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A COMPARISON OF THE LINEAR, QUADRATIC AND CUBIC MINDLIN STRIP ELEMENTS FOR THE ANALYSIS OF THICK AND THIN PLATES

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Abstract—The behaviour of the linear, quadratic and cubic elements of the Mindlin plate strip family for thick and very thin plate analysis is investigated in this paper. Selective integration techniques are used to ensure the good behaviour of the elements when dealing with thin plates. Numerical results showing the convergence and accuracy of the elements for the analysis of plates of a wide range of thicknesses are given. The general performance of the three elements is discussed in detail. In particular, the linear element with a single integration point seems to be the best value strip element for practical purposes.

INTRODUCTION

The finite strip method, which combines one dimensional finite elements [1] with harmonic expansions, has been extensively used in the last few years to analyse a wide range of plate and bridge deck problems [2, 3]. Initial finite strip plate elements were developed following classical Kirchhoff thin plate theory [4, 5]. These elements work well for thin situations but are unable to reproduce shear effects when the thickness of the plate is thick, as is the case in prestressed slabs and many practical concrete box girde bridge situations.

In addition, Kirchhoff plate elements present several other well known problems due to the need of continuity of deflections and its first derivatives between adjacent elements (C_1 continuity) which in most cases leads to nonconforming elements which do not satisfy such criterion [1]. This disadvantage can, however, be ignored in many cases, and most Kirchhoff plate elements can be used "illegaly" with success for practical thin plate bending analysis.

One of the ways of overcoming the above problems is using Mindlin's plate theory [6] to derive plate and plate strip elements. In Mindlin plate theory slopes and deflections are independent variables, therefore, continuity requirements do not involve first derivatives of deflections and slopes and deflections must be continuous independently (Co continuity) [1]. In addition, Mindlin's theory includes the effect of transverse shear deformation and thus is applicable to thick, thin and sandwich plate problems [7, 9]. Mindlin's plate elements worked well initially for thick and moderately thin situations. However, despite its theoretical versatility, Mindlin's elements did not behave well for thin and very thin plate problems and overstiff unrealistic numerical results were obtained in many cases [10]. This spurious behaviour is nowadays fully understood, and it can be easily explained if it is noticed that when the plate becomes very thin, the shear terms of the stiffness matrix become very big in comparison with the bending ones, and tend to dominate the solution [1].

One of the more successfull ways of overcoming this problem is to relax the shear constraint integrating the shear terms of the stiffness matrix with a quadrature order less than that needed for its exact integration. This procedure, commonly known as "selective integration" technique, has become very popular and it has been extensively used in the last few years for the development of many accurate plate and shell elements valid for both thick and very thin situations [11, 14].

Extensions of Mindlin's plate theory to derive a family of Mindlin's strip elements was the next logical step and many solutions for thick and moderately thin plate and bridge problems have been reported [15–17]. The "selective integration" concept was first introduced in the Mindlin strip formulation by Hinton and Zienkiewicz [18] who presented the lower member of the family and showed its applicability for the linear analysis of thick and moderately thin simply supported square plates under uniform loading.

In this paper the behaviour of the selective integration family of Mindlin strip elements for the analysis of thick and very thin plates is investigated. Attention is particularly focussed in the study of the linear, quadratic and cubic elements. In the last part of the paper examples showing the convergence and accuracy of the elements for various thick and very thin plate bending problems are given. Numerical results obtained for all the elements are compared and finally the "best value" linear Mindlin plate strip element is suggested.

In the next section some theoretical background is briefly presented.

MINDLIN STRIP FORMULATION

For a single, quadratic or cubic Mindlin strip as those shown in Figs. 1 (a–c), the mid-plane deflection and rotations are written as the sum of a finite series of products of interpolation functions in the x direction and harmonic functions in the y direction.

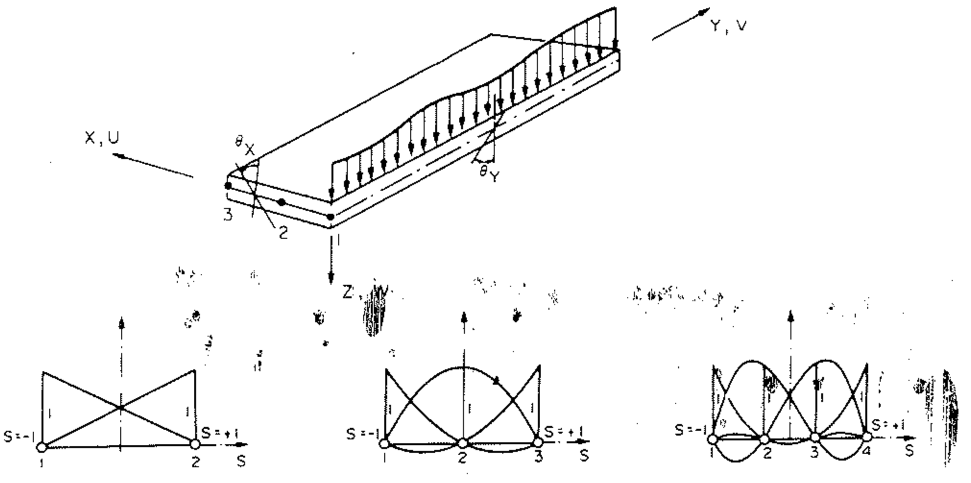
For a strip with k nodes the displacements are expressed as the sum of a series of n terms as

$$\delta = \sum_{l=1}^{n} \sum_{i=1}^{k} \mathbf{N}_i^l \mathbf{a}_i^l \tag{1}$$

in which the shape function matrix $\mathbf{N}_i^l$ associated with node i for the lth harmonic term in the series is

$$\mathbf{N}_i^l = \begin{bmatrix} N_i S_l & 0 & 0 \\ 0 & N_i S_l & 0 \\ 0 & 0 & N_i C_l \end{bmatrix} \tag{2}$$

where $S_l = \sin(l\pi y/b)$ and $C_l = \cos(l\pi y/b)$, the mid-plane



$$N_1 = \frac{1-S}{2} \qquad N_2 = \frac{1+S}{2}$$

$$N_1 = \frac{S^2-S}{2} \qquad N_2 = 1-S^2 \qquad N_3 = \frac{S^2+S}{2}$$

$$N_1 = -\frac{9}{16}\left(S^3-S^2-\frac{1}{9}S+\frac{1}{9}\right)$$

$$N_2 = \frac{27}{16}\left(S^3-\frac{S^2}{3}-S+\frac{1}{3}\right)$$

$$N_3 = -\frac{27}{16}\left(S^3+\frac{S^2}{3}-S-\frac{1}{3}\right)$$

$$N_4 = \frac{9}{16}\left(S^3+S^2-\frac{S}{9}-\frac{1}{9}\right)$$

Fig. 1. Shape functions for linear, quadratic and cubic Mindlin strip elements.

displacement vector is

$$\delta = [w, \theta x, \theta y]^T \tag{3a}$$

the vector of nodal parameters of node i for the lth harmonic is

$$\mathbf{a}_i^l = [w_i^l, \theta_{xi}^l, \theta_{yi}^l]^T \tag{3b}$$

N_i are linear, quadratic or cubic one dimensional shape functions (see Fig. 1) and b is the element length in the y direction.

The displacements given by eqn (1) impose *a priori* a set of boundary conditions at $y=0$ and b which are summarised as

$$w = \theta_x = \frac{\partial w}{\partial x} = \frac{\partial \theta_x}{\partial x} = \frac{\partial \theta_y}{\partial y} = 0 \tag{4}$$

this shows that, at either end, the strip is simply supported and that the cross-section is free to warp longitudinally but cannot distort transversely.

Using Mindlin's plate theory [6] and proceeding in a standard finite element manner [1] the generalized strain vector, $\boldsymbol{\epsilon}$ can be expressed in terms of the displacement parameters $\mathbf{a}_i^l$ as

$$\boldsymbol{\epsilon} = \sum_{l=1}^{n} \sum_{i=1}^{k} \mathbf{B}_i^l \mathbf{a}_i^l \tag{5}$$

where

$$\boldsymbol{\epsilon} = \left[-\frac{\partial \theta_x}{\partial x}, -\frac{\partial \theta_y}{\partial y}, -\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right), \frac{\partial w}{\partial x} - \theta_x, \frac{\partial w}{\partial y} - \theta_y \right]^T \tag{6}$$

and $\mathbf{B}_i^l$ is the strain matrix for node i for the lth harmonic term which can be written in terms of the bending and transverse shear strain components as

$$\mathbf{B}_i^l = [\mathbf{B}_{b_i}^l, \mathbf{B}_{S_i}^l]^T \tag{7}$$

with

$$\mathbf{B}_{b_i}^l = \begin{bmatrix} 0 & -\frac{\partial N_i}{\partial x} S_l & 0 \\ 0 & 0 & N_i \frac{l\pi}{b} S_l \\ 0 & -N_i \frac{l\pi}{b} S_l & -\frac{\partial N_i}{\partial x} C_l \end{bmatrix} \tag{8}$$

$$\mathbf{B}_{S_i}^l = \begin{bmatrix} \frac{\partial N_i}{\partial x} S_l & -N_i S_l & 0 \\ N_i \frac{l\pi}{b} C_l & 0 & -N_i C_l \end{bmatrix}. \tag{9}$$

Also, the linear relationship between stress resultants and generalized strains can be written in the standard form [1]

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon} \tag{10}$$

where

$$\boldsymbol{\sigma} = [M_x, M_y, M_{xy}, Q_x, Q_y]^T \tag{11}$$

and $\mathbf{D}$ is the elasticity matrix which can be written as [1]:

$$\mathbf{D} = \begin{bmatrix} Et^3\mathbf{D}_b & \mathbf{0} \\ \mathbf{0} & Et\mathbf{D}_S \end{bmatrix} \tag{12}$$

where for an isotropic plate

$$\mathbf{D}_b = \frac{1}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{13}$$

and

$$\mathbf{D}_S = \frac{1}{2\alpha(1+\nu)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{14}$$

In above, t is the plate thickness, E the Young's modulus, ν the coefficient of Poisson and α a modification factor to allow cross-sectional warping. ($\alpha = 6/5$ for a rectangular cross section.)

The total potential energy can be written as [1]

$$\Pi = \frac{1}{2}\int\int_{\Omega} \boldsymbol{\epsilon}^T \boldsymbol{\sigma}\, d\Omega - \int\int_{\Omega} \bar{\mathbf{t}}^T \boldsymbol{\delta}\, d\Omega. \tag{15}$$

In eqn (15) only the energy due to uniformly distributed loading $\bar{\mathbf{t}}$, has been considered for simplicity.

Expressing also the loading $\bar{\mathbf{t}}$, as the sum of harmonic series in the longitudinal direction as

$$\bar{\mathbf{t}} = \sum_{l=1}^{n} \begin{bmatrix} S_l & 0 & 0 \\ 0 & S_l & 0 \\ 0 & 0 & C_l \end{bmatrix} \bar{\mathbf{t}}^l \tag{16a}$$

with

$$\bar{\mathbf{t}} = [\bar{t}_w, \bar{m}_{\theta_x}, \bar{m}_{\theta_y}]^T$$

and

$$\bar{\mathbf{t}}^l = [\bar{t}_w^l, \bar{m}_{\theta_x}^l, \bar{m}_{\theta_y}^l]^T \tag{16b}$$

and substituting eqns (5), (10) and (16) in eqn (15) the total potential energy can be rewritten in the form

$$\Pi = \frac{1}{2}\int\int_{\Omega}\left(\sum_{l=1}^{n}\sum_{i=1}^{k} \mathbf{B}_i^l \mathbf{a}_i^l\right)^T \mathbf{D}\left(\sum_{m=1}^{n}\sum_{j=1}^{k} \mathbf{B}_j^m \mathbf{a}_j^m\right) d\Omega$$

$$- \int\int_{\Omega} \sum_{l=1}^{n} (\bar{\mathbf{t}}^l)^T \sum_{m=1}^{n}\sum_{i=1}^{k} \mathbf{N}_i^m \mathbf{a}_i^m \, d\Omega. \tag{17}$$

The strip equilibrium equations are then obtained minimizing the total potential energy Π with respect to the displacement parameters $\mathbf{a}_i^l$ as

$$\frac{\partial \Pi}{\partial \mathbf{a}_i^l} = 0. \tag{18}$$

It can be shown [1, 2] that using the ortogonality properties of the harmonic series eqn (18) yields simply

$$\sum_{i=1}^{k} \mathbf{k}_{ij}^{ll} \mathbf{a}_j^l = \mathbf{f}_i^l \tag{19}$$

where

$$\mathbf{k}_{ij}^{ll} = \frac{b}{2}\int_{L} (\mathbf{B}_i^l)^T \mathbf{D}\mathbf{B}_j^l \, dx \tag{20}$$

and

$$\mathbf{f}_i^l = \frac{b}{2}\int (\mathbf{N}_i^l)\mathbf{t}^l \, dx \tag{21}$$

and the harmonic terms thus uncouple. This implies that for the lth harmonic term a strip equilibrium equation can be formed, involving only the stiffness matrix, displacement parameter vector and equivalent nodal loads for that harmonic. For this lth harmonic the total equilibrium equations, involving all stripes can be assembled in the usual manner [1] giving:

$$\begin{bmatrix} \mathbf{k}^{11} & & & \\ & \mathbf{k}^{22} & & \\ & & \ddots & \\ & & & \mathbf{k}^{nn} \end{bmatrix} \begin{Bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^n \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \vdots \\ \mathbf{f}^n \end{Bmatrix} \tag{22}$$

and a solution found for the displacement parameters $\mathbf{a}_i^l (i = 1, k)$ This procedure is repeated for all harmonics 1 to n, allowing the displacements to be founded from eqn (1), and the stresses, $\boldsymbol{\sigma}$ from eqns (10) and (5) for any longitudinal position, y, and transverse position x.

Behaviour of Mindlin plate strip elements for thin plate situations: Locking effect

The behaviour of Mindlin strip elements with varying plate thickness can be easily explained if we note that the element stiffness matrix for the lth harmonic, eqn (20), can be rewritten using eqn (7) and (14) as

$$\mathbf{k}_{ij}^{ll} = \frac{b}{2}\left[\int (\mathbf{B}_{b_i}^l)^T Et^3 \mathbf{D}_b \mathbf{B}_{b_j}^l \, dx + \int (\mathbf{B}_{S_i}^l)^T Et\mathbf{D}_S \mathbf{B}_{S_j}^l \, dx\right]. \tag{23}$$

If the Young's modulus and the thickness are constant throughout the plate, eqn (23) can be rewritten as

$$\mathbf{k}_{ij}^{ll} = Et^3[\mathbf{k}_{ij}^{ll}]_1 + Et\,[\mathbf{k}_{ij}^{ll}]_2 \tag{24}$$

where

$$[\mathbf{k}_{ij}^{ll}]_1 = \frac{b}{2}\int (\mathbf{B}_{b_i}^l)^T \mathbf{D}_b \mathbf{B}_{b_j}^l \, dx \tag{25a}$$

$$[\mathbf{k}_{ij}^{ll}]_2 = \frac{b}{2}\int (\mathbf{B}_{S_i}^l]^T \mathbf{D}_S \mathbf{B}_{S_j}^l \, dx. \tag{25b}$$

Following the arguments of previous section, a typical assembled stiffness equation for the lth harmonic could be written as

$$\mathbf{k}^{ll}\mathbf{a}^l = [Et^3\mathbf{k}_1^{ll} + Et\,\mathbf{k}_2^{ll}]\mathbf{a}^l = \mathbf{f}^l \tag{26}$$

where $Et^3\,\mathbf{k}_1^{ll}$ and $Et\,\mathbf{k}_2^{ll}$ are respectively the bending and shear matrix contributions to the total stiffness matrix $\mathbf{k}^{ll}$ for the lth harmonic term.

If the thickness, t, is small, t^3 becomes negligible with respect to t and eqn (26) yields approximately

$$\mathbf{k}^{ll}\mathbf{a}^l \simeq Et\,\mathbf{k}_2^{ll}\,\mathbf{a}^l = \mathbf{f}^l \tag{27}$$

The "exact" solution for the displacement of a thin plate can be obtained using Kirchhoff finite strip theory [2] in the form

$$Et^3\,\mathbf{k}_{k_{ch}}^{ll}\,\mathbf{a}_{k_{ch}}^l = \mathbf{f}_{k_{ch}}^l \simeq \mathbf{f}^l \tag{28}$$

where $\mathbf{k}_{k_{ch}}^{ll}$, $\mathbf{a}_{k_{ch}}^l$ and $\mathbf{f}_{k_{ch}}^l$ are the stiffness matrix, the displacement vector and the nodal forces vector for the lth harmonic term using Kirchhoff plate theory [2]. Since $\mathbf{f}^l_{k_{ch}} = \mathbf{f}^l$ from eqns (28) and (27) we obtain:

$$Et\,\mathbf{k}_2{}^{ll}\,\mathbf{a}^l = Et^3\,\mathbf{k}^{ll}_{k_{ch}}\,\mathbf{a}^l_{k_{ch}} \qquad (29)$$

or

$$\mathbf{k}_2{}^{ll}\,\mathbf{a}^l = t^2\,\mathbf{k}^{ll}_{k_{ch}}\,\mathbf{a}^l_{k_{ch}} \qquad (30)$$

and when $t \to 0$ since $\mathbf{k}^{ll}_{k_{ch}}$ and $\mathbf{a}^l_{k_{ch}}$ are finite, eqn (30) yields finally

$$\mathbf{k}_2{}^{ll}\,\mathbf{a}^l \to 0 \qquad (31)$$

which implies that the only solution possible is $\mathbf{a}^l \to 0$ (commonly known as locking effect [10, 11, 21]) unless matrix $\mathbf{k}_2{}^{ll}$ is singular.

Therefore, if Mindlin strip plate elements are used for the analysis of thin plates unrealistic overstiff results will be obtained in most cases unless some precautions are taken.

The selective integration family of Mindlin strip plate elements

There are a variety of techniques to avoid the effect of locking:

(1) The use of formulations based on classical thin plate theory which, as mentioned previously, they require C_1 continuity and they cannot deal with transverse shear deformation [2, 3].

(2) To impose the constraint of zero shear deformation at the integration points at the element level prior to assembly [19].

(3) The use of special mixed formulations which retain C_0 continuity at the cost of increasing the number of variables [20].

(4) The other technique for avoiding locking and the one which is studied in this paper is the use of selective integration of the terms in the element striffness matrix.

The basic idea of the "selective integration" technique is to relax the constraint imposed on $\mathbf{k}^{ll}$ by the effect of the shear terms' making $\mathbf{k}_2{}^{ll}$ singular (or nearly singular) by under integrating the coefficients of $\mathbf{k}_2{}^{ll}$ in the numerical integration of the integrals which appear in the stiffness matrix $\mathbf{k}^{ll}$. The rest of the stiffness matrix, i.e. the terms of $\mathbf{k}_1{}^{ll}$ can be integrated exactly, and thus the process is called "selective integration", or also can be underintegrated, which is usually termed "reduced integration".

In Fig. 2 it can be seen the number of Gaussian integrating points necessary for the exact (full) integration of $\mathbf{k}_1{}^{ll}$ and $\mathbf{k}_2{}^{ll}$, and that needed for selective and reduced integration.

It can be shown [21] that the singularity, or its absence, in the finite element stiffness matrices depends on the number of independent relations used at each integrating point. Thus, if the total number of independent relations (integrating points X number of strain components at each point) is less than the number of available degrees of freedom, then singularity certainly will exist.

This simple theorem provides the number of integrating points necessary to get the complete singularity of $\mathbf{k}_2{}^{ll}$. However, this will obviously be a mesh and boundary conditions dependent problem and it must be studied for each particular case.

In the remaining of the paper the performance of the linear, quadratic and cubic elements of the Mindlin strip family is investigated. First, the ability of those elements to make $\mathbf{k}_2{}^{ll}$ singular for a series of examples using different integration rules is discussed in the next section. Second, the accuracy of the elements for the analysis of a wide range of thick and thin plate bending problems is checked. Finally in the last section of the paper some conclussions about the general behaviour of the elements are given.

Singularity of $\mathbf{k}_2{}^{ll}$

To asses the ability of the elements in making $\mathbf{k}_2{}^{ll}$ singular using different orders of integration, a series of checks on the number of available degrees of freedom, d, (total number of degrees of freedom less number of constraints) and the total number of independent relations (number of elements, e, $\times$ number of integrating points, n, $\times$ number of strain components in $\mathbf{k}_2{}^{ll}$, $s = 2$) for three square plates with different boundary conditions have been carried out. Results are shown in Fig. 3. Only one half of the plate has been considered assuming symmetrical loading. Singularity of $\mathbf{k}_2{}^{ll}$ is ensured in all cases when $d > n \times e \times s$. This has been marked in the Fig. 3 with a square.

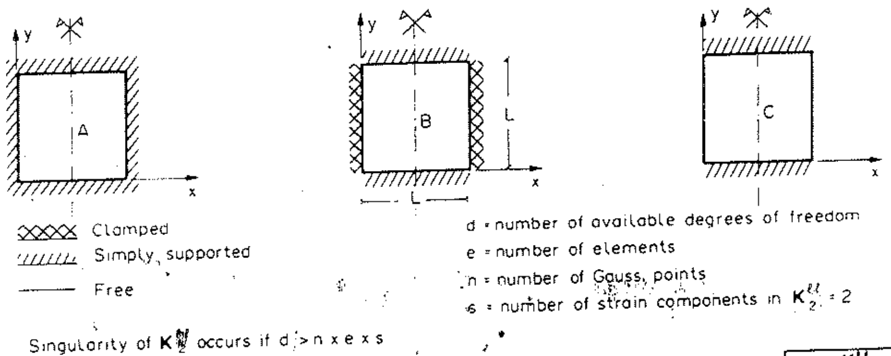
Some important conclusions can be drawn from the numbers displayed in Fig. 3.

(1) The *cubic element* using $n = 4$ (exact) and $n = 3$ (reduced or selective) Gauss points for the numerical integration of $\mathbf{k}_2{}^{ll}$ gives for all the three cases and all the meshes $d > n \times e \times s$. Therefore, the singularity of $\mathbf{k}_2{}^{ll}$ is always satisfied and good results should be expected when using this element for thin plate situations.

(2) the *quadratic element* with 2 Gauss points gives $\mathbf{k}_2{}^{ll}$ singular for all cases and all meshes. However, if exact integration is used ($n = 3$) we can see there are cases

Element \ Integration	Full		Selective		Reduced	
	K_1^{ll}	K_2^{ll}	K_1^{ll}	K_2^{ll}	K_1^{ll}	K_2^{ll}
o——o	2	2	2	1	1	1
o——o——o	3	3	3	2	2	2
o—o—o—o	4	4	4	3	3	3

Fig. 2. Number of Gaussian integrating points for $\mathbf{K}_1{}^{ll}$ and $\mathbf{K}_2{}^{ll}$ with full, selective and reduced integration for the linear, quadratic and cubic Mindlin strip elements.



Linear	K_2^{ll} d A	B	C	n x e x s n=2	n=1
	3	2	5	4	2
	6	5	8	8	[4]
	9	8	11	12	[6]
	12	11	14	16	[8]
	15	14	17	20	[10]

Quadratic	K_2^{ll} d A	B	C	n x e x s n=3	n=2
	6	5	8	6	[4]
	12	11	14	12	[8]
	18	17	20	18	[12]
	24	23	26	24	[16]
	30	29	32	30	[20]

Cubic	K_2^{ll} d A	B	C	n x e x s n=4	n=3
	9	8	11	[8]	[6]
	18	17	20	[16]	[12]
	27	26	29	[24]	[18]
	36	35	38	[32]	[24]
	45	44	47	[40]	[30]

Note: Only half plate has been considered assuming symmetric loading

Fig. 3. Study of the singularity of $\mathbf{K}_2^{ll}$ with the number of elements and the integration rule for three square plates with different boundary conditions.

when $d \leq n \times e \times s$ therefore in those cases overstiff results might be expected.

(3) The *linear element* with only 1 Gauss point ensures for all the three cases and for all the meshes, (with the exception of case *B* with one single element) a complete singularity of $\mathbf{k}_2^{ll}$. Exact integration ($n = 2$) gives exactly the opposite and the criterion for singularity is *never* satisfied.

It is important to mention here that a similar series of checks have been performed on the singularity of the full matrix $\mathbf{k}^{ll}$ with exact, selective and reduced integration for the same examples, giving always *a non singular matrix* $\mathbf{k}^{ll}$ *for all cases*, thus ensuring the existeness of the numerical solution.

Zero energy modes

One of the drawbacks of using reduced or selective integration techniques is the fact that in some cases they can excite internal zero energy nodes in the element which, if they are able to propagate in a mesh, they can polute the solution and lead in some cases to unrealistic results [14, 23].

A rigorous method of determining whether an element has any internal zero energy modes is to evaluate the number of zero eigenvalues associated with the stiffness matrix.

The number of zero valued eigenvalues for a single strip element or group of elements should be equal to zero due to the fact that the implicit boundary conditions of the strip formulation already unrestrain the element against movement. Any extra zero valued eigenvalue will be associated with an spurious zero energy mode.

In this work it has been checked that only the linear element with reduced integration has one possible zero energy mode. It is very important to mention here that this mode is *not* propagable and it is eliminated when a mesh of two or more linear elements is considered. In conclusion, for practical purposes no danger of polluted solutions due to the propagation of zero energy modes can be expected for any of the three elements studied in this paper.

Numerical examples

The accuracy of the linear, quadratic and cubic Mindlin strip elements for thick and thin plate bending analysis will be checked in the series of examples showed next.

Example 1. *Convergence with the number of harmonics.* To get an estimate of the number of harmonic terms needed to get complete convergence for each solution, a simple supported square plate like that of case A of Fig. 3, under uniform distributed loading has been analysed with the three element for two different thickness ratios of $(t/L) = 0.1$ and 0.01 respectively. Results of the percentage of error of the central deflection and bending moment *Mx* with respect to the final solution for a mesh of 3 elements for the linear quadratic and cubic elements using full, selective or reduced integration are the same and these are shown in Fig. 4. It can be noticed that for both thicknesses 5 non zero harmonics are enough to get an error of less than 0.2% in both deflection and bending moments with respect to the *final* solution (note that the even harmonic terms are zero due to the symmetry of the loading [1–3].

It can be seen in the same figure that *one harmonic* term gives an error *less than* 1.5% in deflections and 3% in bending moments and that is accurate enough for many practical engineering problems. However in all the examples shown next 9 *non zero harmonic* terms have been taken.

Example 2. *Convergence with the number of strips.* Figures 5–7 show the convergence of the central deflection and the central bending moments *Mx* and *My* with the number of strips for the same simple supported

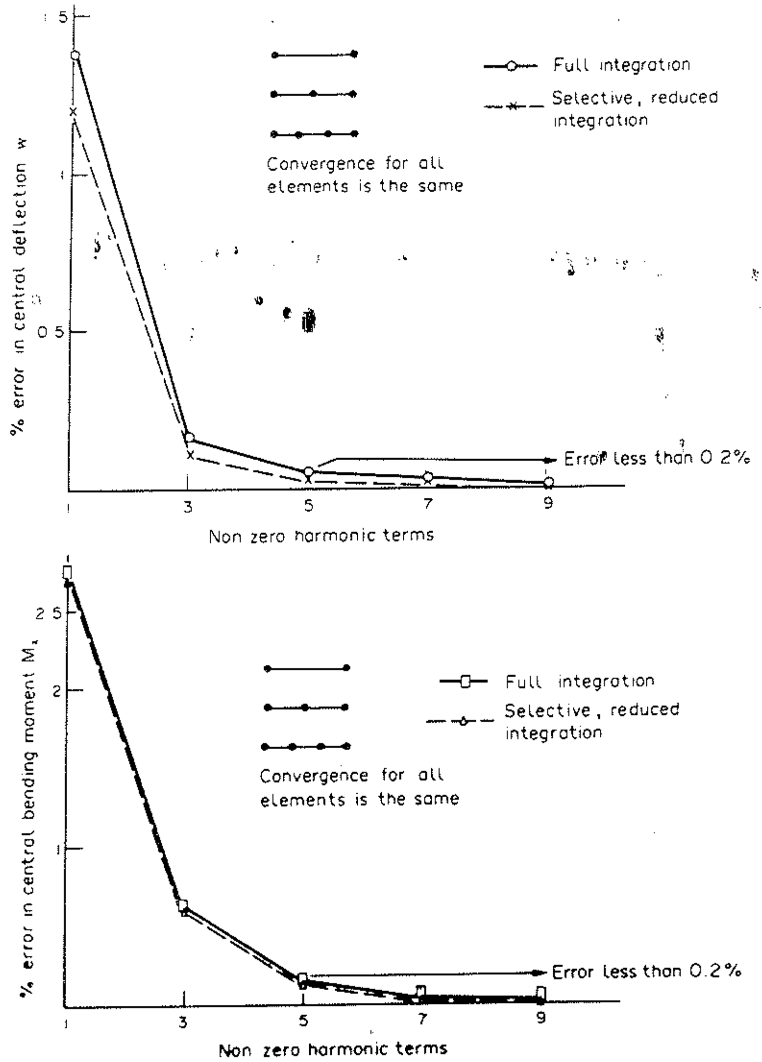


Fig. 4. Simply supported square plate under uniformly distributed loading. Convergence study with number of non zero harmonics for $t/L = 0.1$.

square plate of the previous example for three different ratios of (t/L) of 0.1, 0.01 and 0.001 respectively. Percentages of error with respect to the theoretical "exact" solution [23] for the linear, quadratic and cubic elements with exact, selective and reduced integration are shown. Also in the same figures the number of available degrees of freedom for each solution is shown. This gives a more realistic idea of the cost of the numerical computations.

Thus, it can be seen that results for the linear element with selective and reduced integration are extremely good with only 6 degrees of freedom (less than 1% in the central deflection and less that 6% error in the bending moments at the centre of the plate for any of the three thicknesses studied) whereas the quadratic and cubic elements need 12 and 9 degrees of freedom respectively to get the same order of accuracy (see Fig. 8). This suprisingly accuracy of the linear element with one single integrating point for all integrals is by no means new, and indeed it has been already reported by Zienkiewicz *et al.* [13] and Vykutil [24], both in the context of axisymmetric shell analysis.

Convergence rate to the theoretically exact solution is good for the higher order cubic and parabolic elements as expected.

Nevertheless, in Fig. 8 it can be seen that for practical purposes the ratio degrees of freedom/accuracy of the solution is favourable to the simple and economical linear element with one integration point which seems to be the "best value" element for this type of problems.

On the other hand, results obtained with the linear element with exact integration for the small thickness cases ($t = 0.01$ and $t = 0.001$) are unrealistically overstiff as expected from the non singularity of matrix $\mathbf{k}_2''$ discussed in previous section. No other locking effects are observed for any other case in this example again in agreement with the theoretical predictions discussed earlier.

Example 3. This example analyses the preliminary conclusions drawn previously about the good or bad performance of the three elements with exact selective and reduced integration for thin plate situations. Figure 9 shows the value of the central deflection of a square

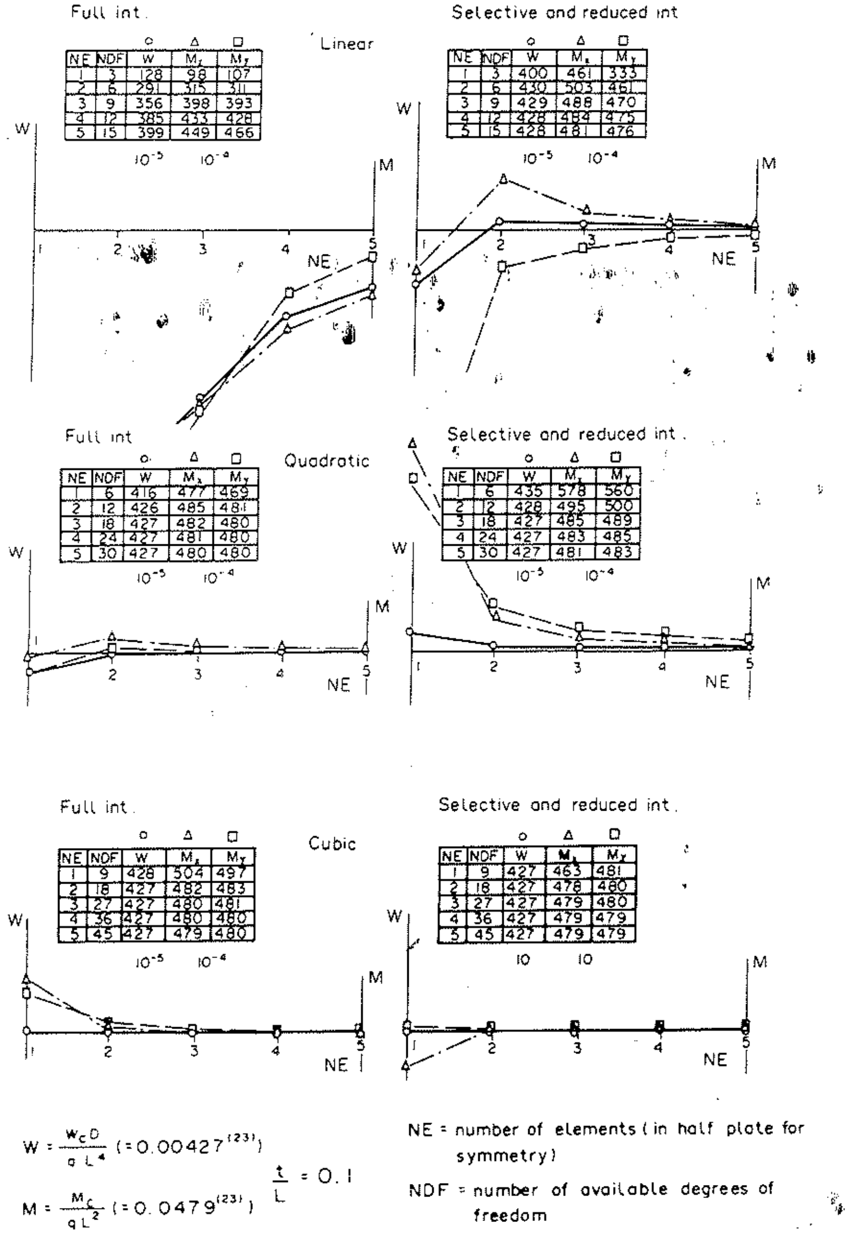


Fig. 5. Simply supported square plate under uniformly distributed loading. Convergence study of central deflection and bending moments with number of elements and degrees of freedom for $t/L = 0.1$. Nine non-zero harmonic terms taken.

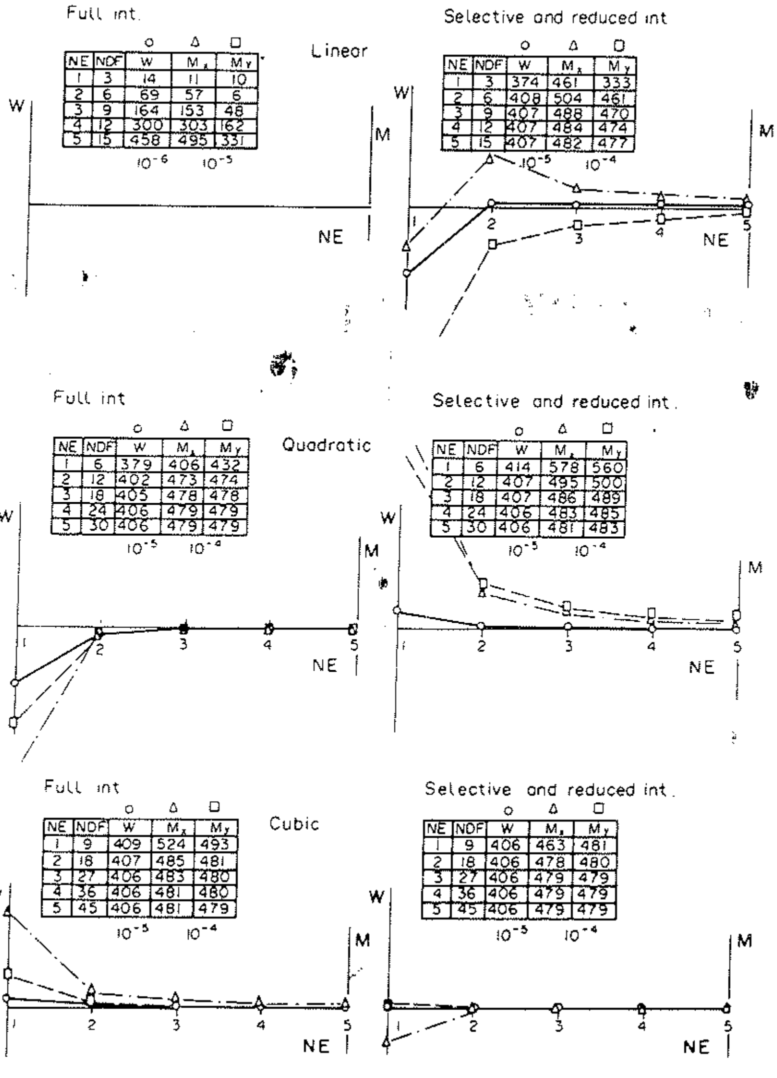
plate under uniform loading with boundary conditions of the type A, B or C of Fig. 3 for a wide range of ratios thicknesses/width of the plate. In case C the value showed is the deflection at the center of one of the free sides. The general conclusions which can be drawn from these figures are the following:

(1) The cubic element with exact, selective or reduced integration behaves well in all the three cases for thick ($t/L = 0.1$) and a wide range of thin and very thin plate situations. This agrees with the predictions made previously.

(2) The quadratic element with selective and reduced integration behaves well in all cases for a wide range of plate thicknesses. If exact integration is used, results for the deflections for cases A and C are good as expected from the singularity of $\mathbf{k}_2''$ (see Fig. 3), but for case B overstiff results for thin and very thin plates are obtained as again it was predicted by the fact that matrix $\mathbf{k}_2''$ it is clearly not singular.

(3) The linear element with selective and reduced integration behaves well in all cases for thick, moderately thin and very thin plates. On the other hand if exact integration is used, overstiff unrealistic results are obtained in all cases when the thickness of the plate is small. This again agrees with the theory presented previously.

It is interesting to plot the distribution of moments and shear forces along the center of the plate for different thicknesses. Figure 10 shows the bending moments Mx along the center of a simple supported square plate under



$W = \frac{W_c D}{q L^4}$ $(= 0.00406^{(23)})$

$M = \frac{M_c}{q L^4}$ $(= 0.0479^{(23)})$

$\frac{t}{L} = 0.01$

NE = number of elements (in half plate for symmetry)

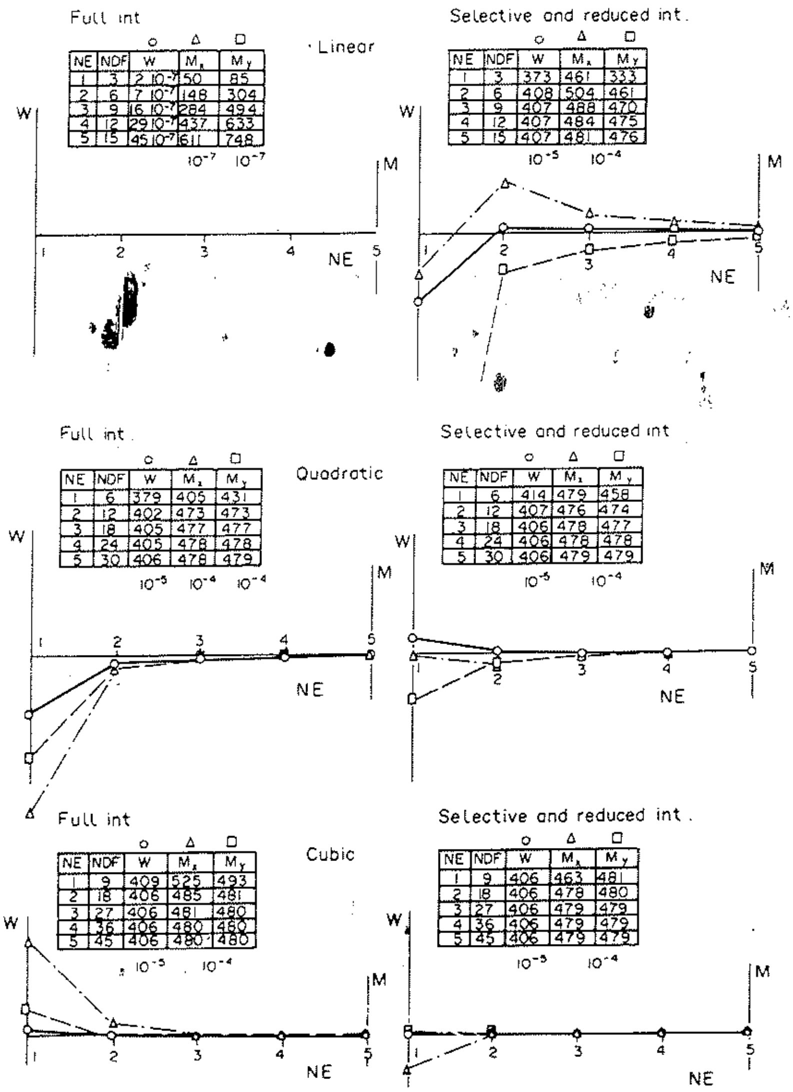
NDF = number of available degrees of freedom

Fig. 6. Simply supported square plate under uniformly distributed loading. Convergence study of central deflection and bending moments with number of elements and degrees of freedom for $t/L = 0.01$. Nine non zero harmonic terms taken.

uniform loading for the linear, quadratic and cubic elements with exact selective and reduced integration for three ratios of $t/L = 0.1$, 0.01 and 0.001. Results showed at the nodes have been extrapolated directly from the Gauss points using polynomial expansions. We can see that the moment distribution obtained is good in all cases except for the linear element with exact integration for $(t/L) = 0.01$, and 0.001. The shear forces along the center of the plate are shown in Fig. 11. We note that wrong results are obtained with the linear element with exact integration for all cases whereas selective and reduced integration gives the right results. Results for the shear force for the quadratic element with full integration present oscillations which are eliminated if selective or reduced integration is used. This poor performance of the quadratic element is due to the spurious shear straining modes excited by the quadratic shear terms of the stiffness matrix. One of the ways of eliminating these modes is the one presented by Mawenya [25] which suggests to apply a least squares linear smoothing to the shear strain matrix. This is in fact identical to use selective integration since to reduce the order of integration of the shear terms implies smoothing of the shear strain matrix [25].

This explains why results for the shear forces obtained at the Gaussian points using selective or reduced integration are correct (see Fig. 11). More accurate stresses at the nodes can be obtained using any of the well-known local or global smoothing techniques [26] which in most cases also eliminate the oscillations mentioned above.

It is worth noting that results for the shear force obtained with the cubic element are correct in all cases. This shows that the cubic shear terms do not give rise to spurious shear straining modes and the element can be considered "safe" in this context.



$$W = \frac{W_c D}{q L^4} \; (= 0.00406^{(23)})$$

$$M = \frac{M_c}{q L^2} \; (= 0.0479^{(23)})$$

$$\frac{t}{L} = 0.001$$

NE = number of elements (in half plate for symmetry)

NDF = number of available degrees of freedom

Fig. 7. Simply supported square plate under uniformly distributed loading convergence study of central deflection and bending moments with number of elements and degrees of freedom for $t/L = 0.001$. Nine non zero harmonic terms taken.

	F	S	R
o—o	—	9	[6]
o—o—o	12	12	12
o—o—o—o	9	9	9

Fig. 8. Simple supported square plate: Uniform loading. Number of available degrees of freedom necessary to get an error of less than 1% in central deflection and 6% in central bending moments for the linear, quadratic and cubic Mindlin strip elements with full, selective and reduced integration.

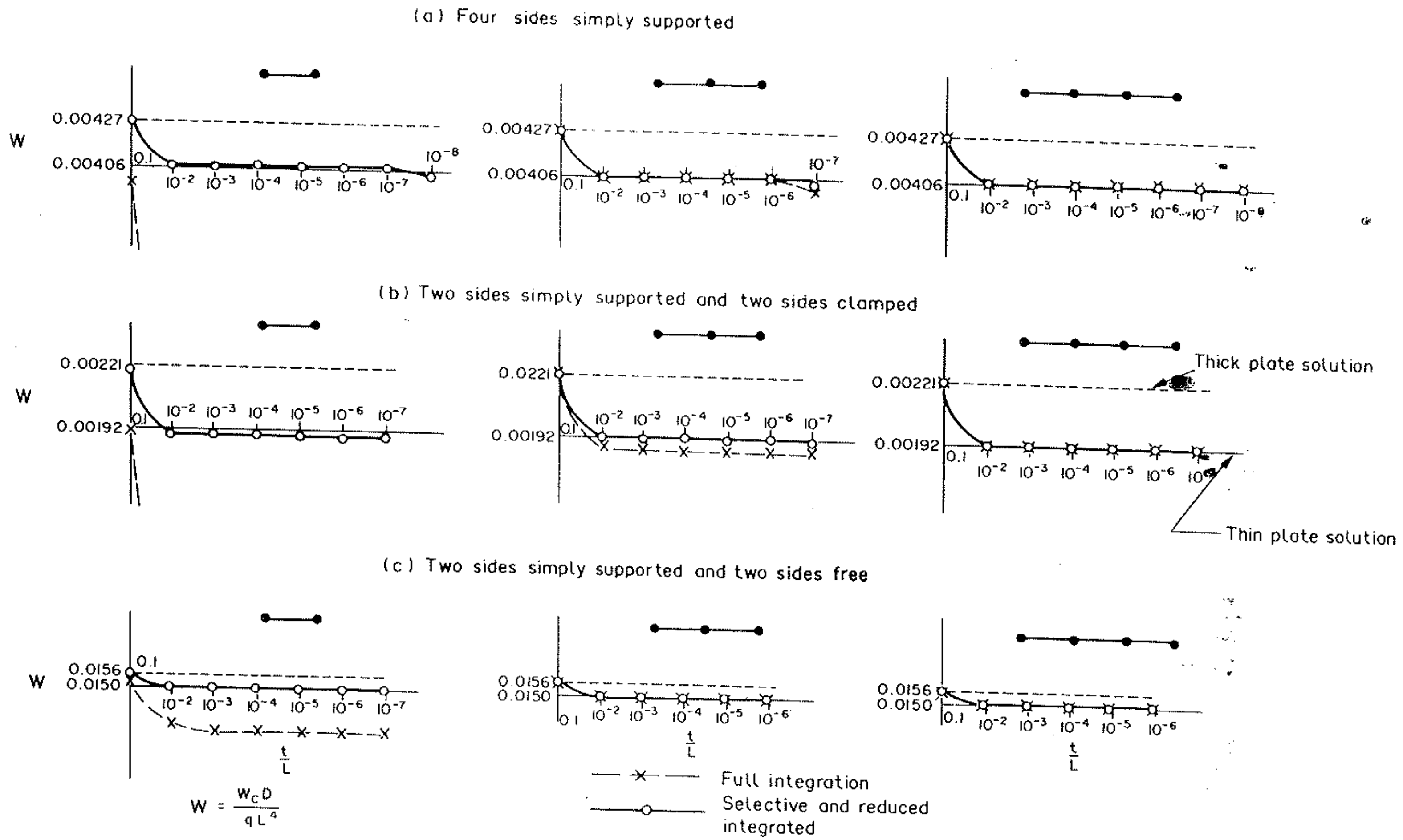


Fig. 9. Square plate under uniformly distributed loading. Central deflection vs t/l for the linear, quadratic and cubic strip elements with full, selective and reduced integration.

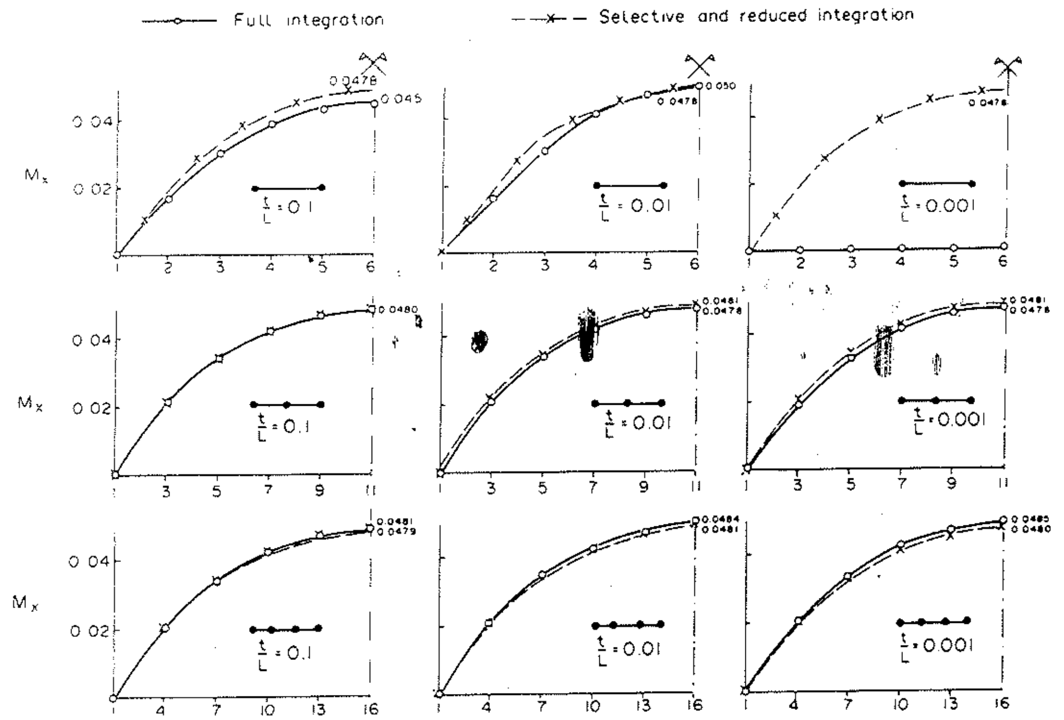


Fig. 10. Simply supported square plate. Uniform loading. Distribution of M_x along the mid section for the linear, quadratic and cubic Mindlin strip elements with full, selective and reduced integration.

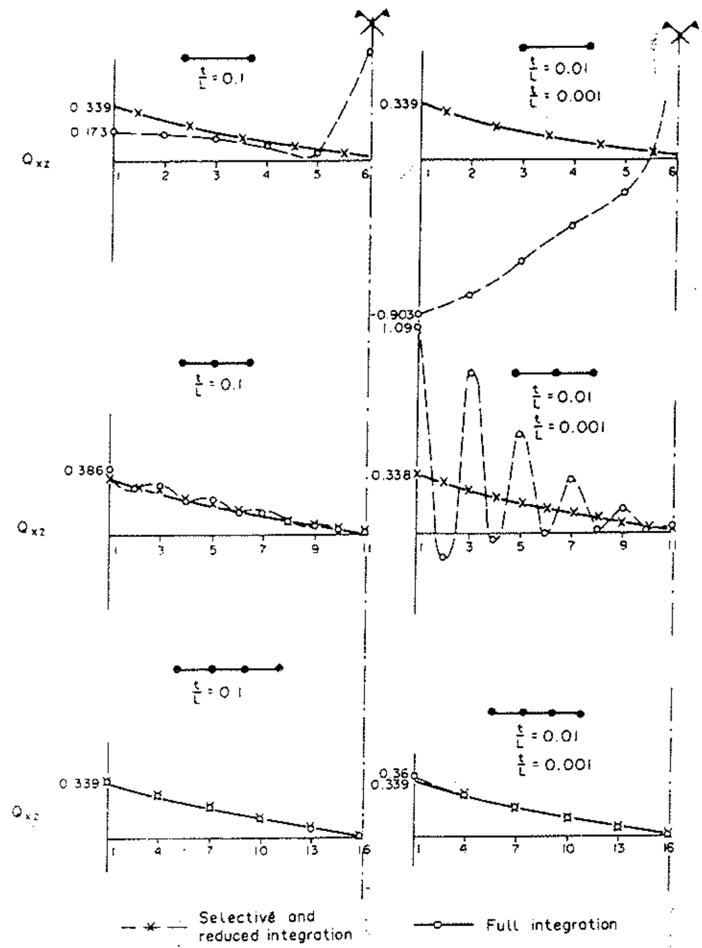
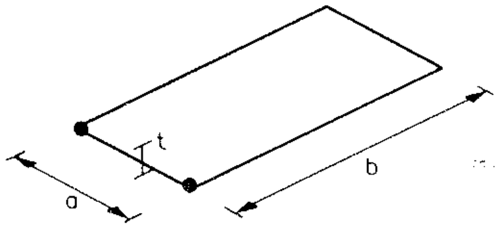


Fig. 11. Simply supported square plate. Uniformly loading. Distribution of Q_{xz} along the mid section for the linear, quadratic and cubic Mindlin strip elements with full, selective and reduced integration.



$$
\mathbf{K}^{ll} = \frac{abEt}{24(1+\nu)}
\begin{bmatrix}
5\left(\frac{1}{a^2}+\frac{l^2\pi^2}{4b^2}\right) & \frac{2.5}{a} & -1.25\frac{l\pi}{b} & 5\left(-\frac{1}{a^2}+\frac{l^2\pi^2}{4b^2}\right) & \frac{2.5}{a} & -1.25\frac{l\pi}{b} \\
 & \frac{t^2}{1-\nu}\left(\frac{1}{a^2}+\frac{1-\nu}{8}\frac{l^2\pi^2}{b^2}\right)+1.25 & \frac{t^2}{1-\nu}\frac{3\nu-1}{4a}\frac{l\pi}{b} & \frac{-2.5}{a} & \frac{t^2}{1-\nu}\left(-\frac{1}{a^2}+\frac{1-\nu}{8}\frac{l^2\pi^2}{b^2}\right)+1.25 & \frac{t^2}{1-\nu}\frac{1+\nu}{4a}\frac{l\pi}{b} \\
 & & \frac{t^2}{1-\nu}\left(\frac{l^2\pi^2}{4b^2}+\frac{1-\nu}{2a^2}\right)+1.25 & -1.25\frac{l\pi}{b} & -\frac{t^2}{1-\nu}\frac{1+\nu}{4a}\frac{l\pi}{b} & \frac{t^2}{1-\nu}\left(\frac{l^2\pi^2}{4b^2}-\frac{1-\nu}{2a^2}\right)+1.25 \\
 & \text{Symmetric} & & 5\left(\frac{1}{a^2}+\frac{l^2\pi^2}{4b^2}\right) & -\frac{2.5}{a} & -1.25\frac{l\pi}{b} \\
 & & & & \frac{t^2}{1-\nu}\left(\frac{1}{a^2}+\frac{1-\nu}{8}\frac{l^2\pi^2}{b^2}\right)+1.25 & \frac{t^2}{1-\nu}\frac{1-3\nu}{4a}\frac{l\pi}{b} \\
 & & & & & \frac{t^2}{1-\nu}\left(\frac{l^2\pi^2}{4b^2}+\frac{1-\nu}{2a^2}\right)+1.25
\end{bmatrix}
$$

E = Young modulus
ν = Poisson's ratio

Fig. 12. Linear Mindlin strip element: Explicit form of the stiffness matrix for the lth harmonic term obtained using one single gaussian integrating point.

CONCLUSIONS

(1) The *linear, parabolic* and *cubic* elements of the Mindlin plate strip family *with selective or reduced integration behave well* in the analysis of thick, moderately thin and very thin plates.

(2) The performance of the *cubic strip element with exact integration is also good*, however reduced integration is recommended whenever the cubic element is used, for obvious computing economy reasons.

(3) The *quadratic strip* element *with exact integration* presents an *unrelieable behaviour* for thick and thin plate analysis. Thus, whereas deflections and bending moments agree in most cases with theoretical values, the shear forces at the Gauss points are oscillatory unless some smoothing precautions are taken. This together with, again, economy reason makes the use of the quadratic element with exact integration not advisable for plate bending analysis.

(4) The behaviour of the *linear srip element with exact integration* is *bad* in most cases and it should never be used in the context of plate bending analysis.

(5) The *linear strip element with one single Gaussian integrating point* seems to be probably the "*best value*" Mindlin plate strip element of the three studied in this paper. Its behaviour is excellent for both thick and thin plate analysis and it has the lowest ratio degrees of freedom/accuracy of all. The big simplicity of the element (an explicit form of the element stiffness matrix is easily obtained and this is shown in Fig. 12) adds another point in its favour and it makes the linear Mindlin strip element with reduced integration probably the more promising strip element for practical plate bending, bridge deck and axisymmetric shell analysis [27].

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