springer; 2020. p. 257–89 doi: 10.1007/978-3-030-53905-4_9.

33. Moysis L, Gupta MK, Mishra V, Marwan M, Volos C. Observer design for rectangular descriptor systems with incremental quadratic constraints and nonlinear outputs—application to secure communications. Int J Robust Nonlinear Control. 2020;30(18):8139–58. doi:10.1002/rnc.5233.
34. Darouach M, Boutat Baddas L, Zerrougui M. H_∞ observers design for a class of nonlinear singular systems. Automatica. 2011;47(11):2517–25. doi:10.1016/j.automatica.2011.08.037.
35. Yang C, Zhang Q, Chou JH, Yin F. H_∞ observer design for descriptor systems with slope-restricted nonlinearities. Asian J Control. 2012;14(4):1133–40. doi:10.1002/asjc.459.
36. Aliyu MDS, Boukas EK. $\mathcal{H}_\infty$ filtering for nonlinear singular systems. IEEE Trans Circuits Syst I Regul Pap. 2012;59(10):2395–404. doi:10.1109/tcsi.2012.2189038.
37. Liu L, Wen Q, Li Y, Fu D, Fu Z. State and adaptive disturbance observer co-design for incrementally quadratic nonlinear descriptor systems with nonlinear outputs. Int J Robust Nonlinear Control. 2023;33(17):10678–97. doi:10.1002/rnc.6908.
38. Trinh H, Fernando T. Functional observers for dynamical systems. Vol. 420. Berlin/Heidelberg, Germany: Springer Science & Business Media; 2011.
39. Wang HS, Yung CF, Chang FR. H_∞ control for nonlinear descriptor systems. Vol. 326. Berlin/Heidelberg, Germany: Springer; 2006.
40. Tunga PK, Jaiswal J, Tomar NK. Functional observers for descriptor systems with unknown inputs. IEEE Access. 2023;11(6):19680–9. doi:10.1109/access.2023.3249099.
41. Jaiswal J, Berger T, Tomar NK. Existence conditions for functional ODE observer design of descriptor systems revisited. J Frankl Inst. 2024;361(8):106848. doi:10.1016/j.jfranklin.2024.106848.
42. Ríos-Ruiz C, Osorio Gordillo GL, Darouach M, Souley Ali H, Astorga-Zaragoza CM. Design of finite-time convergent functional observer for descriptor systems: application to fault estimation. Asian J Control. 2024;26(6):2843–53. doi:10.1002/asjc.3392.
43. Liu HY, Duan ZS. Unknown input observer design for systems with monotone nonlinearities. IET Control Theory Appl. 2012;6(12):1941–7.
44. Piziak R, Odell PL. Matrix theory: from generalized inverses to Jordan form. New York, NY, USA: CRC Press; 2007.
45. Barnett S, Cameron RG. Introduction to mathematical control theory. Oxford, UK: Clarendon Press; 1985.
46. Boyd S, El Ghaoui L, Feron E, Balakrishnan V. Linear matrix inequalities in system and control theory. Philadelphia, PA, USA: SIAM; 1994.
47. Sjöberg J. Some results on optimal control for nonlinear descriptor systems [master's thesis]. Linkö̈ping, Sweden: Department of Electrical Engineering, Linköpings Universitet; 2006.