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ABSTRACT

This study explores the existence, uniqueness (EU), and Hyers-Ulam stability (HUS) for a class of weighted fractional Itô-Doob stochastic integral equations (WFIDSIEs). To establish these crucial properties, we utilize the Banach fixed-point theorem (BFPT) as a foundational tool, alongside a variety of essential mathematical inequalities. These inequalities offer valuable insights into the underlying structure and behavior of WFIDSIEs, particularly in the context of fractional and stochastic dynamics. By leveraging these techniques, we demonstrate the conditions under which solutions to these equations exist and are unique, as well as how small perturbations affect the stability of these solutions in the Hyers-Ulam sense. Our results contribute to a deeper understanding of the stability and solvability of WFIDSIEs, which are important in modeling complex systems exhibiting both stochasticity and memory effects. Additionally, these findings open avenues for further research into the robustness of solutions in fractional stochastic models across various applied disciplines, such as finance, engineering, and biology.

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1 Introduction

Fractional calculus is an advanced area of mathematical analysis that broadens the principles of differentiation and integration to include non-integer (fractional) orders. Unlike classical calculus, which focuses on integer-order derivatives and integrals, fractional calculus enables calculations of arbitrary order, offering a more flexible framework for mathematical modeling [1–4]. This discipline has numerous applications across fields such as physics, engineering, control theory, and signal processing, particularly in systems that exhibit memory effects and hereditary behavior. By utilizing fractional derivatives, researchers can more accurately model complex phenomena like anomalous diffusion, viscoelasticity, and fractal processes, enhancing the precision and effectiveness of scientific and engineering solutions.

Fractional derivatives and integrals extend the traditional concepts of differentiation and integration to non-integer orders, allowing for a more generalized and flexible approach to mathematical

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analysis. Unlike standard derivatives, which measure instantaneous rates of change, fractional derivatives account for historical behavior, making them useful for modeling systems with memory effects. Similarly, fractional integrals provide a way to smooth functions over a broader range, offering deeper insights into complex processes. These concepts have applications in various fields, including physics, engineering, and finance, where they help describe anomalous diffusion, viscoelastic materials, and other phenomena that cannot be accurately captured by classical calculus (see [5–7]).

Weighted fractional integrals and derivatives [8–10] are a generalized form of fractional integrals and derivatives that incorporate weighted functions to emphasize specific aspects of a function's behavior. These derivatives provide additional flexibility by allowing variable influence over different regions of a function, making them particularly useful for modeling systems with spatial or temporal heterogeneity. They are widely applied in physics, engineering, and signal processing, where they help describe complex dynamics such as anomalous diffusion, viscoelasticity, and nonlocal processes with varying intensities.

In recent years, the idea of Hyers-Ulam stability (HUS) has been extended to various advanced mathematical contexts, including fractional differential equations (FDEs) and stochastic fractional differential equations (SFDEs). The application of Hyers-Ulam stability to these equations involves investigating whether approximate solutions to FDEs can be adjusted to yield exact solutions under small perturbations, which is crucial for both theoretical and practical purposes in modeling real-world systems. Furthermore, the study of HUS in the context of SFDEs is relatively new but rapidly evolving [11–13]. It aims to determine the stability of solutions to these equations when small random perturbations are introduced, offering insights into the robustness of systems under uncertain conditions. Researchers are increasingly exploring the HUS in both deterministic and stochastic fractional models to ensure reliable approximations and to improve the understanding of complex dynamic systems in the presence of fractional and stochastic influences.

Despite the individual progress in weighted fractional operators and the stability of SFDEs, a significant gap remains at their intersection. To the best of our knowledge, there is no existing research on the HUS for WFIDSIEs. This class of equations combines three critical aspects: stochastic noise (via the Itô-Doob integral), long-range memory (via fractional calculus), and variable memory intensity (via the weight functions). Establishing the fundamental properties of existence, uniqueness, and stability for such equations is therefore a necessary and non-trivial step towards their reliable application in modeling complex real-world systems.

This work is designed to bridge the aforementioned gap. Our main contributions are succinctly summarized as follows:

- We establish the first EU results for solutions to a broad class of WFIDSIEs. This is achieved by formulating the problem within a suitable Banach space of stochastic processes and rigorously applying the BFPT.
- We prove that the solutions to WFIDSIEs are Hyers-Ulam stable. This is demonstrated by leveraging the Gronwall inequality to show that small perturbations in the equation lead to only proportionally small deviations in the solutions, thereby guaranteeing the robustness of the model.
- Our results serve as a non-trivial generalization of the work in [13] by incorporating general weighted functions.

The organization of this paper is as follows: Section 2 introduces the necessary preliminaries, including key notations and fundamental concepts. In Section 3, we investigate the EU and HUS of solutions to the WFIDSIE, utilizing the BFPT and the Gronwall inequality as central tools. This section provides a detailed analysis of how these methods can be applied to establish the desired properties of the solutions. Finally, Section 4 presents three illustrative examples that highlight the utility of our results.

2 Preliminaries and Equation Description

Set $\mathrm{S} > 0$ and $\mathcal{V} = \{\mathbb{Y}, \widetilde{\mathfrak{X}}, \widetilde{\mathbb{M}} = (\widetilde{\mathfrak{X}}_{\mathrm{t}})_{0\le \mathrm{t}\le \mathrm{S}}, \mathcal{P}\}$ be a complete probability space and $\mathbb{C}(\mathrm{t})$ be a standard Brownian motion.

For $q \ge 2$, set $\mathbb{Y}^q_{\mathrm{t}} = L^q(\mathbb{Y}, \widetilde{\mathfrak{X}}_{\mathrm{t}}, \mathcal{P})$ the space of all $\widetilde{\mathfrak{X}}_{\mathrm{t}}$-measurable and q-th integrable functions $f : \mathbb{Y} \to \mathbb{R}$ with:

$$\|f\|_q = \left(\mathbb{E}\left(|f|^q\right)\right)^{\frac{1}{q}}.$$

Definition 1: [8] Let $\varepsilon > 0$ and Ω be a positive continuous function, the weighted fractional integral of $f \in L^1(I)$ is given by

$$I_a^{\varepsilon,\Omega} f(\rho) = \frac{\Omega^{-1}(\rho)}{\Gamma(\varepsilon)} \int_a^\rho (\rho - \ell)^{\varepsilon-1}\Omega(\ell) f(\ell)\, d\ell,$$

where $\Omega^{-1}(\rho) = \dfrac{1}{\Omega(\rho)}$.

Remark 1: In the case when $\Omega(\rho) = e^{c\rho}$ with $c > 0$, we obtain the tempered fractional integral (see [14]).

Consider the following integral equations:

$$\mathrm{X}(\mathrm{t}) = \mathrm{a} + \int_0^{\mathrm{t}} \mathrm{F}_1\big(\ell, \mathrm{X}(\ell)\big) d\ell + \sum_{i=2}^{n} \Omega_i^{-1}(\mathrm{t}) \int_0^{\mathrm{t}} (\mathrm{t} - \ell)^{\mathrm{c}_i - 1}\, \Omega_i(\ell) \mathrm{F}_i\big(\ell, \mathrm{X}(\ell)\big) d\ell + \int_0^{\mathrm{t}} g\big(\ell, \mathrm{X}(\ell)\big) d\mathbb{C}(\ell), \quad (1)$$

where $\mathrm{a} \in \mathbb{R}$, $0 < \mathrm{c}_i < 1$, $\Omega_i \in C\big([0, \mathrm{S}], (0, \infty)\big)$ for $2 \le i \le n$, $\mathrm{t} \in [0, \mathrm{S}]$ and $g, \mathrm{F}_i : [0, \mathrm{S}] \times \mathbb{R} \longrightarrow \mathbb{R}$ for $1 \le i \le n$ are measurables.

Let consider the following hypothesis:

$\mathcal{H}_1$: There is $\bar{C} > 0$ such that

$$|g(\mathrm{t}, \gamma_1) - g(\mathrm{t}, \gamma_2)| \vee |\mathrm{F}_1(\mathrm{t}, \gamma_1) - \mathrm{F}_1(\mathrm{t}, \gamma_2)| \vee \ldots \vee |\mathrm{F}_n(\mathrm{t}, \gamma_1) - \mathrm{F}_n(\mathrm{t}, \gamma_2)| \le \bar{C}|\gamma_1 - \gamma_2|, \quad (2)$$

for all $(\mathrm{t}, \gamma_1, \gamma_2) \in [0, \mathrm{S}] \times \mathbb{R} \times \mathbb{R}$.

$\mathcal{H}_2$: There is $d > 0$ such that:

$\operatorname*{ess\,sup}_{\mathrm{t}\in[0,\mathrm{S}]} |g(\mathrm{t}, 0)| \le d$, $\operatorname*{ess\,sup}_{\mathrm{t}\in[0,\mathrm{S}]} |\mathrm{F}_i(\mathrm{t}, 0)| \le d$ for $i \in \{1, 2, \ldots, n\}$.

Henceforth, we assume that $q > \max\limits_{1\le j\le n} \left\{\dfrac{1}{\mathrm{c}_j}\right\}$.

3 Main Results

We define $\mathcal{V}^q([0,S])$ as the set of all $\widetilde{\mathbb{M}}$-adapted measurable processes X such that $\sup_{t\in[0,S]} \|X(t)\|_q < \infty$. Let the norm $\|\cdot\|_{\mathcal{V}^q}$ on $\mathcal{V}^q([0,S])$ defined by:

$\|X\|_{\mathcal{V}^q([0,S])} = \sup_{t\in[0,S]} \|X(t)\|_q$.

We have $(\mathcal{V}^q([0,S]), \|\cdot\|_{\mathcal{V}^q([0,S])})$ is a Banach space.

Let $\mathcal{K}_a : \mathcal{V}^q([0,S]) \to \mathcal{V}^q([0,S])$ given by

$$(\mathcal{K}_a X)(t) = a + \int_0^t F_1\big(\ell, X(\ell)\big)d\ell + \int_0^t g\big(\ell, X(\ell)\big)d\mathbb{C}(\ell) + \sum_{i=2}^{n} \Omega_i^{-1}(t) \int_0^t (t-\ell)^{c_i-1}\,\Omega_i(\ell)F_i\big(\ell, X(\ell)\big)d\ell, \tag{3}$$

for every $t \in [0,S]$.

We note by $M_i = \max_{s\in[0,S]}\big(\Omega_i(s)\big)$ and $m_i = \min_{s\in[0,S]}\big(\Omega_i^{-1}(s)\big)$.

Lemma 1: *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are satisfied, therefore $\mathcal{K}_a$ is well defined for every* $a \in \mathbb{R}$.

Proof: Let $X \in \mathcal{V}^q([0,S])$, we get for $t \in [0,S]$,

$$||(\mathcal{K}_a X)(t)||_q^q \le (n+2)^{q-1}\Big[||a||_q^q + \|\int_0^t F_1\big(\ell, X(\ell)\big)d\ell\|_q^q + \|\int_0^t g\big(\ell, X(\ell)\big)d\mathbb{C}(\ell)\|_q^q + \sum_{i=2}^{n} \Omega_i^{-q}(t)\|\int_0^t (t-\ell)^{c_i-1}\,\Omega_i(\ell)F_i\big(\ell, X(\ell)\big)d\ell\|_q^q. \tag{4}$$

By the Hölder inequality, we obtain

$$\begin{aligned}\|\int_0^t F_1\big(\ell, X(\ell)\big)d\ell\|_q^q &\le \mathbb{E}\left(\int_0^t |F_1\big(\ell, X(\ell)\big)|d\ell\right)^q \\ &\le t^{q-1}\mathbb{E}\int_0^t |F_1\big(\ell, X(\ell)\big)|^q d\ell \\ &\le S^{q-1}\mathbb{E}\int_0^t |F_1\big(\ell, X(\ell)\big)|^q d\ell \\ &\le S^{q-1}\mathbb{E}\int_0^t |F_1\big(\ell, X(\ell)\big) - F_1\big(\ell, 0\big) + F_1\big(\ell, 0\big)|^q d\ell. \end{aligned} \tag{5}$$

Using the fact that $|a+b|^q \le 2^{q-1}\left(|a|^q + |b|^q\right)$ for all $a, b \in \mathbb{R}$, we get

$$\begin{aligned}\|\int_0^t F_1\big(\ell, X(\ell)\big)d\ell\|_q^q &\le 2^{q-1}S^{q-1}\mathbb{E}\int_0^t \Big(\bar{C}^q|X(\ell)|^q + |F_1\big(\ell, 0\big)|^q\Big)d\ell \\ &\le (2S)^{q-1}\bar{C}^q S||X||_{\mathcal{V}^q([0,S])}^q + (2S)^{q-1}\int_0^S |F_1\big(\ell, 0\big)|^q d\ell. \end{aligned} \tag{6}$$

We obtain

$$\begin{aligned}
&\Omega_i^{-q}(\mathrm{t})\|\int_0^{\mathrm{t}}(\mathrm{t}-\ell)^{\mathrm{c}_i-1}\,\Omega_i(\ell)\mathrm{F}_i\big(\ell,\mathrm{X}(\ell)\big)d\ell\|_q^q\\
&\quad\le \Omega_i^{-q}(\mathrm{t})\mathbb{E}\left(\int_0^{\mathrm{t}}(\mathrm{t}-\ell)^{\mathrm{c}_i-1}\,\Omega_i(\ell)|\mathrm{F}_i\big(\ell,\mathrm{X}(\ell)\big)|d\ell\right)^q\\
&\quad\le \frac{M_i^q}{m_i^q}\mathbb{E}\left(\left(\int_0^{\mathrm{t}}(\mathrm{t}-\ell)^{\frac{(\mathrm{c}_i-1)q}{q-1}}d\ell\right)^{q-1}\int_0^{\mathrm{t}}|\mathrm{F}_i\big(\ell,\mathrm{X}(\ell)\big)|^q d\ell\right)\\
&\quad\le \frac{M_i^q\mathrm{S}^{\mathrm{c}_iq-1}(q-1)^{q-1}}{m_i^q(\mathrm{c}_iq-1)^{q-1}}\int_0^{\mathrm{t}}\|\mathrm{F}_i\big(\ell,\mathrm{X}(\ell)\big)\|_q^q d\ell\\
&\quad\le \frac{M_i^q\mathrm{S}^{\mathrm{c}_iq-1}(2q-2)^{q-1}}{m_i^q(\mathrm{c}_iq-1)^{q-1}}\left(\bar{C}^q\mathrm{S}\|\mathrm{X}\|^q_{\mathcal{V}^q([0,\mathrm{S}])}+\int_0^{\mathrm{S}}|\mathrm{F}_i\big(\ell,0\big)|^q d\ell\right).
\end{aligned}$$

Combining Theorem 7.1 of [15] with Hölder's inequality yields

$$\begin{aligned}
\|\int_0^{\mathrm{t}}g\big(\ell,\mathrm{X}(\ell)\big)d\mathbb{C}(\ell)\|_q^q &\le M_q\mathbb{E}\left(\int_0^{\mathrm{t}}|g\big(\ell,\mathrm{X}(\ell)\big)|^2 d\ell\right)^{\frac{q}{2}}\\
&\le M_q\mathbb{E}\left(\int_0^{\mathrm{t}}|g\big(\ell,\mathrm{X}(\ell)\big)|^q d\ell\right)\left(\int_0^{\mathrm{t}}d\ell\right)^{\frac{q-2}{2}}\\
&\le M_q\mathrm{S}^{\frac{q-2}{2}}\mathbb{E}\left(\int_0^{\mathrm{t}}|g\big(\ell,\mathrm{X}(\ell)\big)|^q d\ell\right)\\
&\le M_q2^{q-1}\mathrm{S}^{\frac{q}{2}}\big(\bar{C}^q\|\mathrm{X}\|^q_{\mathcal{V}^q([0,\mathrm{S}])}+d^q\big),
\end{aligned} \tag{7}$$

where

$$M_q=\left(\frac{(q-1)q}{2}\right)^{\frac{q}{2}}\mathrm{S}^{\frac{q-2}{2}}.$$

Therefore, $||\mathcal{K}_{\mathrm{a}}\mathrm{X}(\mathrm{t})||_{\mathcal{V}^q([0,\mathrm{S}])}<\infty$. □

Theorem 1: *Under the conditions $\mathcal{H}_1$ and $\mathcal{H}_2$, the solution to (1) exists and is unique.*

Proof: For $\rho>0$, we consider $\|.\|_\rho$ the norm on $\mathcal{V}^q([0,\mathrm{S}])$ defined by

$$\|\mathrm{X}\|_\rho^q=\operatorname*{ess\,sup}_{\mathrm{t}\in[0,\mathrm{S}]}\left(\frac{\mathbb{E}|\mathrm{X}(\mathrm{t})|^q}{\exp(\rho\mathrm{t})}\right).$$

We have $\|.\|_{\mathcal{V}^q([0,\mathrm{S}])}$ and $\|.\|_\rho$ are equivalent.

We will use the BFPT to prove the existence and uniqueness of the solution of Eq. (1).

At every time $t \in [0, S]$,

$$\begin{aligned}
&\left\|\mathcal{K}_a X(t) - \mathcal{K}_a \widehat{X}(t)\right\|_q^q \\
&\leq (n+1)^{q-1}\Big[\left\|\int_0^t \left(F_1\left(\ell, X(\ell)\right) - F_1\left(\ell, \widehat{X}(\ell)\right)\right) d\ell\right\|_q^q \\
&+ \left\|\int_0^t \left(g\left(\ell, X(\ell)\right) - g\left(\ell, \widehat{X}(\ell)\right)\right) d\mathbb{C}(\ell)\right\|_q^q \\
&+ \sum_{i=2}^{n} \left\|\int_0^t \frac{(t-\ell)^{c_i-1}}{\Omega_i(t)} \Omega_i(\ell) \left(F_i\left(\ell, X(\ell)\right) - F_i\left(\ell, \widehat{X}(\ell)\right)\right) d\ell\right\|_q^q\Big].
\end{aligned} \tag{8}$$

Applying Hölder's inequality under assumption $\mathcal{H}_1$ yields

$$\begin{aligned}
\|\int_0^t \left(F_1\left(\ell, X(\ell)\right) - F_1\left(\ell, \widehat{X}(\ell)\right)\right) d\ell\|_q^q &\leq t^{q-1}\mathbb{E}\left(\int_0^t |F_1\left(\ell, X(\ell)\right) - F_1\left(\ell, \widehat{X}(\ell)\right)|^q d\ell\right) \\
&\leq \bar{C}^q S^{q-1} \int_0^t \|X(\ell) - \widehat{X}(\ell)\|_q^q d\ell,
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
&\left\|\int_0^t \frac{(t-\ell)^{c_i-1}}{\Omega_i(t)} \Omega_i(\ell) \left(F_i\left(\ell, X(\ell)\right) - F_i\left(\ell, \widehat{X}(\ell)\right)\right) d\ell\right\|_q^q \\
&\leq \frac{M_i^q}{m_i^q}\mathbb{E}\left(\left(\int_0^t (t-\ell)^{\frac{q(c_i-1)}{q-1}} d\ell\right)^{q-1} \int_0^t |F_i\left(\ell, X(\ell)\right) - F_i\left(\ell, \widehat{X}(\ell)\right)|^q d\ell\right) \\
&\leq \frac{M_i^q \bar{C}^q S^{c_i q-1}(q-1)^{q-1}}{m_i^q (c_i q-1)^{q-1}} \int_0^t \|X(\ell) - \widehat{X}(\ell)\|_q^q d\ell.
\end{aligned} \tag{10}$$

Applying Theorem 7.1 of [15] under assumption $\mathcal{H}_1$ together with Hölder's inequality yields

$$\begin{aligned}
\left\|\int_0^t \left(g\left(\ell, X(\ell)\right) - g\left(\ell, \widehat{X}(\ell)\right)\right) d\mathbb{C}(\ell)\right\|_q^q &\leq M_q \mathbb{E}\left|\int_0^t |g\left(\ell, X(\ell)\right) - g\left(\ell, \widehat{X}(\ell)\right)|^2 d\ell\right|^{\frac{q}{2}} \\
&\leq M_q \bar{C}^q S^{\frac{q-2}{2}} \int_0^t \|X(\ell) - \widehat{X}(\ell)\|_q^q d\ell.
\end{aligned} \tag{11}$$

Therefore, we get

$$\left\|\mathcal{K}_a X(t) - \mathcal{K}_a \widehat{X}(t)\right\|_q^q \leq \lambda \int_0^t \|X(\ell) - \widehat{X}(\ell)\|_q^q d\ell, \tag{12}$$

where

$$\lambda = (n+1)^{q-1}\left[\bar{C}^q S^{q-1} + \sum_{i=2}^{n} \bar{C}^q S^{c_i q-1} \frac{M_i^q}{m_i^q}\left(\frac{q-1}{c_i q-1}\right)^{q-1} + \bar{C}^q M_q S^{\frac{q-2}{2}}\right].$$

Then,

$$
\begin{aligned}
\left\|\mathcal{K}_a X(t) - \mathcal{K}_a \widehat{X}(t)\right\|_q^q &\leq \lambda \int_0^t \|X(\ell) - \widehat{X}(\ell)\|_q^q d\ell \\
&\leq \lambda \int_0^t \frac{\|X(\ell) - \widehat{X}(\ell)\|_q^q}{\exp(\rho\ell)} \exp(\rho\ell) d\ell \\
&\leq \lambda \|X - \widehat{X}\|_\rho^q \int_0^t \exp(\rho\ell) d\ell \\
&\leq \frac{\lambda \exp(\rho t)}{\rho} \|X - \widehat{X}\|_\rho^q.
\end{aligned} \tag{13}
$$

Thus,

$$
\left\|\mathcal{K}_a X - \mathcal{K}_a \widehat{X}\right\|_\rho \leq \left(\frac{\lambda}{\rho}\right)^{\frac{1}{q}} \|X - \widehat{X}\|_\rho. \tag{14}
$$

Therefore, $\mathcal{K}_a$ is contractive for some $\rho > 0$. Hence, the Eq. (1) has a unique solution. □

The concept of Ulam-Hyers stability is defined as follows:

Definition 2: [16] Let

$$
\psi(t) = y(0) + \int_0^t F_1\big(\ell, y(\ell)\big) d\ell + \sum_{i=2}^{n} \Omega_i^{-1}(t) \int_0^t (t-\ell)^{c_i - 1}\, \Omega_i(\ell) F_i\big(\ell, y(\ell)\big) d\ell
$$
$$
+ \int_0^t g\big(\ell, y(\ell)\big) d\mathbb{C}(\ell), \ t \in [0, S],
$$

with $0 < c_i < 1$ for $2 \leq i \leq n$, $y(0) = a \in \mathbb{R}$ and the functions $F_i, g : [0, S] \times \mathbb{R} \longrightarrow \mathbb{R}$, $1 \leq i \leq n$, are measurables.

Eq. (1) is Ulam-Hyers stable if there is $\Delta > 0$ such that, for every $\epsilon > 0$ and for every $y \in \mathcal{V}^q([0, S])$, with

$$
\|y(t) - \psi(t)\|_q^q \leq \epsilon, \ \forall\, t \in [0, S], \tag{15}
$$

there is a solution $\varphi \in \mathcal{V}^q([0, S])$ of (1), with $\varphi(0) = y(0)$, and

$$
\|y(t) - \varphi(t)\|_q^q \leq \Delta\epsilon, \ \forall\, t \in [0, S].
$$

Theorem 2: *Under $\mathcal{H}_1$ and $\mathcal{H}_2$, the Eq. (1) is UHS.*

Proof: We will use the Gronwall inequality to prove the UHS of Eq. (1).

Set $\epsilon > 0$ and $y \in \mathcal{V}^q([0, S])$ satisfies (15). Let $\varphi \in \mathcal{V}^q([0, S])$ the solution of (1) with $\varphi(0) = y(0)$, then

$$
\varphi(t) = \varphi(0) + \int_0^t F_1\big(\ell, \varphi(\ell)\big) d\ell + \sum_{i=2}^{n} \Omega_i^{-1}(t) \int_0^t (t-\ell)^{c_i - 1}\, \Omega_i(\ell) F_i\big(\ell, \varphi(\ell)\big) d\ell
$$
$$
+ \int_0^t g\big(\ell, \varphi(\ell)\big) d\mathbb{C}(\ell). \tag{16}
$$

Thus,

$$\begin{aligned}
&\mathbb{E}|y(\mathrm{t})-\varphi(\mathrm{t})|^q \\
&\quad \leq \mathbb{E}|y(\mathrm{t})-\psi(\mathrm{t})+\psi(\mathrm{t})-\varphi(\mathrm{t})|^q \\
&\quad \leq (n+2)^{q-1}\Big[\epsilon+\left\|\int_0^{\mathrm{t}}\left(\mathrm{F}_1\big(\ell,\varphi(\ell)\big)-\mathrm{F}_1\big(\ell,y(\ell)\big)\right)d\ell\right\|_q^q \\
&\quad +\left\|\int_0^{\mathrm{t}}\left(g\big(\ell,\varphi(\ell)\big)-g\big(\ell,y(\ell)\big)\right)d\mathbb{C}(\ell)\right\|_q^q \\
&\quad +\sum_{i=2}^{n}\left\|\int_0^{\mathrm{t}}\frac{(\mathrm{t}-\ell)^{c_i-1}}{\Omega_i(\mathrm{t})}\Omega_i(\ell)\left(\mathrm{F}_i\big(\ell,\varphi(\ell)\big)-\mathrm{F}_i\big(\ell,y(\ell)\big)\right)d\ell\right\|_q^q\Big].
\end{aligned} \tag{17}$$

Following the approach used in the proof of Theorem 1, we obtain

$$\mathbb{E}|y(\mathrm{t})-\varphi(\mathrm{t})|^q \leq (n+2)^{q-1}\epsilon+(n+2)^{q-1}\tilde{\lambda}\int_0^{\mathrm{t}}\mathbb{E}|y(\ell)-\varphi(\ell)|^q\,d\ell,$$

where the constant $\tilde{\lambda}$ is defined for consistency as:

$$\tilde{\lambda}=\bar{C}^q\left[\mathrm{S}^{q-1}+\sum_{i=2}^{n}\mathrm{S}^{c_iq-1}\frac{M_i^q}{m_i^q}\left(\frac{q-1}{c_iq-1}\right)^{q-1}+M_q\mathrm{S}^{\frac{q-2}{2}}\right].$$

Applying Gronwall's inequality yields

$$\mathbb{E}|y(\mathrm{t})-\varphi(\mathrm{t})|^q \leq (n+2)^{q-1}\epsilon e^{(n+2)^{q-1}\tilde{\lambda}\mathrm{t}},\ \forall \mathrm{t}\in[0,\mathrm{S}]. \tag{18}$$

Hence,

$$\mathbb{E}|y(\mathrm{t})-\varphi(\mathrm{t})|^q \leq N\epsilon,\ \forall \mathrm{t}\in[0,\mathrm{S}], \tag{19}$$

where $N=(n+2)^{q-1}e^{(n+2)^{q-1}\tilde{\lambda}\mathrm{S}}$.

Thus, Eq. (1) is UHS. □

Remark 2: For the case when $\Omega_i(\ell)=\exp(\frac{1-\upsilon_i}{\upsilon_i}\ell)$ with $0<\upsilon_i\leq 1$, we get the results given in [17].

4 Examples

To verify our theoretical findings, we examine three representative cases.

Example 1: Let consider the Eq. (1) for $\mathrm{S}=3$, $n=2$,

$$\mathrm{F}_1(\mathrm{t},\mathrm{X})=5\mathrm{X}\sin(\mathrm{t})$$

$$\mathrm{F}_2(\mathrm{t},\mathrm{X})=6(\mathrm{X}+1)\cos(\mathrm{t}^2+1)$$

$$g(\mathrm{t},\mathrm{X}(\mathrm{t}))=3\mathrm{X}\cos(\mathrm{t}).$$

We have

$$|\mathrm{F}_1(\mathrm{t},\mathrm{X}_1)-\mathrm{F}_1(\mathrm{t},\mathrm{X}_2)|\leq 5|\mathrm{X}_1-\mathrm{X}_2|, \tag{20}$$

$$|\mathrm{F}_2(\mathrm{t},\mathrm{X}_1)-\mathrm{F}_2(\mathrm{t},\mathrm{X}_2)|\leq 6|\mathrm{X}_1-\mathrm{X}_2|, \tag{21}$$

and

$$|g(t, X_1) - g(t, X_2)| \leq 3|X_1 - X_2|. \tag{22}$$

Thus, assumption $\mathcal{H}_1$ is satisfied for $\bar{C} = 6$. Additionally, for $1 \leq i \leq 2$, we get

$$\operatorname*{ess\,sup}_{t\in[0,2]} |F_i(t, 0)| \leq 6 \tag{23}$$

and

$$\operatorname*{ess\,sup}_{t\in[0,2]} |g(t, 0)| \leq 6. \tag{24}$$

Therefore, assumption $\mathcal{H}_2$ holds for $d = 6$.

Thus, the equation is UHS.

Example 2: Let consider the Eq. (1) for $S = 7$, $n = 4$,

$$F_1(t, X) = (t + 1)X$$

$$F_2(t, X) = 3\cos(X)$$

$$F_3(t, X) = (t^2 + 1)\cos(X)$$

$$F_4(t, X) = t\sin(X)$$

$$g(t, X(t)) = \frac{\cos(t)\arctan(X)}{2}.$$

We have

$$|F_1(t, X_1) - F_1(t, X_2)| \leq 8|X_1 - X_2|, \tag{25}$$

$$|F_2(t, X_1) - F_2(t, X_2)| \leq 3|X_1 - X_2|, \tag{26}$$

$$|F_3(t, X_1) - F_3(t, X_2)| \leq 50|X_1 - X_2|, \tag{27}$$

$$|F_4(t, X_1) - F_4(t, X_2)| \leq 7|X_1 - X_2|, \tag{28}$$

and

$$|g(t, X_1) - g(t, X_2)| \leq \frac{1}{2}|X_1 - X_2|. \tag{29}$$

Thus, assumption $\mathcal{H}_1$ holds for $\bar{C} = 50$. Additionally, for $1 \leq i \leq 4$, we get

$$\operatorname*{ess\,sup}_{t\in[0,5]} |F_i(t, 0)| \leq 50 \tag{30}$$

and

$$\operatorname*{ess\,sup}_{t\in[0,5]} |g(t, 0)| \leq \frac{1}{2}. \tag{31}$$

Therefore, assumption $\mathcal{H}_2$ holds for $d = 50$.

Thus, the equation is UHS.

Example 3: Let consider the Eq. (1) for $S = 2$, $n = 3$,

$F_1(t, X) = 2\sin(X)$

$F_2(t, X) = 3\cos(X)\cos(t^4)$

$F_3(t, X) = 3X$

$g(t, X(t)) = 2\cos(X)$.

We have

$$|F_1(t, X_1) - F_1(t, X_2)| \leq 2|X_1 - X_2|, \tag{32}$$

$$|F_2(t, X_1) - F_2(t, X_2)| \leq 3|X_1 - X_2|, \tag{33}$$

$$|F_3(t, X_1) - F_3(t, X_2)| \leq 3|X_1 - X_2|, \tag{34}$$

and

$$|g(t, X_1) - g(t, X_2)| \leq 2|X_1 - X_2|. \tag{35}$$

Thus, assumption $\mathcal{H}_1$ holds for $\bar{C} = 3$. Additionally, for $i \in \{1, 2, 3\}$, we get

$$\operatorname*{ess\,sup}_{t\in[0,1]} |F_i(t, 0)| \leq 3, \tag{36}$$

and

$$\operatorname*{ess\,sup}_{t\in[0,1]} |g(t, 0)| \leq 2. \tag{37}$$

Therefore, assumption $\mathcal{H}_2$ holds for $d = 3$.

Thus, the equation is UHS.

5 Conclusion

In this study, we have successfully applied the BFPT to establish both the existence and uniqueness of solutions for WFIDSIE. By combining classical stochastic analysis techniques with the Gronwall inequality, we have demonstrated the HUS of these equations. Our approach not only provides a solid theoretical foundation but also ensures that small perturbations in the system do not compromise the stability of the solutions. To illustrate the applicability and effectiveness of our results, we have presented three detailed examples in the final section.

From a perspective of potential applications, the flexibility of the WFIDSIE framework makes it a promising candidate for modeling complex real-world systems with memory and stochasticity. For instance, in financial mathematics, our equation could model an asset price process where the standard drift and diffusion terms capture short-term trends and volatility, while the weighted fractional integrals model the persistent, time-decaying influence of past market shocks. The weight functions Ω_i provide a crucial tool to calibrate the memory kernel to empirical data, offering a more realistic alternative to simple power-law decay. The existence, uniqueness, and stability results proven in this work are essential prerequisites for the reliable use of such models in practice, ensuring that pricing and risk management strategies derived from them are robust and well-defined.

Looking forward, this work opens several avenues for further research. A key direction will be the development of numerical schemes for WFIDSIEs to facilitate simulation-based analysis and provide graphical illustrations of solution stability. Furthermore, significant theoretical challenges remain in relaxing the Lipschitz condition ($\mathcal{H}_1$) to include non-Lipschitz or multiplicative noise structures, and in extending the framework to higher-dimensional systems. Future work will focus on these open problems and on applying these models to concrete problems across various scientific domains.

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