# A review of some finite element families for thick and thin plate and shell analysis

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# A REVIEW OF SOME FINITE ELEMENT FAMILIES FOR THICK AND THIN PLATE AND SHELL ANALYSIS

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## SUMMARY

This paper describes a number of triangular and quadrilateral plate and shell elements derived via Reissner-Mindlin plate theory and mixed interpolation. It is shown how by introducing the adequate constrains the original thick plate elements evolve into DK forms adequate for thin situations only. This evolution process allows to revisite some classical elements like the Morley triangle and also to derive simple plate and shell triangles and quadrilaterals with only translational degrees of freedom as nodal variables.

## INTRODUCTION

Considerable effort has been put in recent years in the development of plate and shell elements based on the so called Reissner-Mindlin plate theory [1]. The attractive feature of this theory is that it allows for an independent approximation of the deflection and the rotation fields, thus overcoming one of the main drawacks of standard Kirchhoff's theory [1]. Moreover, shear deformation is naturally taken into account in Reissner-Mindlin theory and, thus, the corresponding plate and shell elements are in principle applicable to both thick and thin situations. It is however well known that overstiff solutions are obtained when Reissner-Mindlin elements are used to solve very thin cases. This "locking" effect is due to the progressibly increasing influence of the shear stiffness terms as the thickness reduces. This leads to an undesirable larger global stiffness which tends to an infinite value in the thin limit. Locking was originally overcome by means of the so called selective integration (SI) techniques [1] which basically use a reduced quadrature for integrating the shear stiffness terms. This simple procedure can in some occassions modify the proper rank of the global stiffness matrix leading to the appearance of spurious mechanisms. The more popular alternative to SI techniques is the use of mixed interpolations where the deflection, the rotations and the shear forces (and sometimes also the bending moments) are independently interpolated. The analogy of this procedure with SI techniques was soon established and it has opened a wide scope for development of new plate and shell element families which can "safely" operate in both thick and thin regimes [1].

Reissner-Mindlin theory can be taken as the starting point for the development of "pure" thin plate (or shell) elements, i.e. elements satisfying

Kirchhoff's orthogonality conditions for the normal vector. A well known technique is based on the introduction of Kirchhoff's constraints at a number of discrete element points so that the transverse shear strain is effectively zero over the element. Some of the so called DK elements, like the DK triangle [1, 2, 3], have enjoied great popularity in the last decade among plate and shell practitioners .

This paper reviews the derivation of different Reissner-Mindlin and DK element families for plate and shell analysis. It is shown in particular how mixed Reissner-Mindlin thick plate elements "degenerate" into thin DK forms in a natural and simple manner. The first element family starts from compatible quadratic triangle and quadrilateral elements based on Reissner-Mindlin theory and a mixed interpolation. By introducing adequate Kirchhoff's constraints these element evolve naturally into standard DK triangular and quadrilateral forms.

The second element family starts from simple incompatible linear triangular and quadrilateral Reissner-Mindlin elements. The introduction of Kirchhoff's constraints leads in this case to the well known Morley triangle [4] and also to its corresponding quadrilateral form. The introduction of new constrains on the rotations field leads to the simplest elements of this family, i.e. the linear triangle and bi-linear quadrilateral with the deflection as the only nodal variable [5, 15].

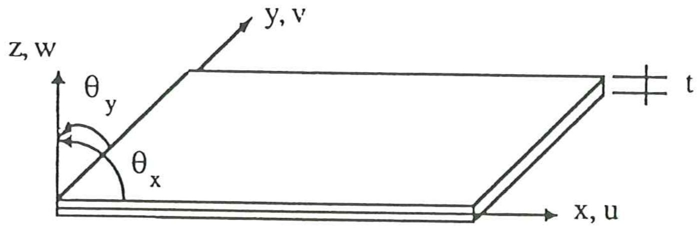


Figure 1. Definition of deflection and rotations in a plate

## COMPATIBLE PLATE ELEMENT FAMILIES. FROM QUADRATIC REISSNER - MINDLIN ELEMENTS TO THE DKT AND DKQ

Figure 1 shows the geometry of a plate with the sign convention for the deflection $w$ and the two rotations $\theta_x$ and $\theta_y$.

Table I shows the basic equations of Reissner-Mindlin plate theory [1] defining the curvature and shear strain fields, the constitutive relationships and the principle of virtual work for a distributed loading $q$.

An independent finite element interpolation will now be assumed for the deflection, the rotations and the shear strains as

$$w = \sum_i N_{w_i} w_i, \quad \boldsymbol{\theta} = \sum_i N_{\theta_i} \boldsymbol{\theta}_i, \quad \boldsymbol{\gamma} = \sum_i N_{\gamma_i} \gamma_i \tag{1}$$

Displacement field

$$\mathbf{u} = \left[w, \boldsymbol{\theta}^T\right]^T \quad , \quad \boldsymbol{\theta} = \left[\theta_x, \theta_y\right]^T$$

Curvature field

$$\boldsymbol{\kappa} = \left[\kappa_x, \kappa_y, \kappa_{yz}\right]^T = \left[-\frac{\partial \theta_x}{\partial x}, -\frac{\partial \theta_y}{\partial y}, -\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right)\right]^T = \mathbf{L}\boldsymbol{\theta}$$

Shear strain field

$$\boldsymbol{\gamma} = [\gamma_x, \gamma_y]^T = \left[\frac{\partial w}{\partial x} - \theta_x, \frac{\partial w}{\partial y} - \theta_y\right]^T$$

Constitutive relationships

$$\mathbf{m} = [M_x, M_y, M_{xy}]^T = \mathbf{D}_b \boldsymbol{\kappa}$$
$$s = [Q_x, Q_y]^T = \mathbf{D}_s \boldsymbol{\gamma}$$

$$\mathbf{D}_b = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu^2}{2} \end{bmatrix} \quad , \quad \mathbf{D}_s = \alpha G t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad \alpha = 5/6$$

Principle of virtual work (PVW)

$$\int\!\!\int_A \left[\delta\boldsymbol{\kappa}^T \mathbf{m} + \delta\boldsymbol{\gamma}^T s\right] dA = \int\!\!\int_A \delta w q dA$$

Table I. Basic equations of Reissner-Mindlin plate theory

where $w_i$, $\boldsymbol{\theta}_i$ and $\gamma_i$ are nodal values of the deflection, the rotation and the transverse shear strains, respectively and $N_{w_i}$, $N_{\theta_i}$ and $N_{\gamma_i}$ are the corresponding interpolating functions.

The conditions which must be satisfy these three fields to give a stable and locking free solution are [1, 6-8]

$$n_\theta + n_w \geq n_\gamma \quad ; \quad n_\gamma \geq n_w \tag{2}$$

where $n_w$, $n_\theta$ and $n_\gamma$ denote the number of variables involved in the interpolation of each field (after discounting the prescribed boundary values). Condition (2) must be satisfied for any element (or patch of elements) as a necessary (although not always sufficient) requirement for stability of the solution, whereas the convergence should always be verified via the patch test [1].

## TQQL quadratic triangle

The first element considered is a 6 node Reissner-Mindlin plate triangle (Figure 2) with the following interpolation fields:

1) A complete quadratic field is used to interpolate the deflection and the rotations in terms of the nodal values in the standard manner
2) A linear interpolation for the transverse shear strains is defined in the natural coordinate system as

$$\begin{aligned} \gamma_\xi &= \alpha_1 + \alpha_2 \xi + \alpha_3 \eta \\ \gamma_\eta &= \alpha_4 + \alpha_5 \xi + \alpha_6 \eta \end{aligned} \tag{3}$$

The parameters $\alpha_i$ are obtained by sampling the shear strains at the six Gauss points along the sides. This allows to express $\gamma_\xi$ and $\gamma_\eta$ in terms of the six tangential shear strains $\gamma_s$ along the element sides.

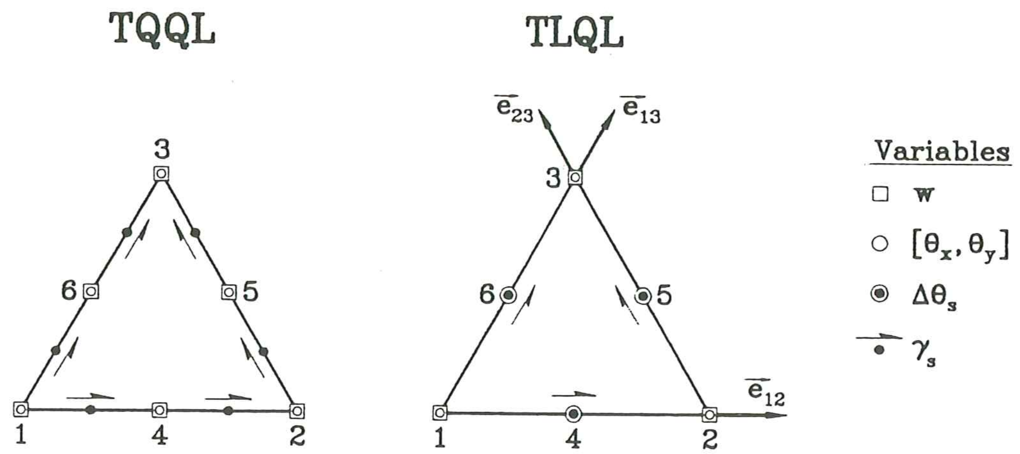


Figure 2. TQQL and TLQL Reissner-Mindlin plate elements

The relationship between the tangential shear strains and the nodal displacements is obtained at element level by imposing the condition $\gamma_s - \frac{\partial w}{\partial s} + \theta_s = 0$ along the element sides in a weighted residual sense as

$$\int_l W \left[ \gamma_s - \frac{\partial w}{\partial s} + \theta_s \right] ds = 0 \tag{4}$$

where $s$ denotes the side coordinate, $l$ is the side length and $W$ are appropriate weighting functions. The simplest choice is point collocation, however other alternatives are possible [1, 6-8]. Transforming the natural shear strains to the cartesian system gives the final expression between the shear strain vector $\boldsymbol{\gamma}$ of Table I and the nodal displacement vector for the element $\boldsymbol{a}^{(e)}$ as

$$\gamma = \bar{\mathbf{B}}_s \mathrm{a}^{(e)} \tag{5}$$

Matrix $\bar{\mathbf{B}}_s$ is termed substitute transverse shear strain matrix (sometimes called also B-bar matrix). The detailed derivation of $\bar{\mathbf{B}}_s$ can be found in [1, 6-8].

The element bending and shear stiffness matrices are obtained now by

$$\mathbf{K}_b^{(e)} = \int_{A^{(e)}} \mathbf{B}_s^T \mathbf{D}_b \mathbf{B}_s dA \quad ; \quad \mathbf{K}_s^{(e)} = \int_{A^{(e)}} \bar{\mathbf{B}}_s^T \mathbf{D}_s \bar{\mathbf{B}}_s dA \tag{6}$$

where $A^{(e)}$ is the element area, $\bar{\mathbf{B}}_s$ is the substitute shear strain matrix of eq. (5) and $\mathbf{B}_b$ is the standard bending strain matrix given by $\mathbf{B}_b = [\mathbf{B}_{b_1}, \mathbf{B}_{b_2} \dots \mathbf{B}_{b_n}]$ where $n$ is the number of element nodes and $\mathbf{B}_{b_i} = \mathbf{L}\mathbf{N}_{\theta_i}$ (see Table I for definition of $\mathbf{L}$).

The computation of the integrals in (6) requires in this case a 3 point numerical quadrature.

This element was originally developed by Zienkiewicz et al. [7] and it is termed here TQQL (for **T**riangle, **Q**uadratic deflection, **Q**uadratic rotations and **L**inear transverse shear strain fields). The TQQL element satisfies eq. (2) for all meshes and it behaves well in all examples analyzed [6-9] although a too flexible behaviour was found for coarse meshes. This can be improved as shown next.

## TLQL quadratic triangle

An enhanced version of the TQQL element can be derived by constraining the normal rotation to vary linearly along the sides (Figure 2). This idea originally proposed in [7] was the basis towards the derivation of a new plate triangle with the following assumed fields:

1) The deflection varies linearly as $w = \sum_{i=1}^{3} L_i w_i$

2) The following incomplete quadratic interpolation is used for the rotations

$$\boldsymbol{\theta} = \sum_{i=1}^{3} L_i \boldsymbol{\theta}_i + 4L_1L_2\mathbf{e}_{12}\Delta\theta_{s_4} + 4L_2L_3\mathbf{e}_{23}\Delta\theta_{s_5} + 4L_1L_3\mathbf{e}_{13}\Delta\theta_{s_6} \tag{8}$$

In above $L_i$ are the standard linear shape functions of the 3 node triangle, $\Delta\theta_{s_i}$ is a hierarchical tangential rotation at the mid-side points (Figure 2) and $\mathbf{e}_{ij}$ are unit vectors along the side directions. Eq (8) defines a linear variation of the normal rotation along the sides, whereas the tangential rotation varies quadratically.

3) The transverse shear strains vary linearly as [8, 10]

$$\begin{Bmatrix} \gamma_\xi \\ \gamma_\eta \end{Bmatrix} = \begin{bmatrix} 1-\eta & -\eta\sqrt{2} & \eta \\ \xi & \xi\sqrt{2} & 1-\xi \end{bmatrix} \begin{Bmatrix} \gamma_s^{12} \\ \gamma_s^{23} \\ \gamma_s^{13} \end{Bmatrix} = \mathbf{S}\boldsymbol{\gamma}_s \tag{9}$$

where $\gamma_s^{ij}$ are the tangential shear strains at the mid point of side $ij$. Eq.(4) is now used to obtain the relationship between the tangential shear strains and the nodal displacements. Choosing $W = 1$ in (4) gives [8]

$$\gamma_s^{ij} = \frac{1}{l_\xi^i}(w_j - w_i) - \frac{l_{ij}}{2l_\xi^i} e_{ij}^T(\boldsymbol{\theta}_i + pmb\theta_j) - \frac{2}{3}\Delta\theta_{s_k}\frac{l_{ij}}{l_\xi^i} \tag{10}$$

where $l_\xi^1 = l_\xi^3 = 1$ and $l_\xi^2 = \sqrt{2}$, $l_{ij}$ is the length of side $ij$ and $k = 3+i$. Combining eqs. (9) and (10) and transforming the natural strains to the cartesian system gives finally an expression identical to (5). Full details of the derivation of matrix $\bar{\mathbf{B}}_s$ in this case can be found in [6, 8]. It can verified that the so called TLQL element (Linear $w$, Quadratic $\boldsymbol{\theta}$ and Linear $\gamma$) satisfies eqs. (2) for all cases. Also note that a 3 point quadrature is required for integrating all terms of the stiffness matrix to prevent spurious mechanisms. The extension of the TLQL to the shell case is straightforward and details can be found in [6, 16]. The performance of the TLQL element is excellent for plate and shell analysis and many examples are given in [6, 8, 11]. This element is the basis for the derivation of a 9 d.o.f. DK plate triangle as shown is next section.

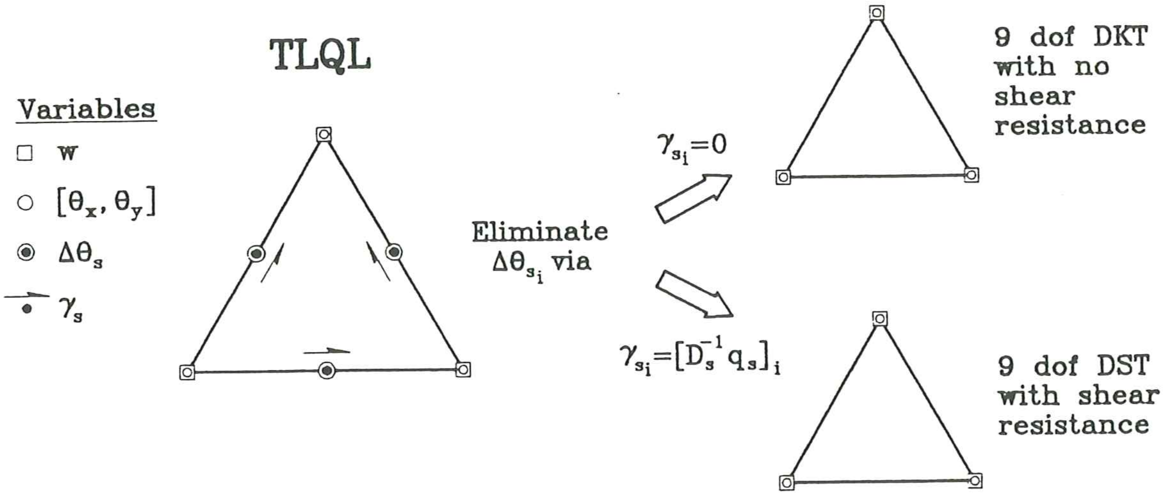


Figure 3. DKT and DST plate elements

## Derivation of a DKT plate triangle

The TLQL evolves naturally into a 9 d.o.f. DK plate triangle (hereafter termed DKT) as follows. The transverse shear strains are made zero over the element by constraining the tangential shear strains at the element mid-sides to a zero value (Figure 3). This allows to eliminate the hierarchical side rotations in terms of the side degrees of freedom as

$$\gamma_s^{ij} = 0 \rightarrow \Delta\theta_{s_k} = \frac{3}{2l_{ij}}(w_j - w_i) - \frac{3}{4}\mathbf{e}_{ij}^T(\boldsymbol{\theta}_i + \boldsymbol{\theta}_j) \tag{11}$$

Substituting (11) into (8) gives the new rotation field as

$$\boldsymbol{\theta} = \sum_{i=1}^{3} \mathbf{N}_i \mathbf{a}_i^{(e)} \quad ; \quad \mathbf{a}_i^{(e)} = \begin{Bmatrix} w_i \\ \theta_{x_i} \\ \theta_{y_i} \end{Bmatrix} \tag{12}$$

with

$$N_1 = \left[ -\left( \frac{6L_1L_2}{l_{12}} \mathbf{e}_{12} + \frac{6L_1L_3}{l_{13}} \mathbf{e}_{13} \right), (L_1 - 3L_1L_2 + 3L_1L_3)\, \mathbf{I}_2 \right]$$

$$N_2 = \left[ \left( \frac{6L_1L_2}{l_{12}} \mathbf{e}_{12} - \frac{6L_2L_3}{l_{23}} \mathbf{e}_{23} \right), (L_2 - 3L_2L_3 + 3L_1L_2)\, \mathbf{I}_2 \right]$$

$$N_3 = \left[ \left( \frac{6L_1L_3}{l_{13}} \mathbf{e}_{13} + \frac{6L_2L_3}{l_{23}} \mathbf{e}_{23} \right), (L_3 - 3L_2L_3 + 3L_1L_3)\, \mathbf{I}_2 \right] \tag{13}$$

where $\mathbf{I}_2$ is the $2 \times 2$ unit matrix. Eq. (12) allows to obtain the element *bending stiffness matrix* using eq. $(6)_1$. Again a 3 point quadrature is required in this case. It is interesting to note that the resulting element is identical to the popular DKT element originally presented by Batoz et al. [2, 3] although the derivation shown here is much simpler.

## Derivation of a 9 d.o.f. triangle with shear deformation

The hierarchical mid-side rotation in the TLQL element can be eliminated by equaling the mid-side tangential shear strains to those given by the standard bending moment-shear equilibrium relationship, i.e.

$$\boldsymbol{\gamma} = \mathbf{D}_s^{-1}\mathbf{q} = -\mathbf{D}_s^{-1}\mathbf{L}^T\mathbf{m} = -\mathbf{D}_s^{-1}\mathbf{L}^T\mathbf{D}_b\mathbf{B}_b\mathbf{a}^{(e)} \tag{14}$$

where $\mathbf{L}$ can be deduced from Table I.

Eq. (14) can now be particularized to give the tangential shear strain at the mid-side points. Equaling the resulting expression to that given by eq.(10) provides the three equations necessary for eliminating the three hierarchical rotations $\Delta\theta_{s_k}$. The resulting 9 d.o.f. element - termed DST (for Discrete Shear Triangle) is incompatible and it incorporates shear deformation effects (Figure 3). This approach was originally proposed by Batoz et al. [13, 14] and Katili [18] to derive similar plate elements.

## Some elements of the Reissner-Mindlin and DK quadrilateral family

The simplest Reissner-Mindlin (mixed) plate quadrilateral is the four node with bilinear interpolation of deflections and rotations and the following linear shear strain field

$$\gamma_\xi = \alpha_1 + \alpha_2\eta \quad , \quad \gamma_\eta = \alpha_3 + \alpha_4\xi \tag{15}$$

Parameters $\alpha_i$ can be obtained via eq. (4). The simplest choice for $W$ in this case is point collocation. The integration of the stiffness matrix requires a full $2 \times 2$ quadrature. This element termed here QLLL (for Quadrilateral and Linear deflection, rotation and shear fields) was originally proposed by Dvorkin and Bathe [12], it satisfies eqs. (2) (except for a four element patch) and it behaves very well for thick and thin plate and shell analysis. A detailed description of this element can be found in [1, 6, 8, 12].

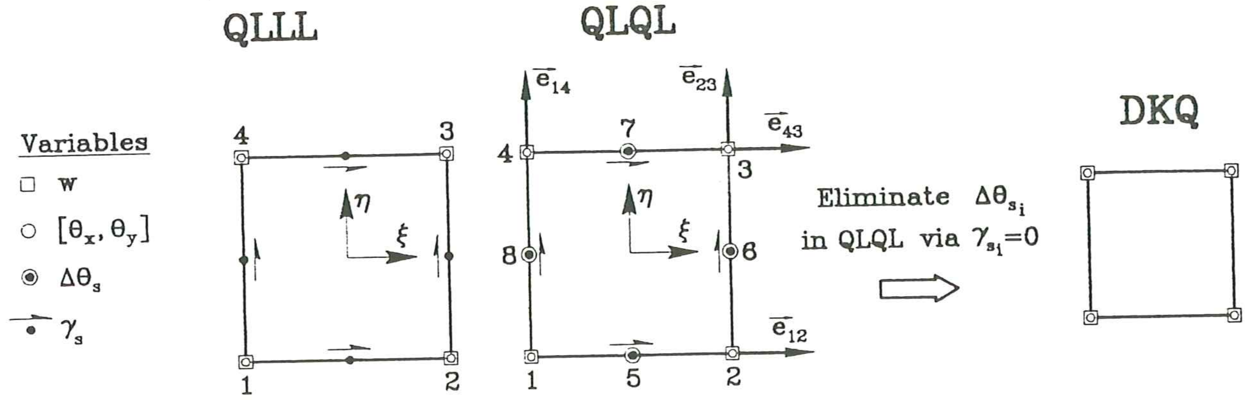


Figure 4. QLLL, QLQL and DKQ plate elements

### 16 d.o.f. QLQL plate quadrilateral

This is an element with identical features to the TLQL of a previous section. The interpolating fields are the following:

1) The deflections are bi-linearly interpolated in terms of the corner values in the standard manner [1].
2) The following incomplete quadratic field is chosen for the rotations

$$\theta = \sum_{i=1}^{4} N_i \boldsymbol{\theta}_i + \frac{1}{2} f(\xi)(1-\eta)\mathbf{e}_{12}\Delta\theta_{s_5} + \frac{1}{2} f(\eta)(1+\xi)\mathbf{e}_{23}\Delta\theta_{s_6} + \\ + \frac{1}{2} f(\xi)(1+\eta)\mathbf{e}_{43}\Delta\theta_{s_7} + \frac{1}{2} f(\eta)(1-\xi)\mathbf{e}_{14}\Delta\theta_{s_8} \quad (16)$$

where $N_i$ are bilinear shape functions, $f(x) = 1 - x^2$ and $\mathbf{e}_{ij}$ and $\Delta\theta_{s_i}$ have the same meaning as in eq. (8).

3) The transverse shear strains are assumed to vary linearly as is eq. (15) for the QLLL element. A $2 \times 2$ quadrature is used for all terms of the stiffness matrix.

Examples of the good behaviour of this element can be found in [6, 8, 11].

### 12 d.o.f. DK and DS plate quadrilaterals

The QLQL plate element is the basis for deriving a DK quadrilateral with 12 d.o.f. The approach follows precisely the lines explained for deriving the DKT in a previous section; i.e. the condition of zero tangential shear strain along the four element sides is used to eliminate the hierarchical tangential side rotations $\Delta\theta_{s_i}$ in terms of the corner degrees of freedom. Details of the resulting interpolating functions for the rotation field can be found in [6].

The procedure explained previously to derive a 3 node triangle incorporating shear effects (DST element) can be followed now again to obtain an equivalent DSQ 12 d.o.f. quadrilateral. This element has been formulated by Oñate and Castro [11] and Katili [18]. However its efficiency is not comparable to the simpler QLLL element.

## Linear 9 d.o.f. TLLL Reissner-Mindlin plate triangle

Oñate et al. [10] have recently proposed a simple lower order plate triangle based on the following fields:

1) The deflection is linearly interpolated in terms of the corner values.
2) The rotations are *also* linearly interpolated in terms of the mid-side node values as (Figure 5)

$$\boldsymbol{\theta} = \sum_{i=4}^{6} N_i \boldsymbol{\theta}_i \quad ; \quad \boldsymbol{\theta}_i = [\theta_{x_i}, \theta_{y_i}]^T \tag{17}$$

with

$$N_4 = 1 - 2\eta \quad , \quad N_5 = 2\xi + 2\eta - 1 \quad , \quad N_6 = 1 - 2\xi$$

For convenience the element displacement vector is now defined as $\mathbf{a}^{(e)} = [w_1, w_2, w_3, \boldsymbol{\theta}_4^T, \boldsymbol{\theta}_5^T, \boldsymbol{\theta}_6^T]^T$. Eq. (17) defines an incompatible rotation field with interelemental compatibility satisfied at the mid-side nodes only. The good performance of the element is however ensured via satisfaction of the patch test.

3) The assumed transverse shear strains field is also linear and it coincides with that expressed by eq.(9) for the TLQL element. The form of the $\bar{\mathbf{B}}_s$ matrix is simply given in this case by [10]

$$\bar{\mathbf{B}}_s = \mathbf{J}^{-1}\mathbf{S} \begin{bmatrix} -1 & 1 & 0 & x_{12} & y_{12} & 0 & 0 & 0 & 0 \\ 0 & -a & a & 0 & 0 & ax_{23} & ay_{23} & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & x_{13} & y_{13} \end{bmatrix} \tag{18}$$

where $a = 1/\sqrt{2}$, $x_{ij} = x_i - x_j$ $y_{ij} = y_i - y_j$, $\mathbf{S}$ is given by eq.(9) and $\mathbf{J}$ is the standard Jacobian matrix.
Examples of the good behaviour of this element (termed TLLL due to the linearity of all fields) for plate and shell analysis can be found in [6, 10, 15].

## The Morley element revisited

The DK version of the TLLL element is simply obtained by constraining the three tangential shear strains along the sides to a zero value. This allows to eliminate the three tangential side rotations giving a DK triangle with 6 d.o.f. (three corner deflections and three normal rotations along the sides (Figure 5)).

It can be verified that this element is *identical* to the classical Morley triangle [4]. Note however that the derivation follows here a different approach. The resulting bending strain matrix is extremely simple and its expression is given below.

$$\mathbf{B}_b = \begin{bmatrix} (a_{12} - a_{13}) & (a_{23} - a_{12}) & (a_{13} - a_{23}) & c_{12} & c_{23} & -c_{13} \\ (a_{13} - a_{12}) & (a_{12} - a_{23}) & (a_{23} - a_{13}) & b_{12} & b_{23} & -b_{13} \\ (d_{13} - d_{12}) & (d_{12} - d_{23}) & (d_{23} - d_{13}) & -2a_{12} & -2a_{23} & -2a_{13} \end{bmatrix} \tag{19}$$

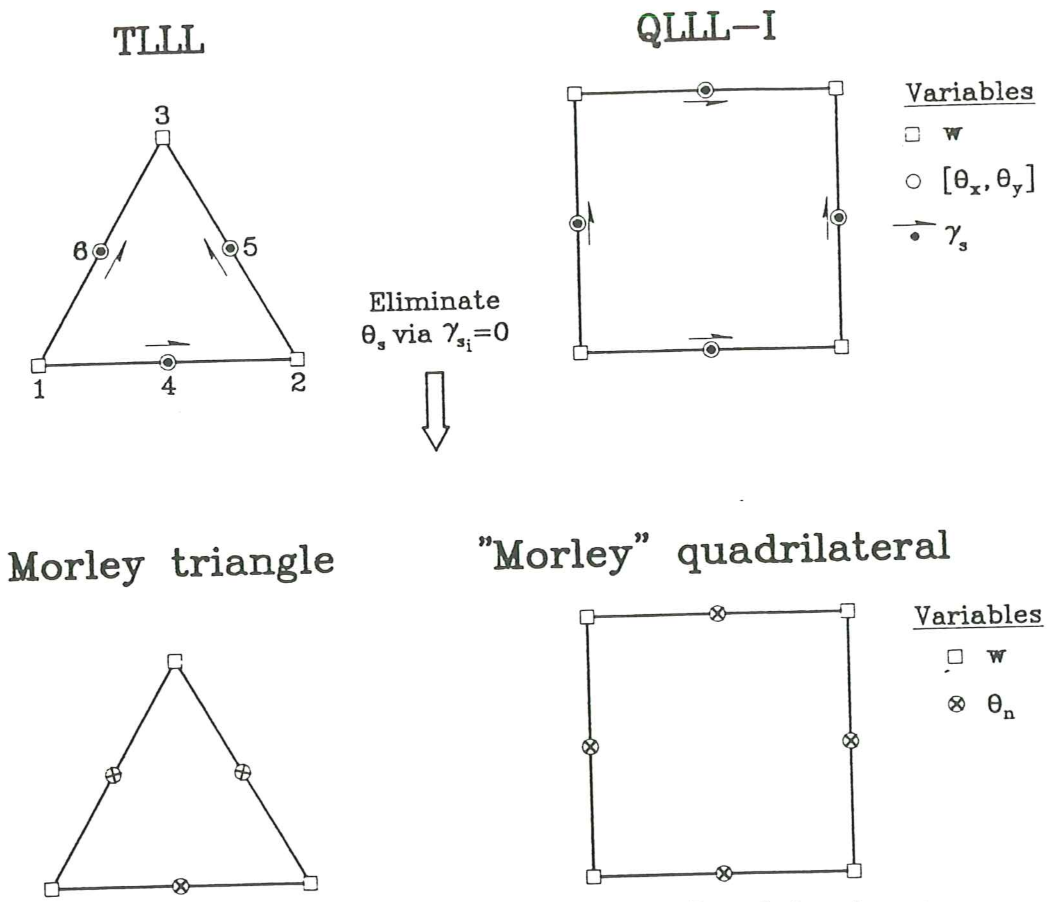


Figure 5. Some incompatible triangular and quadrilateral plate elements

with

$$a_{ij} = \frac{x_{ij} y_{ij}}{l_{ij}^2}, \qquad b_{ij} = \frac{x_{ij}^2}{l_{ij}}, \qquad c_{ij} = \frac{y_{ij}^2}{l_{ij}}, \qquad d_{ij} = \frac{x_{ij}^2 - y_{ij}^2}{l_{ij}^2}$$

$$x_{ij} = x_i - x_j, \qquad y_{ij} = y_i - y_j, \qquad l_{ij} = (x_{ij}^2 + y_{ij}^2)^{1/2} \tag{20}$$

The element bending stiffness matrix is obtained by eq.$(6)_1$. One point quadrature suffices in this case.

## From the incompatible 12 d.o.f. linear quadrilateral to a 8 d.o.f. Morley quadrilateral

The extension of the incompatible TLLL element to a new 12 d.o.f. incompatible thick plate quadrilateral is straight-forward. This element (termed here QLLL-I) has now four corner deflections defining a bilinear deflection field and two rotations at each mid-side point defining an incompatible linear rotation field (Figure 5). The shear strains are linearly iterpolated in terms of the tangential values at the mid-side nodes as for the original QLLL element (see eq. (15)).

The condition of zero tangential shear strain at each mid-side node allows to eliminate the tangential rotations at these nodes leading to an 8 d.o.f. thin plate quadrilateral. This element can be considered a quadrilateral version of the classical Morley triangle (Figure 5).

The cost-efficiency of these two elements still needs to be verified through numerical experiments.

## Derivation of thin plate triangles and quadrilaterals with one degree of freedom per node

Oñate and Cervera [5] have recently presented a procedure for deriving thin plate bending elements with the deflection as the only nodal variable. The starting point is the following mixed set of equilibrium equations

$$\int\int_A \delta\boldsymbol{\kappa}^T \mathbf{D}_b \boldsymbol{\kappa} dA = \int\int_A \delta w q dA \tag{21}$$

$$\int\int_A \mathbf{W}^T [\boldsymbol{\kappa} - \hat{\mathbf{L}} w] dA = 0 \tag{22}$$

Eq.(21) expresses the PVW for the thin plate case, whereas eq.(22) defines the curvature-deflection relationship in a weighted residual sense with $\hat{\mathbf{L}} = [-\frac{\partial^2}{\partial x^2}, -\frac{\partial^2}{\partial y^2}, -2\frac{\partial^2}{\partial x \partial y}]^T$. Integrating by parts eq. (22) and choosing the weighting functions $\mathbf{W} = \mathbf{I}$ over appropriate subdomains $A_s$ where $\mathbf{I}$ is the $3 \times 3$ unit matrix leads to

$$\int\int_{A_s} \boldsymbol{\kappa} dA = \int_{\Gamma_s} \left[ n_x \frac{\partial w}{\partial x}, \; n_y \frac{\partial w}{\partial y}, \; n_x \frac{\partial w}{\partial y} + n_y \frac{\partial w}{\partial x} \right]^T d\Gamma \tag{23}$$

Eq. (23) relates the curvature field within $A_s$ with the deflection gradient along its boundary $\Gamma_s$. Obviously, these gradients are discontinuous when a $C_0$ continuous field is chosen for the deflection and some problems arise in the computation of the boundary integral in (23). These problems can be overcome by smoothing the deflection gradient over element patches (the simplest option being nodal averaging). Further details on the smoothing procedure can be found in [5].

The discretized systems of equations is obtained now by choosing two independent interpolating fields for the deflection and the curvatures as

$$w = \mathbf{N}_w \bar{\mathbf{w}}^{(e)} \quad \text{and} \quad \boldsymbol{\kappa} = \mathbf{N}_\gamma \bar{\boldsymbol{\kappa}}^{(e)} \tag{24}$$

The simplest option is the choice of a $C_0$ continuous field for $w$ and a discontinuous field for $\boldsymbol{\kappa}$. However many other alternatives are possible [5]. Substituting (24) into (23) allows to obtain the discretized curvature-deflection relationship as $\boldsymbol{\kappa} = \mathbf{B}_b \bar{\mathbf{w}}^{(e)}$ Substituting this expression into (21) gives the final stiffness equations relating external forces and nodal deflections as $\mathbf{K}_b \bar{\mathbf{w}} = \mathbf{f}$ where the bending stiffness matrix is given by eq. $(6)_1$. Full details of this approach can be found in [5].

The simplest elements of this family are the three node triangle and the four node quadrilateral with 3 and 4 d.o.f., respectively (termed hereafter BPT and BPQ for **B**asic **P**late **T**riangle and **Q**uadrilateral respectively, Figure 6). A description of the BPT including the explicit form of its stiffness matrix can be found in [5]. Both BPT and BPQ plate elements can be derived from the Morley triangle and quadrilateral elements presented in previous sections, simply by constraining the normal rotation at the mid-side nodes to take the following value

$$\theta n_i = \frac{1}{2}\left[\frac{\partial w}{\partial n}\bigg|_i^1 + \frac{\partial w}{\partial n}\bigg|_i^2\right] \tag{25}$$

where $\frac{\partial w}{\partial n}|_i^1$ and $\frac{\partial w}{\partial n}|_i^2$ denote the values of the discontinuous normal deflection gradient at the two elements sharing the mid-side node $i$.

Obviously many other smoothing alternatives are possible and some are discussed in [15] where new thin plate and flat shell elements with only translational degrees of freedom are proposed.

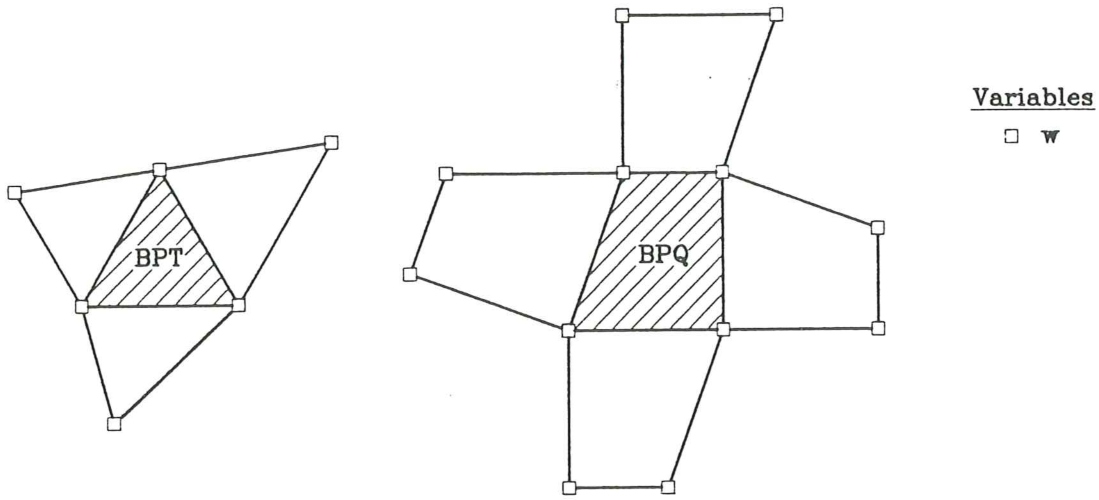


Figure 6. BPT and BPQ elements and patches used for smoothing of the discontinuous deflection gradient along the sides

## CONCLUDING REMARKS

This paper shows the potential of combining Reissner-Mindlin theory and mixed interpolations for deriving different element families adequate for thick and thin plate (and shell) analysis.

Obviously the list of elements presented here is by no means exhaustive and many new and interesting elements are available. Among these we note the simple Q4BL and T3BL plate elements developed by Zienkiewicz et al. [19] and Taylor and Auricchio [20] using an interpolation linking the deflection and rotations fields, and the family of plate and shell elements developed by Van Keulen et al. [21, 22] using a mixed hybrid approach.

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