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THE INVERSE TRANSIENT PROBLEM OF IDENTIFYING THE LAW OF CHANGE IN THE CROSS-SECTIONAL AREA OF AN ELASTIC BAR

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Abstract. The main goal of investigations is to obtain solutions for new inverse transient problems of elastic bars. The research objective is to develop and realize new methods, approaches and algorithms for solving inverse transient problems of bar mechanics. The direct transient task for elastic bar consists in determining elastic displacements, which satisfies a given equation of transient oscillations in partial derivatives and some given initial and boundary conditions. The solution of inverse problem with a completely unknown space-time law of load distribution is based on the method of influence functions. With its application the inverse problem is reduced to solving a system of integral equations of the Volterra type of the first kind in time with respect to the sought external axial load of the elastic bar. To solve it, the method of mechanical quadratures is used in combination with the Tikhonov regularization method.

1 INTRODUCTION

An elastic homogenous isotropic bar of finite length is considered, the left end of which is rigidly fixed and the right end of the bar is free. A concentrated time-dependent load is applied to the free end of the bar. The bar has variable cross-sectional area where the coordinate distribution law is unknown and should be identified in the process of solving the inverse problem. It's assumed that the displacements are known in some vicinity of the bar's free end. In practice this information can come from sensors for measuring longitudinal displacements, installed in several sections in vicinity of the free end of the bar. To construct a method for solving the inverse problem it is first necessary to obtain solutions to the direct problem where space is known and it's required to identify transient displacements of the elastic bar.

The methodology for solving the direct problem is based on the principle of superposition, in which displacements and contact stresses are related by means of integral operator with respect to the spatial variable and time [1]-[23]. Its core is influence function for the elastic bar. This function represents a fundamental solution relating to the differential equation of movement for the considering bar. The influence function is calculated with the help of Laplace transform in time and expansions in Fourier in the system of eigenfunctions.

In inverse problem it's required to calculate cross-sectional variable area based on sensor data. The solution of inverse problem is reduced to the solution of Volterra autonomous integral equations of the first kind being incorrect in J. Hadamard sense due to kernel of integral operator degeneracy. To regularize the inverse problem Tikhonov method is applied leading to the system of integral equations with nondegenerate kernels.

A numerical-analytical algorithm based on the method of mean rectangles in combination with Tikhonov regularization is developed and realized on a computer to solve resolvent integral equations.

2 THE SOLUTION OF DIRECT TRANSIENT PROBLEM

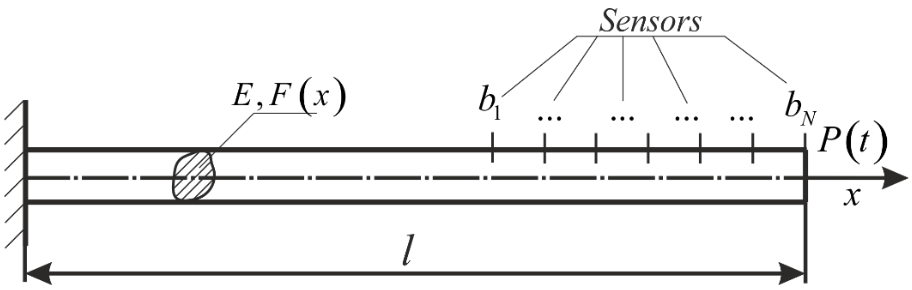

Figure 1: *Loading Diagram for The Bar*

An elastic isotropic bar of finite length is considered which left end is rigidly fixed and right end is under external load $P(t)$ (Figure 1). Initial conditions are zero. The direct problem includes an identification of the elastic bar displacements.

The mathematical formulation of direct problem includes motion equation of variable cross-section homogenous bar, boundary conditions and zero initial conditions [13]:

$$\begin{gathered} \left[EF(x)u'\right]' = \rho F(x)\ddot{u}, \\ u\big|_{x=0} = 0, \;\; EF(x)u'\big|_{x=l} = P(t), \\ u\big|_{t=0} = \dot{u}\big|_{t=0} = 0, \end{gathered} \tag{1}$$

where u - longitudinal bar movement, E - Young's module, $F(x)$ - cross-section area, ρ - bar density.

We'll introduce the following dimensionless values (diemnsional parameters are marked with stroke):

$$x = \frac{x'}{l},\; u = \frac{u'}{l},\; \tau = \frac{ct}{l},\; c^2 = \frac{E}{\rho},\; F = \frac{F'}{F_0},\; P = \frac{P'}{EF_0}, \tag{2}$$

where τ – nondimensional time, c – longitudinal wave velocity in bar, F_0 - any constant characteristic area. Then, equations, initial and boundary conditions (1) in dimensionless record take the following form:

$$\begin{gathered}\left[F(x)u'\right]' = F(x)\ddot{u},\\ u\big|_{x=0} = 0,\ u'\big|_{x=l} = P(\tau),\\ u\big|_{\tau=0} = \dot{u}\big|_{\tau=0} = 0.\end{gathered} \tag{3}$$

Here and elsewhere a dot over any value shall mean its time derivative τ, and stroke – on coordinate x.

The solution of problem (3) will be represented as:

$$u(x,\tau) = u_0(x,\tau) + \varepsilon u_1(x,\tau) + o\left(\varepsilon^2\right), \tag{4}$$

where $\varepsilon > 0$ - small parameter.

Cross-section equation $F(x)$ is also may be represented as:

$$F(x) = 1 + \varepsilon F_1(x) + o\left(\varepsilon^2\right), \tag{5}$$

where the first term of the right-hand corresponds to the bar with singular dimensionless cross-sectional area or sectional area F_0 in diemnsional problem. Besides, obviously, if we neglect smallness second order terms on ε in (5), then the equality holds $F_1(1) = 0$.

Substituting (4) and (5) in (3), and and equating in coefficients ε^0 ε^1, we'll come to points of functions $u_0(x,\tau)$ and $u_1(x,\tau)$:

$$\begin{gathered}u_0'' = \ddot{u}_0,\\ u_0\big|_{x=0} = 0,\ u_0'\big|_{x=1} = P(\tau),\\ u_0\big|_{\tau=0} = \dot{u}_0\big|_{\tau=0} = 0.\end{gathered} \tag{6}$$

$$\begin{gathered}u_1'' = \ddot{u}_1 - F_1'(x)u_0'\\ u_1\big|_{x=0} = 0,\ u_1'\big|_{x=1} = 0,\\ u_1\big|_{\tau=0} = \dot{u}_1\big|_{\tau=0} = 0.\end{gathered} \tag{7}$$

We'll seek the solution of the problem (6) as expansion in terms of proper functions. We may preliminary reduce the problem (6) to the problem with homogenous boundary conditions [13]. So, the solution of problem (6) is as follows:

$$u_0 = \sum_{n=1}^{\infty} w_{0n}(\tau)\sin(\lambda_n x) - xP(\tau), \tag{8}$$

where

$$w_{0n}(\tau) = a_n P(\tau) - a_n \lambda_n \int_0^\tau P(t) \sin\left[\lambda_n (\tau - t)\right] dt,$$

$$a_n = 2\int_0^1 x \sin \lambda_n x dx = \frac{2}{\lambda_n^2}(-1)^n,$$

$$\lambda_n = \frac{2n-1}{2}\pi.$$

Further, we solve problem (7), including found u_0 and seeking area:

$$u_1'' = \ddot{u}_1 + q(x,\tau),\ q(x,\tau) = -F_1'(x) u_0',$$
$$u_1\big|_{x=0} = 0,\ u_1'\big|_{x=1} = 0, \quad (9)$$
$$u_1\big|_{\tau=0} = \dot{u}_1\big|_{\tau=0} = 0.$$

According to [14], influence function and solution of direct problem for (9) is as follows:

$$u_1(x,\tau) = \sum_{n=1}^{\infty} I_n(\tau) \frac{\sin(\lambda_n x)}{\lambda_n},\ I_n(\tau) = \int_0^\tau G_n(\tau - t) q_n(t) dt, \quad (10)$$
$$G_n(\tau) = \sin \lambda_n \tau.$$

$$q(x,\tau) = \sum_{n=1}^{\infty} q_n(\tau) \sin \lambda_n x,\ q_n(\tau) = 2\int_0^1 q(x,\tau) \sin \lambda_n x dx.$$

As an example we may consider the problem on the influence of fixed load to the bar executed from rust-resisting steel 10X17H13M2T. Besides, $c = 5047.55$, bar dimensionless length is equal 1, $\varepsilon = 0.1$, $N = 3$, bar area is $F(x) = 1 + \varepsilon \sin \pi x$

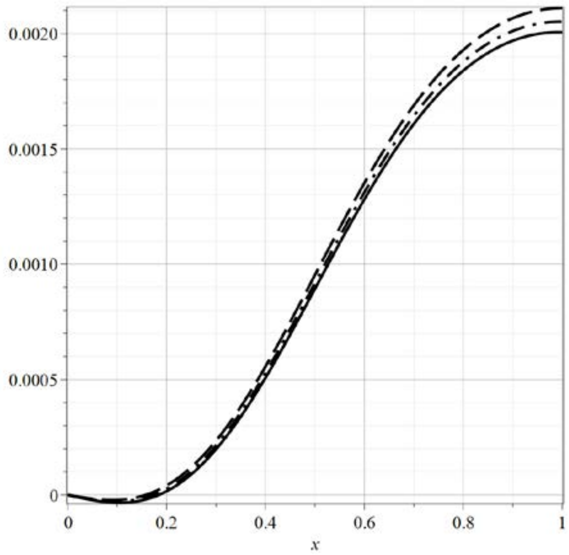

Figure 2: Bar Movement at the Moment of Time $\tau = 8$

Here, the solid graph – movements $u_0(x,\tau)$, dashed line – movements $u_0(x,\tau)+\varepsilon u_1(x,\tau)$, dash-and-dot line corresponds to four members of Fourier series for function $u_0(x,\tau)+\varepsilon u_1(x,\tau)$. As can be seen from obtained results, to solve direct and further inverse we may limit by three members of Fourier series and three movement sensors.

3 THE SOLUTION OF THE INVERSE TRANSIENT PROBLEM

The inverse problem includes an identification of coefficients $q_n(\tau)$ of (11) series.

Suppose, on a certain bar segment N sensors are installed, measuring bar movement values $U_1(\tau)=u_1(b_1,\tau), U_2(\tau)=u_1(b_2,\tau),..., U_N(\tau)=u_1(b_N,\tau)$ depending on time τ (Figure 1), where b_n, $n=\overline{1,N}$ - sensor installation coordinates. Restricting the first N members, out of (11) we'll get N integral performances

$$U_k(\tau)=\sum_{n=1}^{N} I_n(\tau)a_{kn},\ a_{kn}=\frac{\sin b_k\lambda_n}{\lambda_n},\ k=1,..,N, \tag{11}$$

Forming the system of algebric equations towards integral Volterra operators $I_n(\tau)$, $n=\overline{1,N}$.

$$\mathbf{U}=\mathbf{AI},\ \mathbf{A}=(a_{kn})_{N\times N},\ \mathbf{U}=[U_k(\tau)]_{N\times 1},\ \mathbf{I}=[I_n(\tau)]_{N\times 1}. \tag{12}$$

Solving this system, we'll get the following vector $\mathbf{I}$

$$\mathbf{I}=\mathbf{U}^*, \tag{13}$$

where $\mathbf{U}^*=\mathrm{A}^{-1}\mathbf{U}=[U_n^*(\tau)]_{N\times 1}$.

Vector-matrix equality (14) is equal to N autonomous Volterra integral equations of the first kind to desired series coefficients (11):

$$I_n(\tau)=U_n^*(\tau),\ n=1,...,N. \tag{14}$$

As known, if $G_n(0)=0$, equations (15) will be incorrect following J. Hadamard [15].So, to solve this problem (15) we'll use A.N. Tikhonov regularization method [16]-[17].

3.1 Numerical Solution for Volterra Integral Equation of the First Kind

To solve the equations (15) we'll use the midpoint rule.

We'll fix a certain final time T. Split time integration segment $[0,T]$ on M equal parts with even pitch $h=\frac{T}{M}$. For each time moment $\tau_m=hm$ the equation (15) we'll replace by numerical analogue using the method of mean rectangles:

$$\begin{gathered} u_{nm}^* \approx h\sum_{k=1}^{m} G_{nmk}q_{nk},\ m=1,...,M \\ u_{nm}^*=u_{nm}^*(\tau_m),\ G_{nmk}=G_n(\tau_m-t_k), \\ q_{nk}=q_n(t_k),\ t_k=h\frac{2k-1}{2}. \end{gathered} \tag{15}$$

As a result, we'll come to the system of linear algebraic equations towards q_{nk}, being the values of the sought coefficients $q_n(\tau)$ at the moments of time t_k, $k=1,...,M$:

$$\mathbf{G}_n\mathbf{Q}_n = \mathbf{U}_n^*. \tag{16}$$

3.2 Inverse Problem Regularization

Due to incorrectness of the problem (15) matrix $\mathbf{G}_n$ is incorrectly conditioned, so, the system of equations (17) is solved using Tikhonov regularization method [16]-[17]. Besides, (17) is replaced by the problem of identifying minimum of Tikhonov functional:

$$\Omega_\alpha(\tau) = \left|\mathbf{G}_n\tau - \mathbf{U}_n^*\right|^2 + \alpha|\tau|^2.$$

We may represent [16]-[17], that the problem of Tikhonov functional minimization is reduced to the solution of the other system of algebraic equations:

$$\left(\mathbf{G}_n^T\mathbf{G}_n + \alpha\mathbf{E}\right)\tilde{\mathbf{Q}}_n = \mathbf{G}_n^T\mathbf{U}_n^*, \tag{17}$$

where α – small positive parameter of regularization selecting by any optimal method, $\tilde{\mathbf{Q}}_n$ – quasi-solution vector for the equation (17).

3.3 Inverse Solution Example

Similarly to the direct problem the bar is made of steel with the same dimensional and dimensionless parameters.

Dimensionless coordinates of sensor installation ($N=3$):

$$b_1 = 0.8,\ b_2 = 0.9,\ b_3 = 1.$$

The number of time steps $M=100$, Sensor operating hours $T=5$, small regularization parameter $\alpha = 10^{-5}$.

Let's analyze the problem with the specified right-hand part for the equation (11) (Figure 3). Cross-section area, specified for identifying bar movements, shown in Figure 3: $F(x) = 1 + \varepsilon\sin\pi x$.

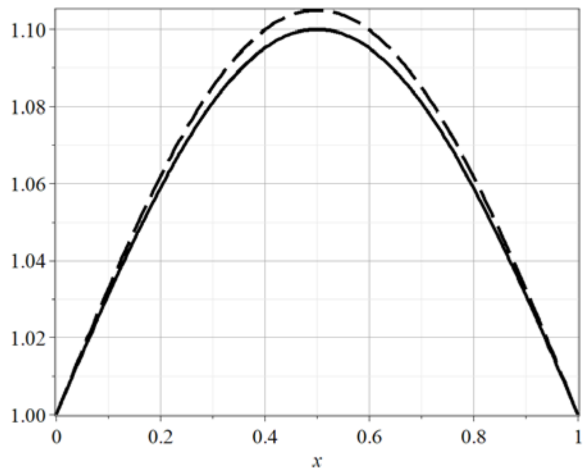

Figure 3: The comparison of inverse solution to sectional area, specified for the direct solution

Here, dash line is a reconstructed cross-sectional area, solid graph is a cross-sectional area, specified for identifying the bar movements.

4 HEADINGS

This paper proposes the method and algorithm of direct and inverse transient problem solution for the elastic bar of finite length identifying cross-sectional variable area. A numerical-analytical algorithm based on the method of mean rectangles in combination with Tikhonov regularization is developed and realized on a computer to solve resolvent integral equations [16]-[17] и [18]-[19]. Sample calculations are specified.

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