THE INVERSE TRANSIENT PROBLEM OF IDENTIFYING THE LAW OF CHANGE IN THE CROSS-SECTIONAL AREA OF AN ELASTIC BAR

G.V. FEDOTENKOV^{1,2}, E.I. STAROVOITOV³ AND YA.A. VAHTEROVA^{1,2}

¹ Moscow Aviation Institute (National Research University)
Volokolamsk highway, 4, , 125993, Moscow

² Lomonosov Moscow State University
Institute of Mechanics
1 Michurinsky Ave., 119192, Moscow

³ Department of Building Mechanics,
Belarusian State University of Transport,
34 Kirov Str., 246653, Gomel, Republic of Belarus

Email: greghome@mail.ru, edstar0@yandex.by, yana-vahterova@mail.ru

Key words: inverse problem, Fourier series, influence function, integral equations, quadrature formulas, Tikhonov regularization.

Abstract. The main goal of investigations is to obtain solutions for new inverse transient problems of elastic bars. The research objective is to develop and realize new methods, approaches and algorithms for solving inverse transient problems of bar mechanics. The direct transient task for elastic bar consists in determining elastic displacements, which satisfies a given equation of transient oscillations in partial derivatives and some given initial and boundary conditions. The solution of inverse problem with a completely unknown space-time law of load distribution is based on the method of influence functions. With its application the inverse problem is reduced to solving a system of integral equations of the Volterra type of the first kind in time with respect to the sought external axial load of the elastic bar. To solve it, the method of mechanical quadratures is used in combination with the Tikhonov regularization method.

1 INTRODUCTION

An elastic homogenous isotropic bar of finite length is considered, the left end of which is rigidly fixed and the right end of the bar is free. A concentrated time-dependent load is applied to the free end of the bar. The bar has variable cross-sectional area where the coordinate distribution law is unknown and should be identified in the process of solving the inverse problem. It's assumed that the displacements are known in some vicinity of the bar's free end. In practice this information can come from sensors for measuring longitudinal displacements, installed in several sections in vicinity of the free end of the bar. To construct a method for solving the inverse problem it is first necessary to obtain solutions to the direct problem where

space is known and it's required to identify transient displacements of the elastic bar.

The methodology for solving the direct problem is based on the principle of superposition, in which displacements and contact stresses are related by means of integral operator with respect to the spatial variable and time [1]-[23]. Its core is influence function for the elastic bar. This function represents a fundamental solution relating to the differential equation of movement for the considering bar. The influence function is calculated with the help of Laplace transform in time and expansions in Fourier in the system of eigenfunctions.

In inverse problem it's required to calculate cross-sectional variable area based on sensor data. The solution of inverse problem is reduced to the solution of Volterra autonomous integral equations of the first kind being incorrect in J. Hadamard sense due to kernel of integral operator degeneracy. To regularize the inverse problem Tikhonov method is applied leading to the system of integral equations with nondegenerate kernels.

A numerical-analytical algorithm based on the method of mean rectangles in combination with Tikhonov regularization is developed and realized on a computer to solve resolvent integral equations.

2 THE SOLUTION OF DIRECT TRANSIENT PROBLEM

Figure 1: Loading Diagram for The Bar

Register for free at <https://www.scipedia.com> to download the version without the watermark

An elastic isotropic bar of finite length is considered, which left end is rigidly fixed and right end is under external load $P(t)$ (Figure 1). Initial conditions are zero. The direct problem includes an identification of the elastic bar displacements.

The mathematical formulation of direct problem includes motion equation of variable cross-section homogenous bar, boundary conditions and zero initial conditions [13]:

$$\begin{aligned} [EF(x)u']' &= \rho F(x)\ddot{u}, \\ u|_{x=0} &= 0, \quad EF(x)u'|_{x=l} = P(t), \\ u|_{t=0} &= \dot{u}|_{t=0} = 0, \end{aligned} \quad (1)$$

where u - longitudinal bar movement, E - Young's module, $F(x)$ - cross-section area, ρ - bar density.

We'll introduce the following dimensionless values (dimensional parameters are marked with stroke):

$$x = \frac{x'}{l}, \quad u = \frac{u'}{l}, \quad \tau = \frac{ct}{l}, \quad c^2 = \frac{E}{\rho}, \quad F = \frac{F'}{F_0}, \quad P = \frac{P'}{EF_0}, \quad (2)$$

where τ – nondimensional time, c – longitudinal wave velocity in bar, F_0 – any constant characteristic area. Then, equations, initial and boundary conditions (1) in dimensionless record take the following form:

$$\begin{aligned} [F(x)u']' &= F(x)\ddot{u}, \\ u|_{x=0} &= 0, \quad u'|_{x=l} = P(\tau), \\ u|_{\tau=0} &= \dot{u}|_{\tau=0} = 0. \end{aligned} \quad (3)$$

Here and elsewhere a dot over any value shall mean its time derivative τ , and stroke – on coordinate x .

The solution of problem (3) will be represented as:

$$u(x, \tau) = u_0(x, \tau) + \varepsilon u_1(x, \tau) + o(\varepsilon^2), \quad (4)$$

where $\varepsilon > 0$ – small parameter.

Cross-section equation $F(x)$ is also may be represented as:

$$F(x) = 1 + \varepsilon F_1(x) + o(\varepsilon^2), \quad (5)$$

where the first term of the right-hand corresponds to the bar with singular dimensionless cross-sectional area or sectional area F_0 in dimensional problem. Besides, obviously, if we neglect smallness second order terms on ε in (5), then the equality holds $F_1(1) = 0$.

Substituting (4) and (5) in (3), and equating in coefficients ε^0 , ε^1 , we'll come to points of functions $u_0(x, \tau)$ and $u_1(x, \tau)$:

$$\begin{aligned} u_0'' &= \ddot{u}_0, \\ u_0|_{x=0} &= 0, \quad u_0'|_{x=1} = P(\tau), \end{aligned} \quad (6)$$

$$u_0|_{\tau=0} = \dot{u}_0|_{\tau=0} = 0.$$

Register for free at <https://www.scipedia.com> to download the version without the watermark

$$\begin{aligned} u_1' &= \ddot{u}_1 - F_1(x)u_0, \\ u_1|_{x=0} &= 0, \quad u_1'|_{x=1} = 0, \\ u_1|_{\tau=0} &= \dot{u}_1|_{\tau=0} = 0. \end{aligned} \quad (7)$$

We'll seek the solution of the problem (6) as expansion in terms of proper functions. We may preliminary reduce the problem (6) to the problem with homogenous boundary conditions [13]. So, the solution of problem (6) is as follows:

$$u_0 = \sum_{n=1}^{\infty} w_{0n}(\tau) \sin(\lambda_n x) - xP(\tau), \quad (8)$$

where

$$w_{0n}(\tau) = a_n P(\tau) - a_n \lambda_n \int_0^\tau P(t) \sin[\lambda_n(\tau-t)] dt,$$

$$a_n = 2 \int_0^1 x \sin \lambda_n x dx = \frac{2}{\lambda_n^2} (-1)^n,$$

$$\lambda_n = \frac{2n-1}{2} \pi.$$

Further, we solve problem (7), including found u_0 and seeking area:

$$u_1'' = \ddot{u}_1 + q(x, \tau), \quad q(x, \tau) = -F_1'(x) u_0',$$

$$u_1|_{x=0} = 0, \quad u_1'|_{x=1} = 0,$$

$$u_1|_{\tau=0} = \dot{u}_1|_{\tau=0} = 0.$$
(9)

According to [14], influence function and solution of direct problem for (9) is as follows:

$$u_1(x, \tau) = \sum_{n=1}^{\infty} I_n(\tau) \frac{\sin(\lambda_n x)}{\lambda_n}, \quad I_n(\tau) = \int_0^\tau G_n(\tau-t) q_n(t) dt,$$

$$G_n(\tau) = \sin \lambda_n \tau.$$
(10)

$$q(x, \tau) = \sum_{n=1}^{\infty} q_n(\tau) \sin \lambda_n x, \quad q_n(\tau) = 2 \int_0^1 q(x, \tau) \sin \lambda_n x dx.$$

As an example we may consider the problem on the influence of fixed load to the bar executed from rust-resisting steel 10X17H13M2T. Besides, $c = 5047.55$, bar dimensionless length is equal 1, $\varepsilon = 0.1$, $N = 3$, bar area is $F(x) = 1 + \varepsilon \sin \pi x$.

Figure 2: Bar Movement at the Moment of Time $\tau = 8$

Here, the solid graph – movements $u_0(x, \tau)$, dashed line – movements $u_0(x, \tau) + \varepsilon u_1(x, \tau)$, dash-and-dot line corresponds to four members of Fourier series for function $u_0(x, \tau) + \varepsilon u_1(x, \tau)$. As can be seen from obtained results, to solve direct and further inverse we may limit by three members of Fourier series and three movement sensors.

3 THE SOLUTION OF THE INVERSE TRANSIENT PROBLEM

The inverse problem includes an identification of coefficients $q_n(\tau)$ of (11) series.

Suppose, on a certain bar segment N sensors are installed, measuring bar movement values $U_1(\tau) = u_1(b_1, \tau), U_2(\tau) = u_1(b_2, \tau), \dots, U_N(\tau) = u_1(b_N, \tau)$ depending on time τ (Figure 1), where $b_n, n = \overline{1, N}$ - sensor installation coordinates. Restricting the first N members, out of (11) we'll get N integral performances

$$U_k(\tau) = \sum_{n=1}^N I_n(\tau) a_{kn}, \quad a_{kn} = \frac{\sin b_k \lambda_n}{\lambda_n}, \quad k = \overline{1, N}, \quad (11)$$

Forming the system of algebraic equations towards integral Volterra operators $I_n(\tau), n = \overline{1, N}$.

$$\mathbf{U} = \mathbf{A}\mathbf{I}, \quad \mathbf{A} = (a_{kn})_{N \times N}, \quad \mathbf{U} = [U_k(\tau)]_{N \times 1}, \quad \mathbf{I} = [I_n(\tau)]_{N \times 1}. \quad (12)$$

Solving this system, we'll get the following vector $\mathbf{I}$

$$\mathbf{I} = \mathbf{U}^*, \quad (13)$$

where $\mathbf{U}^* = \mathbf{A}^{-1}\mathbf{U} = [U_n^*(\tau)]_{N \times 1}$.

Vector-matrix equality (14) is equal to N autonomous Volterra integral equations of the first kind to desired series coefficients (11):

$$I_n(\tau) = U_n^*(\tau), \quad n = \overline{1, N}. \quad (14)$$

Register for free at <https://www.scipedia.com> to download the version without the watermark

As known, if $G_n(0) = 0$, equations (15) will be incorrect following J. Hadamard [15]. So, to solve this problem (15) we'll use A.N. Tikhonov regularization method [16]-[17].

3.1 Numerical Solution for Volterra Integral Equation of the First Kind

To solve the equations (15) we'll use the midpoint rule.

We'll fix a certain final time T . Split time integration segment $[0, T]$ on M equal parts with even pitch $h = \frac{T}{M}$. For each time moment $\tau_m = hm$ the equation (15) we'll replace by numerical analogue using the method of mean rectangles:

$$\begin{aligned} u_{nm}^* &\approx h \sum_{k=1}^m G_{nmk} q_{nk}, \quad m = \overline{1, M} \\ u_{nm}^* &= u_{nm}^*(\tau_m), \quad G_{nmk} = G_n(\tau_m - t_k), \\ q_{nk} &= q_n(t_k), \quad t_k = h \frac{2k-1}{2}. \end{aligned} \quad (15)$$

As a result, we'll come to the system of linear algebraic equations towards q_{nk} , being the values of the sought coefficients $q_n(\tau)$ at the moments of time t_k , $k=1, \dots, M$:

$$\mathbf{G}_n \mathbf{Q}_n = \mathbf{U}_n^*. \quad (16)$$

3.2 Inverse Problem Regularization

Due to incorrectness of the problem (15) matrix $\mathbf{G}_n$ is incorrectly conditioned, so, the system of equations (17) is solved using Tikhonov regularization method [16]-[17]. Besides, (17) is replaced by the problem of identifying minimum of Tikhonov functional:

$$\Omega_\alpha(\tau) = \|\mathbf{G}_n \tau - \mathbf{U}_n^*\|^2 + \alpha \|\tau\|^2.$$

We may represent [16]-[17], that the problem of Tikhonov functional minimization is reduced to the solution of the other system of algebraic equations:

$$(\mathbf{G}_n^T \mathbf{G}_n + \alpha \mathbf{E}) \tilde{\mathbf{Q}}_n = \mathbf{G}_n^T \mathbf{U}_n^*, \quad (17)$$

where α – small positive parameter of regularization selecting by any optimal method, $\tilde{\mathbf{Q}}_n$ – quasi-solution vector for the equation (17).

3.3 Inverse Solution Example

Similarly to the direct problem the bar is made of steel with the same dimensional and dimensionless parameters.

Dimensionless coordinates of sensor installation ($N=3$):

$$b_1 = 0.8, \quad b_2 = 0.9, \quad b_3 = 1.$$

The number of time steps $M=100$, Sensor operating hours $T=5$, small regularization parameter $\alpha = 10^{-5}$.

Let's analyze the problem with the specified right-hand part for the equation (11) (Figure 3).

Register for free at <https://www.scipedia.com> to download the version without the watermark

Cross-section area, specified for identifying bar movements, shown in Figure 3.

$$F(x) = 1 + \varepsilon \sin \pi x.$$

Figure 3: The comparison of inverse solution to sectional area, specified for the direct solution

Here, dash line is a reconstructed cross-sectional area, solid graph is a cross-sectional area, specified for identifying the bar movements.

4 HEADINGS

This paper proposes the method and algorithm of direct and inverse transient problem solution for the elastic bar of finite length identifying cross-sectional variable area. A numerical-analytical algorithm based on the method of mean rectangles in combination with Tikhonov regularization is developed and realized on a computer to solve resolvent integral equations [16]-[17] и [18]-[19]. Sample calculations are specified.

The reported study was funded by RFBR, project number 20-38-90043 and by RFBR and BRFR, project number 20-58-00023.

REFERENCES

- [1] Serdyuk A.O., Serdyuk D.O., Fedotenkov G.V. *Non-Stationary Deflection Function for the unlimited anisotropic plate* // Samara State Technical University bulletin. «Physico-mathematical Sciences Series. 2021. T. 25, No. 1, p. 111-126. doi: 10.14498/vsgtu1793.
- [2] N.A. Lokteva, D. O. Serdyuk, P. D. Skopintsev *Non-stationary influence function for an unbounded anisotropic Kirchhoff-Love shell* // Journal of Applied Engineering Science, 2020. Vol. 18, No. 4, pp. 737 – 744. DOI: 10.5937/jaes0-28205
- [3] Okonechnikov A. S., Tarlakovsky D. V., Fedotenkov G. V. *Spatial non-stationary contact problem for a cylindrical shell and absolutely rigid body* // Mechanics of Solids. — 2020. — Vol. 55, no. 3. — P. 366–376.
- [4] Okonechnikov A. S., Tarlakovskiy D. V., Fedotenkov G. V. *Space Non-stationary Contact problem for cylindrical shell and Rigid Body* // Proceedings of the Russian Academy of Sciences. Solid Mechanics. — 2020. — No. 3. — P. 80–91.
- [5] *Non-Stationary Stress-Strain State of the Composite Cylindrical Shell* / N. A. Lokteva, D. O. Serdyuk, P. D. Skopincev, G. V. Fedotenkov // Composite and Structural Mechanics. — 2020. — V. 26, No. 4. — P. 544–559., DOI: 10.33113/mkmk.ras.2020.26.04.544_559.08
- [6] Fedotenkov G. V., Tarlakovsky D. V. *Modeling of free regular and irregular plate oscillations under unsteady elastodiffusive perturbations* // Acta Mechanica. — 2021., DOI: 10.1007/s00707-020-02879-1
- [7] Tarlakovskii D. V., Lam N. V. *Propagation of non-stationary antisymmetric kinematic perturbations from a spherical cavity in cosserat medium* // PNRPU Mechanics Bulletin. — 2020. — no. 4. — P. 201–210. DOI: 10.15593/perm.mech/2020.4.17
- [8] Fam D. T., Tarlakovskii D. V. *Dynamic Bending of Infinite Electromagnetoelastic Bar* // Saratov University Bulletin. New Series. Series: Mathematics. Mechanics. Informatics. — 2020. — V. 20, No. 4. — P. 493–501., DOI: 10.18500/1816-9791-2020-20-4-493-501
- [9] Rabinskiy, L.N., Tushavina, O.V., Formalev, V.F. *Mathematical modeling of heat and mass transfer in shock layer on dimmed bodies at aerodynamic heating of aircraft*// Asia Life Sciences, 2019, (2), p. 897–911.
- [10] Egorova, O.V., Kurbatov, A.S., Rabinskiy, L.N., Zhavoronok, S.I. *Modeling of the dynamics of plane functionally graded waveguides based on the different formulations of the plate theory of I. N. Vekua type*, Mechanics of Advanced Materials and Structures, 2021, 28(5), P. 506–515, <https://doi.org/10.1080/15376494.2019.1578008>
- [11] Babaytsev, A.V., Orekhov, A.A., Rabinskiy, L.N. *Properties and microstructure of AlSi10Mg samples obtained by selective laser melting*// Nanoscience and Technology: An International Journal, 2020, 11(3), p. 213–222.

Register for free at <https://www.scipedia.com> to download the version without the watermark

- [12] Antufev, B.A., Egorova, O.V., Rabinskiy, L.N. Quasi-static stability of a ribbed shell interacting with moving load// INCAS Bulletin, 2019, 11, p. 33–39.
- [13] Kireenkov, A.A. Modelling of the force state of contact of a ball rolling along the boundaries of two rails. (2021) AIP Conference Proceedings, 2343, article No. 120002, DOI: 10.1063/5.0047955
- [14] Kireenkov, A.A. Preface: Mathematical Models and Investigations Methods of Strongly Nonlinear Systems, (2021) AIP Conference Proceedings, 2343, article No. 120001, DOI: 10.1063/5.0047954
- [15] Kireenkov, A.A., Zhavoronok, S.I. Numeric-analytical methods of the coefficients definition of the rolling friction model of the pneumatic aviation tire (2021) 8th International Conference on Computational Methods for Coupled Problems in Science and Engineering, COUPLED PROBLEMS 2019, pp. 204-212.
- [16] Kireenkov, A.A., Ramodanov, S.M. Combined dry friction models in the case of random distribution of the normal contact stresses inside contact patches (2021) 8th International Conference on Computational Methods for Coupled Problems in Science and Engineering, COUPLED PROBLEMS 2019, pp. 176-182.
- [17] Kireenkov, A.A., Zhavoronok, S.I., Nushtaev, D.V. On tire models accounting for both deformed state and coupled dry friction in a contact spot (2021) Computer Research and Modeling, 13 (1), pp. 163-173., DOI: 10.20537/2076-7633-2021-13-1-163-173
- [18] Kireenkov, A.A., Zhavoronok, S.I. Anisotropic Combined Dry Friction in Problems of Pneumatics' Dynamics. (2020) Journal of Vibration Engineering and Technologies, 8 (2), pp. 365-372., DOI: 10.1007/s42417-019-00140-1
- [19] Bolshakov, P.V., Sachenkov, O.A. Destruction simulation for the inhomogeneous body by finite element method using computed tomography data // Russian Journal of Biomechanics, 2020, 24(2), стр. 248–258, DOI: 10.15593/RZhBiomeh/2020.2.12
- [20] Kharin, N., Vorob'yev, O., Bol'shakov, P., & Sachenkov, O. (2019). Determination of the orthotropic parameters of a representative sample by computed tomography. Paper presented at the Journal of Physics: Conference Series, , 1158(3) doi:10.1088/1742-6596/1158/3/032012
- [21] Gerasimov, O., Kharin, N., Vorob'yev, O., Semenova, E., & Sachenkov, O. (2019). Determination of the mechanical properties distribution of the sample by tomography data. Paper presented at the Journal of Physics: Conference Series, , 1158(2) doi:10.1088/1742-6596/1158/2/022046
- [22] Kuznetsov, I., Salakhutdinov, M., Shakirzyanov, F., Khaydarovand, L., & Aripov, D. (2020). An investigation of the influence of the reduced elasticity modulus on strength of pultruded lfp members in bending. Paper presented at the IOP Conference Series: Materials Science and Engineering, , 890(1) doi:10.1088/1757-899X/890/1/012047
- [23] Kayumov, R. A., Tazyukov, B. F., Muhamedova, I. Z., & Shakirzyanov, F. R. (2019). Identification of the elastic characteristics of a composite material based on the results of tests for the stability of panels made from it. Uchenye Zapiski Kazanskogo Universiteta.Seriya Fiziko-Matematicheskie Nauki, 161(1), 75-85. doi:10.26907/2541-7746.2019.1.75-85
- [24] Gorshkov A.G., Medvedskii A.L., Rabinski L.N., Tarlakovskii D.V. *Waves in Continuous Media*. – M.: FIZMATLIT, 2004. 472 p.
- [25] Vahterova Y. A., Fedotenkov G. V. *The inverse problem of recovering an unsteady linear load for an elastic rod of finite length* // Journal of Applied Engineering Science. — 2020. — Vol. 18, no. 4. — P. 687–692.
- [26] Hadamard J., *Le probleme de Cauchy et les equations aux derivers particlee lineaires hyperbolique*, Paris: Hermann, 1932.
- [27] Tikhonov A.N., Arsenin V.Y., *Methods of Solving Incorrect Problems*. 2-nd ed. – M.: Nauka: The main office of physico-mathematical literature, 2-nd ed., 1979, p. 285.
- [28] Tikhonov A.N., Goncharski A.V., Stepanov V.V., Yagola A.G., *Numerical Methods of Solving Incorrect Problems*, M.: Nauka, 1990, p. 232.

Register for free at <https://www.scipedia.com> to download the version without the watermark

- [29] Fedotenkov G. V., Tarlakovsky D. V., Vahterova Y. A. *Identification of non-stationary load upon timoshenko beam* // Lobachevskii Journal of Mathematics. — 2019. — Vol. 40, no. 4. — P. 439–447., DOI: 10.1134/S1995080219040061
- [30] Vakhterova Y. A., Serpicheva E. V., Fedotenkov G. V. *Inverse Problem of Identifying Transient Load for Timoshenko Bar* // Tula State University Bulletin. Technical Sciences. — 2017. — No 4. — p. 82–92.

Register for free at <https://www.scipedia.com> to download the version without the watermark