

PDE-BASED AND SOLUTION-DEPENDENT PARAMETERIZATION FOR ISOGOMETRIC ANALYSIS

Zulfiqar Ali¹, Weiyin Ma¹

¹ Department of Mechanical Engineering, City University of Hong Kong, Hong Kong, China
zzali2-c@my.cityu.edu.hk, mewma@cityu.edu.hk

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Abstract. This paper presents some results for PDE-based and solution-dependent parameterization of computational domains for isogeometric analysis (IGA) using non-uniform rational B-splines (NURBS). The final parameterization is produced based on the solution of a partial differential equation (PDE) that is solved using isogeometric collocation method (IGA-C) with Dirichlet boundary condition being the input boundary of the final desired computational domain for IGA, namely the IGA-C-PDE method for domain parameterization. The theory of PDE guarantees that the mapping between physical and transformed region will be one-to-one. In addition, we also apply intuitive position and ratio constraints while solving the PDE to achieve solution-dependent parameterization. While one may use any general PDE with any constraint, the PDEs and additional constraints selected in our case are such that the resulting solution can be efficiently solved through a system of linear equations with or without additional linear constraints. This approach is different from typical existing parameterization methods in IGA that are often solved through an expensive nonlinear optimization processes. The results show that the proposed method can efficiently produce satisfactory analysis-suitable parameterization.

1 Introduction

Isogeometric collocation method (IGA-C) was introduced as an alternative approach to conduct engineering analysis without the need of efficient integration rules, as required in the standard isogeometric Galerkin methods (IGA-G). The main idea of IGA-C rests on the discretization of the governing partial differential equations in strong form at designated collocation positions which leads to the use of reduced number of evaluations for setting up the system of equations to only one per degree of freedom [1, 2]. The advantage of low computational cost with promising results attracts many researchers across the globe to use IGA-C method in various applications including phase-field modeling [3], contact problems [4] and nonlinear elasticity [4]. This paper presents some results for solution-dependent parameterization of computation domains using a PDE-based IGA-C method [5] with intuitive position and ratio constraints for IGA, which corresponds to the mesh generation process in Finite Element Analysis (FEA), and has a key impact on analysis result and efficiency. One of the significant challenges towards IGA is the construction of an analysis suitable parameterization from a given CAD boundary representation [1]. Generally in IGA, the parameterization of the computational domain is determined by the representation of the computational domain whose parameters are degrees of the basis functions, knot vectors and control points of the B-spline or NURBS model. For 2D and 3D IGA problems, the knot vectors and the

degrees of the computational domain are often determined by the given boundary curves/surfaces. The quality of the parameterization of the computational domain is determined by the knots and the distribution of the control points. A basic requirement of the resulting parameterization for IGA as discussed in [6] is that an analysis-suitable parameterization of the computational domain should have no self-intersections i.e. the mapping from the parametric domain to physical domain should be injective. Also, the iso-parametric elements should be as far as possible uniform and orthogonal. Most parameterization methods in the literature are based on the minimization of a quality functional. Commonly used quality functionals include area orthogonality, Liao, Winslow, uniformity and harmonic energy [6]. One of the latest contributions for efficient parameterization of planar domains for IGA is reported in [7], where the domain is mapped to an equivalent convex domain using harmonic functions. Similarly, an IGA-suitable planar B-spline parameterization of the computational domain with high genus and more complex boundary curves is reported in [8]. Xu *et al.* also presented various approaches for the parameterization of the computational domain in IGA such as constrained optimization methods and shape optimization methods for generating analysis-aware parameterization with a limitation that the solution is problem-dependent [9, 10]. Martin *et al.* proposed a method to fit a genus-0 triangular mesh by B-spline volume parameterization, based on discrete volumetric harmonic functions that can be used to build computational domains for 3D IGA problems [11]. A general framework for constructing planar parameterization has also been introduced in IGA using linear and non-linear constraints [12]. Falini *et al.* also proposed planar domain parameterization using THB-splines [13]. However, if we look in FEA, a commonly used method for mesh generation is based on solutions of partial differential equations (PDEs) [14]. The solution of an elliptical PDE with Dirichlet boundary conditions on all boundaries produces the grid points / mesh vertices. The theory of PDEs assures that the mapping between physical and parametric regions will be one-to-one. There are several advantages in elliptical mesh generation, such as inherent smoothness in the solution. However, a limitation of elliptical methods is that some non-uniform grid elements may appear near convex (concave) regions of the boundary. Motivated from the elliptical grid generation in FEA, we developed a framework for analysis-suitable parameterization of computational domains by solving a PDE system using IGA-C methods. Moreover, we also introduce additional linear constraints to produce solution-dependent parameterization. We test the parameterization results for 2D heat conduction problems to show the effectiveness of the proposed method.

2 Isogeometric collocation methods for solving PDEs

2.1 B-splines and NURBS

Non-Uniform Rational B-spline (NURBS) are the basis functions widely used to represent the geometry for analysis in IGA. A B-spline of polynomial degree p or order $k = p + 1$ is defined by a knot vector $\Xi = \{\xi_1, \xi_2, \dots, \xi_{n+k}\}$, with n the number of basis functions. B-spline basis functions $\{\hat{N}_i(\xi)\}_{i=1}^n$ can be uniquely defined in a recursive manner as:

$$\hat{N}_{i,1}(\xi) = \begin{cases} 1, & \xi_i \leq \xi < \xi_{i+1} \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

$$\hat{N}_{i,k}(\xi) = \frac{\xi - \xi_i}{\xi_{i+k+1} - \xi_i} \hat{N}_{i,k-1}(\xi) + \frac{\xi_{i+k} - \xi}{\xi_{i+k} - \xi_{i+1}} \hat{N}_{i+1,k-1}(\xi), \quad k > 1. \quad (2)$$

Given two univariate B-spline basis $\{\hat{N}_i(\xi)\}_{i=1}^{n_\xi}$ and $\{\hat{N}_i(\eta)\}_{i=1}^{n_\eta}$ respectively of order k_ξ and k_η , and associated to the two knot vectors $\Xi_\xi = \{\xi_1, \xi_2, \dots, \xi_{n_\xi+k_\xi}\}$ $\Xi_\eta = \{\eta_1, \eta_2, \dots, \eta_{n_\eta+k_\eta}\}$, a bivariate NURBS basis function is defined as:

$$\hat{R}_{i,j}(\xi, \eta) = \frac{w_{ij}\hat{N}_{i,k_\xi}(\xi)\hat{N}_{j,k_\eta}(\eta)}{\sum_{i=1}^{n_\xi}\sum_{j=1}^{n_\eta}w_{ij}\hat{N}_{i,k_\xi}(\xi)\hat{N}_{j,k_\eta}(\eta)} \quad \forall(\xi, \eta) \in \hat{\Omega}, \quad (3)$$

where w_{ij} is the weight associated to the basis function $\hat{R}_{i,j}$. NURBS surfaces $\mathbf{S}(\xi, \eta)$ are defined by:

$$\mathbf{S}(\xi, \eta) = \sum_{i=1}^{n_\xi}\sum_{j=1}^{n_\eta}S_{ij}\hat{R}_{i,j}(\xi, \eta) = \sum_{l=1}^nS_l\hat{R}_l(\xi, \eta), \quad (\xi, \eta) \in [\xi_{k_\xi}, \xi_{n_\xi+1}] \times [\eta_{k_\eta}, \eta_{n_\eta+1}], \quad (4)$$

where $\{S_l\}_{l=1}^n = \{S_{ij}\}_{i=1}^{n_\xi}\}_{j=1}^{n_\eta}$ represent the set of $n = n_\xi \times n_\eta$ control points for the geometry. Using the isogeometric approach, the solution domain $\mathbf{u}(\xi, \eta)$ is approximated by the same basis functions that represent the geometry such that:

$$\mathbf{u}(\xi, \eta) = \sum_{i=1}^{n_\xi}\sum_{j=1}^{n_\eta}u_{ij}\hat{R}_{i,j}(\xi, \eta) = \sum_{l=1}^nu_l\hat{R}_l(\xi, \eta) \quad (5)$$

where u_l is the solution control points corresponding to the geometry control points S_l .

2.2 The general procedure of IGA-C for solving PDEs

We assume the strong form of the boundary value problem is given by:

$$\begin{cases} \mathcal{D}\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \mathcal{G}\mathbf{u} = \mathbf{g} & \text{on } \partial\Omega \end{cases} \quad (6)$$

where $\mathbf{u} : \Omega \rightarrow \mathbb{R}$ represents the unknown solution, \mathcal{D} is a linear differential operator, \mathcal{G} is a vector operator, $\mathbf{f} : \Omega \rightarrow \mathbb{R}$ and $\mathbf{g} : \partial\Omega \rightarrow \mathbb{R}$ are known functions, and $\partial\Omega$ stands for the boundary of the computational domain Ω of the PDE. As an example, one can take $\mathcal{D}\mathbf{u} = \nabla^2\mathbf{u}$. The goal here is to find a solution $\mathbf{u} : \Omega \rightarrow \mathbb{R}$ meeting the strong form of BVP. To solve this problem by the collocation method, two sets of collocation points need to be considered, a set \bar{S}_i^{int} , $i = 1, \dots, m^{int}$ in the interior of Ω and a set \bar{S}_i^{bnd} , $i = 1, \dots, m^{bnd}$ on $\partial\Omega$, where m^{int} and m^{bnd} are the number of collocation points chosen in the interior and on the boundary, respectively. Then, the collocation solution shown in Eq. 5 is required to satisfy:

$$\begin{cases} \mathcal{D}\mathbf{u}(\bar{S}_i^{int}) = \mathbf{f}(\bar{S}_i^{int}) & \text{in } \Omega, \quad i = 1, \dots, m^{int} \\ \mathcal{G}\mathbf{u}(\bar{S}_i^{bnd}) = \mathbf{g}(\bar{S}_i^{bnd}) & \text{on } \partial\Omega, \quad i = 1, \dots, m^{bnd} \end{cases} \quad (7)$$

This will leads to a system of linear equations with linear constraints

$$\mathbf{K}\mathbf{u} = \mathbf{F} \quad (8)$$

In Eq. (8), $\mathbf{K} \in \mathbb{R}^{(m \times n)}$ is a stiffness matrix, $\mathbf{u} \in \mathbb{R}^{(n \times 1)}$ is the solution vector, and $\mathbf{F} \in \mathbb{R}^{(m \times 1)}$ load vector acting upon the system, The total number of equations m is usually equal to the number of unknowns n and a unique solution for \mathbf{u} can be found. If the total number of equations is more than the number of unknowns, i.e. $m > n$, Eq. (8) can then be solved using a constrained linear least-squares algorithm

with exact satisfaction of desired boundary constraints similar to that reported in [15]. The traditional IGA-C uses the so-called "Greville abscissa" to define the collocation points that are obtained via knots averaging. For a given knot vector Ξ_ξ and Ξ_η the associated Greville points can be calculated as:

$$\begin{aligned}\bar{\xi}_i &= \frac{\xi_{i+1} + \xi_{i+2} + \dots + \xi_{i+p}}{p} \quad \text{for } i = 1, \dots, n_\xi, \\ \bar{\eta}_j &= \frac{\eta_{j+1} + \eta_{j+2} + \dots + \eta_{j+q}}{q} \quad \text{for } j = 1, \dots, n_\eta,\end{aligned}\quad (9)$$

The collocation points $\chi_{ij} \in \hat{\Omega}$ as a tensor product structure can be defined as:

$$\bar{\chi}_{ij} = (\bar{\xi}_i, \bar{\eta}_j), \quad i = 1, \dots, m_\xi; \quad j = 1, \dots, m_\eta, \quad (10)$$

Collocation points in physical domain can be defined using a push-forward operator as:

$$\bar{S}_{ij} = \mathbf{S}(\bar{\chi}_{ij}), \quad i = 1, \dots, m_\xi; \quad j = 1, \dots, m_\eta. \quad (11)$$

3 IGA-C with additional constraints for PDE-based parameterization

In this section, we briefly highlight the procedure to produce 2D-parameterization of a desired computational domain (Ω_1) via solving a Laplace PDE on a simple initial computational domain (Ω_0). Four arbitrary boundary curves $\mathbf{X} = \bigcup_{i=1}^4 X_i$ is imposed as a Dirichlet boundary conditions on a (Ω_0) domain and Laplace PDE is solved to find the parameterization of (Ω_1) domain. The Laplace PDE in general form along with Dirichlet boundary conditions can be written as:

$$\begin{cases} \nabla^2 \mathbf{u} = 0 & \text{in } \Omega_0 \\ \mathbf{u} = \mathbf{X} & \text{on } \partial\Omega_0 \end{cases} \quad (12)$$

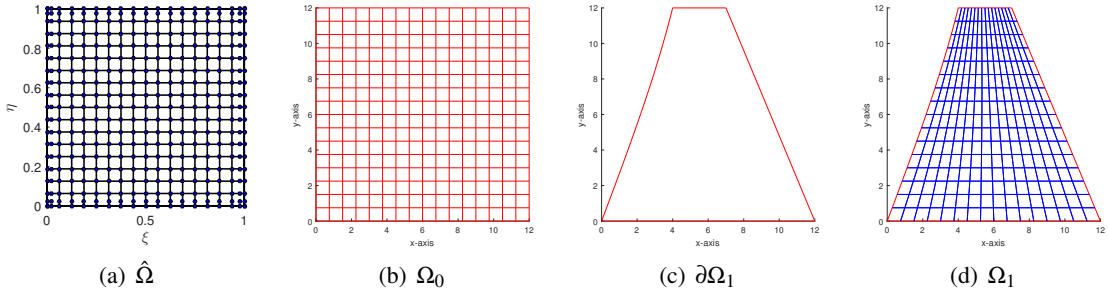


Figure 1: IGA-C-PDE based parameterization process: (a) the parametric domain for IGA-C; (b) an initial computational domain; (c) input boundary of a desired computational domain; and (d) the resulting parameterization of the desired computational domain / physical domain

While the target application of the resulting parameterization will be for IGA, the solution of PDE, i.e., the desired parameterization of Ω_1 domain for IGA, is also produced using the IGA-C method with given initial parameterization of a simple Ω_0 domain. To solve the PDE as shown of Eq. (12) by collocation method, a set of collocation points $\{\bar{\chi}_{ij}\} = \{(\bar{\xi}_i, \bar{\eta}_j)\}$, for $i = 1, \dots, n_\xi, j = 1, \dots, n_\eta$, in parametric

domain are defined as Greville abscissa. $\tilde{\chi}_{ij}$ can be further represented in physical domain as \bar{S}_{ij} via mapping as shown in Eq. (11). After introducing all collocation points $\tilde{\chi}_{ij} = (\bar{\xi}_i, \bar{\eta}_j)$ or $\bar{S}_{ij} = \mathbf{S}(\tilde{\chi}_{ij})$ in Eq. (12), we obtain the final system of equations as:

$$\begin{cases} \frac{\partial^2 \mathbf{u}(\bar{S}_{ij})}{\partial x^2} + \frac{\partial^2 \mathbf{u}(\bar{S}_{ij})}{\partial y^2} = 0, & \text{for } i = 2, \dots, n_\xi - 1; j = 2, \dots, n_\eta - 1 \\ \mathbf{u}(\bar{\xi}_i, \bar{\eta}_1) = \mathbf{X}_1(\bar{\xi}_i), \mathbf{u}(\bar{\xi}_i, \bar{\eta}_{n_\eta}) = \mathbf{X}_3(\bar{\xi}_i), & \text{for } i = 1, \dots, n_\xi \\ \mathbf{u}(\bar{\xi}_1, \bar{\eta}_j) = \mathbf{X}_2(\bar{\eta}_j), \mathbf{u}(\bar{\xi}_{n_\xi}, \bar{\eta}_j) = \mathbf{X}_4(\bar{\eta}_j), & \text{for } j = 2, \dots, n_\eta - 1 \end{cases} \quad (13)$$

which leads to a system of linear equations with linear constraints in the same form as shown in Eq. (8) with $m = n$. We will solve two Laplace PDE with respect to x and y coordinates for finding parameterization.

In this study, we also introduce additional linear constraints while solving the Laplace PDE in order to produce good parameterization with well-behaved Jacobian distribution. In the final solution of the constrained PDE, we ensure exact satisfaction for both the boundary constraints and additional constraints, while other parameters of the parameterization, mainly the interior control vertices, are further solved in the least squares sense. Hence, we are solving linear system of equations using constrained linear least squares methods in which the additional constraints are strictly satisfied as show in Eq 14.

$$\begin{cases} \mathbf{K}\mathbf{u} = \mathbf{F} \\ \text{s.t. } \mathbf{C}\mathbf{u} = \mathbf{d} \end{cases} \quad (14)$$

here \mathbf{C} is the constrained matrix and \mathbf{d} is a virtual loading vector.

3.1 Position constraints at selected locations

When solving the PDE, we can apply additional position constraints that can be either for fixing selected positions of the computational domain, for directly fixing a selected number of control points of the final parameterization, or for fixing both positions of domain points or control points. In this case, the PDE system shown in Eq. 12 is extended as Eq. 15:

$$\begin{cases} \nabla^2 \mathbf{u} = 0 & \text{in } \Omega_0 \\ \mathbf{u} = \mathbf{X} & \text{on } \partial\Omega_0 \\ \mathbf{u}(\xi, \eta) = \mathbf{X}(\xi, \eta) & \text{in } \Omega_0 \\ \mathbf{u}_{IJ} = \hat{\mathbf{u}}_{IJ} & \text{for } 1 < I < n_\xi; 1 < J < n_\eta \end{cases} \quad (15)$$

3.2 A ratio constraint for controlling local degrees of freedom

The ratio among selected control points can also be used as a constraint to control the local degrees of freedom (DOFs), i.e., density of DOFs per unit dimension, for final parameterization. The PDE with respective ratio constraints for the basic topological structure is given in the following equation and is shown in Fig. 2(a),

$$\begin{cases} \nabla^2 \mathbf{u} = 0 & \text{in } \Omega_0 \\ \mathbf{u} = \mathbf{X} & \text{on } \partial\Omega_0 \\ (V_1 - V_0) = \lambda(V_2 - V_0) & \forall V_0 \in \Omega_0 \end{cases} \quad (16)$$

Here, $\{V_0, V_1, V_2\}$ are topologically aligned control points, either horizontally, vertically or diagonally aligned with respect to position V_0 . Some of possible schemes or topological structures are shown in

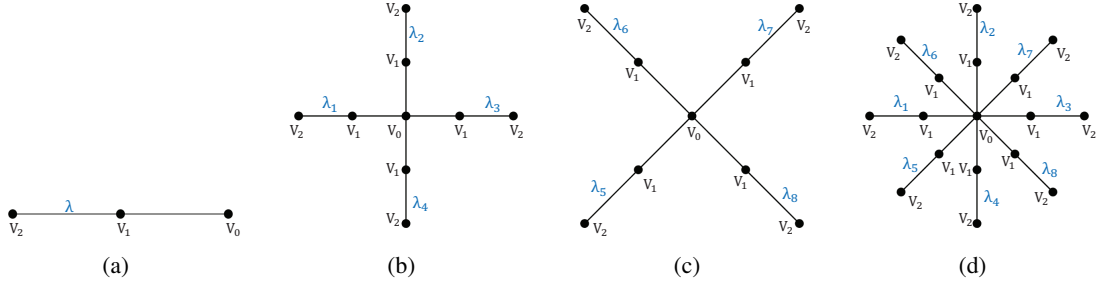


Figure 2: Basic schemes to control the ratio between control points. (a) basic Topology (b) scheme-I (c) scheme-II (d) scheme-III

Fig 2. The value of λ is defined as the distance ratio among these control points as shown in Eq. 16. When $\lambda < 0.5$, $\lambda = 0.5$ or $\lambda > 0.5$, one achieves local converging, uniform or diverging distribution of the control points, respectively. Or in other words, the local DOFs will increase with $\lambda < 0.5$ and the local DOFs will decrease with $\lambda > 0.5$, which uniform distribution of control points is achieved when $\lambda = 0.5$. The ratio constraint can be applied at positions of either control points or domain points.

4 Numerical results for parameterization with position and ratio constraints

4.1 An example with fixed position constraint

The first example is for producing a parameterization from a annulus boundary as shown in Fig 3(c). The parametric domain for (ξ, η) is defined as a simple square domain with $\hat{\Omega} = [0, 1]^2$ while the initial computational domain (x, y) is defined as a simple square domain $\Omega_0 = [0, 4]^2$ as shown in shown in Fig 3. In this case, the simple IGA-C-PDE method produces non-uniform parameterization near concave and

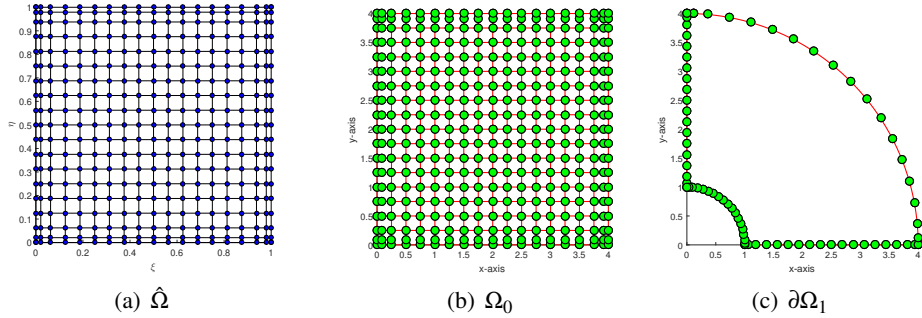


Figure 3: Initial parameterization of the computational domain using $p = q = 3$ uniform k -multiple end knots and with $n_\xi \times n_\eta = 361$ control points consistent with the given boundary curves for the annulus geometry: (a) parametric domain with knot lines and collocation points; (b) computational domain with control points; and (c) input concave boundary with control points.

convex boundaries (see Fig. Fig 4(d)-(f)), while the Coon's mesh produces a uniform parameterization (see Fig. Fig 4(a)-(c)). However, the Jacobian is satisfied in both of the parameterization as shown in Fig 4. By further applying a position constraint in fixing the center control point at the center position of the physical domain a uniform and regular parameterization is also achieved using the IGA-C-PDE

method (see Fig. Fig 4(g)-(i)). For constrained IGA-C-PDE as discussed in the previous section, a least squares solution is obtained while exactly meeting all constraints, including the boundary curves and the center position of the physical domain. This constraint can be very useful for constructing a clustering of control points at positions of the physical domain for achieving solution dependent parameterization.

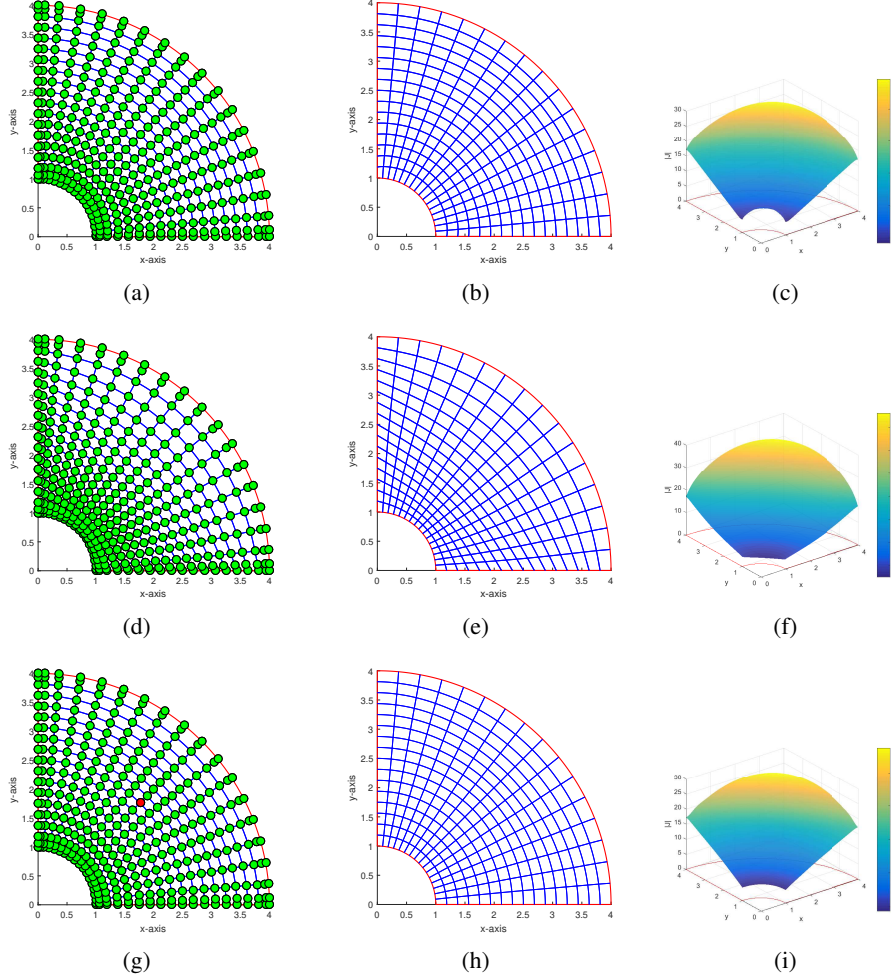


Figure 4: Resulting parameterization for the annulus geometry using Coon's method and our method based on IGA-C-PDE: (a)-(c) parameterization using the Coon's method; (d)-(f) parameterization using the unconstrained IGA-C-PDE method; (g)-(i) parameterization using the IGA-C-PDE method with center position constraints; (left column) parameterization with control points; (middle column) parameterization with parametric curves on the surface at knot lines; and (right column) Jacobian distribution over resulting computational domain.

4.2 An example for controlling the local DOFs using a ratio constraint

The distance ratio of control points is a constraint that can be useful to control the density of the control points in a local region. When $\lambda = 0.5$, the resulting parameterization will be uniformly distributed. An example in this case is first shown in Fig. 5 using a square computational domain for the three schemes

discussed in the previous section using an uniform $\lambda = 0.5$ ratio. Smooth and uniform parameterization is achieved in such cases. In this and other examples in this subsection with λ -ratio control, the initial parameterization of the computational domain is defined with $(\xi, \eta) \in \hat{\Omega} = [0, 1]^2$ using $p = q = 3$ with uniform k -multiple end knots and with $n_\xi \times n_\eta = 361$ control points consistent with the given boundary curves for the square domain: The initial computational domain (x, y) is defined as a simple square domain $\Omega_0 = [0, 6]^2$, while the final parameterization is defined for a square domain as well with $\Omega_1 = [0, 3]^2$.

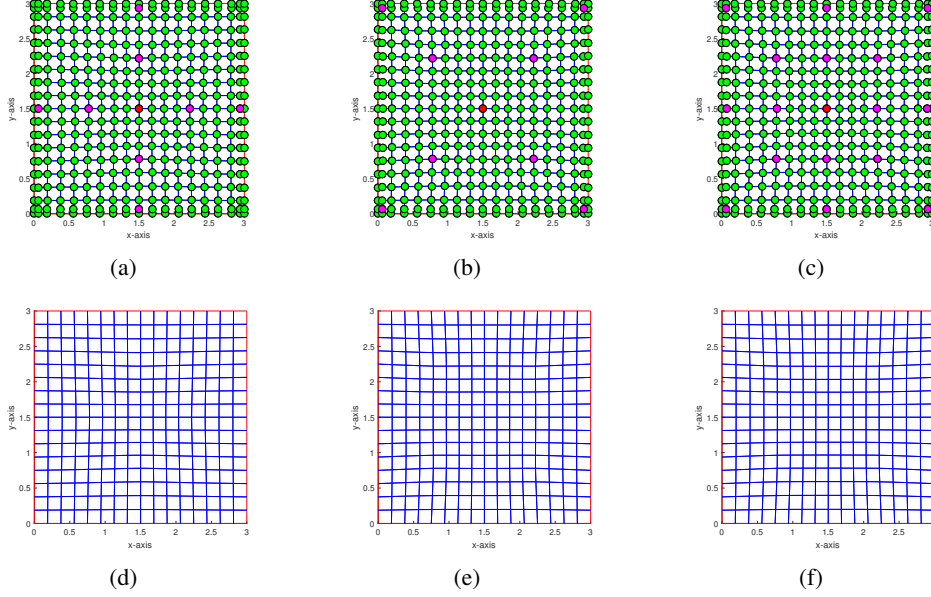


Figure 5: IGA-C-PDE based uniform parameterization from different ratio constraints (scheme-I, scheme-II and scheme-III), but with the same ratio uniform $\lambda = 0.5$: (upper row) resulting computational domain with control points and highlighted V_0, V_1, V_2 ; (bottom row) resulting computational domain with parameter lines; (left, middle and right columns) parameterization of the resulting computational domain from scheme-I, scheme-II and scheme-III, respectively.

The distribution of the inner control points can be controlled using the λ ratio parameter and we can purposefully control the degrees of freedom in a local region of V_0 for IGA with either higher or lower density as shown in Fig. 6. Combined with the position constraints, one can achieve centralized or decentralized and local clustered distribution of control points, i.e., with either higher density or lower density in terms of the local degrees of freedom for isogeometric analysis in any region centered at V_0 . In addition, by selecting the distance of from V_1, V_2 with reference to V_0 , the size of the region for achieving either higher or lower density of DOFs for IGA can also be controlled. One example with different position constraints for V_0 , but with the same λ -ration is shown in Fig. 7.

5 Application for thermal conduction analysis

The advantage of the new constraints that are introduced along with the elliptical PDE parameterization provides the flexibility in manipulating the parameterization in a desired manner. The constrained IGA-C-PDE method is thus particularly useful for constructing solution dependent parameterization. In the

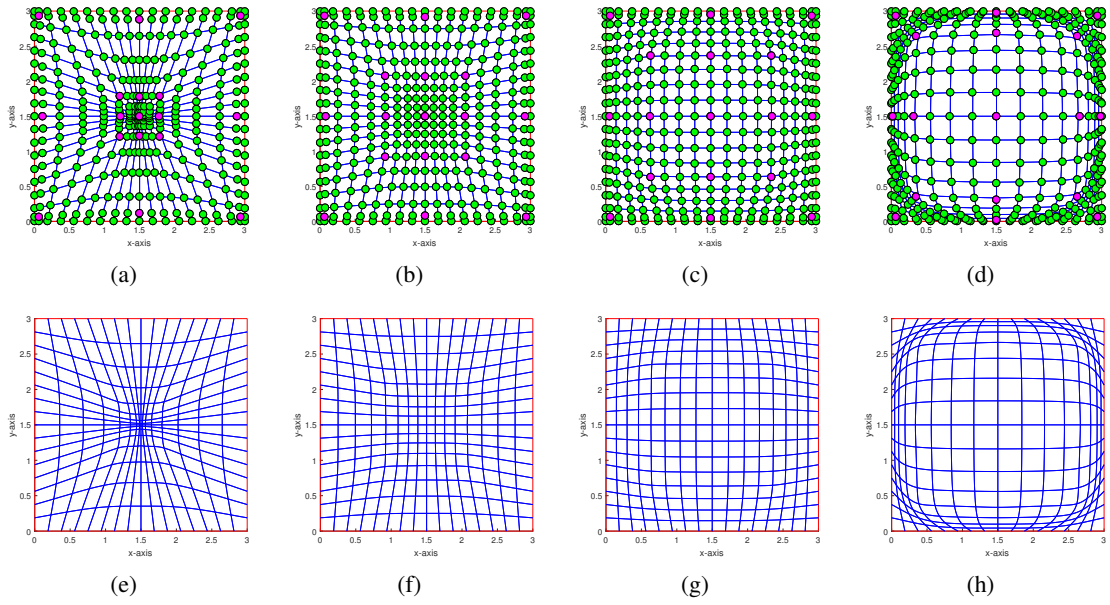


Figure 6: IGA-C-PDE based parameterization with different ratio λ constraints and with free V_0 : (top and middle rows) resulting parameterization of the computational domain with control points and knot lines display, respectively; (columns from left to right) resulting parameterization with ratio constraints at $\lambda = 0.2$, $\lambda = 0.4$, $\lambda = 0.6$ and $\lambda = 0.8$, respectively.

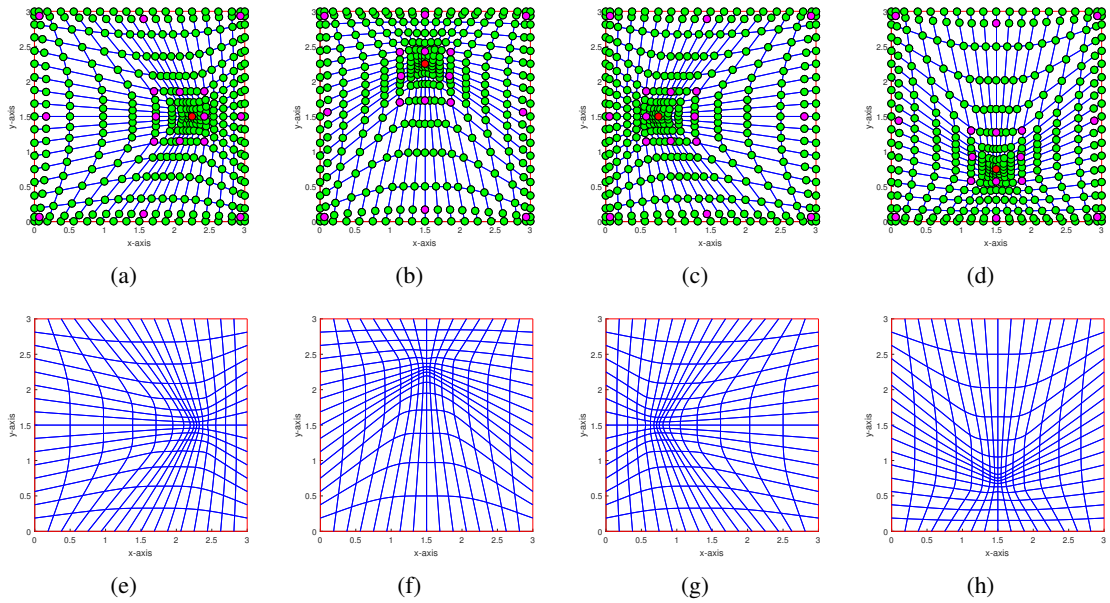


Figure 7: IGA-C-PDE based parameterization with fixed ratio $\lambda = 0.25$ constraints and with different position constraints applied at V_0 : (top and middle rows) resulting parameterization of the computational domain with control points and knot lines display, respectively; (columns from left to right) resulting parameterization with different position constraints at $V_0 = (2.25, 1.5)$, $V_0 = (1.5, 2.25)$, $V_0 = (0.75, 1.5)$ and $V_0 = (1.5, 0.75)$, respectively.

following, we show an examples in using the resulting parameterization for thermal conduction analysis. Given a physical domain enclosed by boundary $\Omega = \Omega_D \cup \Omega_N$, we consider the following thermal conduction problem

$$\begin{cases} \nabla[k(\mathbf{X})\nabla T(\mathbf{X})] = f(\mathbf{X}) & \text{in } \Omega \\ T(\mathbf{X}) = T_0(\mathbf{X}) & \text{on } \partial\Omega_D \\ k(\mathbf{X})\frac{\partial T}{\partial n}(\mathbf{X}) = q(\mathbf{X}) & \text{on } \partial\Omega_N \end{cases} \quad (17)$$

where \mathbf{X} stands for Cartesian coordinates, T represents the temperature field, k is the thermal conductivity, T_0 specifies the imposed temperature on boundary $\partial\Omega_D$, q is the thermal flux, and ∇T is the temperature gradient with $\nabla = \frac{\partial}{\partial \mathbf{X}}$. In order to see the importance of solution dependent parameteriza-

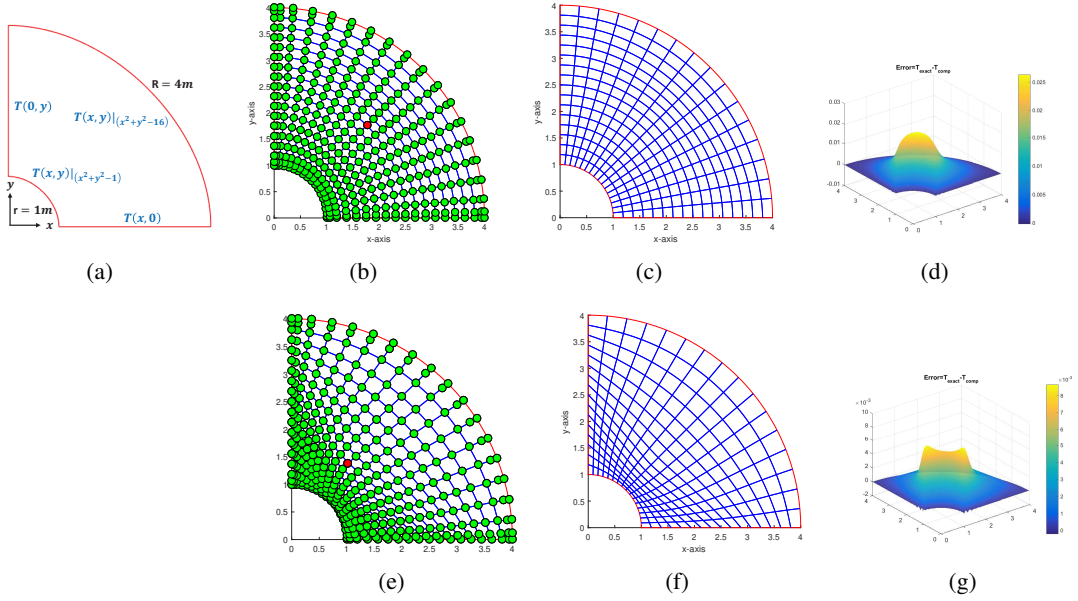


Figure 8: Parameterization using $p = q = 3$ with k -multiple end uniform knots and with $n_\xi \times n_\eta = 361$ for heat conduction test: (a) boundary conditions with $T(x, y) = \frac{1}{x+y} \cdot e^{-\frac{1}{x+y}}$; (b)-(d) solution using IGA-C-PDE based parameterization-I, a uniform regular parameterization; (e)-(g) solution using IGA-C-PDE based parameterization-II, a parameterization with purposefully shifted control vertices; (left and middle columns) the respective resulting computational domain; and (right column) the respective solution errors.

tion, we assume an annulus physical domain with quarter circular shape represented by a cubic B-spline surface. We have purposefully designed a temperature distribution function as the theoretical solution of the thermal conduction problem and the temperature function has highly varying local region near the inner circular edge of the annulus domain.

$$T(x, y) = \frac{1}{x+y} \cdot e^{-\frac{1}{x+y}} \quad (18)$$

Two parameterizations as shown in Fig. 17 are used for solving the heat conduction problem with exact solution given in Eq. (18) over the Annulus domain $[0, 4] \times [0, 4]$. Parameterization-I is a regular and uniform mesh while parameterization-II is motivated from the solution and design accordingly.

The solution is also produced using isogeometric collocation (IGA-C) method. As shown in Fig. 17, parameterization-II achieves a better and more accurate solution compared with that of parameterization-I. Although, parameterization-I is more uniform but for this particular problem we find that cluster of control points near the inner circular curve produces better solution with less error.

6 Conclusions

In this study, we apply the idea of PDE-based grid generation in FEA for analysis suitable (AS) parameterization in IGA based on isogeometric collocation (IGA-C) methods. Given an arbitrary boundary, the Laplace PDE is applied to produce a smooth parameterization of the computational domain defined by the input boundary for IGA applications. The PDE for parameterization is also solved using IGA-C methods with input boundary geometry as Dirichlet conditions and with inner control points of the computational domain as unknowns of the PDE. Additional position and ratio constraints along with the arbitrary input domain boundary have also been introduced in order to produce solution-dependent parameterization. With the present constrained IGA-C-PDE methods, B-splines/NURBS are used for parameterization and the solution can be efficiently solved using constrained linear least squares solutions, which is more efficient compared with most other known methods using non-linear optimization for analysis suitable parameterization for IGA. In our future work, the IGA-C-PDE method can also be extended for parameterization using other modeling schemes with complex domain boundaries.

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