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ON THE DISCRETIZATION OF DISCONTINUOUS SOURCES OF HYPERBOLIC BALANCE LAWS

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Abstract. We focus on a toy problem which corresponds to a simplification of a boiling two-phase flow model. This model is a hyperbolic system of balance laws with a source term defined as a discontinuous function of the unknown. Several discretizations of this source terms are studied, and we illustrate their capacity to capture steady states.

1 INTRODUCTION

The present work present numerical approaches adapted to hyperbolic balance laws with a source term defined as a discontinuous function of the unknown. These constructions are described on the following toy problem ([9])

$$\partial_t u - \partial_x u = S^u(h(u,v)), \qquad S^u(h) = \begin{cases} a & \text{if} \quad h \le 0, \\ b & \text{if} \quad h > 0, \end{cases} \tag{1a}$$

$$\partial_t v + \partial_x v = S^v(h(u,v)), \qquad S^v(h) = \begin{cases} c & \text{if} \quad h \le 0, \\ d & \text{if} \quad h > 0, \end{cases} \tag{1b}$$

$$h(u,v) = u + v.$$

This system corresponds to a simplification of a 4-equation drift-flux model ([7, 1, 10]) for 1D boiling two-phase flow given by

$$\partial_t \begin{pmatrix} \alpha\rho_v \\ \rho \\ \rho u \\ \rho e \end{pmatrix} + \partial_x \begin{pmatrix} \alpha\rho_v u \\ \rho u \\ \rho u^2 + p \\ (\rho e + p)u \end{pmatrix} = \begin{pmatrix} \Gamma \\ 0 \\ 0 \\ \phi \end{pmatrix}, \quad \Gamma = \begin{cases} 0 & \text{if } h \le h^{eb}, \\ K\phi & \text{otherwise}, \end{cases} \tag{2}$$

where the enthalpy h depends on $(\alpha\rho_v, \rho, \rho u, \rho e)$ and the heat source $\phi > 0$ and the evaporation constant $K > 0$.

System (1) retains the main difficulties of (2), i.e. the discontinuity of the source term w.r.t. the unknown, but it is reduced to two equations and the fluxes and the enthalpy were linearized and normalized for simplicity.

This system was studied at a continuous level in [9] where a framework was proposed for well-posedness studies. Now, we aim at tackling the difficulties emerging at the numerical level when discretizing it.

2 POSITION OF THE PROBLEM

The main issue emerging when discretizing (1) is the appearance of artifacts, typically spurious oscillations. These already appear when considering steady solutions, and we illustrate these issue by applying a naive scheme to capture a steady solution.

2.1 The continuous steady state

We first construct a steady solution to (1) in a domain $x \in [-1,1]$. We fix

$$a = 0, \qquad b = -\frac{3}{5}, \qquad c = \frac{2}{5}, \qquad d = 0.$$

Then, one verifies that

$$u_0(x) = \begin{cases} -ax & \text{if } x < 0, \\ -bx & \text{otherwise,} \end{cases} \qquad v_0(x) = \begin{cases} cx & \text{if } x < 0, \\ dx & \text{otherwise,} \end{cases}$$

is a time-independent solution to (1) with the boundary conditions $u(x=1) = u_0(x=1)$ and $v(x=-1) = v_0(x=-1)$.

The well-posedness of (1) was studied in [9] and this test does not satisfy the conditions provided in Proposition 3.5 in this reference. These were only sufficient conditions, but not necessary ones and, in practice, one easily verifies that the Cauchy problem possesses a unique generalized solution. We even exhibit some stability property of this equilibrium.

Proposition 1. *There exists a convex entropy-entropy flux pair* $(\mathcal{H}, \mathcal{G})$ *such that*

$$\partial_t \mathcal{H}(u,v) + \partial_x \mathcal{G}(u,v) = \mathcal{D}(u,v) \leq 0,$$

such that $\mathcal{H}(u,v) \geq 0$ *and* $\mathcal{H}(u,v) = 0 \Leftrightarrow (u,v) = (u_0, v_0)$.

Proof. Writing (u^ϵ, v^ϵ) the solution to (1) with a perturbed initial condition $(u_0 + \epsilon\delta u, v_0 + \epsilon\delta v)$, one computes for all (x,t)

$$\partial_t(u^\epsilon - u_0) - \partial_x(u^\epsilon - u_0) = \begin{cases} (a-b) & \text{if } x > 0 \text{ and } (u^\epsilon + v^\epsilon)(x,t) < 0, \\ (b-a) & \text{if } x \leq 0 \text{ and } (u^\epsilon + v^\epsilon)(x,t) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\partial_t(v^\epsilon - v_0) + \partial_x(v^\epsilon - v_0) = \begin{cases} (c-d) & \text{if } x > 0 \text{ and } (u^\epsilon + v^\epsilon)(x,t) < 0, \\ (d-c) & \text{if } x \leq 0 \text{ and } (u^\epsilon + v^\epsilon)(x,t) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then summing the first equation multiplied by $(u^\epsilon - u)/3$ with the second equation multiplied by $(v^\epsilon - v)/2$, and using the exact value of u_0, v_0, a, b, c and d yields

$$\partial_t \left[\frac{(v^\epsilon - v_0)^2}{4} + \frac{(u^\epsilon - u_0)^2}{6} \right] + \partial_x \left[\frac{(v^\epsilon - v_0)^2}{4} - \frac{(u^\epsilon - u_0)^2}{6} \right]$$
$$= \begin{cases} \frac{1}{5}(u^\epsilon + v^\epsilon) - \frac{3x}{10} & \text{if } x > 0 \text{ and } (u^\epsilon + v^\epsilon)(x,t) < 0, \\ -\frac{1}{5}(u^\epsilon + v^\epsilon) + \frac{2x}{10} & \text{if } x \leq 0 \text{ and } (u^\epsilon + v^\epsilon)(x,t) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Especially, the values on the right-hand-side are non-positive and the equilibrium is stable. □

One remarks that some values $(u, v) \neq (u_0, v_0)$ satisfy $\mathcal{D}(u, v) = 0$ and the solution is not necessarily attracted back to the equilibrium (u_0, v_0).

2.2 Non-existence of a discrete steady state with a centered source term

First, let us write an explicit discretization of (1) with upwind fluxes and centered source

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^n - u_i^n}{\Delta x} = S^u(h_i^n), \tag{3a}$$

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + \frac{v_i^n - v_{i-1}^n}{\Delta x} = S^v(h_i^n), \tag{3b}$$

where $h_i^n = u_i^n + v_i^n$.

Proposition 2. *Suppose $a - d < 0$ and $b - c > 0$, then the scheme* (3) *possesses no discrete steady state with source switch.*

Proof. By contradiction, let us assume that a discrete steady state is reached. Then $u_i^n = u_i$ and $v_i^n = v_i$, and this rewrites

$$u_i = u_{i+1} + \Delta x S^u(h_i), \qquad v_i = v_{i-1} + \Delta x S^v(h_i).$$

For simplicity, consider only the two cells in the middle of the domain where the source switches value. This system has for unknowns u_1^n, u_2^n, v_1^n, v_2^n and has for boundary conditions (given) u_3^n and v_0^n. Rewriting this system with $h_1^n = u_1^n + v_1^n$ and $h_2^n = u_2^n + v_2^n$ as unknowns reads

$$\begin{aligned} h_1 &= (u_3 + v_0) + \Delta x \left[(S^u + S^v)(h_1) + S^u(h_2) \right], \\ h_2 &= (u_3 + v_0) + \Delta x \left[(S^u + S^v)(h_2) + S^v(h_1) \right]. \end{aligned}$$

where $u_3 + v_0$ is a constant. We may differentiate four cases

$$(h_1, h_2) = \begin{cases} \big((u_3 + v_0) + \Delta x(2a + c), & (u_3 + v_0) + \Delta x(a + 2c)\big) \quad \text{if} \quad h_1 < 0, \quad h_2 < 0, \\ \big((u_3 + v_0) + \Delta x(a + c + b), & (u_3 + v_0) + \Delta x(b + d + c)\big) \quad \text{if} \quad h_1 < 0, \quad h_2 \geq 0, \\ \big((u_3 + v_0) + \Delta x(b + d + a), & (u_3 + v_0) + \Delta x(a + c + d)\big) \quad \text{if} \quad h_1 \geq 0, \quad h_2 < 0, \\ \big((u_3 + v_0) + \Delta x(2b + d), & (u_3 + v_0) + \Delta x(b + 2d)\big) \quad \text{if} \quad h_1 \geq 0, \quad h_2 \geq 0. \end{cases} \tag{4}$$

The first and the last correspond to the cases with a constant source and are not those of interest here. Computing $h_1 - h_2$ in the two middle cases provides

$$\begin{aligned} \frac{h_1 - h_2}{\Delta x} &= a - d \qquad \text{if} \quad h_1 < 0, \quad h_2 \geq 0, \\ \frac{h_1 - h_2}{\Delta x} &= b - c \qquad \text{if} \quad h_1 \geq 0, \quad h_2 < 0. \end{aligned}$$

Based on the expected signs of h_1 and h_2, we violate these two conditions when

$$a - d > 0, \qquad b - c < 0.$$

If neither of those two conditions are satisfied, then there exists no discrete steady state with source switch. □

Especially, these conditions are not satisfied with the values given in the last subsection.

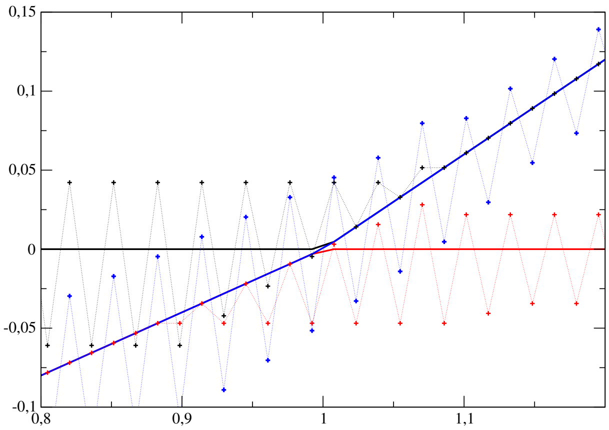

Figure 1: Exact solution (and initial condition) in straight line and discrete one with (3) in dotted line with pluses. The solution u are in black, v are in red and $u+v$ in blue.

2.2.1 Numerical test

The numerical results obtained with the centered scheme on the test case of the Subsection 2.1 are given in Fig. 1. These are given for an initial condition $u_i^0 = u_0(x_i)$ and $v_i^0 = v_0(x_i)$ with parameters $x_i = (i - \frac{1}{2})\Delta x$ and $\Delta x = L/N$ with a number of cells $N = 128$. The final time is $T = 1$ and a CFL condition of $\Delta t = 0.95\Delta x$ was used. The non-existence of a discrete source term is characterized here by oscillations. In practice, the enthalpy h in the middle of the domain alternates between a positive and a negative value, which creates an oscillation. This oscillation is afterward transported at velocity $+1$ with the unknown v and velocity -1 with the unknown u.

3 FIRST ALTERNATIVE SCHEMES

Some techniques from the literature offers the well-balanced property, here capturing the steady states. We present some of them here that we adapt to the present problem.

A first family of approaches was proposed in [2] as extensions of [6] with source terms. These approaches were shown to be well-adapted to capture equilibria.

They were both developed for a generic non-linear hyperbolic balance law

$$\partial_t U + \partial_x F(U) = S(U),$$

in an explicit finite volume formalism

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}) + \Delta t \mathcal{S}_i^n.$$

3.1 Flux-difference splitting approach

The flux-difference splitting approach leads to fixing

$$\mathcal{F}_{i+\frac{1}{2}}^n = \frac{1}{2}\left[F(U_i^n) + F(U_{i+1}^n) - A_{i+\frac{1}{2}}^n(U_{i+1}^n - U_i^n)\right]$$

where the matrix $A^n_{i+\frac{1}{2}} = A(U^n_{i+1}, U^n_i)$ satisfies the homogeneity property $A(U,U)U = F(U)$. One popular manner of discretizing the source term following this flux-difference splitting yields $\mathcal{S}^n_i = S^{n,R}_{i-\frac{1}{2}} + S^{n,L}_{i+\frac{1}{2}}$ with

$$\begin{aligned} S^{n,R}_{i+\frac{1}{2}} &= \frac{1}{2}\left[Id - \left|A^n_{i+\frac{1}{2}}\right|(A^n_{i+\frac{1}{2}})^{-1}\right] S^R(U^n_{i+1}, U^n_i), \\ S^{n,L}_{i-\frac{1}{2}} &= \frac{1}{2}\left[Id + \left|A^n_{i-\frac{1}{2}}\right|(A^n_{i-\frac{1}{2}})^{-1}\right] S^L(U^n_i, U^n_{i-1}), \end{aligned}$$

where the interface source terms also satisfy the homogeneity properties $S^R(U,U) = S(U)$ and $S^L(U,U) = S(U)$.

Applying this scheme with Roe matrix (in the linear case $F(U) = AU$, then $A(U,U) = A$ and it equivals to upwind fluxes) to (1) yields

$$u^{n+1}_i = u^n_i\left(1 - \frac{\Delta t}{\Delta x}\right) + \frac{\Delta t}{\Delta x}u^n_{i+1} + \Delta t S^{u,R}(U^n_{i+1}, U^n_i), \tag{5a}$$

$$v^{n+1}_i = v^n_i\left(1 - \frac{\Delta t}{\Delta x}\right) + \frac{\Delta t}{\Delta x}v^n_{i-1} + \Delta t S^{v,L}(U^n_i, U^n_{i-1}). \tag{5b}$$

Following the computations of [9] in the Riemann problem case, we suggest defining at the interfaces

$$S^{u,R}(U^n_{i+1}, U^n_i) = S^u(u^n_{i+1} + v^n_i), \qquad S^{v,L}(U^n_{i+1}, U^n_i) = S^v(u^n_{i+1} + v^n_i). \tag{5c}$$

In a stationnary framework, one obtains

$$\begin{aligned} u_1 &= u_2 + \Delta x S^u(u_2 + v_1), \qquad & u_2 &= u_3 + \Delta x S^u(u_3 + v_2), \\ v_1 &= v_0 + \Delta x S^v(u_1 + v_0), \qquad & v_2 &= u_3 + \Delta x S^v(u_2 + v_1), \end{aligned}$$

and eventually writing $h_1 = u_1 + v_0$, $h_2 = u_2 + v_1$ and $h_3 = u_3 + v_2$ provides

$$\begin{array}{lll} h_1 & = v_0 + u_2 + \Delta x S^u(h_2) \;\; = & v_0 + u_3 + \Delta x(S^u(h_2) + S^u(h_3)), \\ h_2 & = & v_0 + u_3 + \Delta x(S^u(h_3) + S^v(h_1)), \\ h_3 & = v_1 + u_3 + \Delta x S^v(h_2) \;\; = & v_0 + u_3 + \Delta x(S^v(h_2) + S^v(h_1)). \end{array}$$

This leads to

$$\frac{h_2 - h_1}{\Delta x} = S^v(h_1) - S^u(h_2), \qquad \frac{h_3 - h_2}{\Delta x} = S^v(h_2) - S^u(h_3).$$

When expecting a change of sign either between h_1 and h_2 or between h_2 and h_3, we fall back onto the same issue seen in Section 2.

3.2 Flux-vector splitting approach

In a flux-vector splitting approach, we decompose the numerical flux

$$\mathcal{F}^n_{i+\frac{1}{2}} = A^+_{i+\frac{1}{2}} U^n_{i+1} + A^-_{i+\frac{1}{2}} U^n_i,$$

where the matrices $A^{\pm}_{i+\frac{1}{2}} = A^{\pm}(U^n_{i+1}, U^n_i)$ satisfies the homogeneity properties

$$\left[A^+(U,U) + A^-(U,U)\right] U = F(U), \qquad Sp(A^{\pm}(U,V)) \subset \mathbb{R}^{\pm}.$$

One popular manner of discretizing the source term following this flux-vector splitting consists in decomposing similarly

$$\mathcal{S}^n_i = \frac{1}{2}\left[S^n_{i-\frac{1}{2}} + S^n_{i+\frac{1}{2}}\right], \qquad S^n_{i+\frac{1}{2}} = B^+_{i+\frac{1}{2}} S(U^n_{i+1}) + B^-_{i+\frac{1}{2}} S(U^n_i), \tag{6}$$

where the matrices $B^{\pm}_{i+\frac{1}{2}} = B^{\pm}(U^n_{i+1}, U^n_i)$ satisfy the homogeneity property

$$B^+(U,U) + B^-(U,U) = Id.$$

Applying this scheme by fixing $A^{\pm} = (A \pm |A|)/2$ as the positive, resp. negative, part of the Roe matrix to (1) yields similarly

$$u^{n+1}_i = u^n_i\left(1 - \frac{\Delta t}{\Delta x}\right) + \frac{\Delta t}{\Delta x} u^n_{i+1} + \frac{\Delta t}{2}\left(B^+_{i+\frac{1}{2}} S(U^n_{i+1}) + (B^-_{i+\frac{1}{2}} + B^+_{i-\frac{1}{2}}) S(U^n_i) + B^-_{i+\frac{1}{2}} S(U^n_{i-1})\right)_1,$$

$$v^{n+1}_i = v^n_i\left(1 - \frac{\Delta t}{\Delta x}\right) + \frac{\Delta t}{\Delta x} v^n_{i-1} + \frac{\Delta t}{2}\left(B^+_{i+\frac{1}{2}} S(U^n_{i+1}) + (B^-_{i+\frac{1}{2}} + B^+_{i-\frac{1}{2}}) S(U^n_i) + B^-_{i+\frac{1}{2}} S(U^n_{i-1})\right)_2.$$

An intuitive choice for $B^{\pm}$ consists in choosing $A^{\pm}A^{-1}$ which reads $B^+_{i+\frac{1}{2}} U^n_i = u^n_i$ and $B^-_{i+\frac{1}{2}} U^n_i = v^n_i$ and therefore

$$u^{n+1}_i = u^n_i\left(1 - \frac{\Delta t}{\Delta x}\right) + \frac{\Delta t}{\Delta x} u^n_{i+1} + \frac{\Delta t}{2}\left(S^u(U^n_{i+1}) + S^u(U^n_i)\right), \tag{7a}$$

$$v^{n+1}_i = v^n_i\left(1 - \frac{\Delta t}{\Delta x}\right) + \frac{\Delta t}{\Delta x} v^n_{i-1} + \frac{\Delta t}{2}\left(S^v(U^n_i) + S^v(U^n_{i-1})\right). \tag{7b}$$

In a stationnary framework, writing $h_i = u_i + v_i$, one obtains

$$u_1 = u_3 + \frac{\Delta x}{2}(S^u(h_3) + 2S^u(h_2) + S^u(h_1)), \qquad u_2 = u_3 + \frac{\Delta x}{2}(S^u(h_3) + S^u(h_2)),$$

$$v_2 = v_0 + \frac{\Delta x}{2}(S^v(h_2) + 2S^v(h_1) + S^v(h_0)), \qquad v_1 = v_0 + \frac{\Delta x}{2}(S^v(h_1) + S^v(h_0)),$$

and eventually

$$h_1 = (v_0 + u_3) + \frac{\Delta x}{2}(S^u(h_3) + S^v(h_0)) + \frac{\Delta x}{2}(2S^u(h_2) + S^u(h_1) + S^v(h_1))$$

$$h_2 = (v_0 + u_3) + \frac{\Delta x}{2}(S^u(h_3) + S^v(h_0)) + \frac{\Delta x}{2}(S^v(h_2) + 2S^v(h_1) + S^u(h_2)).$$

This leads to

$$\frac{h_2 - h_1}{\Delta x} = (S^v - S^u)(h_2) + (S^v - S^u)(h_1),$$

or equivalently

$$(Id - \Delta x(S^v - S^u))(h_2) = (Id + \Delta x(S^v - S^u))(h_1).$$

The function $Id \pm \Delta x(S^v - S^u)$ are discontinuous in 0 and do not cross each other. Therefore, there still exists no stationnary solutions.

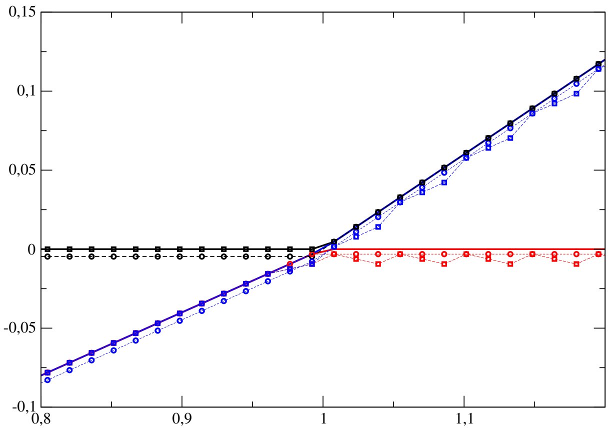

Figure 2: Exact solution (and initial condition) in straight line and discrete ones with (5) in dashed line with squares and with (7) with circles. The solution u are in black, v are in red and $u+v$ in blue.

3.3 Numerical test

Again, these two schemes are tested with the parameters of Subsection 2.2.1. The numerical results are given in Fig. 2. We observe again oscillations with the flux-vector splitting scheme (7), but those are of smaller amplitude than with the centered source. On this test case, the flux-difference scheme (5) seems to have a better behavior.

4 TWO OTHER WELL-BALANCED DISCRETIZATIONS

4.1 An integral finite difference scheme

One essential reason for the upwind discretizations of the source term presented in the last section to fail to capture steady states lies in the choice of approximation used. These schemes are closely related to the approximate Riemann solver approaches but extended with a source term. However, it was shown in [9] that (1) with a Heavyside initial condition, i.e. Riemann problems, do not possess a steady state at the continuous level. In practice, the solution always evolves in time. In this previous work, only one type of configuration was shown to create steady states, it was those associated with a subcharacteristic boiling curve, i.e. when the boiling front $\{(x,t)$ s.t. $(u+v)(x,t)=0\}$ propagates slower than the characteristic speeds. Then, instead of using a finite volume approach based on approximating the solution by constants in each cell, we use a finite difference approach where the solution is approximated by continuous piecewise affine functions (the exact generalized solution was shown to be $W^{1,\infty}$). Following the characteristics, this yields

$$u(x_i,t^{n+1}) = u(x_i+\Delta t,t^n) + \int_0^{\Delta t} S^u(u+v)(x_i+(\Delta t-\tau),t^n+\tau)d\tau, \tag{8a}$$

$$v(x_i,t^{n+1}) = v(x_i-\Delta t,t^n) + \int_0^{\Delta t} S^v(u+v)(x_i-(\Delta t-\tau),t^n+\tau)d\tau. \tag{8b}$$

Approximating $u(.\,,t^n)$ and $v(.\,,t^n)$ by a continuous piecewise affine function passing in u_i^n, resp. v_i^n, at x_i, and approximating $\tau \mapsto (u+v)(x_i+(\Delta t-\tau))$ and $\tau \mapsto (u+v)(x_i-(\Delta t-\tau))$ by constants in the integral provides (3). Let us modify this last approximation. We compute similarly

$$u_i^{n+1} = u_i^n\left(1-\frac{\Delta t}{\Delta x}\right) + u_{i+1}^n\frac{\Delta t}{\Delta x} + \Delta t S_i^{u,n}, \tag{9a}$$

$$v_i^{n+1} = v_i^n\left(1-\frac{\Delta t}{\Delta x}\right) + v_{i-1}^n\frac{\Delta t}{\Delta x} + \Delta t S_i^{v,n}, \tag{9b}$$

where u_i^n and v_i^n approximate $u(x_i,t^n)$ and $v(x_i,t^n)$ and $S_i^{u,n}$ and $S_i^{v,n}$ approximate the integrals in (8). The functions in these integrals are constant by part (they switch if $u+v$ changes sign). Following the computations in [9], we suggest the following approximation that mimics this change of sign (w designate either u and v)

$$S_i^{u,n} = \tilde{S}_{+,i}^{u,n}\beta_i^{u,n} + \left(a+b-\tilde{S}_{+,i}^{u,n}\right)\left(1-\beta_i^{u,n}\right), \quad S_i^{v,n} = \tilde{S}_{-,i}^{v,n}\beta_i^{v,n} + \left(c+d-\tilde{S}_{-,i}^{v,n}\right)\left(1-\beta_i^{v,n}\right), \tag{9c}$$

$$\tilde{S}_{\pm,i}^{w,n} = S^w\left((u_i^n+v_i^n)\left(1-\frac{\Delta t}{\Delta x}\right) + (u_{i\pm1}^n+v_{i\pm1}^n)\frac{\Delta t}{\Delta x}\right), \tag{9d}$$

and where the coefficients β are either 1 if no jump occurs along the respective characteristics or $\beta \in]0,1[$ and they are computed to capture the time of switch of sign of h along the characteristics.

To keep this writing short, only the computations to obtain β^u are described below, those for β^v follows by similar computations. Following the characteristics, we have

$$\begin{aligned}
&(u+v)(x_i+\Delta t(1-\beta), t^n+\beta\Delta t) = 0,\\
&\quad u(x_i+\Delta t(1-\beta), t^n+\beta\Delta t) = u(x_i+\Delta t, t^n) + \beta\Delta t S^u\left((u+v)\left(x_i+\Delta t, t^n\right)\right),\\
&\quad v(x_i+\Delta t(1-\beta), t^n+\beta\Delta t) = v(x_i+(1-2\beta)\Delta t, t^n) + \beta\Delta t S^v\left((u,v)\left(x_i+(1-2\beta)\Delta t, t^n\right)\right),
\end{aligned}$$

and where we approximate down the characteristics

$$\begin{aligned}
w(x_i+\alpha\Delta t, t^n) &\approx \tilde{w}(x_i+\alpha, t^n) := w_i^n(1-\alpha\frac{\Delta t}{\Delta x}) + w_{i+1}^n\alpha\frac{\Delta t}{\Delta x},\\
w(x_i-\alpha\Delta t, t^n) &\approx \tilde{w}(x_i-\alpha, t^n) := w_i^n(1-\alpha\frac{\Delta t}{\Delta x}) + w_{i-1}^n\alpha\frac{\Delta t}{\Delta x},\\
S^w\left((u+v)\left(x_i+\alpha\Delta t, t^n\right)\right) &\approx S^w\left((\tilde{u}+\tilde{v})\left(x_i+\alpha\Delta t, t^n\right)\right).
\end{aligned}$$

Defining $\gamma^\pm = \min\left(1, \max\left(0, \pm\frac{u_i^n+v_i^n}{(u_{i\pm1}^n+v_{i\pm1}^n)-(u_i^n+v_i^n)}\right)\right)$ which corresponds to the potential location where $(u+v)(x+\gamma\Delta x, t^n)=0$, then

$$S^w\left((\tilde{u}+\tilde{v})\left(x_i\pm\alpha, t^n\right)\right) = \begin{cases} S^w(u_{i\pm1}^n+v_{i\pm1}^n) & \text{if} \quad \alpha \geq \gamma^\pm,\\ S^w(u_i^n+v_i^n) & \text{otherwise.}\end{cases}$$

Eventually, this leads to

$$\beta_i^{u,n} = 1 \text{ if } sign\left((\tilde{u}+\tilde{v})(x_i+\Delta t, t^n)\right) = sign\left(\tilde{u}(x_i+\Delta t, t^n) + \tilde{v}(x_i-\Delta t, t^n) + \Delta t(\tilde{S}_{+,i}^{u,n}+\tilde{S}_{-,i}^{v,n})\right).$$

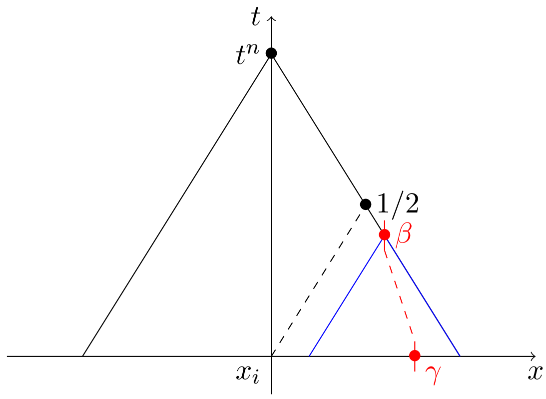

Figure 3: Representation of the differnent quantities for the computation of β^u.

Otherwise, $\beta_i^{u,n} = \tilde{\beta}_i^{u,n}$ if $\tilde{\beta}_i^{u,n} \le \min(\frac{1}{2}, \frac{1-\gamma^+}{2})$ where

$$\tilde{\beta}_i^{u,n} = \frac{u_i^n\left(1 - \frac{\Delta t}{\Delta x}\right) + u_{i+1}^n \frac{\Delta t}{\Delta x} + v_{i+1}^n}{\Delta t \left(\tilde{S}_{+,i}^{u,n} + \tilde{S}_{+,i}^{v,n} - 2\frac{v_{i+1}^n - v_i^n}{\Delta x}\right)}.$$

Otherwise, $\beta_i^{u,n} = \tilde{\beta}_i^{u,n}$ if $\frac{1-\gamma^+}{2} \le \tilde{\beta}_i^{u,n} \le \frac{1}{2}$ where

$$\tilde{\beta}_i^{u,n} = \frac{u_i^n\left(1 - \frac{\Delta t}{\Delta x}\right) + u_{i+1}^n \frac{\Delta t}{\Delta x} + v_{i+1}^n}{\Delta t \left(\tilde{S}_{+,i}^{u,n} + \tilde{S}_{i}^{v,n} - 2\frac{v_{i+1}^n - v_i^n}{\Delta x}\right)}.$$

Otherwise, $\beta_i^{u,n} = \tilde{\beta}_i^{u,n}$ if $\frac{1}{2} \le \tilde{\beta}_i^{u,n} \le \frac{1+\gamma^-}{2}$ where

$$\tilde{\beta}_i^{u,n} = \frac{u_i^n\left(1 - \frac{\Delta t}{\Delta x}\right) + u_{i+1}^n \frac{\Delta t}{\Delta x} + v_i^n(1 + \frac{\Delta t}{\Delta x}) - v_{i-1}^n \frac{\Delta t}{\Delta x}}{\Delta t \left(\tilde{S}_{+,i}^{u,n} + \tilde{S}_{i}^{v,n} - 2\frac{v_i^n - v_{i-1}^n}{\Delta x}\right)}.$$

Otherwise, $\beta_i^{u,n} = \tilde{\beta}_i^{u,n}$ if $\frac{1+\gamma^-}{2} \le \tilde{\beta}_i^{u,n} \le 1$ where

$$\tilde{\beta}_i^{u,n} = \frac{u_i^n\left(1 - \frac{\Delta t}{\Delta x}\right) + u_{i+1}^n \frac{\Delta t}{\Delta x} + v_i^n(1 + \frac{\Delta t}{\Delta x}) - v_{i-1}^n \frac{\Delta t}{\Delta x}}{\Delta t \left(\tilde{S}_{i,+}^{u,n} + \tilde{S}_{i,-}^{v,n} - 2\frac{v_i^n - v_{i-1}^n}{\Delta x}\right)}.$$

These correspond to the exact integrals if the solution at time t^n was indeed continuous piecewise affine passing at the (u_i^n, x_i) and (v_i^n, x_i). By construction, the underlying approximation (continuous piecewise affine) is an exact continuous steady state for (1). Therefore, its discrete data (u_i^n, v_i^n) also captures a discrete steady state.

4.2 A Finite Volume approach with a localized source

Another approach is often prefered as it remains the approximate Riemann solver framework. It was based on the idea of Greenberg and Leroux [5] (see also [4, 3]) and consists in approximating the source term by a distribution of the form $S(t)\delta_{x_{i+1/2}}$ which is defined at the interfaces.

This approach is very popular for the construction of well-balanced schemes capturing steady states in a non-linear framework.

In this spirit, we aim to approximate the solution of the system

$$\partial_t u - \partial_x u - S^u(u,v)\partial_x y = 0, \tag{10a}$$
$$\partial_t v + \partial_x v - S^v(u,v)\partial_x z = 0, \tag{10b}$$
$$\partial_t y = 0 = \partial_t z, \tag{10c}$$

where the exact $a = x = b$ are also approximated. We consider Heaviside initial condition for every component. The non-conservative products are understood by regularizing

$$y^\epsilon(x,t=0) = y_L + (y_R - y_L)\left(\frac{x}{\epsilon}\mathbf{1}_{[0,\epsilon[}(x) + \mathbf{1}_{[\epsilon,+\infty[}(x)\right)(x),$$
$$z^\epsilon(x,t=0) = z_R + (z_L - z_R)\left(\frac{x}{-\epsilon}\mathbf{1}_{]-\epsilon,0]}(x) + \mathbf{1}_{]-\infty,-\epsilon]}(x)\right),$$

and taking the limit $\epsilon \to 0$. Integrating (10a) on $[0,\Delta t]\times[-\Delta x,-\epsilon]$ and (10b) on $[0,\Delta t]\times[\epsilon,\Delta x]$ and computing the limit provides

$$\frac{u^* - u_L}{\Delta t} - \frac{u(x=0^-) - u_L}{\Delta x} = 0, \qquad \frac{v^* - v_R}{\Delta t} + \frac{v_R - v(x=0^+)}{\Delta x} = 0,$$

where $u^* \approx u(x,t^{n+1})$ for $x \in [-\Delta t, 0]$ and $v^* \approx v(x,t^{n+1})$ for $x \in [0,\Delta t]$ and we need to give a sense to the flux terms $u(x=0^-)$ and $v(x=0^+)$.

Integrating (10a) over $[-\Delta x, 0]$ and taking the limit $\epsilon \to 0$ (see [5]), those fluxes satisfy

$$y_R - y_L = \begin{cases} -\frac{u(x=0^-)-u_L}{S^u(u_R,v_L)} & \text{if} \quad sign(u_L + v_L + (y_R - y_L)S^u(u_R,v_L)) = sign(u_L+v_L), \\ -\frac{u(x_L)-u_L}{S^u(u_R,v_L)} - \frac{u(x=0^-)-u(x_L)}{a+b-S^u(u_R,v_L)} & \text{otherwise}, \end{cases}$$
$$z_R - z_L = \begin{cases} \frac{v_R-v(x=0^+)}{S^v(u_R,v_L)} & \text{if} \quad sign(u_R + v_R + (z_R - z_L)S^v(u_R,v_L)) = sign(u_R+v_R), \\ \frac{v_R-v(x_R)}{S^v(u_R,v_L)} - \frac{v(x_R)-v(x=0^+)}{c+d-S^v(u_R,v_L)} & \text{otherwise}, \end{cases}$$

where x_R and x_L locate the zeros of $u+v$, i.e. such that

$$u(x_L) + v_L = u_L + v_L + (y_R - y(x_L))S^u(u_R,v_L) = 0,$$
$$u_R + v(x_R) = u_R + v_R + (z_R - z(x_x))S^v(u_R,v_L) = 0.$$

Eventually, integrating (10a) and (10b) on $[0,\Delta t]\times[x_{i-1/2}, x_{i+1/2}]$ provides the scheme

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{f^+(u_{i+1}^n, v_{i+1}^n, u_i^n, v_i^n) - f^-(u_i^n, v_i^n, u_{i-1}^n, v_{i-1}^n)}{\Delta x} = 0, \tag{11a}$$
$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + \frac{g^+(u_{i+1}^n, v_{i+1}^n, u_i^n, v_i^n) - g^-(u_i^n, v_i^n, u_{i-1}^n, v_{i-1}^n)}{\Delta x} = 0, \tag{11b}$$

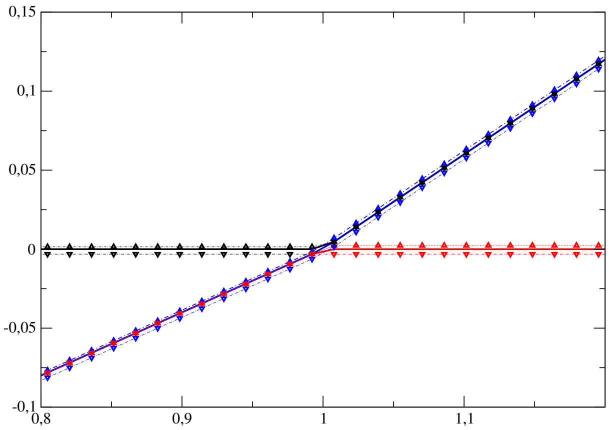

Figure 4: Exact solution (and initial condition) in straight line and discrete ones with (9) in dashed line with up triangles and with (11) with down triangles. The solution u are in black, v are in red and $u+v$ in blue.

with

$$f^-(u_R, v_R, u_L, v_L) = -u_R, \qquad g^+(u_R, v_R, u_L, v_R) = v_L, \tag{11c}$$

$$f^+(u_R, v_R, u_L, v_L) = -u_R - \begin{cases} S^u(u_R+v_L)\Delta x \\ \quad \text{if} \quad sign(u_R+v_R+\Delta x S^u(u_R+v_L)) = sign(u_R+v_R), \\ \Delta x \left[-\frac{u_R-u_L}{\Delta x} + (a+b-S^u(u_R+v_L))\right] \\ \quad +\frac{a+b}{S^u(u_R+v_L)}u_L + \left(\frac{a+b}{S^u(u_R+v_L)} - 1\right)v_L \quad \text{otherwise.} \end{cases} \tag{11d}$$

$$g^-(u_R, v_R, u_L, v_L) = v_L + \begin{cases} S^v(u_R, v_L)\Delta x \\ \quad \text{if} \quad sign(u_L+v_L+\Delta x S^v(u_R+v_L)) = sign(u_L+v_L), \\ \Delta x \left[\frac{v_R-v_L}{\Delta x} + (c+d-S^v(u_R+v_L))\right] \\ \quad +\left(\frac{c+d}{S^u(u_R+v_L)} - 1\right)u_R + \frac{c+d}{S^v(u_R+v_L)}v_R \quad \text{otherwise.} \end{cases} \tag{11e}$$

As for the last scheme, this one is obtained from the exact solution of a continuous solution which a priori possesses a steady state. Therefore, it may possess a discrete steady state.

4.3 Numerical tests

These last two schemes are tested with the numerical parameters of Subsection 2.1 and the values are plotted on Fig. 4. As expected, these two schemes capture a steady state that is consistent with the exact one.

5 CONCLUDING REMARKS AND COMMENTS

In this short paper, some discretizations of discontinuous source terms in a hyperbolic balance law were proposed. The existence or not of discrete steady states with each of them was studied and illustrated on a test case.

If not all the schemes presented capture steady states, their use is not necessarily prohibited. Indeed, we have observed numerically that some of those still had a good behavior even if they were not well-balanced (the flux difference splitting from [2]). Furthermore, we want to highlight that the considered numerical test was specifically design to trigger oscillations. All the schemes, even with a centered source term, have a good behavior in most numerical experiments performed by the author. Especially, no steady configurations satisfying the criteria from [9] were found to trigger such numerical artifacts. The stability of the equilibrium in such cases is probably stronger than the one described in Subsection 2.1.

Perspectives of this work include the extension and application of this approach to the full non-linear model (2) for boiling flows and a complete stability analysis of the equilibria in this case and at the discrete level. Study of these steady states in a non-linear framework were also provided in [8]

REFERENCES

[1] A. Bergeron and I. Toumi. Assessment of the FLICA-IV code on rod bundle experiments. *Proceedings of ICONE-6, San Diego, California, USA*, 1998.

[2] A. Bermudez and M. E. Vazquez. Upwind methods for hyperbolic conservation laws with source terms. *Comput. & Fluids*, 23(8):1049–1071, 1994.

[3] L. Gosse. *Computing Qualitatively Correct Approximations of Balance Laws*, volume 2. Springer-Verlag, 2013.

[4] L. Gosse and A.-Y. Leroux. Un schéma-équilibre adapté aux lois de conservation scalaires non-homogènes. *C. R. Acad. Sci. Paris*, 323(5):543–546, 1996.

[5] J. M. Greenberg and A. Y. Leroux. A well-balanced scheme for the numerical processing of source terms in hyperbolic equations. *SIAM J. Numer. Anal.*, 33(1):1–16, 1994.

[6] A. Harten, P. Lax, and B. Van Leer. On upstream differencing and gudonov-type schemes for hyperbolic conservation laws. *SIAM Rev.*, 25(1):35–61, 1983.

[7] M. Ishii and T. Hibiki. *Thermo-fluid dynamics of two-phase flows.* Springer, 2011.

[8] T. Pichard. Existence of steady two-phase flows with discontinuous boiling effects. *AIMS, Proceedings of HYP18 conference*, 2019.

[9] T. Pichard, N. Aguillon, B. Després, E. Godlewsky and M. Ndjinga. Existence and uniqueness of generalized solutions to hyperbolic systems with linear fluxes and stiff sources. *J. Hyp. Diff. Eqt.*, 18(3):653–700, 2021.

[10] I. Toumi, A. Bergeron, D. Gallo, E. Royer, and D. Caruge. FLICA-4: a three-dimensional two-phase flow computer code with advanced numerical methods for nuclear applications. *Nuclear Engineering and Design*, 200, 2000.