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ANALYSIS OF STOCHASTIC DYNAMIC RESPONSES BASED ON THE REDUCED-DIMENSIONAL PROBABILITY EVOLUTION EQUATION UNDER THE INFLUENCE OF ADDITIVE GAUSSIAN WHITE NOISE

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Abstract. The solution of derivative moments in the reduced-dimensional probability evolution equation (RDPEE) remains a challenging problem in contemporary research, particularly for complex systems. This paper introduces the Cell Renormalization Method (CRM), which dissects both the state and probabilistic spaces, and applies it for the numerical calculation of the derivative moments at cell-centered coordinates. Subsequently, an efficient nonlinear regression technique, grounded in Bayesian inference and known as Gaussian Process Regression (GPR), is employed to obtain the continuous curve of derivative moments within the state space. Finally, by solving the RDPEE using the path integral approach, the final probability distribution can be easily recovered. Numerical examples demonstrate the efficacy of the proposed method.

1 INTRODUCTION

Over the past few decades, stochastic dynamic response analysis has emerged as a critical tool in structural disaster mitigation, particularly in the context of seismic and wind-resistant design. A widely adopted method for conducting such analyses involves formulating a probability evolution equation that governs the response's probability density function. White noise, recognized as one of the most fundamental forms of stochastic excitation, has been applied in structural seismic analysis since the 1940s [1]. Subsequently, numerous researchers have explored methods to determine structural responses subjected to white noise excitation [2, 3]. Despite these advancements, deriving the probability density function of structural responses under white noise remains a pivotal and challenging issue in the stochastic analysis of complex structural systems.

It is well-established that the probability density function (PDF) of a system's response to Gaussian white noise excitation is governed by the Fokker–Planck–Kolmogorov (FPK) equation . Despite its theoretical significance, the FPK equation can only be solved exactly for a limited class of simple systems due to its inherent mathematical complexity. This limitation underscores the necessity of developing and employing numerical approaches for practical applications. Commonly used numerical techniques for solving the FPK equation include the finite difference method [4], the finite element method [5], random walk simulations [6], and various series expansion methods [7, 8], among others. However, significant challenges persist. One major obstacle is the high dimensionality of FPK equations encountered in real-world engineering problems, which severely limits the accuracy and efficiency of conventional numerical methods. Although dimensionality reduction techniques, such as stochastic averaging [9, 10, 11, 12], have been introduced to address this issue, they still fall short of fully meeting the demands of complex engineering scenarios. Moreover, the applicability of the FPK equation is confined to Markovian systems, rendering it unsuitable for non-Markovian systems, especially those characterized by intricate behaviors like fractional-order dynamics.

A key focus in the aforementioned research lies in reducing the dimensionality of the FPK equation. Recent advancements in its mathematical formulation [13, 14] have provided promising new avenues for applying this approach within stochastic dynamics. By introducing probability fluxes and performing direct integration on the high-dimensional FPK equation, it becomes possible to derive a simplified two-dimensional version that only involves the joint probability density function of displacement and velocity for the relevant physical variables [15]. This reduction has shown considerable success in accurately capturing the response distributions of Markovian systems. Nonetheless, solving non-Markovian problems using this framework remains a significant challenge. Furthermore, a novel class of Reduced Dimension Probability Evolution Equations (RDPEE) has emerged in recent years, offering an alternative reduction strategy. Notably, it has been proven that RDPEE remains applicable as long as the underlying stochastic process satisfies the condition of first-order continuity [16, 17, 18].

It is evident that accurately constructing the first-order derivative moments in the RDPEE framework is fundamental to achieving reliable solutions. These moments are central to the precision of the entire method. Currently, Monte Carlo Simulation (MCS) is the most commonly adopted approach for estimating these moments [15, 16, 17, 18]. However, MCS faces inherent limitations: using too few samples leads to poor accuracy and significant result variability, while increasing the number of samples substantially raises computational costs, reducing overall efficiency. This trade-off makes it challenging to balance precision and efficiency effectively. Therefore, there is a pressing need for more efficient numerical strategies tailored to solving RDPEE. In earlier studies, a cell renormalization technique [13, 14], which partitions the state and probability spaces, was introduced to estimate the first-order derivative moments numerically. Following this, polynomial least squares fitting was used to generate continuous surfaces for these moments. However, this approach requires predefining the form of the polynomials, limiting its adaptability, especially when the analytical form of the derivative moments is unknown or highly complex. In this research, we adopt a similar framework for solving RDPEE, but enhance the accuracy of derivative moment estimation by introducing Gaussian Process Regression (GPR) [19], grounded in Bayesian inference. GPR allows for flexible and precise construction of continuous surfaces for derivative moments across the state space without relying on predefined functional forms. Once the derivative moments are accurately modeled, the joint and marginal probability density functions of the system response can be computed using numerical techniques such as the finite difference method [4] or the path integral approach [20, 21, 22]. Due to its superior accuracy, the path integral method has attracted significant attention, with researchers striving to develop more efficient implementations. Initially, its use was confined to Markovian and low-dimensional systems due to computational constraints, as the method's efficiency diminishes with increasing system dimensionality.

The remaining sections of this paper are organized as follows: Section 2 provides the Problem formulation. Section 3 provides the proposed method to solve RDPEE. In Section 4, numerical example is employed to demonstrate the high accuracy and efficiency of proposed method. Some conclusions are provided in Section 5.

2 PROBLEM FORMULATION

2.1 Probability evolution equation

The probability evolution equation can be given by [20]

$$\frac{\partial p_{\boldsymbol{X}}\left(\boldsymbol{x},t\right)}{\partial t}=\sum_{l_1=1}^{+\infty}\cdots\sum_{l_n=1}^{+\infty}\frac{\left(-1\right)^{l_1}}{l_1!}\cdots\frac{\left(-1\right)^{l_n}}{l_n!}\frac{\partial^{\left(l_1+\cdots+l_n\right)}\left[\alpha_{l_1+\cdots+l_n}\left(\boldsymbol{x},t\right)p_{\boldsymbol{X}}\left(\boldsymbol{x},t\right)\right]}{\partial x_1^{l_1}\cdots\partial x_n^{l_n}} \tag{1}$$

According to the Lyu-Chen theorem [17], when a stochastic process $\boldsymbol{X}(t)$ fulfills the condition of first-order continuity, all higher-order derivative moments beyond the second order vanish. As a result, the probability evolution equation for a first-order continuous process $\boldsymbol{X}(t)$ can be expressed as:

$$\frac{\partial p_{\boldsymbol{X}}\left(\boldsymbol{x},t\right)}{\partial t}=-\sum_{i=1}^{n}\frac{\partial\left[\mu_i\left(\boldsymbol{x},t\right)p_{\boldsymbol{X}}\left(\boldsymbol{x},t\right)\right]}{\partial x_i}+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial\left[\sigma_{ij}\left(\boldsymbol{x},t\right)p_{\boldsymbol{X}}\left(\boldsymbol{x},t\right)\right]}{\partial x_i\partial x_j} \tag{2}$$

where $\mu_i\left(\boldsymbol{x},t\right)$ and $\sigma_{ij}\left(\boldsymbol{x},t\right)$ denote the first and second order derivate moments, respectively, which can be defined by

$$\begin{cases}\mu_i\left[\boldsymbol{X}\left(t\right),t\right]=\frac{\mathbb{E}\left[\mathrm{d}X_i(t)|\boldsymbol{X}(t)=\boldsymbol{x}\right]}{\mathrm{d}t}=\lim\limits_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}\left[\Delta X_i\left(t\right)|\boldsymbol{X}\left(t\right)=\boldsymbol{x}\right]\\ \sigma_{ij}\left[\boldsymbol{X}\left(t\right),t\right]=\frac{\mathbb{E}\left[\mathrm{d}X_i(t)\mathrm{d}X_j(t)|\boldsymbol{X}(t)=\boldsymbol{x}\right]}{\mathrm{d}t}=\lim\limits_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}\left[\Delta X_i\left(t\right)\Delta X_j\left(t\right)|\boldsymbol{X}\left(t\right)=\boldsymbol{x}\right]\end{cases} \tag{3}$$

2.2 Reduced dimension probability evolution equation

By integrating over so variables other than the physical quantity of interest, and using the method of partial integration, the Reduced dimension probability evolution equation (RDPEE) can be given by

$$\begin{aligned}\frac{\partial p_{X_lX_k}\left(x_l,x_k,t\right)}{\partial t}=&-\frac{\partial\left[\mu_l\left(x_l,x_k,t\right)p_{X_lX_k}\left(x_l,x_k,t\right)\right]}{\partial X_l}-\frac{\partial\left[\mu_k\left(x_l,x_k,t\right)p_{X_lX_k}\left(x_l,x_k,t\right)\right]}{\partial X_k}\\&+\frac{1}{2}\frac{\partial^2\left[\sigma_{ll}\left(x_l,x_k,t\right)p_{X_lX_k}\left(x_l,x_k,t\right)\right]}{\partial X_l^2}+\frac{1}{2}\frac{\partial^2\left[\sigma_{lk}\left(x_l,x_k,t\right)p_{X_lX_k}\left(x_l,x_k,t\right)\right]}{\partial X_l\partial X_k}\\&+\frac{1}{2}\frac{\partial^2\left[\sigma_{kl}\left(x_l,x_k,t\right)p_{X_lX_k}\left(x_l,x_k,t\right)\right]}{\partial X_k\partial X_l}+\frac{1}{2}\frac{\partial^2\left[\sigma_{kk}\left(x_l,x_k,t\right)p_{X_lX_k}\left(x_l,x_k,t\right)\right]}{\partial X_k^2}\end{aligned} \tag{4}$$

where the first and second order derivate moments can be expressed as

$$\begin{cases} \mu_l\left(x_l,x_k,t\right)=\lim\limits_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}\left[\Delta X_l\left(t\right)\left|X_l\left(t\right)=x_l,X_k\left(t\right)=x_k\right.\right] \\ \mu_k\left(x_l,x_k,t\right)=\lim\limits_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}\left[\Delta X_k\left(t\right)\left|X_l\left(t\right)=x_l,X_k\left(t\right)=x_k\right.\right] \\ \sigma_{ll}\left(x_l,x_k,t\right)=\lim\limits_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}\left[\left[\Delta X_l\left(t\right)\right]^2\left|X_l\left(t\right)=x_l,X_k\left(t\right)=x_k\right.\right] \\ \sigma_{lk}\left(x_l,x_k,t\right)=\lim\limits_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}\left[\Delta X_l\left(t\right)\Delta X_k\left(t\right)\left|X_l\left(t\right)=x_l,X_k\left(t\right)=x_k\right.\right] \\ \sigma_{kl}\left(x_l,x_k,t\right)=\lim\limits_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}\left[\Delta X_k\left(t\right)\Delta X_l\left(t\right)\left|X_l\left(t\right)=x_l,X_k\left(t\right)=x_k\right.\right] \\ \sigma_{kk}\left(x_l,x_k,t\right)=\lim\limits_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}\left[\left[\Delta X_k\left(t\right)\right]^2\left|X_l\left(t\right)=x_l,X_k\left(t\right)=x_k\right.\right] \end{cases} \tag{5}$$

3 THE SOLUTION OF PROPOSED METHOD

3.1 The cell renormalized method

The first order derivate moments of cell-centered coordinates can be written as

$$\begin{aligned} \mu_l\left[\left(z_l,\dot{z}_l\right)=c^{(i,j)},t\right] &= \mathbb{E}_{\Omega_{\Theta}}\left[\ddot{Z}_l\left(\boldsymbol{\theta},t\right)\middle|Z_l\left(\boldsymbol{\theta},t\right)=z_l,\ \dot{Z}_l\left(\boldsymbol{\theta},t\right)=\dot{z}_l\right] \\ &\approx \mathbb{E}\left\{\ddot{Z}_l\left(\boldsymbol{\theta},t\right)\ \middle|\ \left[Z_l\left(\boldsymbol{\theta},t\right),\dot{Z}_l\left(\boldsymbol{\theta},t\right)\right]\in\Omega^{(i,j)}\right\} \\ &= \sum_{q=1}^{N}\ddot{Z}_l\left(t\right)\times\Pr\left(\boldsymbol{\Theta}=\boldsymbol{\theta}_q\middle|\left[Z_l\left(\boldsymbol{\theta},t\right),\dot{Z}_l\left(\boldsymbol{\theta},t\right)\right]\in\Omega^{(i,j)}\right) \end{aligned} \tag{6}$$

where $\Pr\left(\boldsymbol{\Theta}=\boldsymbol{\theta}_q,|,\left[Z_l\left(\boldsymbol{\theta},t\right),\dot{Z}_l\left(\boldsymbol{\theta},t\right)\right]\in\Omega^{(i,j)}\right)$ denotes the probability that the realization of all basic random variables corresponds to the q-th sample $\boldsymbol{\theta}_q$, given that the system state lies within the region $\Omega^{(i,j)}$ of the state space.

According to Bayes rule, Eq. (6) can be rewritten as

$$\begin{aligned} \mu_l\left[\left(z_l,\dot{z}_l\right)=c^{(i,j)},t\right] &= \mathbb{E}_{\Omega_{\Theta}}\left[\ddot{Z}_l\left(\boldsymbol{\theta},t\right)\middle|Z_l\left(\boldsymbol{\theta},t\right)=z_l,\ \dot{Z}_l\left(\boldsymbol{\theta},t\right)=\dot{z}_l\right] \\ &\approx \mathbb{E}\left\{\ddot{Z}_l\left(\boldsymbol{\theta},t\right)\middle|\left[Z_l\left(\boldsymbol{\theta},t\right),\dot{Z}_l\left(\boldsymbol{\theta},t\right)\right]\in\Omega^{(i,j)}\right\} \\ &= \sum_{q=1}^{N}\ddot{Z}_l\left(\boldsymbol{\theta},t\right)\times\Pr\left(\boldsymbol{\Theta}=\boldsymbol{\theta}_q\middle|\left[Z_l\left(\boldsymbol{\theta},t\right),\dot{Z}_l\left(\boldsymbol{\theta},t\right)\right]\in\Omega^{(i,j)}\right) \\ &= \sum_{q=1}^{N^{\Omega^{(i,j)}}}\ddot{Z}_l\left(\boldsymbol{\theta},t\right)\times\frac{P_q}{\sum_{q=1}^{N^{\Omega^{(i,j)}}}P_q} \end{aligned} \tag{7}$$

3.2 Gaussian process regression

It is known that the discrete conditional expectations can be calculated via Eq. (7). To obtained the continuous conditional expectation surfaces, the Gaussian process regression (GPR) is employed, which can be denoted as

$$\begin{aligned} \mathbb{E}\left[\boldsymbol{y}_*\left|\boldsymbol{y}\right.\right] &= \mathbb{E}\left[\boldsymbol{w}+\mathbf{K}_*^{\mathrm{T}}\left(\mathbf{K}+\sigma_n^2\mathbf{E}\right)^{-1}\boldsymbol{y}\left|\boldsymbol{y}\right.\right] \\ &= \mathbb{E}\left[\boldsymbol{w}\left|\boldsymbol{y}\right.\right]+\mathbb{E}\left[\mathbf{K}_*^{\mathrm{T}}\left(\mathbf{K}+\sigma_n^2\mathbf{E}\right)^{-1}\boldsymbol{y}\left|\boldsymbol{y}\right.\right] \\ &= \mathbf{K}_*^{\mathrm{T}}\left(\mathbf{K}+\sigma_n^2\mathbf{E}\right)^{-1}\boldsymbol{y} \end{aligned} \tag{8}$$

where the covariance matrix $\mathbf{K}$ can be given by Define the square exponential function as the kernel function (covariance function), which can be expressed as

$$k\left(\boldsymbol{x}, \boldsymbol{x}'\right) = \sigma_f^2 \exp\left[-\frac{\left\|\boldsymbol{x} - \boldsymbol{x}'\right\|^2}{2l^2}\right] \tag{9}$$

and

$$\begin{cases} \mathbf{K} = k\left(\boldsymbol{x}_i, \boldsymbol{x}_j\right) \\ \mathbf{K}_* = \mathbf{K}_*^{\mathrm{T}} = k\left(\boldsymbol{x}_{*i}, \boldsymbol{x}_j\right) = k\left(\boldsymbol{x}_i, \boldsymbol{x}_{*j}\right) \\ \mathbf{K}_{**} = k\left(\boldsymbol{x}_{*i}, \boldsymbol{x}_{*j}\right) \end{cases} \tag{10}$$

Finally, the value of hyperparameters can be obtained via minimizing the negative natural logarithm

$$-\ln\left[\Pr\left(\boldsymbol{y}\left|\boldsymbol{X}, \boldsymbol{\psi}\right.\right)\right] = \frac{1}{2}\boldsymbol{y}^{\mathrm{T}}\left(\mathbf{K} + \sigma_n^2\mathbf{E}\right)^{-1}\boldsymbol{y} + \frac{1}{2}\ln\left|\mathbf{K} + \sigma_n^2\mathbf{E}\right| + \frac{N}{2}\log 2\pi \tag{11}$$

3.3 Path integral solution

Once the continuous conditional expectation surface is obtained, the eventual PDF can be given by Path integral solution such as

$$\begin{cases} p_{Z\dot{Z}}\left(z, \dot{z}, t_h\right) = \displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p_{Z\dot{Z}}\left(z, \dot{z}, t_h \mid z', \dot{z}', t_{h-1}\right) p_{Z\dot{Z}}\left(z', \dot{z}', t_{h-1}\right) \mathrm{d}z' \,\mathrm{d}\dot{z}' \\ p_{Z\dot{Z}}\left(z_0, \dot{z}_0, t_0\right) = p_{Z\dot{Z}}\left(z_0, \dot{z}_0, 0\right) = \delta\left(z - z_0\right)\delta\left(\dot{z} - \dot{z}_0\right) \end{cases} \tag{12}$$

$$p_{Z\dot{Z}}\left(z, \dot{z}, t + \Delta t \mid z', \dot{z}', t\right) = \frac{\delta\left(z - z' - \dot{z}'\Delta t\right)}{\sqrt{2\pi\sigma\Delta t}} \exp\left[-\frac{\left(\dot{z} - \dot{z}' - \mu\left(z', \dot{z}', t\right)\Delta t\right)^2}{2\sigma\Delta t}\right] \tag{13}$$

and

$$\begin{cases} p_Z\left(z, t\right) = \int_{-\infty}^{+\infty} p_{Z\dot{Z}}\left(z, \dot{z}, t\right) \mathrm{d}\dot{z} \\ p_{\dot{Z}}\left(\dot{z}, t\right) = \int_{-\infty}^{+\infty} p_{Z\dot{Z}}\left(z, \dot{z}, t\right) \mathrm{d}z \end{cases} \tag{14}$$

where $h = 1, 2, \ \ldots\ , N_t$.

4 NUMERICAL EXAMPLE

In the first example, a SDOF system is employed to validate the advantages of proposed method, whose motion's differential equation can be represented as

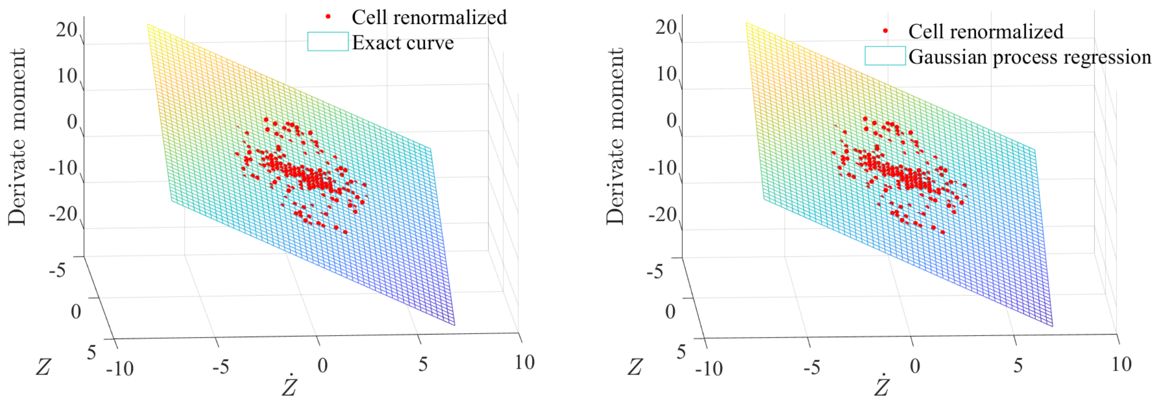
$$m\ddot{z} + c\dot{z} + kz = m\xi\left(t\right) \tag{15}$$

where m stands for mass, c is damping coefficient and k denotes the stiffness coefficient. $\xi\left(t\right)$ is Gaussian white noise, which can be given as $m = 1,\ c = 2,\ k = 3$ in this example.

The RDPEE can be given by

$$\frac{\partial p_{Z\dot{Z}}\left(z, \dot{z}, t\right)}{\partial t} = -\dot{z}\frac{\partial\left[p_{Z\dot{Z}}\left(z, \dot{z}, t\right)\right]}{\partial z} - \frac{\partial\left[\mu\left(z, \dot{z}, t\right) p_{Z\dot{Z}}\left(z, \dot{z}, t\right)\right]}{\partial\dot{z}} + \frac{D}{2}\frac{\partial^2 p_{Z\dot{Z}}\left(z, \dot{z}, t\right)}{\partial\dot{z}^2} \tag{16}$$

With $D = 0.01$, a stochastic harmonic function is employed to generate Gaussian white noise excitation, characterized by a dimension $d = 10$. Using Sobol sequences, 300 sample points are produced, corresponding to 300 required deterministic simulations. Subsequently, finite element analysis is conducted to compute the displacement, velocity, and hysteretic displacement responses for each sample. The cell renormalization method is then applied, wherein the continuous state space is discretized into 900 distinct cells. This method utilizes probabilistic partitioning to assign probability weights to each sample point, allowing for the calculation of the derivative moments $\mu(z, \dot{z}, t)$ at the centroid of every cell. Following this, Gaussian Process Regression (GPR) is implemented to interpolate these first-order derivative moments over the entire state space, resulting in a smooth, continuous surface of derivative moments evolving over time. The Reduced Dimension Probability Evolution Equation (RDPEE) is then solved via the path integral method, enabling efficient determination of the response's probability distribution. Fig. 1 illustrates the first-order derivative moments at selected time instance, as computed by the cell renormalization method and refined through GPR fitting. The standard deviations of displacement and velocity are presented in Fig. 2, demonstrating strong agreement between the results of the proposed method and those obtained through Monte Carlo Simulation (MCS). Furthermore, Fig. 3 displays the joint probability density functions (PDFs) of displacement and velocity. To evaluate the performance of the proposed approach, the marginal PDF of displacement is shown in Fig. 4 (a), revealing a close match between the reconstructed PDFs and those from MCS with 10^5 samples. Additionally, cumulative distribution functions (CDFs) are plotted on a logarithmic scale in Fig. 4 (b), further confirming the near-perfect alignment between the proposed method and the MCS histograms.

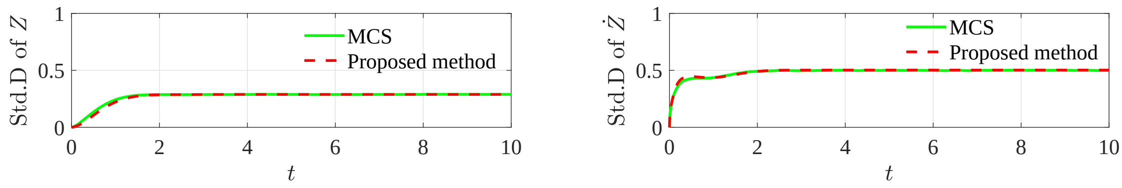


(a) Derivate moment (b) Derivate moment

Figure 1: Derivate moment for Example 1

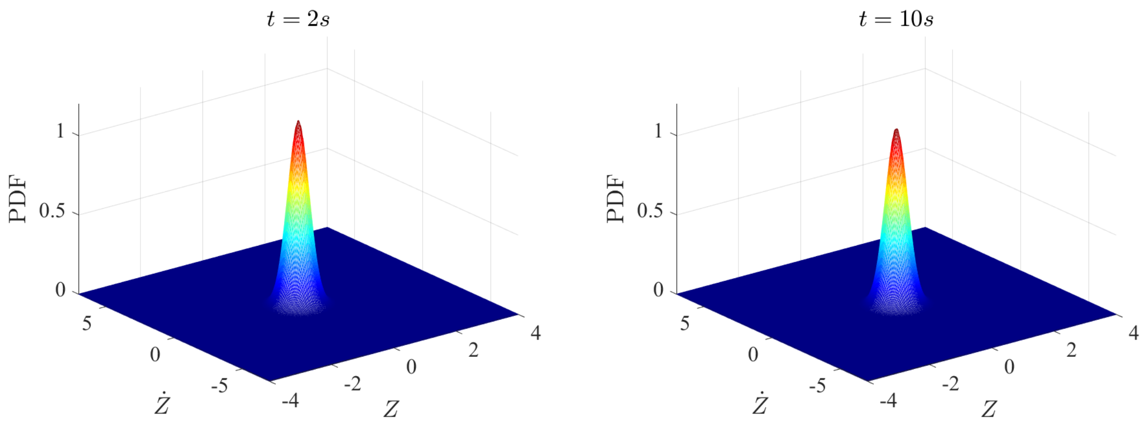
5 CONCLUSIONS

This paper presents a novel approach for solving the Reduced Dimension Probability Evolution Equation (RDPEE) in the context of stochastic dynamics driven by additive Gaussian white noise, utilizing a cell renormalization technique. The method involves partitioning both the state space and probability space to evaluate the first-order derivative moments at the centers of discretized cells. To extend these discrete results into a continuous representation across the state space, Gaussian Process Regression (GPR) is employed to interpolate the mean values of the derivative moments with exceptional precision, particularly in systems exhibiting



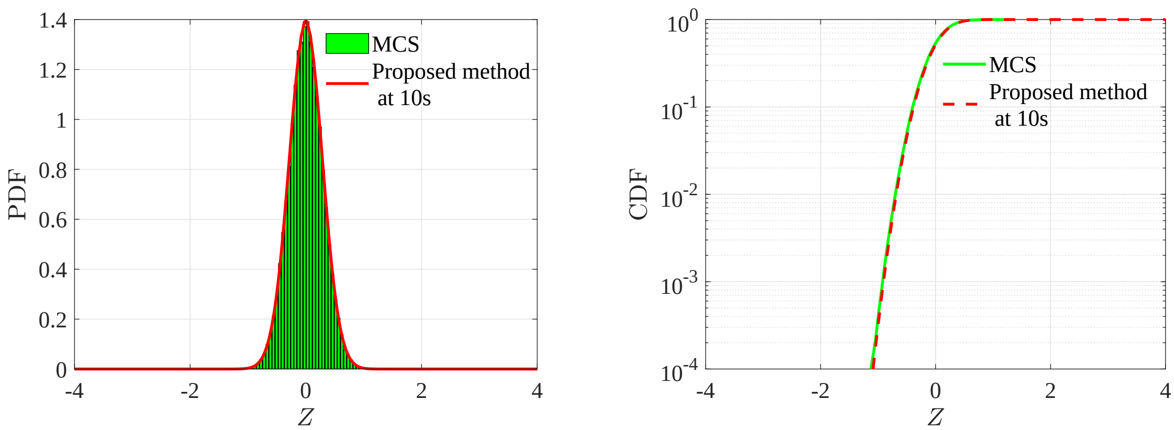
(a) Std.D (b) Std.D

Figure 2: The standard deviation of displacement and velovity for Example 1



(a) PDF (b) PDF

Figure 3: The joint PDF for Example 2



(a) PDF (b) PDF

Figure 4: The marginal PDF and CDF for Example 1

nonlinear behavior. After establishing a continuous surface for the first-order derivative moments, the probability distribution of the system response is determined using the path integral method. The effectiveness and accuracy of the proposed methodology are validated through the numerical example.

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