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CONTROLLING NONLINEAR ELASTIC SYSTEMS IN STRUCTURAL DYNAMICS

Timo Ströhle and Peter Betsch

Institute of Mechanics, Karlsruhe Institute of Technology
Otto-Ammann-Platz 9, 76131 Karlsruhe, Germany
{timo.stroehle, peter.betsch}@kit.edu, http://www.ifm.kit.edu/

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Abstract. This contribution deals with the feedforward control of continuous mechanical systems. After introducing a general formulation of such problems and adressing the limitations of the commonly used semi-discrete method, two numerical methods are presented that resolve these limitations.

1 INTRODUCTION

The main task of inverse dynamics is the determination of forces acting on mechanical systems, such that a desired motion of particular points of the system considered is achieved. For spatial continuous systems this often leads to the following quasilinear partial differential equation

$$A(r,s,t)\partial_t^2 r(s,t) - \partial_s(B(r,s,t)\partial_s r(s,t)) = C(r,s,t) \quad (s,t) \in \Omega \tag{1}$$

with coefficients A, B and C allowed to depend on the space and time variable s and t as well as on the solution r itself. The functions $r : S \times T \mapsto \mathbb{R}^d$ and $f : T \mapsto \mathbb{R}^d$, with spatial dimension $d \in \{1,2,3\}$, have to satisfy the initial conditions

$$r(s,0) = r_0(s),\ \partial_t r(s,0) = v_0(s) \quad s \in S = [0,1]$$

and the boundary conditions

$$B\partial_s r(0,t) = f(t),\ B\partial_s r(1,t) = 0,\ r(1,t) = \gamma(t) \quad t \in T = \mathbb{R}^+$$

Usually, to solve such problems, a spatially discrete form of (1) is considered together with the algebraic constraint at hand. The main problem is, that the index of the resulting differential algebraic equations can be quite large hindering their numerical solution.

In our contribution we consider an alternative approach to the inverse dynamics of flexible mechanical systems. In contrast to a sequential discretization in space and time we apply a simultaneous space-time discretization of the problem at hand.

After having addressed the problems of spatial discretization in Section 2, the method of characteristics and a space-time finite element formulation for inverse dynamics will be introduced in Section 3 and Section 4 respectively. In Section 5 the inverse dynamics of a nonlinear elastic rope will be investigated.

2 SEMI-DISCRETIZATION

The partial differential equation in (1) can be transformed into an ordinary differential equation through a approximation of the solution in space by applying e.g. finite elements. Therefore, equation (1) is multiplied with a sufficiently smooth test function and integrated over the spatial domain S:

$$\int_S w \cdot \partial_s(B\partial_s r)\,ds + \int_S w \cdot C\,ds = \int_S w \cdot A\partial_t^2 r\,ds$$

Integrating the first integral by parts

$$\int_S w \cdot \partial_s(B\partial_s r)\,ds = [w \cdot (B\partial_s r)]_S - \int_S \partial_s w B\partial_s r\,ds$$

leads together with the boundary conditions

$$B\partial_s r(0,t) = f(t), \qquad B\partial_s r(1,t) = 0$$

to a weak formulation of the problem at hand:

$$\int_S w \cdot A\partial_t^2 r\,ds + \int_S \partial_s w B\partial_s r\,ds = \int_S w \cdot C\,ds + [w \cdot (B\partial_s r)]_S$$

Approximation of the test and trial functions with piecewise continuous Lagrangian polynomials of order p:

$$r(s) \approx \sum_{j=1}^{p+1} L_j(s) r_j\,; \qquad w(s) \approx \sum_{i=1}^{p+1} L_i(s) w_i$$

yields together with the positive definite matrices

$$M = \int_S L_i(s) A^h L_j(s)\,ds \quad \text{and} \quad K = \int_S \partial_s L_i(s) B^h \partial_s L_j(s)\,ds$$

and the right-handside

$$F = \int_S L_i(s) C^h\,ds - L_i(s)|_{s=0} f(t)$$

a system of ordinary differential equations of second order:

$$M\partial_t^2 r + Kr = F \tag{2}$$

Introducing the servo constraint

$$r(1,t) - \gamma(t) = 0$$

the control problem can be described with the developed system of differential algebraic equations. Regarding the differentiation index of the DAE at hand, in the following a linear scalar problem is considered. Thus the coefficients in (1) are only dependent on the space and time variables s and t and $r \in \mathbb{R}$. Additionally the matrix M shall be lumped in the following. Choosing one finite element in space the following DAE can be investigated exemplarily:

$$\frac{1}{2}A\partial_t^2 r_1 + B(r_1 - r_2) = f(t) \quad , \quad \frac{1}{2}A\partial_t^2 r_2 - B(r_1 - r_2) = 0 \quad , \quad r_2 - \gamma(t) = 0 \tag{3}$$

Differentiating the algebraic equation in (3) twice and inserting it into the two differential equations yields:

$$\frac{1}{2}A\partial_t^2 r_1 + B(r_1 - \gamma(t)) = f(t) \quad , \quad \frac{1}{2}A\partial_t^2 \gamma(t) - B(r_1 - \gamma(t)) = 0 \tag{4}$$

Differentiating the algebraic equation in (4) twice again and inserting it into the remaining differential equation yields a totally algebraic equation for the unknown function $f(t)$:

$$f(t) = \frac{A^2}{4B}\partial_t^4 \gamma(t) + A\partial_t^2 \gamma(t) \tag{5}$$

The actuating force $f(t)$ can be completely formulated by the given function $\gamma(t)$ and its derivatives, or in other words, the underlying differential equation can be transformed into an algebraic equation without integrating the differential equation. Such systems are called differentially flat systems with a flat output $\gamma(t)$ ([2, 10, 21, 7]). This property of differential equations can be traced back to [14]. Differentiating equation (5) with respect to time again an ordinary differential equation for the unknown function $f(t)$ is developed:

$$\partial_t f(t) = \frac{A^2}{4B}\partial_t^5 \gamma(t) + A\partial_t^3 \gamma(t)$$

Five differentiations are therefore needed to transform the DAE into an ODE. This means the underlying DAE has a differentiation index of five ([17]).

Unfortunately, a spatial discretization by n elements leads to a differentiation index of $2n+3$ and the servo-constraint has to be C^{2n+1} continuous which is obviously not feasible for a sufficiently reliable spatial discretization ([19, 4]).

3 METHOD OF CHARACTERISTICS

In this section the method of characteristics is applied for the control problem at hand. The method of characteristics is based on a geometric interpretation of first order quasilinear partial differential equations ([1, 9, 20, 16]). For this, the problem at hand (1) can be transformed by introducing the functions $q(s,t) = \partial_t r(s,t)$ and $p(s,t) = \partial_s r(s,t)$:

$$A\partial_t q - \partial_s(Bp) = C \tag{6}$$

$$B\partial_t p - B\partial_s q = 0 \tag{7}$$

With $B\partial_t(p) = \partial_t(Bp) - \partial_t Bp$ equation (7) can be written as

$$\partial_t(Bp) - B\partial_s q = \partial_t Bp \tag{8}$$

Together with $\partial_t B(p(s,t)) = \partial_p B \partial_t p$ and $\partial_t p = \partial_s q$ it follows then:

$$\partial_t(Bp) - B\partial_s q = \partial_p B p \partial_s q \tag{9}$$

The two equations (7) and (9) are forming with $H = BI + \partial_p B p$ a system of first order partial differential equations:

$$\begin{bmatrix} AI & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} q \\ Bp \end{bmatrix}_{,t} - \begin{bmatrix} 0 & I \\ H & 0 \end{bmatrix} \begin{bmatrix} q \\ Bp \end{bmatrix}_{,s} = \begin{bmatrix} C \\ 0 \end{bmatrix} \tag{10}$$

Introducing the column vectors $z \in \mathbb{R}^{2d}$ and $F \in \mathbb{R}^{2d}$ and the square matrices $D \in \mathbb{R}^{2d\times 2d}$ and $E \in \mathbb{R}^{2d\times 2d}$, equation (10) can be written more compactly:

$$D\partial_t z + E\partial_s z = F \tag{11}$$

Let $s = k(t)$ be an initial line along which the solution $z = z(k(t), t) = z_0(t)$ is given. This line is called a characteristic line if the derivatives of the solution z cannot be determined uniquely through the differential equation with given information on the line. This means, the following system of equations

$$\begin{aligned} D\partial_t z + E\partial_s z &= F \\ \partial_t z + \partial_s z \frac{d}{dt}k(t) &= \frac{d}{dt}z_0(t) \end{aligned}$$

or more compactly

$$\left(E - D\frac{d}{dt}k(t)\right)\partial_s z = F - D\frac{d}{dt}z_0(t) \tag{12}$$

cannot be solved uniquely for $\partial_s z$ and $\partial_t z$ if the determinant of the coefficient matrix

$$E - D\frac{d}{dt}k(t) \tag{13}$$

as well as the determinant of the matrices in which in each case one column has been exchanged with the right-hand side $F - D\frac{d}{dt}z_0(t)$ vanishes according to Cramers rule. For the problem at hand, there are $i \in \{1, 2\}$ families of characteristic lines with the direction in the space-time domain of wave propagation

$$\left(\frac{ds}{dt}\right)_{ij} = (-1)^{j+1} c_i(s, t, p) \tag{14}$$

Each family of lines have $j \in \{1, 2\}$ members, where $j = 1$ yields the forward propagating directions and $j = 2$ the backward propagating directions with respect to s. Along these characteristic lines $s = k(t)$, the solution z solves the following ordinary differential equation along these very same lines: Along these characteristic lines $s = k(t)$, the solution x solves together with

$$n - B(p)p = 0$$

the following ordinary differential equation along these very same lines.

$$U_i \cdot \left(\frac{dn}{dt}\right)_i - (-1)^{j+1} c_i V_i \cdot \left(\frac{dq}{dt}\right)_i - (-1)^{j+1} c_i W_i = 0 \tag{15}$$

Where $U_i \in \mathbb{R}^d, V_i \in \mathbb{R}^d$ and $W_i \in \mathbb{R}$ in general are functions of s, t and the solution z itself.

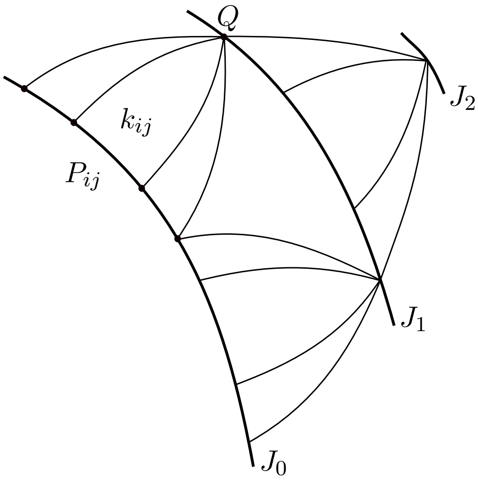

Figure 1: Characteristic Net

The implemented numerical procedure can be described as follows: Suppose the curve J_0 is an initial curve, on which the solution z is fully known. Then, the solution in point Q on the curve J_1 depends on the solution in the points P_{ij} and can be computed by solving the system of ordinary differential equations (14) and (15) along the characteristic lines k_{ij}, see Figure (1). To this end we have to suppose that the curves J_n are nowhere characteristic. Once the solution at the point Q on J_1 is computed, the solution between the major characteristics ($i = 1$) can be interpolated ([9]).

Using finite differences for solving these equations, we get the following system of algebraic equations

$$\left(\frac{s\big|_Q - s\big|_{P_{ij}}}{t\big|_Q - t\big|_{P_{ij}}}\right) + (-1)^j c_i\big|_{P_{ij}} = 0$$

$$U_i\big|_{P_{ij}} \left(\frac{n\big|_Q - n\big|_{P_{ij}}}{t\big|_Q - t\big|_{P_{ij}}}\right) + (-1)^j (c_i V_i)\big|_{P_{ij}} \left(\frac{q\big|_Q - q\big|_{P_{ij}}}{t\big|_Q - t\big|_{P_{ij}}}\right) + (-1)^j (c_i W_i)\big|_{P_{ij}} = 0$$

The boundary and initial conditions specified in (1) can then be applied directly at the nodes of the characteristic net. This graphical-numerical approach is often called Massaus method ([18]).

Remark (Riemann Invariants). *A function $z \in \mathbb{R}^{2d} \mapsto h(z) \in \mathbb{R}^d$ is a preserved quantity along characteristic lines if*

$$\partial_\xi h(z(\xi)) = 0 \tag{16}$$

Therein the coordinate ξ along characteristic lines is introduced. Together with equation (11) and setting therein for simplicity $F = 0$, the following eigenvalue problem can be established:

$$(D^{-1}E)^T \partial_z h = c_i \partial_z h$$

Finding the eigenvalues c_i and the corresponding eigenfunctions, the preserved function $h(z)$ can be found by integrating the eigenfunctions. The function $h(z)$ is called Riemann invariant of the wave equation at hand (cf. [8] for more details).

4 SPACE-TIME FINITE ELEMENT METHOD

In this section a space-time finite element formulation (cf. [6, 13, 12]) is presented which can be applied to the control problem introduced in Section 1. Introducing the function $q(s,t) = \partial_t r(s,t)$ in (1) yields the following system of equations:

$$\begin{aligned} \partial_t r - q &= 0 \\ A\partial_t q - \partial_s(B\partial_s r) &= C \end{aligned} \tag{17}$$

Multiplying each equation in (17) with sufficiently smooth test functions $w_1(s,t)$ and $w_2(s,t)$, respectively, and integrating over the space-time domain $\Omega = S \times T$ yields the following weak formulation:

$$\int_\Omega w_1 \cdot (\partial_t r - q)\, d\Omega = 0 \tag{18}$$

$$\int_\Omega w_2 \cdot (A\partial_t q - \partial_s (B\partial_s r))\, d\Omega = \int_\Omega w_2 \cdot C\, d\Omega \tag{19}$$

Integrating the second term in (19) on the left side by parts

$$\int_\Omega w_2 \cdot \partial_s(B\partial_s r)\, d\Omega = \int_T [w_2 \cdot B\partial_s r]_{s=0}^1\, dt - \int_\Omega \partial_s w_2 \cdot B\partial_s r\, d\Omega \tag{20}$$

we get the following equation for (19):

$$\int_\Omega w_2 \cdot A\partial_t q\, d\Omega - \int_T [w_2 \cdot B\partial_s r]_{s=0}^1\, \mathrm{d}t + \int_\Omega \partial_s w_2 \cdot B\partial_s r\, d\Omega = \int_\Omega w_2 \cdot C\, d\Omega \tag{21}$$

Additionally the servo-constraint $r(s=1,t) = \gamma(t)$ which has to be satisfied for all $t \in T$ can be enforced weakly

$$\int_{\partial\Omega_{t_0}} w_3(t) \cdot (r(1,t) - \gamma(t))\, dt = 0 \tag{22}$$

The task is now to find the unknown functions

$$\begin{aligned} r(s,t) \in V_1 &= \left\{ r \in H^1(\Omega) : r(\partial\Omega_{t_0}) = r_0 \right\} \\ q(s,t) \in V_2 &= \left\{ q \in H^1(\Omega) : q(\partial\Omega_{t_0}) = q_0 \right\} \\ f(t) \in V_3 &= \left\{ f \in H^1(\partial\Omega_{s_0}) : f(t=0) = f_0 \right\} \end{aligned}$$

such that for arbitrary but sufficiently smooth test functions

$$w_1(s,t), w_2(s,t) \in W_1 = \{w_1, w_2 \in H^1(\Omega) : w_1(s,t=0) = 0, w_2(s,t=0) = 0\}$$
$$w_3(t) \in W_2 = \{w_3 \in H^1(T) : w_3(t=0) = 0\}$$

the equations (18), (21) and (22) are together with the boundary

$$B\partial_s r(s=0,t) = f(t) \quad \text{and} \quad B\partial_s r(s=1,t) = h(\partial_t^2 r|_{1,t}, t) \quad t \in T$$

and initial conditions

$$r(s,0) = r_0(s) \quad \text{and} \quad \partial_t r(s,0) = q_0(s) \quad s \in S$$

satisfied. These equations represent a weak form of the control problem. With the isoparametric map $\Phi(\hat{x})$ every event in a space-time finite element $x \in \Omega^e$ can be transformed into an event in the reference element $\hat{x} = \begin{bmatrix}\hat{s} & \hat{t}\end{bmatrix}^T \in \hat{\Omega}$. Together with the Jacobian of the transformation $J = \partial_{\hat{x}} x$ it follows for the contribution of one representative space-time element to the weak form (18), (21) and (22)

$$\int_{\hat{\Omega}} w_1(\partial_{\hat{x}} r \cdot \partial_t \hat{x} - q)\,|J|\; d\hat{\Omega}$$
$$\int_{\hat{\Omega}} w_2(\hat{A}\partial_{\hat{x}} q \cdot \partial_t \hat{x})\,|J|\; d\hat{\Omega} + \int_{\hat{\Omega}} \partial_{\hat{x}} w_2 \partial_s \hat{x} \hat{B} \partial_{\hat{x}} r \partial_s \hat{x}\,|J|\; d\hat{\Omega} + \int_{\partial\hat{\Omega}_{s_0}} w_2 f(t) \frac{h_e}{2} d\hat{t} - \int_{\hat{\Omega}} w_2 \hat{C}\,|J|\; d\hat{\Omega}$$
$$\int_{\partial\hat{\Omega}_{s_1}} w_3(r(1,\hat{t}) - \gamma(\hat{t})) \frac{h_e}{2}\, d\hat{t}$$

Here, h_e is the time element length at $s = 1$. Subsequently, the test and solution functions can be approximated with piecewise continuous Lagrangian interpolation polynomials of $k-$th order.

$$Q_{ij}(\hat{x}) = L_i(\hat{s}) L_j(\hat{t})$$

with:

$$L_i(\cdot) = \prod_{\substack{j=1 \\ j \neq i}}^{k+1} \frac{(\cdot) - (\cdot)_j}{(\cdot)_i - (\cdot)_j}$$

Collecting the tensor product shape functions Q_{ij} in a column vector N and introducing

$$M^t = (\partial_{\hat{x}} N \partial_t \hat{x})\,; \qquad M^s = (\partial_{\hat{x}} N \partial_s \hat{x})$$

the residual contributions to the algebraic system after discretization to solve, can be summarized, using isoparametric mapping

$$x = \Phi(\hat{x}) = \sum_{i=1} N_i(\hat{x}) x_i$$

as follows:

$$w_1^i \int_{\hat{\Omega}} N_i M_j^t \, |J| \; d\hat{\Omega} \, r_j - w_1^i \int_{\hat{\Omega}} N_i N_j \, |J| \; d\hat{\Omega} \, q_j$$

$$w_2^i \int_{\hat{\Omega}} N_i \hat{A}^h M_j^t \, |J| \; d\hat{\Omega} \, q_j + w_2^i \int_{\hat{\Omega}} M_i^s \hat{B}^h M_j^s \, |J| \; d\hat{\Omega} \, r_j + w_2^i \int_{\partial\hat{\Omega}_{s_0}} N_i L_j \frac{h^e}{2} \, d\hat{t} \, f_j - w_2^i \int_{\hat{\Omega}} N_i \hat{C}^h \, |J| \; d\hat{\Omega}$$

$$w_3^i \int_{\partial\hat{\Omega}_{s_1}} L_i N_j \frac{h^e}{2} \, d\hat{t} \, r_j - w_3^i \int_{\partial\hat{\Omega}_{s_1}} L_i \hat{\gamma} \frac{h^e}{2} \, d\hat{t}$$

5 NUMERICAL EXAMPLES

In this section, a linear model of a rod and a nonlinear model of a rope for large elastic deformations are considered.

5.1 Linearly Elastic Bar

A bar with length L, Young's modulus E, mass density ρ and cross-sectional area A_0 fits in the introduced control problem for the coefficients

$$A = \rho A_0, \qquad B = EA_0, \qquad C = 0.$$

In the following a force $f(t)$ at $s = 0$ is searched such that the free end at $s = L$ follows a prescribed trajectory $\gamma(t)$.

$$\gamma(t) = \begin{cases} 0 & t < 1 \\ \frac{1}{2}\sin\left(\frac{\pi}{2}t - \pi\right) + \frac{1}{2} & 1 \leq t \leq 3 \\ 1 & t > 3 \end{cases} \tag{23}$$

For this linear system both presented methods can be applied and due to linearity verified through an analytical solution (see Fig. 2).

Remark (Analytical solution)**.** *An analytical solution to the linear problem at hand with a constant wave propagation* $c^2 = A^{-1}B$ *can be sketched as follows (see [11] for more details). Starting with d'Alemberts solution*

$$r(s,t) = \Phi(t + c^{-1}s) + \Psi(t - c^{-1}s)$$

the prescribed trajectory

$$r(1,t) = \Phi(t + c^{-1}) + \Psi(t - c^{-1}) = \gamma(t)$$

can be differentiated and inserted into the remaining boundary conditions

$$\partial_s r(1,t) = \left(\partial_t \Phi(t+c) - \partial_t \Psi(t-c)\right) c^{-1} = 0$$
$$\partial_s r(0,t) = \left(\partial_t \Phi(t) - \partial_t \Psi(t)\right) c^{-1} = f(t).$$

Eliminating $\partial_t \Phi$ *afterwards yields for the actuating force:*

$$f(t) = (2c)^{-1} \left(\partial_t \gamma(t - c^{-1}) - \partial_t \gamma(t + c^{-1})\right)$$

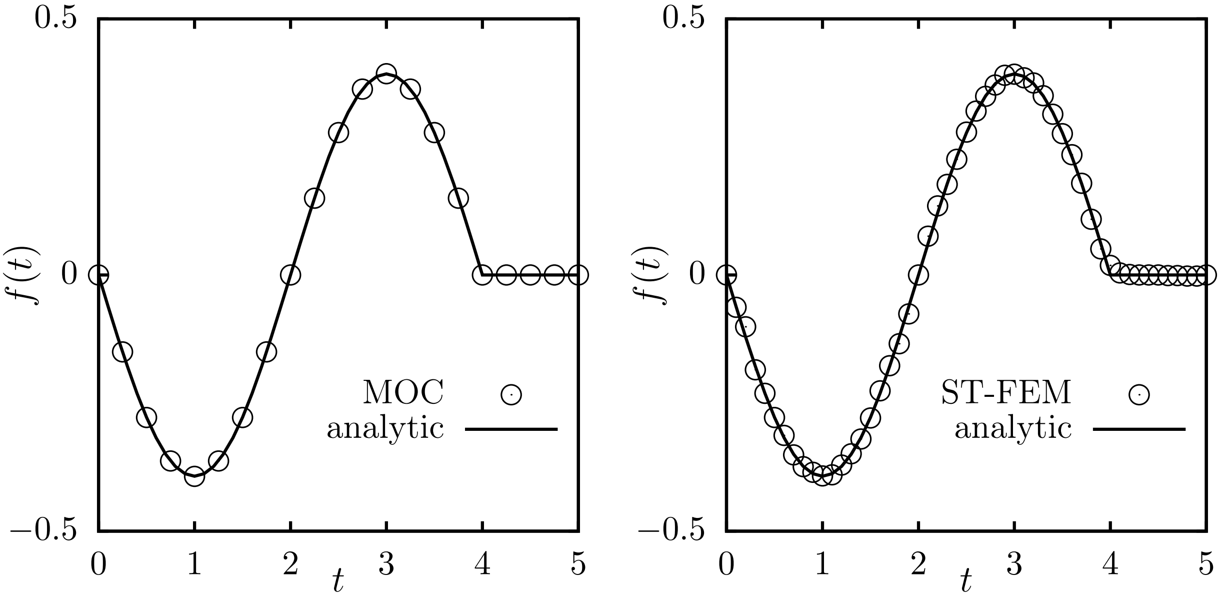

Figure 2: Numerical solution for the actuating force $f(t)$ computed using the method of characteristics (left) and the space-time finite element method (right), compared with the analytical reference solution (solid line)

5.2 Nonlinearly elastic rope

As a second example, an elastic rope undergoing large deformations is considered. The corresponding equations of motion have been taken from [5]. The rope has a mass density ρ, Young's modulus E and in a stress-free reference configuration a length L and cross-sectional area A_0. Introducing a normalized arc length $s \in [0,1]$ in the reference configuration, a spatial configuration is defined through the function $\Omega \mapsto r(s,t) \in \mathbb{R}^d$. Together with the force in the extensible rope $\Omega \mapsto n \in \mathbb{R}^d$ and the body force per unit reference length $\Omega \mapsto b \in \mathbb{R}^d$ the motion of the elastic rope is governed by

$$\partial_s n(s,t) + b(s,t) = \rho A \partial_t^2 r \tag{24}$$

Introducing the stretch $\nu(s,t) = \|\partial_s r(s,t)\|$ and assuming the following constitutive relation

$$n(s,t) = N(s,t)\nu^{-1}\partial_s r \qquad \text{with} \qquad N(s,t) = \frac{EA}{2}(\nu - \frac{1}{\nu}) \tag{25}$$

the inverse dynamics of the considered rope fits into the framework introduced in Section 1 with coefficients

$$A = \rho A_0, \qquad B = N(s,t)\nu^{-1}, \qquad C = b(s,t).$$

Then, following Section 3 with

$$H = BI + \frac{EA}{\nu^4}\partial_s r \otimes \partial_s r$$

the wave propagation speed can be computed

$$c_i = \left(\frac{ds}{dt}\right) = \left(\frac{E}{2\rho}\left(1 + (-1)^i \frac{1}{\nu^2}\right)\right)^{\frac{1}{2}}$$

The $i \in \{1, 2, 3\}$ families of characteristic lines can then be linked to wave propagation in longitudinal ($i = 2$) and transversal ($i = 1, 3$) directions of the rope.

The functions $U_i \in \mathbb{R}^d, V_i \in \mathbb{R}^d$ and $W_i \in \mathbb{R}$ from (15) are in the case of the nonlinear elastic rope for $i \in [1, .., d]$

$$U_i = V_i = p_i \cdot p^T, \qquad W_i = \|p\|$$

Remark (Additional mass at the free end)**.** *Instead of a loose end a mass m can be attached by adding the following ordinary differential equation at $s = 1$:*

$$B\partial_s r(1, t) = m\left(\partial_t q + g\right)$$

In Fig. 3 the actuating forces for a planar rest-to rest maneuver, $\gamma(t) = \begin{bmatrix}\gamma_x(t) & \gamma_y(t)\end{bmatrix}$ with

$$\gamma_x(t) = \gamma_y(t) = \begin{cases} 0 & t < 2 \\ \frac{1}{2}\sin\left(\frac{\pi}{2}(t-3)\right) + \frac{1}{2} & 2 \leq t \leq 4 \\ 1 & t > 4 \end{cases} \tag{26}$$

computed with both the space-time finite element method and the method of characteristics are shown. For the computation, bi-linear Lagrangian shape functions in space-time were applied.

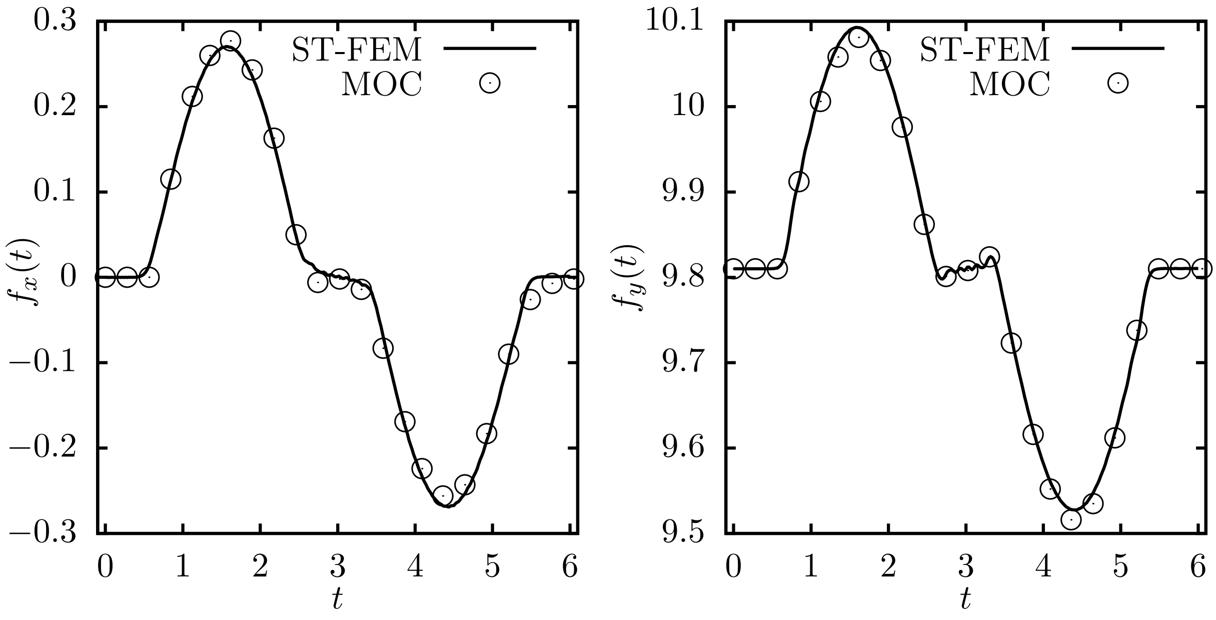

Figure 3: Numerical solution for the components $f_x(t)$ (left) and $f_y(t)$ (right) of the actuating force computed using the method of characteristics (circles) and the space-time finite element method (solid line)

The simulation results of the feedforward control problem at hand are illustrated in Fig. 4. It is worth noting that thus obtained results for the control force $f(t)$ have been checked by using $f(t)$ as prescribed external force in a standard forward simulation based on a semi-discrete model of the rope.

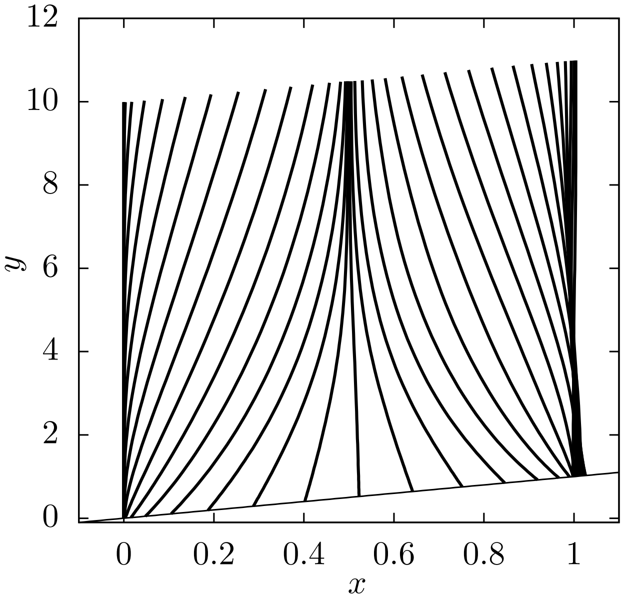

Figure 4: Snapshots of the moving elastic string actuated on the upper end by the computed control force such that the lower end follows the prescribed straight line from the starting point (0,0) to the endpoint (1,1)

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