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A Total Lagrangian Finite Element Formulation for the Geometrically Nonlinear Analysis of Shells

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1. INTRODUCTION

The analysis of structures subjected to large displacements by means of the finite element method has attracted the attention of many researchers in recent years and different publications have been reported in the literature [1]-[8].

In this work the authors suggest an alternative finite element formulation for the analysis of shells which allows for large displacements together with finite rotations. The deformation process of the structure is defined via a total Lagrangian approach. Stresses and strains over the shell surface are defined using a local set of cartesian axes based on the principal curvature directions of the shell middle surface. This allows to obtain useful explicit expressions of the finite element matrices in a simple manner. Additionaly, normals to the midsurface before deformation are assumed to remain straight but not necessarily normal to the midsurface after deformation, thus allowing for shear deformation effects. Finally, it is worth pointing out that no restrictions are made on the magnitude of the curvatures. This is of special interest for the analysis of non-shallow shells using an small number of elements.

The formulation uses two dimensional finite elements for the analysis of 3-D shells. The discretization over the shell thickness is eliminated using what is usually known as "degenerated element technique" [2,4]. This allows for a substantial reduction in the number of variables and eliminates the possibility of ill-conditioning of the element matrices which takes place when using 3-D elements and the thickness of the shell is small.

In the first part of this work the formulation for 3-D shells is presented. Then a series of examples of shells undergoing large displacements are presented.

2. GEOMETRIC DESCRIPTION

The middle surface of the shell can be expressed in parametric form as (see. Fig.1) [9]

$$\underset{\sim}{r}_0 = [\, x_0(\mu_1,\mu_2),\, y_0(\mu_1,\mu_2),\, z_0(\mu_1,\mu_2)\,]^T \tag{1}$$

where μ_1 and μ_2 are the principal curvature lines at point O of the shell middle surface.

Let $\vec{a}$ and $\vec{b}$ be unit vectors tangent to the curvature lines μ_1 and μ_2 in O, respectively, and $\vec{n}$ the normal vector to the middle surface in O. Parameters r,s and t are defined as the lenghts measured along the lines μ_1, μ_2 and along the normal $\vec{n}$, respectively.

A second set of orthogonal vectors $\vec{l}$,$\vec{m}$ and $\vec{n}$ is defined at O, as shown in Fig. 2. Vector $\vec{l}$ is taken as parallel to the global plane xz and tangent to the shell middle surface, $\vec{n}$ is the normal vector, previously defined, and $\vec{m}$ is orthogonal to the plane

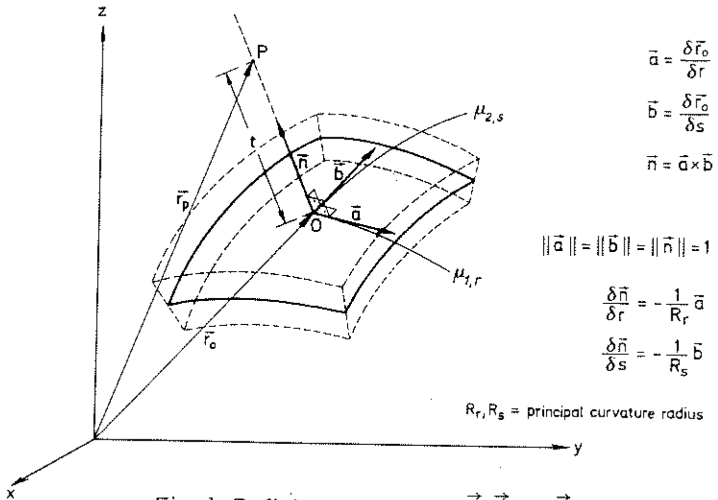


Fig. 1. Definition of vectors $\vec{a}$, $\vec{b}$ and $\vec{n}$

$\vec{n}$-$\vec{l}$ and also tangent to the shell middle surface at point 0. Note that $\vec{l}$ and $\vec{m}$ are not unit vectors and their modulus is $e=(1-n_y^2)^{1/2}$

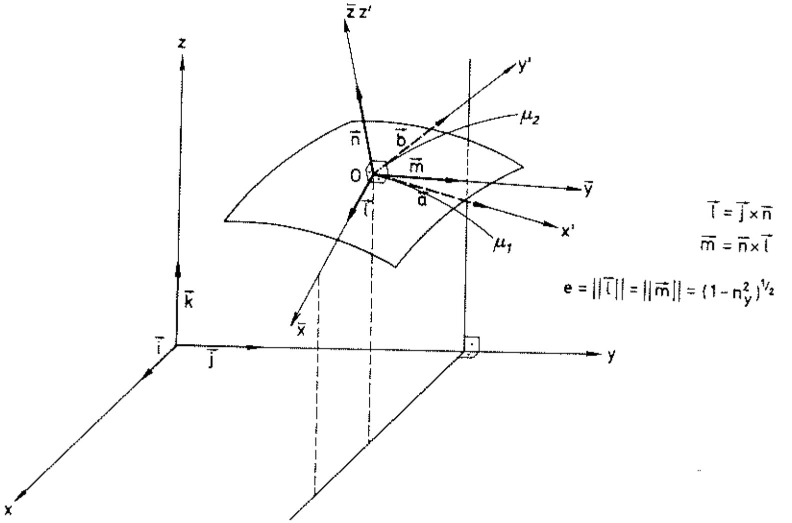


Fig. 2. Definition of vectors $\vec{l}$ and $\vec{m}$

The components of vectors $\vec{a},\vec{b},\vec{n},\vec{l}$ and $\vec{m}$ in the global reference system x,y,z (with associated unit vectors $\vec{i},\vec{j},\vec{k}$, respectively) can be written in matrix form as

$$\underset{\sim}{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} ; \quad \underset{\sim}{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} ; \quad \underset{\sim}{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} ; \quad \underset{\sim}{l} = \begin{bmatrix} l_x \\ l_y \\ l_z \end{bmatrix} ; \quad \underset{\sim}{m} = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} \tag{2}$$

Vectors $\vec{a},\vec{b}$ and $\vec{n}$ define a set of local axes x',y' and z', associated with the shell principal curvature lines. On the other hand, vectors $\vec{l},\vec{m}$ and $\vec{n}$ define a second set of local axes $\bar{x},\bar{y}$ and $\bar{z}$ which is easily identified within the structure ($\bar{x}$ is parallel to plane $\overline{xz}$, etc... see Fig. 2).

A point P over the shell middle surface can be defined by a vector $\vec{r}$ (see Fig. 1) such that we can write, in matrix form

$$\underset{\sim}{r} = \underset{\sim}{r}_p = \underset{\sim}{r}_0 + t\underset{\sim}{n} \tag{3}$$

where t is the distance measured along the normal.

Using eq.(3) and the relationships shown in Fig. 1 the following expressions can be obtained

$$\left[\frac{\partial(x,y,z)}{\partial(r,s,t)}\right] = \underset{\sim}{T}^T \underset{\sim}{R} \tag{4}$$

where $\underset{\sim}{T}$ is the Jacobian matrix of the transformation x'y'z' → xyz given by

$$\underset{\sim}{T} = [\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{n}]^T \tag{5}$$

and matrix $\underset{\sim}{R}$ is the Jacobian matrix of the transformation x'y'z' → rst given by

$$\underset{\sim}{R} = \begin{bmatrix} 1-\frac{t}{R_r} & 0 & 0 \\ 0 & 1-\frac{t}{R_s} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

Additionaly, vectors $\vec{l},\vec{m}$ and $\vec{n}$ can be defined in the system $\vec{a},\vec{b}$ and $\vec{n}$ using the following expression

$$[\underset{\sim}{l},\underset{\sim}{m},\underset{\sim}{n}] = [\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{n}]\,\bar{\underset{\sim}{T}} \tag{7}$$

where

$$\bar{\underset{\sim}{T}} = \begin{bmatrix} b_y & a_y & 0 \\ -a_y & b_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

For the finite element analysis we discretize the middle surface of the shell into a mesh of curved cuadrangular elements as shown in Fig. 3. The geometry of the shell is defined, within each element, in the standard isoparametric form [10] as

$$\underset{\sim}{r} = \sum_{i}^{n_e} N_i(\xi,\eta)\underset{\sim}{r}_{0_i} + \tau \frac{h}{2}\underset{\sim}{n} \tag{9}$$

where $\underset{\sim}{r}_{0_i}$ is the vector of each node of the finite element mesh, n_e the number of nodes per element, $N_i(\xi,\eta)$ is the shape function of node i, and ξ and η the normalized isoparametric coordinates. The third normalized coordinate is defined as

$$\tau = \frac{2t}{h} \tag{10}$$

where h is the thickness of the element. Finally, the normal vector $\vec{n}$ is obtained from the approximate middle surface given in (9) and the expressions (1)-(9). More details about the choice of element will be given in section 7.

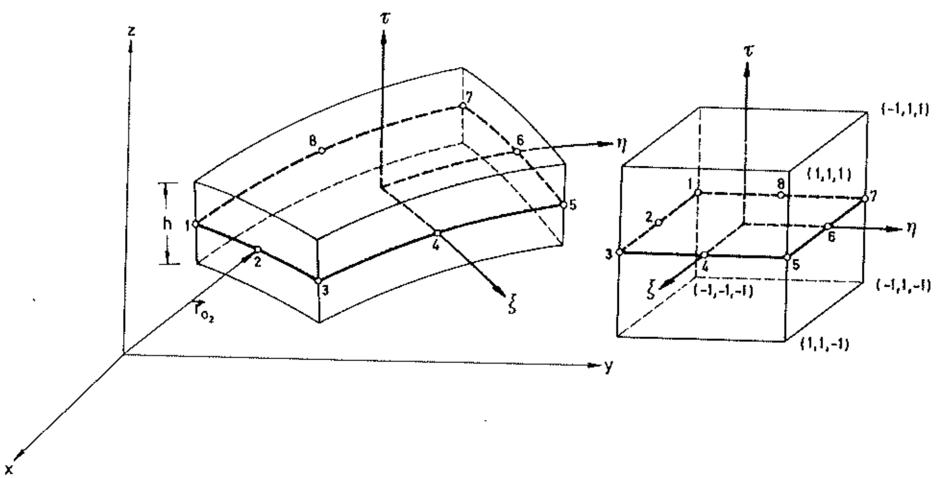


Fig. 3 Tipical eight node isoparametric element. Normalized coordinate system and associated normalized volume.

Eq.(9) transforms the element volume into a cube of unit side defined in the coordinate system ξ, η, τ [10](see Fig. 3.).

3. KINEMATIC DESCRIPTION

The deformation of the structure is based in the following two main assumptions:

a) Normals to the middle surface before deformation remain straight but not necessarily normal to the middle surface after deformation.

b) The lenght of the normal vector does not change during the deformation.

Assumption a) allows to express the displacement vector $\vec{u}$, of a point P (laying over the normal, $\vec{n}$, at a distance t from the corresponding point O over the middle surface (see Fig. 4), in terms of the displacement vector of point O, $\vec{u}_0$, and the relative displacement of the end of the normal vector in O, with respect to this point, $\vec{u}_1$, i.e.

$$\vec{u} = \vec{u}_0 + t\vec{u}_1 \tag{11}$$

We define

$$\underset{\sim}{u}_0 = [u_0, v_0, w_0]^T \quad \text{and} \quad \bar{\underset{\sim}{u}}_1 = [\bar{u}_1, \bar{v}_1, \bar{w}_1]^T \tag{12}$$

where u_0 ,v_0 ,w_0 refer to components of $\vec{u}_0$ in the global system $\vec{i}, \vec{j}, \vec{k}$, respectively, and $\bar{u}_1$,$\bar{v}_1$,$\bar{w}_1$ refer to components of $\vec{u}_1$ on the local system $\vec{l}, \vec{m}, \vec{n}$ (see Fig. 4).

The vector of "fundamental displacements" $\underset{\sim}{p}$ is defined now as

$$\underset{\sim}{p} = [\underset{\sim}{u}_0^T, \underset{\sim}{u}_1^T]^T \tag{13}$$

The components of $\underset{\sim}{p}$ define generically the displacement of any point of the shell. Using the finite element interpolation, it can be written

$$\underset{\sim}{p} = \sum_{1}^{n_e} \underset{\sim}{N}_i \underset{\sim}{p}_i \quad ; \quad \underset{\sim}{N}_i = N_i \underset{\sim}{I}_6 \tag{14}$$

where $\underset{\sim}{p}_i$ is the value of p at node i.

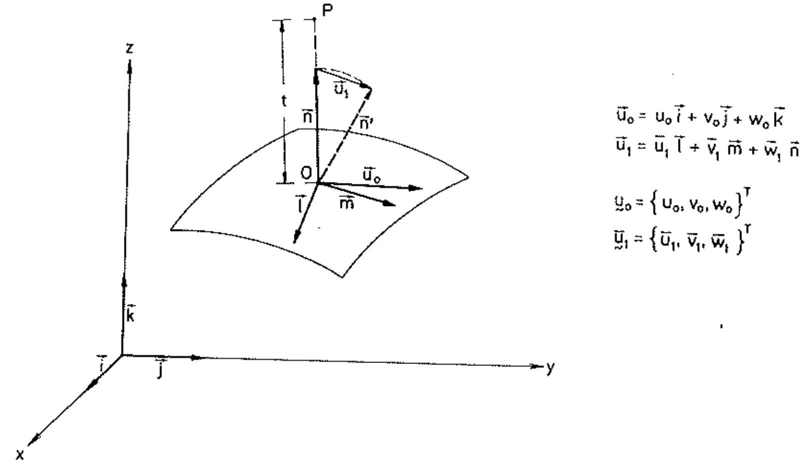
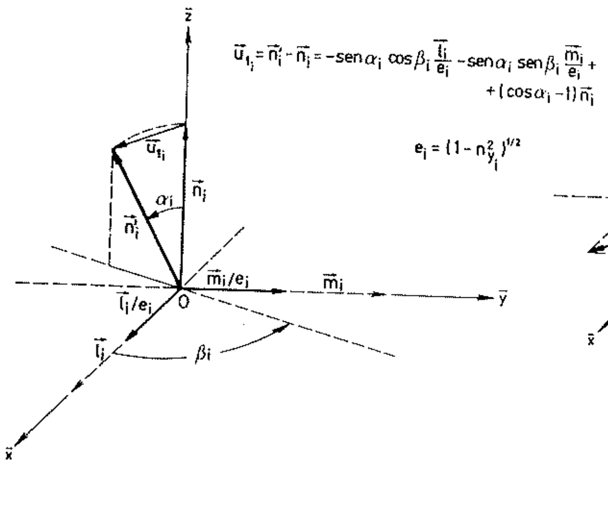


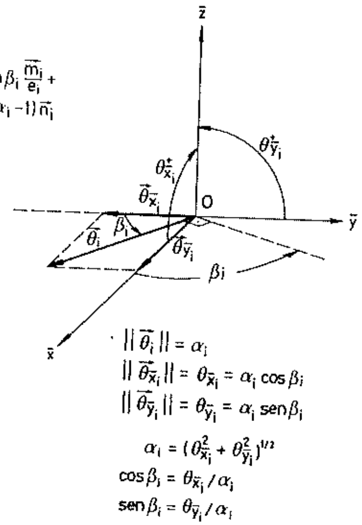
Fig. 4. Definition of displacements.

Assumption b) allows to express the components of $\bar{\underset{\sim}{u}}_1$, at node i, as

$$\bar{\underset{\sim}{u}}_{1_i} = \left[\frac{-\sin\alpha_i \cos\beta_i}{e_i}, \frac{-\sin\alpha_i \sin\beta_i}{e_i}, \cos\alpha_i - 1\right]^T \tag{15}$$

Figs. 5 and 6 show the relationships between angles α_i and β_i and the two components of the rotation vector $\vec{\theta}_i$, which expresses the anti-clockwise rotation of the normal.





We define the vector of displacements of node i as

$$\underset{\sim}{a} = [\underset{\sim}{u}_{0_i}^T, \theta_{\bar{x}_i}, \theta_{\bar{y}_i}]^T \qquad (16)$$

where vector $\underset{\sim}{u}_{0_i}$ is obtained from eq.(12). On the other hand, the relationships linking the rotations and the components of vector $\underset{\sim}{\bar{u}}_1$ can be deduced from eq.(15) and the expressions of Fig. 6.

4. STRAIN FIELD

Let's define $\underset{\sim}{u}' = [u', v', w']$, as the vector containing the components of the displacement vector $\underset{\sim}{u}$ of **any** point of the shell measured in the coordinate system defined by vectors $\vec{a}, \vec{b}$ and $\vec{n}$, corresponding at a **particular** point, 0, laying over the shell middle surface.

The vector of displacement gradients at a point, P, laying over the normal vector in 0 (see Fig. 4), can be defined now as

$$\underset{\sim}{g} = [\underset{\sim}{g}_1^T, \underset{\sim}{g}_2^T, \underset{\sim}{g}_3^T]^T \qquad (17)$$

with

$$\underset{\sim}{g}_1 = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial u'}{\partial x'} \\ \frac{\partial v'}{\partial x'} \\ \frac{\partial w'}{\partial x'} \end{bmatrix}_P \qquad \underset{\sim}{g}_2 = \begin{bmatrix} g_4 \\ g_5 \\ g_6 \end{bmatrix} = \begin{bmatrix} \frac{\partial u'}{\partial y'} \\ \frac{\partial v'}{\partial y'} \\ \frac{\partial w'}{\partial y'} \end{bmatrix}_P \qquad \underset{\sim}{g}_3 = \begin{bmatrix} g_7 \\ g_8 \\ g_9 \end{bmatrix} = \begin{bmatrix} \frac{\partial u'}{\partial z'} \\ \frac{\partial v'}{\partial z'} \\ \frac{\partial w'}{\partial z'} \end{bmatrix}_P \qquad (18)$$

where subscrit P denotes values in point P.

The Green strain vector at point P, (associated to the local directions x',y',z' of Fig. 2) can be written, using eqs.(18), as

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \gamma_{x'y'} \\ \gamma_{x'z'} \\ \gamma_{y'z'} \end{bmatrix} = \begin{bmatrix} g_1 + \frac{1}{2}(g_1^2 + g_2^2 + g_3^2) \\ g_5 + \frac{1}{2}(g_4^2 + g_5^2 + g_6^2) \\ g_2 + g_4 + g_1 g_4 + g_2 g_5 + g_3 g_6 \\ g_2 + g_7 + g_1 g_7 + g_2 g_8 + g_3 g_9 \\ g_6 + g_8 + g_4 g_7 + g_5 g_8 + g_6 g_9 \end{bmatrix} \qquad (19)$$

On the other hand, vector $\underset{\sim}{g}$ can be explicitely obtained in terms of the fundamental displacement vector $\underset{\sim}{p}$, of eq.(13) as [11,12]

$$\underset{\sim}{g} = \begin{bmatrix} C_r \underset{\sim}{T}\frac{\partial}{\partial r} & tC_r(\underset{\sim}{\Delta}_1 + \underset{\sim}{\bar{T}}\frac{\partial}{\partial r}) \\ C_s \underset{\sim}{T}\frac{\partial}{\partial s} & tC_s(\underset{\sim}{\Delta}_2 + \underset{\sim}{\bar{T}}\frac{\partial}{\partial s}) \\ \underset{\sim}{0} & \underset{\sim}{T} \end{bmatrix} \underset{\sim}{p} = \underset{\sim}{L}\,\underset{\sim}{p} \qquad (20)$$

with

$$C_r = \frac{1}{1 - \frac{t}{R_r}} \quad ; \quad C_s = \frac{1}{1 - \frac{t}{R_s}} \qquad (21)$$

$$\underset{\sim}{\Delta}_1 = \frac{1}{R_r}\begin{bmatrix} 0 & ny & -1 \\ -n_y & 0 & 0 \\ b_y & a_y & 0 \end{bmatrix} \qquad \underset{\sim}{\Delta}_2 = \frac{1}{R_s}\begin{bmatrix} n_y & 0 & 0 \\ 0 & n_y & -1 \\ -a_y & b_y & 0 \end{bmatrix} \qquad (22)$$

Substituting eq.(14) in (20) it can be finally written

$$\underset{\sim}{g} = \underset{\sim}{L} \sum_1^{n_e} \underset{\sim}{N}_i \underset{\sim}{p}_i = \sum_1^{n_e} \underset{\sim}{M}_i \underset{\sim}{p}_i \qquad (23)$$

where

$$\underset{\sim}{M}_i = \left[\underbrace{\begin{array}{c} C_r \underset{\sim}{T} \frac{\partial N_i}{\partial r} \\ C_s \underset{\sim}{T} \frac{\partial N_i}{\partial s} \\ \underset{\sim}{0} \end{array}}_{\bar{M}_i} \quad \underbrace{\begin{array}{c} t\, C_r(N_i \underset{\sim}{\Delta}_1 + \frac{\partial N_i}{\partial r} \underset{\sim}{\bar{T}}) \\ t\, C_s(N_i \underset{\sim}{\Delta}_2 + \frac{\partial N_i}{\partial s} \underset{\sim}{\bar{T}}) \\ N_i \cdot \underset{\sim}{\bar{T}} \end{array}}_{\bar{\bar{M}}_i} \right] = [\underset{\sim}{\bar{M}}_i, \underset{\sim}{\bar{\bar{M}}}_i] \qquad (24)$$

Taking into account the linearity of matrix $\underset{\sim}{M}_i$ (independent of the nodal unknowns) we can write

$$\delta \underset{\sim}{g} = \sum_1^{n_e} \underset{\sim}{M}_i \, \delta \underset{\sim}{p}_i \qquad (25)$$

It can be shown that a relationship between fundamental displacements and nodal displacement increments can be obtained in the form [11]

$$\delta \underset{\sim}{p}_i = \underset{\sim}{C}_i \, \delta \underset{\sim}{a}_i \qquad (26)$$

where

$$\underset{\sim}{C}_i = \begin{bmatrix} \underset{\sim}{I}_3 & 0 \\ \underset{\sim}{0} & \underset{\sim}{V}_i \end{bmatrix} \qquad \underset{\sim}{V}_i = \begin{bmatrix} V_{1i} & V_{2i} \\ V_{2i} & V_{3i} \\ V_{4i} & V_{5i} \end{bmatrix} \qquad (27)$$

$$V_{1i} = -\frac{1}{e_i}(\cos\alpha_i \cos^2\beta_i + \frac{\sin\alpha_i}{\alpha_i} \sin^2\beta_i)$$

$$V_{2i} = -\frac{1}{e_i} \sin\beta_i \cos\beta_i \, (\cos\alpha_i - \frac{\sin\alpha_i}{\alpha_i}) \qquad \alpha_i = (\theta_{\bar{x}_i}^2 + \theta_{\bar{y}_i}^2)^{\frac{1}{2}}$$

$$V_{3i} = -\frac{1}{e_i}(\cos\alpha_i \sin^2\beta_i + \frac{\sin\alpha_i}{\alpha_i} \cos^2\beta_i \qquad \cos\beta_i = \frac{\theta_{\bar{x}_i}}{\alpha_i} \tag{28}$$

$$V_{4i} = -\sin\alpha_i \cos\beta_i \qquad \sin\beta_i = \frac{\theta_{\bar{y}_i}}{\alpha_i}.$$

$$V_{5i} = -\sin\alpha_i \sin\beta_i$$

Thus, after substituting eq.(26) in eq.(25), it is finally found

$$\delta \underset{\sim}{g} = \sum_1^{n_e} \underset{\sim}{M}_i \underset{\sim}{C}_i \, \delta \underset{\sim}{a}_i = \sum_1^{n_e} \underset{\sim}{G}_i \, \delta \underset{\sim}{a}_i \tag{29}$$

where matrix $\underset{\sim}{G}_i = \underset{\sim}{M}_i \underset{\sim}{C}_i$ can be explicitely obtained using eqs.(24), (27) and (28).

The relationship between Green strains and nodal displacements is simply obtained from eqs.(19) and (29) as

$$\delta \underset{\sim}{\epsilon} = \underset{\sim}{A} \, \delta \underset{\sim}{g} = \underset{\sim}{A} \sum_1^{n_e} \underset{\sim}{G}_i \, \delta \underset{\sim}{a}_i = \sum_1^{n_e} \underset{\sim}{B}_i \, \delta \underset{\sim}{a}_i \tag{30}$$

Explicit expressions of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ matrices can be found in references [11] and [12].

5. CONSTITUTIVE EQUATIONS

The constitutive relationship must be incorporated to the general formulation in an incremental or rate form. This can be written generically as

$$\delta \underset{\sim}{\sigma} = \underset{\sim}{D}^* \, \delta \underset{\sim}{\varepsilon} \tag{31}$$

where $\underset{\sim}{\sigma} = [\sigma x', \sigma y', \tau x'y', \tau x'y', \tau y'z']^T$ is the second Piola-Kirchoff stress vector.

We will not go here into details of the different alternative forms of matrix $\underset{\sim}{D}^*$ for the various types of non-linear material behaviour. A comprehensive study in this subject can be found in reference [13].

6. DISCRETIZED EQUILIBRIUM EQUATIONS

The simplest procedure to obtain the discretized form of the equilibrium equations in a general way is to make use of the virtual work expression which in the total Lagrangian formulation can be written as

$$\int_V \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma} dv - \int_V \delta \underset{\sim}{u}^T \underset{\sim}{b} dV - \int_\Gamma \delta \underset{\sim}{u}^T \underset{\sim}{t} d\Gamma = 0 \tag{32}$$

where $\underset{\sim}{\varepsilon}$, $\underset{\sim}{u}$ and $\underset{\sim}{\sigma}$ have been defined in previous sections. V and Γ are the undeformed volume and surface of the structure, over which the body forces $\underset{\sim}{b}$ and surface loads $\underset{\sim}{t}$ respectively act. Vectors $\underset{\sim}{b}$ and $\underset{\sim}{t}$ are defined by

$$\underset{\sim}{b} = [b_x, b_y, b_z]^T \quad \text{and} \quad \underset{\sim}{t} = [t_x, t_y, t_z]^T \tag{33}$$

where x,y and z refer to components in the global systems of Fig. 2.

Using eqs.(30) and (11)-(14) we can write the discretized finite element form of eq.(32) in the following standard manner [10].

$$\underset{\sim}{\Psi}(\underset{\sim}{a}) = \underset{\sim}{P}(\underset{\sim}{a}) - \underset{\sim}{R} = \underset{\sim}{0} \tag{34}$$

which for the ith node gives

$$\underset{\sim}{\Psi}_i(\underset{\sim}{a}) = \int_V \underset{\sim}{B}_i^T \underset{\sim}{\sigma} dv - \underset{\sim}{R}_i = \underset{\sim}{0} \tag{35}$$

In eq.(34), $\underset{\sim}{\Psi}(a)$ is the residual force vector and $\underset{\sim}{R}$ is the equivalent nodal force vector due to exterior loads given by

Surface loads

$$\underset{\sim}{R}_i = \int_A [N_i t_x, N_i t_y, N_i t_z, N_i M_{\bar{x}_i}, N_i M_{\bar{y}_i}]^T dA \tag{36}$$

where $M_{\bar{x}_i}$ and $M_{\bar{y}_i}$ are the components of the surface moment in the local axes $\bar{x}$ and $\bar{y}$ of Fig. 2.

Body forces

$$\underset{\sim}{R}_i = \int_V [N_i b_x, N_i b_y, N_i b_z, 0, 0]^T \, dV \tag{37}$$

All the integrals which appear in eqs.(32)-(37) are evaluated assembling the contributions from the different elements following standard finite element procedures [10].

Eq.(34) is a system of non linear equations which must be solved using an iterative numerical technique. From the many existing procedures [14] an standard Newton-Raphson algorithm [10] has been chosen here. Thus, the corresponding displacement increment vector for the nth iteration is calculated by

$$\Delta \underset{\sim}{a}^n = -\underset{\sim}{K}_T(\underset{\sim}{a}^n) \, \underset{\sim}{\psi}(\underset{\sim}{a}^n) \tag{38}$$

from which the updated displacement field can be computed as $\underset{\sim}{a}^{n+1} = \underset{\sim}{a}^n + \Delta \underset{\sim}{a}^n$. Iterations stop when the values of the residual forces are suficiently small.

In eq.(38) $\underset{\sim}{K}_T$ is the, so called, tangent matrix, obtained as $K_T(a) = \frac{\partial \underset{\sim}{\Psi}(a)}{\partial \underset{\sim}{a}}$.

A typical submatrix relating nodes i and j can be obtained as follows

$$\underset{\sim}{K}_{T_{ij}}(\underset{\sim}{a}) = \frac{\partial \Psi_i(\underset{\sim}{a})}{\partial \underset{\sim}{a}_j} \tag{39}$$

Taking the first variation of eq.(35) we obtain

$$\delta\Psi_i(\underset{\sim}{a}) = \sum_{j=1}^{n} \underset{\sim}{K}_{T_{ij}}\, \delta\underset{\sim}{a}_j = \int_v \delta\underset{\sim}{B}_i^T \underset{\sim}{\sigma} dv + \int_v \underset{\sim}{B}_i^T \delta\underset{\sim}{\sigma} dv \tag{40}$$

It can be shown that [12] .

$$\int_v \underset{\sim}{B}_i^T \delta\underset{\sim}{\sigma} dv = \sum_{j=1}^{n} \underset{\sim}{K}_{ij}^L \delta\underset{\sim}{a}_j \tag{41}$$

$$\int_v \delta\underset{\sim}{B}_i^T \underset{\sim}{\sigma} dv = \sum_{j=1}^{n} (\underset{\sim}{K}_{ij}^{\sigma_I} + \underset{\sim}{K}_{ij}^{\sigma_{II}})\, \delta\underset{\sim}{a}_j \tag{42}$$

where the different components of the tangent matrix can be evaluated as follows

a) $\underset{\sim}{K}_{ij}^L$

$$\underset{\sim}{K}_{ij}^L = \int_v \underset{\sim}{B}_i^T \underset{\sim}{D}^* \underset{\sim}{B}_j \, dv \tag{43}$$

where matrix $\underset{\sim}{B}$ can be obtained from eqs.(30)

b) $\underset{\sim}{K}_{ij}^{\sigma_I}$

$$\underset{\sim}{K}_{ij}^{\sigma_I} = \int_v \underset{\sim}{G}_i^T \underset{\sim}{S} \underset{\sim}{G}_j \, dv \tag{44}$$

where matrix $\underset{\sim}{G}$ is obtained from eqs.(29) and matrix $\underset{\sim}{S}$ is given by

$$\underset{\sim}{S} = \begin{bmatrix} \sigma x'\underset{\sim}{I}_3 & \tau x'y'\underset{\sim}{I}_3 & \tau x'z'\underset{\sim}{I}_3 \\ \tau x'y'\underset{\sim}{I}_3 & \sigma y'\underset{\sim}{I}_3 & \tau y'z'\underset{\sim}{I}_3 \\ \tau x'z'\underset{\sim}{I}_3 & \tau y'z'\underset{\sim}{I}_3 & 0 \end{bmatrix} \tag{45}$$

c) $\underset{\sim}{K}_{ij}^{\sigma_{II}}$

$$\underset{\sim}{K}_{ij}^{\sigma_{II}} = 0 \qquad \text{for} \qquad i \neq j \tag{46}$$

$$\underset{\sim}{K}_{ij}^{\sigma_{II}} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{H}_i \end{bmatrix} \tag{47}$$

where $\underset{\sim}{H}$ is obtained as:

$$\underset{\sim}{H}_i(2\times 2) = \int_v \left[\frac{\partial \underset{\sim}{V}_i^T}{\partial\theta_{\bar{x}_i}} \cdot \underset{\sim}{F} \,,\; \frac{\partial \underset{\sim}{V}_i^T}{\partial\theta_{\bar{y}_i}} \cdot \underset{\sim}{F} \right] dv \,,\quad \underset{\sim}{F} = \bar{\bar{\underset{\sim}{M}}}_i^T \underset{\sim}{A}^T \underset{\sim}{\sigma} \tag{48}$$

Explicit expressions of matrices $\frac{\partial \underset{\sim}{V}_i^T}{\partial\theta_{\bar{x}_i}}$ and $\frac{\partial \underset{\sim}{V}_i^T}{\partial\theta_{\bar{y}_i}}$ can be found in references [11] and [12].

7. NUMERICAL COMPUTATION OF THE INTEGRALS AND FINITE ELEMENT CHOSEN

All the different integrals are numerically evaluated using a Gauss-Legendre quadrature 10 which gives

$$\int_v f(x',y',z')\,dv = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} g(\xi,\eta,\zeta)\,d\xi d\eta d\zeta \tag{49}$$

It can be shown [11] that the following relationships exists between differentials of volume:

$$dV = dx'dy'dz' = \left(1-\frac{t}{R_r}\right)\left(1-\frac{t}{R_s}\right)\frac{h}{2}\,\frac{\frac{\partial x_o}{\partial\xi}\frac{\partial y_o}{\partial\eta} - \frac{\partial x_o}{\partial\eta}\frac{\partial y_o}{\partial\xi}}{\frac{\partial x_o}{\partial r}\frac{\partial y_o}{\partial s} - \frac{\partial x_o}{\partial s}\frac{\partial y_o}{\partial r}}\, d\xi\, d\eta\, d\zeta \tag{50}$$

where x_0 and y_0 are components of vector $\underset{\sim}{r}_0$ of eq.(3).

Eq.(50) allows to use the normalized volume in ξ , η and ζ coordinates (see Fig. 3) for the numerical integration of all volume integrals. Same procedure applies for the surface integrals and will not be given here.

With respect to the type of element chosen we have to note that the formulation here presented is, in fact, a "thick" shell formulation, as the terms of shear deformation are included in the analysis. We, therefore, have to be aware that, when using this formulation in the context of very thin shell analysis, some precautions must be taken. It is well known that, for very thin shells or plates, unrealistic overstiff numerical results can be found due to the overestimation of the shear terms which tend to dominate the solution. This numerical error has been studied by many researchers and different alternatives to overcome the problem have been suggested [10]. Here we use one of the simplest procedures which consists in underintegrating the terms due to shear in the numerical integration of the stiffness matrix. This method, commonly known as "reduced integration technique", has been extensively used with success by many authors in the context of thick and thin shell and plate analysis [15]. For the examples presented in the paper, the eight node isoparametric element (Fig. 3) has been chosen with the following integrating rule:

a) A 2x2 Gauss-Legendre rule along directions ξ and η over the middle surface of the shell.

b) A 2 point integration rule over the thickness (τ direction).

8. NUMERICAL EXAMPLES

Example 1. Parabolic Shell

The geometry of the shell and finite element mesh used can be seen in Fig. 7. The problem has been analyzed by a process of incrementing the central deflection from a zero initial value to a value of 50".

Numerical results for the central displacement load elastic curve obtained for different types of boundary conditions at the shell edges are presented in Fig. 8. Numerical results obtained by Wood with para-linear three-dimensional shell elements [7] are also shown for comparison. Note the differences between the boundary conditions imposed by Wood and those used in the present analysis.

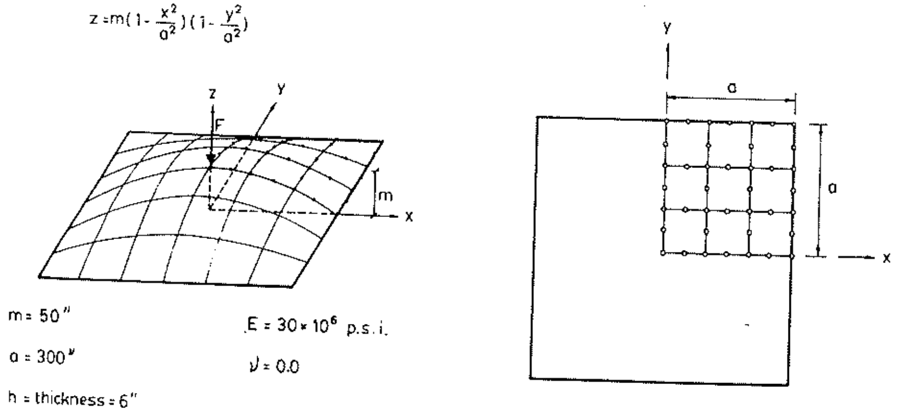


Fig. 7. Parabolic shell under central point load. Geometry, material properties and finite element mesh.

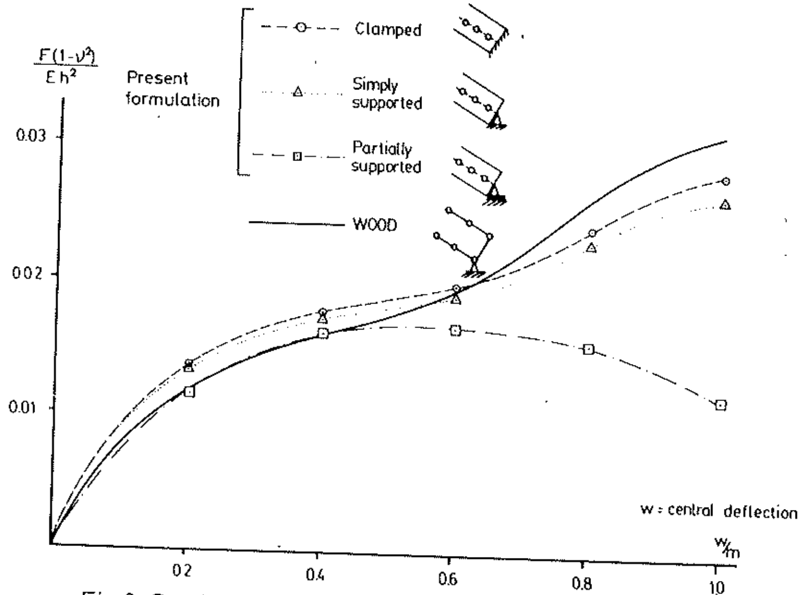


Fig 8. Parabolic shell. Central displacement versus load for different types of boundary conditions.

Example 2. Cylindrical shell.

The cylindrical shell studied can be seen in Fig. 9. Note that only four elements have been used to discretize one quarter of the structure (due to symmetry). The shell is assumed to be simply supported on its straight edges and free on the curved ones.

In the elastic analysis presented an increasing displacement, w, is prescribed in point A and the corresponding vertical reaction force, F, obtained. In Fig. 11 a plot of the value of F and the vertical displacement of point B, at the center of the free edge, versus w is presented for two different values of the shell thickness.

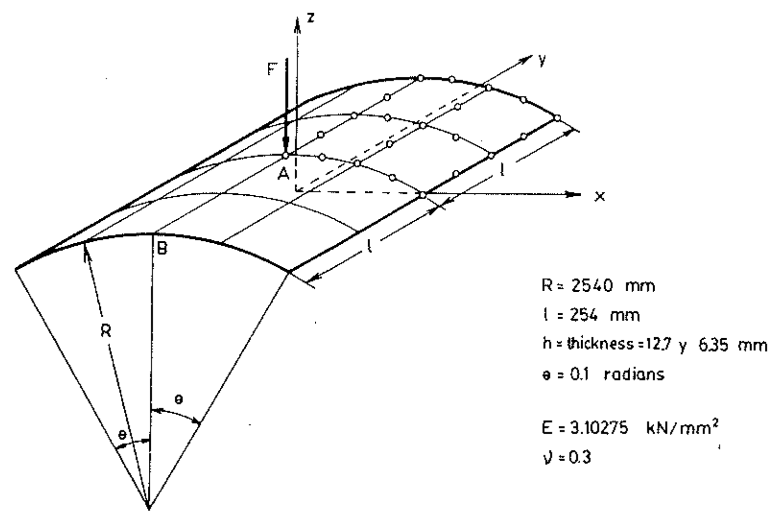


Fig. 9. Cylindrical shell under central point load. Geometry, material properties and finite element mesh.

Note the good agreement between the results obtained with the present formulation and those given by Surana [16] and Sabir and Lock [5] for the same problem.

Example 3. Spherical shell.

The last example is the analysis of a spherical shell with fully clamped edges under an increasing vertical point load acting in the center of the shell, as it can be see in Fig. 10.

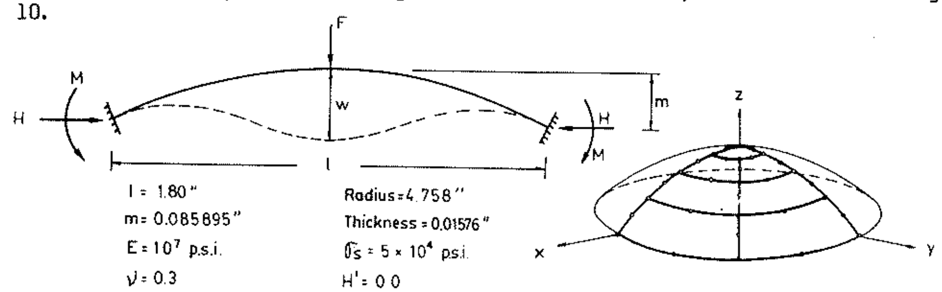


Fig. 10. Spherical shell under central point load. Geometry, material properties and finite element mesh.

Again, due to symmetry, only a quarter of the structure has been considered. A mesh of seven eight noded elements has been used. Results for the load/deflection at the center have been plotted in Fig. 12 and good agreement with results obtained by Wood [7] , with a two-dimensional axi-symmetric formulation, is obtained.

9. FINAL REMARKS

A total Lagrangian finite element formulation for the geometricaly non linear analysis of 3-D shells with large displacement and finite rotations has been presented. The formulation allows for shear deformation effects and large curvatures in the shell surface. Explicit forms of all finite element matrices have been obtained via the use of a local coordinate system based on the principal curvature directions. The

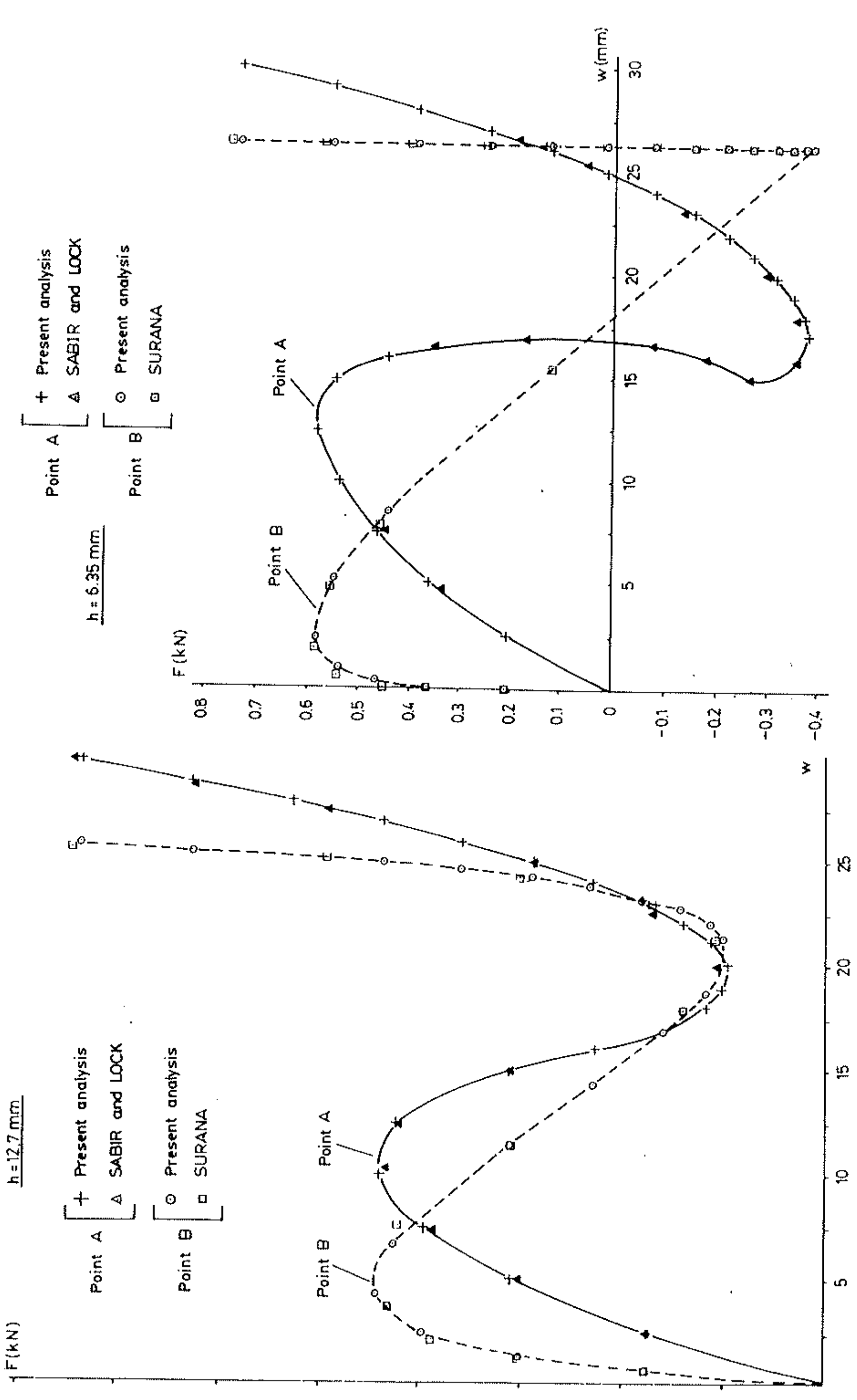


Fig. 11. Cylindrical shell. Deflections versus load for different thickness values.

accuracy of the formulation has been checked in a series of examples of large displacement analysis of shells.

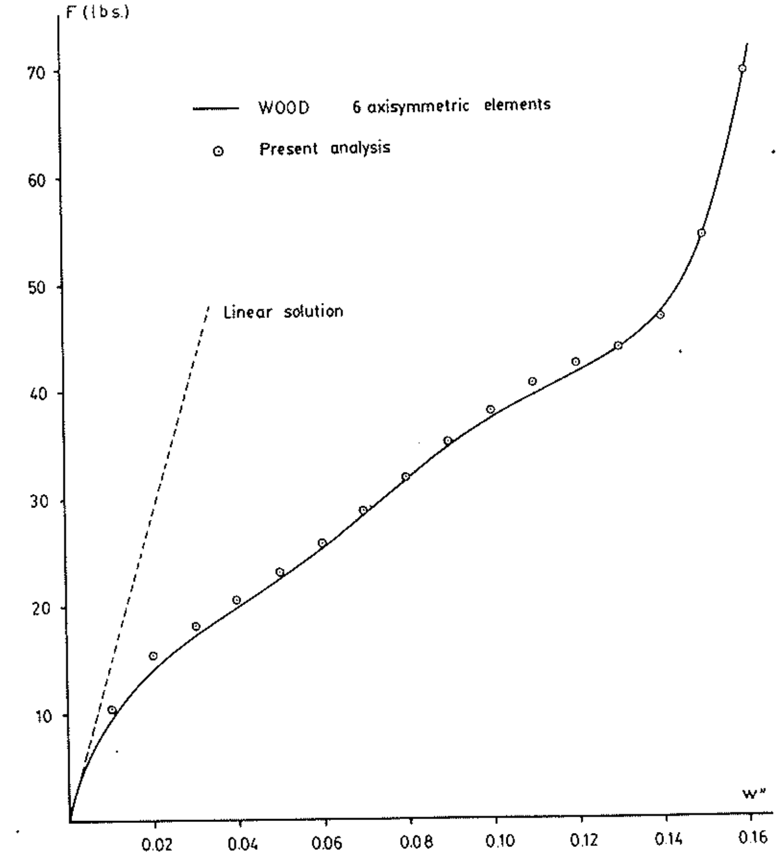


Fig. 12. Spherical shell. Central deflection versus load.

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