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ABSTRACT

This paper aims to estimate the unknown parameters for the exponentiated Rayleigh distribution using Type-I hybrid censored data under a step-stress model. The maximum likelihood and Bayes methods estimate the parameters and acceleration factor. The parameters' approximate confidence intervals are created. The Bayes estimates of the parameters for the squared error and linear exponential loss functions are computed using the Markov Chain Monte Carlo (MCMC) method. Finally, we perform a simulation study to evaluate the effectiveness of the proposed estimators. We provide a real-life data example (Strength data measurement in GPA, for single and impregnated carbon fibers) to explain the obtained results.

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1 Introduction

By exposing a product or material to higher-than-normal stress levels, an accelerated life test (ALT) is used to gather failure data about it rapidly. This speeds up the estimation of a product's life or dependability under typical usage circumstances. When it is used, stressors (such as temperature, voltage, vibration, and humidity) are raised above their typical operating range. The fundamental premise is that the acceleration model (Arrhenius for temperature, Eyring, Inverse Power Law, etc.) correctly forecasts how stress and life would interact. High-stress results are extended to typical application scenarios. When comparing materials or designs rapidly, when failure modes do not alter with greater stress levels, or when actual life testing under normal conditions is too time-consuming, it is utilized. For instance, mechanical components exposed to increased load cycles and electronic components evaluated at high temperatures to forecast thermal aging are two examples.

Partially accelerated life test (PALT) combines both accelerated and normal-use testing, typically when full acceleration is too costly, may cause unrealistic failure modes, or a blend of data is needed to increase confidence in extrapolation models. It's applied as follows: A subset of the test units is run under accelerated conditions. The rest are tested under normal-use conditions. This provides a more robust data set, allowing model validation and better estimation across a broader range of stress levels. It is used when uncertainty exists about how acceleration affects failure modes to validate acceleration models before full deployment, and when budget/time constraints allow only partial acceleration. For example, battery life testing where extreme temperatures may alter chemistry—some units are tested at mild elevations, and aerospace components are tested under both regular and accelerated load profiles to model fatigue.

In summary statement: ALT is ideal when we need rapid reliability estimates and are confident that elevated stress won't introduce new failure mechanisms. PALT, however, is more suitable when we want to balance speed with model validation, using both normal and elevated conditions to ensure the failure modes and extrapolation remain accurate. The key to choosing between them lies in understanding the product's stress-response behavior, time and budget constraints, and the criticality of modeling assumptions.

Interested readers can refer to Nelson [1], Bagdonavicius and Nikulin [2], Showkat et al. [3,4], and Ahmadini et. Al [5].

Type-I and Type-II censoring techniques are the most widely used. When a specific time t on the test is reached, the test is considered to be finished according to Type-I censoring, and failures are not noted after time t. When a certain number of r out of n items have failed, the test is considered to be finished in Type-II censoring. It is also assumed that the failed items are not replaced. The mixture of Type-I and Type-II censoring schemes is called a hybrid censoring scheme (HCS). In Type-I HCS, the life-testing experiment is terminated at a random time $T_1 = \min\{X_{r:n}, T\}$, where $r \in \{1, 2, \dots, n\}$ and $T \in (0, \infty)$ are fixed in advance, when a pre-specified number r out of n items has failed or access to a pre-fixed time in life tests. Epstein [6] was the first to introduce the hybrid censoring technique and to examine the data using the experimental units' assumed exponential lifetime distribution. Most of the literature used hybrid censoring under classical and Bayesian estimation. Classical and Bayesian estimation for the generalized inverted Rayleigh distribution under Step-Stress accelerated life testing data with hybrid censoring were established by Pooja et al. [7]. The Weibull distribution's parameters were estimated using the maximum likelihood and Bayes methods by Kundu [8]. Zhang et al. [9] Bayesian inference of system reliability under the Marshall-Olkin Weibull distribution was reported for a multicomponent stress-strength model. When hybrid censoring was present, Kundu and Pradhan [10] examined the generalized exponential distribution. Balakrishnan and Shafay [11] developed a general method for building Bayes prediction intervals of future observables using Type-I hybrid censored data that has been observed. The Burr XII distribution's parameters were estimated by Rastogi and Tripathi [12] using maximum likelihood and Bayes methods. Point and interval estimates of lognormal parameters under hybrid censored samples are derived by Singh and Tripathi [13]. Tripathi and Rastogi [14] found estimation of the generalized inverted exponential distribution's parameters using hybrid censored samples. Mahmoud and Ghazal [15] gave estimates from the generalized Type-II hybrid censoring based on the exponentiated Rayleigh distribution. Hyun et al. [16] examined a two-parameter log-logistic distribution using hybrid censored data of Type I and Type II. Under Type-I hybrid censored samples, Yunus et al. [17] estimate the parameter of the inverse Rayleigh distribution. Lin et al. [18] provide step-stress test plans for the log-location-scale distribution under Type-I hybrid censoring. Under Type-I Hybrid Censoring, more details can be found in Kazemi and Azizpoor [19] and Rabie et al. [20].

Specifically, the censoring schemes we considered are all variants of Type-I hybrid censoring, characterized by different combinations of the time limit T and r^{th} failure. These schemes were selected because they encompass different censoring behaviors that occur in real-life reliability and survival data collection, such as early stopping due to resource limits, enforced minimum failure observations, or conditional extensions based on data observed until time T. Statistically, they induce different truncation and censoring patterns, which affect the likelihood structure and consequently the estimation and inference procedures. By analyzing these schemes in parallel, we aim to illustrate how different hybrid censoring rules impact the performance of statistical estimators under the same parametric model. This comparative approach also helps practitioners choose an appropriate censoring strategy based on study constraints and objectives.

The lifetime of highly reliable products can be accurately modeled using the exponential Rayleigh distribution, which is used to model lifetime data. This is where its importance becomes clear and why it is used.

By subjecting products to higher than usual stress levels, step-stress partially accelerated life testing was employed to address the issue that many products rarely fail under typical testing conditions within a reasonable testing period.

Because it is often impractical or impossible to monitor all units until they fail, Type-I hybrid censored data is used so that the experiment is terminated either when a predetermined number of failures occur or when a predetermined time is reached, whichever is sooner.

From reviewing previous studies, we find that none of them used step-stress partially accelerated life testing under Type-I hybrid censored data in estimation. This contributes to the development of reliability theory and its practical applications.

This work is structured as follows: The model's description and the test assumptions are provided in Section 2. In Section 3, for both point and interval data, the unknown parameters' maximum likelihood estimates (MLEs) are acquired. Under squared error (SE) and linear exponential (LINEX) loss functions, Section 4 examines Bayes estimates of parameters and the acceleration factor. Section 5 presents MCMC methods, which are used to find Bayes estimates. The simulation study's results are shown in Section 6. In Section 7, a real data set has been utilized as an example. Finally, in Section 8, comments and future work are offered.

2 Model Description and Test Assumptions

Numerous characteristics of the exponentiated Rayleigh distribution are shared by the gamma, Weibull, and exponentiated exponential distributions. It is discovered that both the density function and the exponentiated Rayleigh distribution for the distribution function have closed forms. As such, it is very compatible with data that has been filtered, so reliability analysis and acceptance sampling plans have used lifetime modeling based on the exponentiated Rayleigh (ER) distribution.

The Rayleigh distribution is generalized to produce the ER distribution. The two-parameter Burr type X distribution is another name for it. For more details, see Kayid et al. [21], Tim et al. [22], Fayomi et al. [23], and Ahmad et al. [24].

The probability density function (PDF) of ER is given by:

$$f(x) = 2\alpha\beta x e^{-\beta x^2}\left[1 - e^{-\beta x^2}\right]^{\alpha-1}, \quad x > 0, \alpha, \beta > 0 \tag{1}$$

and the cumulative distribution function (CDF) is

$$F(x) = \left[1 - e^{-\beta x^2}\right]^{\alpha} \tag{2}$$

where α and β are the shape and scale parameters, respectively.

The following cases occur in Type-I HCS:

$$\left.\begin{array}{lll} Scheme\, I: X_{1:n} < X_{2:n} < \ldots < X_{r:n} & \text{if} \quad X_{r:n} \leq T, & \\ Scheme\, II: X_{1:n} < X_{2:n} < \ldots < X_{K:n} & \text{if} \quad T < X_{K:n}, & 0 \leq K \leq n-1 \end{array}\right\} \tag{3}$$

2.1 Test Assumptions

(1) Two levels of stress, s_1 and s_2 (normal and severe) are applied.

(2) The distribution is exponentiated Rayleigh for each stress level.

(3) The total lifetime X of an item is given by:

$$X = \begin{cases} T & if\ T \leq \tau \\ \tau + \lambda^{-1}(T - \tau) & if\ T > \tau \end{cases} \tag{4}$$

where T is the lifetime of an item under normal conditions. According to the literature, DeGroot Goel [25] proposed this model which is called a tampered random variable model.

(4) The failure times $x_i = 1, 2, \ldots, n$ are independent and identically distributed random variables.

2.2 Test Process

(1) Each n test item is first operated under design stress.

(2) If it fails within the time specified limit, it is placed under extreme conditions and operated until it fails, or the experiment is terminated.

Then, the CDF and PDF of the total lifetime X of an item under step-stress partially accelerated life tests (SSPALT) are:

$$F(x) = \begin{cases} \left[1 - e^{-\beta x^2}\right]^{\alpha}, 0 < x < \tau \\ \left[1 - e^{-\beta(\tau + \lambda(x-\tau))^2}\right]^{\alpha}, x \geq \tau \end{cases} \tag{5}$$

$$f(x)=\begin{cases}2\alpha\beta xe^{-\beta x^2}\left[1-e^{-\beta x^2}\right]^{\alpha-1}, & 0<x<\tau,\\ 2\alpha\beta\lambda\left(\tau+\lambda\left(x-\tau\right)\right)e^{-\beta(\tau+\lambda(x-\tau))^2}\left[1-e^{-\beta(\tau+\lambda(x-\tau))^2}\right]^{\alpha-1}, & x\geq\tau\end{cases}\tag{6}$$

where τ is a specified time at which the stress is changed from s_1 to s_2 and λ is an acceleration factor satisfying $\lambda > 1$.

3 Maximum Likelihood Estimation

When Type-I HCS is used, the life-testing experiment ends at a random time $T_2 = \min\{X_{r:n}, T_1\}$, i.e., the experiment is terminated at a predetermined time T_1 if the failure occurs after T_1, otherwise, as soon as the r^{th} failure occurs where $1 \leq r \leq n$. Thus, we may obtain a random number of failures M^* given by,

$$M^*=\begin{cases}Case\ \ I: r, & x_{r:n}\leq\tau\leq T_1,\\ Case\ \ II: r-N_1, & \tau<x_{r:n}\leq T_1,\\ Case\ \ III: N_2, & x_{r:n}>T_1,\end{cases}\tag{7}$$

where N_1 is the number of units which fail before τ, while the number of the units which fail before time T_1 is N_2 and M^* is the number of units that fail before time $T_2 = \min\{X_{r:n}, T_1\}$, see Figs. 1 and 2.

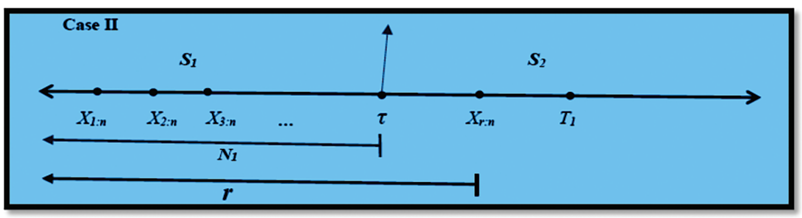

Figure 1: Case *II*

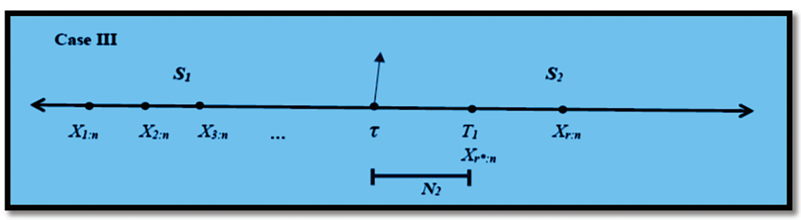

Figure 2: Case *III*

Where Case *II* represents the number of units that fell in the case of acceleration, when the experiment stopped at a pre-specified number, which is $X_{r:n}$ and Case *III* represents the number of units that fell in the case of acceleration, when the experiment stopped at a pre-specified time, which is T_1.

Based on the previous assumptions, the likelihood function under Type-I HCS can be expressed as follows:

For Case I:

$$L(\alpha,\beta|x)=\frac{n!}{(n-r)!}\{1-F(x_{r:n})\}^{n-r}\prod_{i=1}^{n}f(x_{i:n})=\frac{n!}{(n-r)!}(2\alpha\beta)^{n}\left\{1-\left[1-e^{-\beta x^{2}}\right]^{\alpha}\right\}^{n-r}$$
$$\prod_{i=1}^{n}x_{i}e^{-\beta x_{i}^{2}}\left[1-e^{-\beta x_{i}^{2}}\right]^{\alpha-1} \tag{8}$$

This Case will be discarded because the specified number r out of n items is obtained at a normal use condition.

The likelihood function under Type-I HCS can be combined and given for the other Cases, cases II and III as:

$$L(\alpha,\beta,\lambda|x)=\frac{n!}{(n-r^{*})!}\{1-F(T_{2})\}^{n-r^{*}}\prod_{i=1}^{N_1}f(x_{i:n})\prod_{i=N_1+1}^{r^{*}}f(x_{i:n})$$
$$=\frac{n!}{(n-r^{*})!}(2\alpha\beta)^{r^{*}}\lambda^{r^{*}-N_1}\left\{1-\left[1-e^{-\beta(\tau+\lambda(T_2-\tau))^{2}}\right]^{\alpha}\right\}^{n-r^{*}}\prod_{i=1}^{N_1}x_{i}e^{-\beta x_{i}^{2}}\left[1-e^{-\beta x_{i}^{2}}\right]^{\alpha-1}$$
$$\times\prod_{i=N_1+1}^{r^{*}}(\tau+\lambda(x_i-\tau))\,e^{-\beta(\tau+\lambda(x_i-\tau))^{2}}\left[1-e^{-\beta(\tau+\lambda(x_i-\tau))^{2}}\right]^{\alpha-1} \tag{9}$$

where $r^{*}=N_1+M^{*}$, as it runs under accelerated use settings, we only take the likelihood function in Cases II and III into consideration.

Then the log-likelihood function can be written as

$$\ln L(\alpha,\beta,\lambda|x)=\ln\left[\frac{n!}{(n-r^{*})!}\right]+r^{*}\ln(2\alpha\beta)+(r^{*}-N_1)\ln\lambda$$
$$+(n-r^{*})\ln\left\{1-\left[1-e^{-\beta(\tau+\lambda(T_2-\tau))^{2}}\right]^{\alpha}\right\}+\sum_{i=1}^{N_1}\ln x_i-\sum_{i=1}^{N_1}\beta x_i^{2}+(\alpha-1)\sum_{i=1}^{N_1}\ln\left[1-e^{-\beta x_i^{2}}\right]$$
$$+\sum_{i=N_1+1}^{r^{*}}\ln(\tau+\lambda(x_i-\tau))-\beta\sum_{i=N_1+1}^{r^{*}}(\tau+\lambda(x_i-\tau))^{2}+(\alpha-1)\sum_{i=N_1+1}^{r^{*}}\ln\left[1-e^{-\beta(\tau+\lambda(x_i-\tau))^{2}}\right], \tag{10}$$

Then we can get:

$$\frac{\partial\ln L(\alpha,\beta,\lambda|x)}{\partial\alpha}=\frac{r^{*}}{\alpha}-\frac{(n-r^{*})(1-W_1)^{\alpha}\ln(1-W_1)}{1-(1-W_1)^{\alpha}}+\sum_{i=N_1+1}^{r^{*}}\ln\left(1-e^{-\beta x_i^{2}}\right)+\sum_{i=N_1+1}^{r^{*}}\ln(1-W_2)$$

$$\frac{\partial\ln L(\alpha,\beta,\lambda|x)}{\partial\beta}=\frac{r^{*}}{\beta}-\frac{\alpha(n-r^{*})\Psi_1W_1(1-W_1)^{\alpha-1}}{1-(1-W_1)^{\alpha}}-\sum_{i=1}^{N_1}x_i^{2}+(\alpha-1)\sum_{i=1}^{N_1}\frac{x_i^{2}e^{-\beta x_i^{2}}}{1-e^{-\beta x_i^{2}}}$$
$$-\sum_{i=N_1+1}^{r^{*}}\Psi_1+(\alpha-1)\sum_{i=N_1+1}^{r^{*}}\frac{\Psi_2W_2}{1-W_2}$$

$$\frac{\partial \ln L(\alpha,\beta,\lambda|x)}{\partial\lambda}=\frac{r^*-N_1}{\lambda}-\frac{2\alpha\beta\sqrt{\Psi_1}(n-r^*)(T_2-\tau)W_1(1-W_1)^{\alpha-1}}{1-(1-W_1)^{\alpha}}+\sum_{i=N_1+1}^{r*}\frac{x_i-\tau}{\sqrt{\Psi_1}}$$
$$-\beta\sum_{i=N_1+1}^{r^*}2(x_i-\tau)\sqrt{\Psi_1}+(\alpha-1)\sum_{i=N_1+1}^{r^*}\frac{2\beta(x_i-\tau)\sqrt{\Psi_2}W_2}{1-W_2}$$

where $\Psi_1=\{\tau+\lambda(T_2-\tau)\}^2$, $\Psi_2=\{\tau+\lambda(x_i-\tau)\}^2$, $W_1=e^{-\beta\Psi_1}$ and $W_2=e^{-\beta\Psi_2}$.

Now equating $\frac{\partial \ln L(\alpha,\beta,\lambda|x)}{\partial\alpha}=0$, $\frac{\partial \ln L(\alpha,\beta,\lambda|x)}{\partial\beta}=0$ and $\frac{\partial \ln L(\alpha,\beta,\lambda|x)}{\partial\lambda}=0$ yields a system of nonlinear equations in parameters α, β and λ. To find a numerical solution to these equations, the Newton-Raphson method is employed.

Observed Fisher Information Matrix

To calculate the approximate confidence intervals for α, β and λ, asymptotic distributions of the MLEs are used. We can write the asymptotic distribution of the MLEs for α, β and λ by:

$$\left[\left(\hat{\alpha}-\alpha\right),\left(\hat{\beta}-\beta\right),\left(\hat{\lambda}-\lambda\right)\right]\rightarrow N\left(0,I^{-1}(\alpha,\ \beta,\lambda)\right)$$

where $I^{-1}(\alpha,\ \beta,\lambda)$ is the variance-covariance matrix of the parameters α, β and λ. The inverse of the observed Fisher information matrix can be used to estimate it, and it reads as:

$$I\left(\hat{\alpha},\ \hat{\beta},\hat{\lambda}\right)=\begin{pmatrix}\frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\beta} & \frac{\partial^2\ell}{\partial\alpha\partial\lambda}\\ \frac{\partial^2\ell}{\partial\beta\partial\alpha} & \frac{\partial^2\ell}{\partial\beta^2} & \frac{\partial^2\ell}{\partial\beta\partial\lambda}\\ \frac{\partial^2\ell}{\partial\lambda\partial\alpha} & \frac{\partial^2\ell}{\partial\lambda\partial\beta} & \frac{\partial^2\ell}{\partial\lambda^2}\end{pmatrix}_{(\alpha,\beta,\lambda)=\left(\hat{\alpha},\hat{\beta},\hat{\lambda}\right)}$$

Then, for α, β and λ, the approximate $100(1-\gamma)\%$ two-sided CI are provided by:

$$\left[\left\{\hat{\alpha}_{ML}\pm Z_{\gamma/2}\sqrt{I_{11}^{-1}(\hat{\alpha}_{ML})}\right\},\left\{\hat{\beta}_{ML}\pm Z_{\gamma/2}\sqrt{I_{22}^{-1}(\hat{\beta}_{ML})}\right\},\left\{\hat{\lambda}_{ML}\pm Z_{\gamma/2}\sqrt{I_{33}^{-1}(\hat{\lambda}_{ML})}\right\}\right]$$

The upper $100(\gamma/2)^{\text{th}}$ percentile of the standard normal distribution is denoted by $Z_{\gamma/2}$.

4 Bayesian Estimation

In this section, the SE and LINEX loss functions are taken into account to determine the Bayesian estimates (BEs) of the parameters. We applied informative priors to get the BEs.

Assume that the parameters α, β and λ are independent and have informative priors as:

$$\left.\begin{array}{l}\pi_1(\alpha)\propto\alpha^{b_1-1}e^{-\alpha/c_1};\ b_1,\ c_1>0\\ \pi_2(\beta)\propto\beta^{b_2-1}e^{-\beta/c_2};\ b_2,\ c_2>0\\ \pi_3(\lambda)\propto\frac{1}{\lambda},\quad\lambda>1.\end{array}\right\}\qquad(11)$$

Then we can write the joint prior as:

$$\pi(\alpha,\beta,\lambda)\propto\alpha^{b_1-1}\beta^{b_2-1}\lambda^{-1}e^{-\alpha/c_1-\beta/c_2},\quad b_1,b_2,c_1,c_2>0\qquad(12)$$

From Eqs. (9) and (12), the joint posterior density function is

$$\pi^*(\alpha,\beta,\lambda|x) \propto L(\alpha,\beta,\lambda;x)\,\pi(\alpha,\beta,\lambda)$$

$$\propto \alpha^{r*+b_1-1}\beta^{r*+b_2-1}\lambda^{r*-N_1-1}e^{-\alpha/c_1-\beta/c_2}\left\{1-\left[1-e^{-\beta\left(\tau+\lambda\left(T_2-\tau\right)\right)^2}\right]^\alpha\right\}^{n-r*}\prod_{i=1}^{N_1}x_ie^{-\beta x_i^2}\left[1-e^{-\beta x_i^2}\right]^{\alpha-1}$$

$$\times\prod_{i=N_1+1}^{r*}\left(\tau+\lambda\left(x_i-\tau\right)\right)e^{-\beta\left(\tau+\lambda\left(x_i-\tau\right)\right)^2}\left[1-e^{-\beta\left(\tau+\lambda\left(x_i-\tau\right)\right)^2}\right]^{\alpha-1} \tag{13}$$

The BE of the function $\Phi(\alpha,\beta,\lambda)$ of the parameters α, β and λ can be stated as follows based on SE and LINEX loss functions, respectively.

$$\hat{\Phi}_{SE}(\alpha,\beta,\lambda)=E[\Phi(\alpha,\beta,\lambda)|x]=\int_1^\infty\int_0^\infty\int_0^\infty\Phi(\alpha,\beta,\lambda)\,\pi^*(\alpha,\beta,\lambda|x)\,d\alpha d\beta d\lambda$$

$$=\int_1^\infty\int_0^\infty\int_0^\infty\Phi(\alpha,\beta,\lambda)\,\alpha^{r^*+b_1-1}\beta^{r^*+b_2-1}\lambda^{r^*-N_1-1}e^{-\alpha/c_1-\beta/c_2}$$

$$\times\left\{1-\left[1-e^{-\beta\left(\tau+\lambda\left(T_2-\tau\right)\right)^2}\right]^\alpha\right\}^{n-r^*}\prod_{i=1}^{N_1}x_ie^{-\beta x_i^2}\left[1-e^{-\beta x_i^2}\right]^{\alpha-1}$$

$$\times\prod_{i=N_1+1}^{r^*}\left(\tau+\lambda\left(x_i-\tau\right)\right)e^{-\beta\left(\tau+\lambda\left(x_i-\tau\right)\right)^2}\left[1-e^{-\beta\left(\tau+\lambda\left(x_i-\tau\right)\right)^2}\right]^{\alpha-1}d\alpha d\beta d\lambda \tag{14}$$

$$\begin{aligned}\hat{\Phi}_{LINEX}(\alpha,\beta,\lambda)&=-\frac{1}{c}\ln\left[\int_1^\infty\int_0^\infty\int_0^\infty e^{-c\Phi(\alpha,\beta,\lambda)}\pi^*(\alpha,\beta,\lambda|x)\,d\alpha d\beta d\lambda\right]\\&=-\frac{1}{c}\ln\Bigg[\int_1^\infty\int_0^\infty\int_0^\infty e^{-c\Phi(\alpha,\beta,\lambda)}\alpha^{r^*+b_1-1}\beta^{r^*+b_2-1}\lambda^{r^*-N_1-1}e^{-\alpha/c_1-\beta/c_2}\\&\quad\times\left\{1-\left[1-e^{-\beta\left(\tau+\lambda\left(T_2-\tau\right)\right)^2}\right]^\alpha\right\}^{n-r^*}\prod_{i=1}^{N_1}x_ie^{-\beta x_i^2}\left[1-e^{-\beta x_i^2}\right]^{\alpha-1}\\&\quad\times\prod_{i=N_1+1}^{r^*}\left(\tau+\lambda\left(x_i-\tau\right)\right)e^{-\beta\left(\tau+\lambda\left(x_i-\tau\right)\right)^2}\left[1-e^{-\beta\left(\tau+\lambda\left(x_i-\tau\right)\right)^2}\right]^{\alpha-1}d\alpha d\beta d\lambda\Bigg]\end{aligned} \tag{15}$$

where the value of c is arbitrary.

A closed form of this integration is not possible. So, in order to approximate this integral, we use the MCMC approach.

5 MCMC Method

In this section, we examined the Gibbs sampling technique, which determines the credible intervals (CI) and BEs for the SE and LINEX loss functions using the MCMC algorithm. For the parameters

α, β and λ, conditional posterior density functions can be expressed as follows using Eq. (13).

$$\pi^*_\alpha(\alpha|\beta,\lambda;x) \propto \alpha^{r^*+b_1-1} e^{-\alpha/c_1}\left\{1-\left[1-e^{-\beta(\tau+\lambda(T_2-\tau))^2}\right]^\alpha\right\}^{n-r^*}\prod_{i=1}^{N_1} x_i e^{-\beta x_i^2}\left[1-e^{-\beta x_i^2}\right]^{\alpha-1}$$
$$\times\prod_{i=N_1+1}^{r^*}(\tau+\lambda(x_i-\tau))\,e^{-\beta(\tau+\lambda(x_i-\tau))^2}\left[1-e^{-\beta(\tau+\lambda(x_i-\tau))^2}\right]^{\alpha-1} \tag{16}$$

$$\pi^*_\beta(\beta|\alpha,\lambda;x) \propto \beta^{r^*+b_2-1} e^{-\beta/c_2}\left\{1-\left[1-e^{-\beta(\tau+\lambda(T_2-\tau))^2}\right]^\alpha\right\}^{n-r^*}\prod_{i=1}^{N_1} x_i e^{-\beta x_i^2}\left[1-e^{-\beta x_i^2}\right]^{\alpha-1}$$
$$\times\prod_{i=N_1+1}^{r^*}(\tau+\lambda(x_i-\tau))\,e^{-\beta(\tau+\lambda(x_i-\tau))^2}\left[1-e^{-\beta(\tau+\lambda(x_i-\tau))^2}\right]^{\alpha-1} \tag{17}$$

and

$$\pi^*_\lambda(\lambda|\alpha,\beta;x) \propto \lambda^{r^*-N_1-1}\left\{1-\left[1-e^{-\beta(\tau+\lambda(T_2-\tau))^2}\right]^\alpha\right\}^{n-r^*}$$
$$\times\prod_{i=N_1+1}^{r^*}(\tau+\lambda(x_i-\tau))\,e^{-\beta(\tau+\lambda(x_i-\tau))^2}\left[1-e^{-\beta(\tau+\lambda(x_i-\tau))^2}\right]^{\alpha-1} \tag{18}$$

From the Eqs. (16)–(18), it is not possible to analytically reduce the conditional posterior distributions of α, β and λ to known distributions. Therefore, we can solve this problem by using the Metropolis-Hastings algorithm. Refer to Gupta and Upadhyay [26]. We used the following Algorithm 1 to determine the Bayes estimators.

Algorithm 1: MCMC algorithm

1. Start with ($\alpha^{(0)} = \hat{\alpha}_{MLE}$, $\beta^{(0)} = \hat{\beta}_{MLE}$ and $\lambda^{(0)} = \hat{\lambda}_{MLE}$).
2. Set $t = 1$.
3. To generate $\alpha^{(t)}$ from $\pi^*_\alpha\left(\alpha^{(t-1)}|\beta^{(t-1)},\lambda^{(t-1)};x\right)$, apply the Metropolis-Hasting (MHs) algorithm with the proposal distribution $N\left(\alpha^{(t-1)},\sqrt{Var(\hat{\alpha}_{ML})}\right)$, then generate $\beta^{(t)}$ from $\pi^*_\beta\left(\beta^{(t-1)}|\alpha^{(t)},\lambda^{(t-1)};x\right)$ with the proposal distribution $N\left(\beta^{(t-1)},\sqrt{Var(\hat{\beta}_{ML})}\right)$ and $\lambda^{(t)}$ from $\pi^*_\lambda\left(\lambda^{(t-1)}|\alpha^{(t)},\beta^{(t)};x\right)$ with the proposal distribution $N\left(\lambda^{(t-1)},\sqrt{Var(\hat{\lambda}_{ML})}\right)$
4. Set $t = t + 1$
5. Perform steps 2–4 N times.
6. Obtain $\alpha^{(t)}$, $\beta^{(t)}$and $\lambda^{(t)}$, $t = M + 1, \ldots, N$, and at this point, the estimated means of $\Phi(\alpha,\beta,\lambda)$ and $\exp[-c\ \Phi(\alpha,\beta,\lambda)]$ are given, respectively, by

$$E[\Phi(\alpha,\beta,\lambda)|x] = \frac{1}{N-M}\sum_{t=M+1}^{N}\Phi\left(\alpha^{(t)},\beta^{(t)},\ \lambda^{(t)}\right),$$

$$E[\exp(-c\Phi(\alpha,\beta,\lambda)|x)] = \frac{1}{N-M}\sum_{t=M+1}^{N}\exp\left[-c\Phi\left(\alpha^{(t)},\beta^{(t)},\lambda^{(t)}\right)|x\right],$$

where M stands for the burn-in period.

Considering this, the Bayes MCMC point estimates of $\Phi(\alpha,\beta,\lambda)$ based on the SE and the LINEX loss functions are provided, respectively, by

$$\hat{\Phi}_{SE}(\alpha,\beta,\lambda) = E[\Phi(\alpha,\beta,\lambda)|x],\ \hat{\Phi}_{LINEX}(\alpha,\beta,\lambda) = -\frac{1}{c}\ln[E[\exp(-c\Phi(\alpha,\beta,\lambda)|x]].$$

(Continued)

Algorithm 1 (continued)

Repeat steps 1–6 H time and put each estimate in order as $\hat{\alpha}_{SE}^{[1]}, \ldots, \hat{\alpha}_{SE}^{[H]}$, $\hat{\beta}_{SE}^{[1]}, \ldots, \hat{\beta}_{SE}^{[H]}$ and $\hat{\lambda}_{SE}^{[1]}, \ldots, \hat{\lambda}_{SE}^{[H]}$. Then, the $100\,(1-\gamma)\,\%$ credible intervals for α, β and λ are $\left(\hat{\alpha}_{SE}^{\left[\frac{\gamma}{2}H\right]}, \hat{\alpha}_{SE}^{\left[\left(1-\frac{\gamma}{2}\right)H\right]}\right)$, $\left(\hat{\beta}_{SE}^{\left[\frac{\gamma}{2}H\right]}, \hat{\beta}_{SE}^{\left[\left(1-\frac{\gamma}{2}\right)H\right]}\right)$ and $\left(\hat{\lambda}_{SE}^{\left[\frac{\gamma}{2}H\right]}, \hat{\lambda}_{SE}^{\left[\left(1-\frac{\gamma}{2}\right)H\right]}\right)$, respectively.

6 Illustrative Examples

Madi and Raqab [27] prove the assumption that the modified data set for the physical example (Strength data measurement in GPA, for single and impregnated carbon fibers) presented by Badar and Priest [28] follows the ER distributiona

0.101, 0.332, 0.403, 0.428, 0.457, 0.550, 0.561, 0.596, 0.597, 0.645, 0.654, 0.674, 0.718, 0.722, 0.725, 0.732, 0.775, 0.814, 0.816, 0.818, 0.824, 0.859, 0.875, 0.938, 0.940, 1.056, 1.117, 1.128, 1.137, 1.137, 1.177, 1.196, 1.230, 1.325, 1.339, 1.345, 1.420, 1.423, 1.435, 1.443, 1.464, 1.472, 1.494, 1.532, 1.546, 1.577, 1.608, 1.635, 1.693, 1.701, 1.737, 1.754, 1.762, 1.828, 2.052, 2.071, 2.086, 2.171, 2.224, 2.227, 2.425, 2.595, 3.220.

We will choose four Type-I hybrid censoring schemes:

1. Let $n = 63$, $r = 27$, $T_1 = 1.128$ and $\tau = 0.8$, this means that the life test will terminate at $X_{27:63}$ because $X_{27:63} < T_1$.
2. Let $n = 63$, $r = 31$, $T_1 = 1.128$ and $\tau = 0.8$, this means that the life test will terminate at $X_{28:63}$ because $T_1 < X_{31:63}$.
3. Let $n = 63$, $r = 45$, $T_1 = 1.62$ and $\tau = 1.2$, this means that the life test will terminate at $X_{45:63}$ because $X_{45:63} < T_1$.
4. Let $n = 63$, $r = 48$, $T_1 = 1.62$ and $\tau = 1.2$, this means that the life test will terminate at $X_{47:63}$ because $T_1 < X_{48:63}$.

The maximum likelihood and Bayes estimates of α, β and λ. have been computed, and the results are displayed in Table 1.

Table 1: Point estimates under MLE and BEs of α, β and λ.

n	r	Parameters	T	MLE	MCMC	
			T_1		SE	LINEX ($C = 1$)
63	27	α		1.7341	1.5211	1.5684
		β	1.128	1.1721	0.9432	0.9732
		λ		1.6327	1.4921	1.4532
63	31	α		1.6213	1.5118	1.5437
		β	1.128	0.9827	0.8438	0.8743
		λ		1.6199	1.4226	1.4193

(Continued)

Table 1 (continued)

n	r	Parameters	T	MLE	MCMC	
			T_1		SE	LINEX ($C = 1$)
63	45	α		1.4398	1.4875	1.4987
		β	1.62	0.8032	0.7922	0.8132
		λ		1.3945	1.3733	1.3438
63	48	α		1.3143	1.3243	1.3544
		β	1.62	0.7831	0.7825	0.7933
		λ		1.3121	1.3184	1.3021

95% credible intervals for maximum likelihood and Bayes estimates of α, β and λ have been computed, and the results are displayed in Table 2.

Table 2: 95% CIs (credible intervals) under MLE and BEs of the parameters α, β and λ.

n	r	Parameter	t	MLEs			BEs		
			T_1	L	UU	Length	L	U	Length
63	27	α		1.2544	5.1321	3.8777	1.3442	4.8734	3.5292
		β	1.128	0.0044	0.8345	0.8301	0.0052	0.7736	0.7674
		λ		0.2328	3.1432	2.9104	0.4732	2.9654	2.4922
63	31	α		1.2424	5.0325	3.7901	1.5323	4.7232	3.1909
		β	1.128	0.0073	0.7432	0.7359	0.0085	0.6145	0.6060
		λ		0.4314	2.9353	2.5039	0.6212	2.8858	2.2646
63	45	α		1.5321	4.3452	2.8131	1.7321	4.5123	2.7802
		β	1.62	0.0083	0.5425	0.5342	0.0091	0.5239	0.5148
		λ		0.5119	2.2163	1.7044	0.7792	2.4276	1.6484
63	48	α	1.62	1.7311	4.0832	2.3321	1.9263	4.2432	2.3169
		β		0.0095	0.5032	0.4937	0.0128	0.5112	0.4984
		λ		0.6832	2.1545	1.4713	0.8764	2.3298	1.4534

7 Simulation Study

In this section, we used the Algorithm 1 to present a simulation study using different sample sizes.

The mean square errors of maximum likelihood and Bayes estimates of α, β and λ are shown in Tables 3 and 4. The Bayesian credible intervals and the average length (AL) of confidence intervals with coverage percentages (CP) for α, β and λ are shown in Tables 5 and 6.

Table 3: Average estimates and MSEs (mean square errors) (in brackets) of the MLEs and the Bayes estimates for α, β and λ when $\tau = 0.4$, $\mathrm{T} = 0.8$, $\alpha = 1.8$, $\beta = 0.8$, $\lambda = 1.5$, $b_1 = 1.23$, $c_1 = 0.91$, $b_2 = 1.35$, $c_2 = 0.83$ and $c = 1$

n	*r*	Parameters	MLEs (MSE)	MCMC	
				SE (MSE)	LINEX (MSE)
20	10	α	2.8234 (0.8453)	2.5183 (0.6184)	2.5271 (0.6242)
		β	1.4672 (0.6932)	1.4932 (0.4325)	1.4998 (0.4532)
		λ	2.0021 (0.4332)	1.8962 (0.2323)	1.8992 (0.2385)
	15	α	2.5973(0.6462)	2.4392 (0.4955)	2.4483 (0.4985)
		β	1.3932 (0.5142)	1.3071 (0.3899)	1.3099 (0.3873)
		λ	1.9142 (0.2754)	1.7687 (0.1939)	1.7799 (0.1997)
30	20	α	2.4811 (0.5144)	2.4123 (0.4312)	2.4211 (0.4558)
		β	1.3343 (0.4352)	1.2871 (0.3534)	2.2788 (0.3487)
		λ	1.8737 (0.2488)	1.7348 (0.1421)	1.7432 (0.1455)
	25	α	2.4455 (0.4611)	2.3952 (0.4211)	2.3973 (0.4325)
		β	1.2911 (0.3622)	1.2528 (0.3244)	1.2511 (0.3237)
		λ	1.8345 (0.1766)	1.7438 (0.1392)	1.7496 (0.1399)
40	30	α	2.3712 (0.4041)	2.3701 (0.4009)	2.3783 (0.4022)
		β	1.2271 (0.3022)	1.2221 (0.3012)	1.2253 (0.3019)
		λ	1.7155 (0.1177)	1.7099 (0.1101)	1.7141 (0.1134)
	35	α	2.1761 (0.2372)	2.1371 (0.2221)	2.1416 (0.2283)
		β	1.1195 (0.1729)	1.1741 (0.1834)	1.1812 (0.1991)
		λ	1.6811 (0.0801)	1.6771 (0.0714)	1.6793 (0.0787)
50	40	α	2.0551 (0.1986)	2.1001 (0.2111)	2.1104 (0.2192)
		β	1.1022 (0.1341)	1.1732 (0.1442)	1.1783 (0.1421)
		λ	1.6661 (0.0361)	1.7223 (0.0452)	1.5895 (0.0419)
	45	α	1.9855 (0.1291)	2.1934 (0.1933)	2.1994 (0.1945)
		β	0.9653 (0.0951)	1.0021 (0.1055)	1.0069 (0.1077)
		λ	1.6231 (0.0281)	1.7111 (0.0413)	1.7216 (0.0421)

Table 4: Average estimates and MSEs (in brackets) of the MLEs and the Bayes estimates for α, β and λ when$\tau = 0.5$, $\mathrm{T} = 0.9$, $\alpha = 1.4$, $\beta = 0.9$, $\lambda = 1.6$, $b_1 = 1.5$, $c_1 = 0.85$, $b_2 = 1.3$, $c_2 = 0.75$ and $C = 1$

n	*r*	Parameters	MLEs (MSE)	MCMC	
				SE (MSE)	LINEX (MSE)
20	10	α	2.3387 (0.7969)	2.0566 (0.5342)	2.0875 (0.5373)
		β	1.7763 (0.8916)	1.4445 (0.4372)	1.4525 (0.4499)
		λ	2.1561 (0.5573)	1.9932 (0.2054)	1.9985 (0.2178)
	15	α	2.2171 (0.6157)	1.9772 (0.4593)	1.9759 (0.4575)
		β	1.5773 (0.4732)	1.3792 (0.3746)	1.3843 (0.3888)
		λ	2.0541 (0.2635)	1.9149 (0.1257)	1.9121 (0.1224)

(Continued)

Table 4 (continued)

n	r	Parameters	MLEs (MSE)	MCMC	
				SE (MSE)	LINEX (MSE)
30	20	α	2.1183 (0.4765)	1.9154 (0.3833)	1.9162 (0.3892)
		β	1.4443 (0.3955)	1.3154 (0.2654)	1.3155 (0.2744)
		λ	1.9942 (0.1984)	1.8698 (0.0993)	1.8699 (0.0997)
	25	α	1.9861 (0.4246)	1.8932 (0.3766)	1.8976 (0.3785)
		β	1.3625 (0.3551)	1.2983 (0.2433)	1.2989 (0.2488)
		λ	1.9262 (0.1645)	1.8475 (0.0899)	1.8486 (0.0877)
40	30	α	1.8927 (0.3831)	1.8454 (0.3543)	1.8485(0.3582)
		β	1.3161 (0.3245)	1.2679 (0.2344)	1.2788 (0.2476)
		λ	1.8741 (0.0931)	1.8243 (0.0733)	1.8211 (0.0721)
	35	α	1.7543 (0.3288)	1.7445 (0.3147)	1.7577 (0.3193)
		β	1.2232 (0.2299)	1.2014 (0.2198)	1.2344 (0.2177)
		λ	1.8277 (0.0744)	1.8165 (0.0632)	1.8185 (0.0687)
50	40	α	1.5599 (0.1231)	1.5932 (0.1477)	1.6022 (0.1533)
		β	1.1882 (0.1035)	1.2197 (0.1132)	1.2398 (0.1177)
		λ	1.7143 (0.0141)	1.7376 (0.0172)	1.7375 (0.0177)
	45	α	1.5243 (0.0828)	1.5563 (0.0978)	1.5577 (0.0988)
		β	1.1011 (0.0897)	1.1800 (0.0922)	1.1889 (0.0934)
		λ	1.6232 (0.0097)	1.6518 (0.0099)	1.6533 (0.0101)

Table 5: Coverage probabilities (CPs) and lengths of 95% CIs for the parameters α, β, and λ when $\tau = 0.4$, T $= 0.8$, α $= 1.8$, $\beta = 0.8$, $\lambda = 1.5$, $b_1 = 1.23$, $c_1 = 0.91$, $b_2 = 1.35$, $c_2 = 0.83$ and $C = 1$

n	r		MLE		BE	
			Length	CP	Length	CP
20	10	α	1.2155	0.8542	0.9755	0.9143
		β	1.9486	0.8794	1.4954	0.9252
		λ	0.8343	0.8243	0.6943	0.9121
	15	α	1.0945	0.9145	0.9299	0.9554
		β	1.8267	0.8254	1.4555	0.8923
		λ	0.7966	0.9046	0.6332	0.9233
30	20	α	1.0452	0.9223	0.8988	0.9399
		β	1.7291	0.8778	1.4236	0.9184
		λ	0.7633	0.8832	0.5982	0.9473
	25	α	1.0872	0.8577	0.8538	1
		β	1.6321	0.8332	1.3932	0.9022
		λ	0.7291	0.9183	0.5774	0.9933
40	30	α	0.9767	0.9649	0.8182	0.8933
		β	1.4122	0.9521	1.3766	0.9345
		λ	0.6233	0.9233	0.5574	0.9266

(Continued)

Table 5 (continued)

n	r		MLE		BE	
			Length	CP	Length	CP
	35	α	0.7954	0.9235	0.7875	0.9331
		β	1.3843	0.9321	1.3652	0.9544
		λ	0.5599	0.9133	0.5365	1
50	40	α	0.7051	0.9033	0.7232	0.9766
		β	1.2828	0.9224	1.3334	0.9422
		λ	0.4932	0.9127	0.5159	0.9754
	45	α	0.6253	0.9179	0.6513	0.9022
		β	1.2095	0.9432	1.2292	0.9211
		λ	0.4224	0.9733	0.4598	0.9033

Table 6: Lengths and CPs of 95% CIs for parameters α, β and λ when $\tau = 0.5$, T $= 0.9$, $\alpha = 1.4$, $\beta = 0.9$, $\lambda = 1.6$, $b_1 = 1.5$, $c_1 = 0.85$, $b_2 = 1.3$, $c_2 = 0.75$ and $C = 1$

n	r		MLE		BE	
			Length	CP	Length	CP
20	10	α	0.9974	0.8244	0.8784	0.9011
		β	1.7398	0.9192	1.1016	0.9302
		λ	0.9794	0.8954	0.6572	0.9055
	15	α	0.9437	0.9022	0.7911	0.966
		β	1.4684	0.7822	0.9845	0.8676
		λ	0.8392	0.8822	0.5821	0.8699
30	20	α	0.8977	0.9033	0.7499	0.9293
		β	1.2383	0.9022	0.9428	0.9533
		λ	0.7783	0.9103	0.5339	0.9322
	25	α	0.7599	0.8977	0.6928	0.9432
		β	0.9843	0.9026	0.8955	0.9182
		λ	0.5872	0.8445	0.4583	0.8834
40	30	α	0.6566	0.9033	0.6239	0.9432
		β	0.8653	0.8733	0.8436	1
		λ	0.4588	0.9676	0.4277	0.9644
	35	α	0.6166	0.9345	0.5977	0.9234
		β	0.8352	0.9583	0.8184	0.9634
		λ	0.4155	0.9014	0.3998	0.9374
50	40	α	0.5248	0.8567	0.5791	0.8474
		β	0.8122	0.9467	0.7931	0.9388
		λ	0.3239	0.9194	0.3791	1

(Continued)

Table 6 (continued)

n	r		MLE		BE	
			Length	CP	Length	CP
	45	α	0.4188	0.9453	0.5115	0.9543
		β	0.6786	0.9634	0.7651	0.9484
		λ	0.2566	0.9726	0.3222	0.9833

The trace plots represent the posterior distribution of α, β and λ for 10,000 MCMC samples and are displayed in Figs. 3–8. It displays how well the MCMC technique converges. The histogram plots of the generated of α, β and λ are shown in Figs. 9–14. It is shown that the generated posteriors' histograms closely resemble the theoretical posterior density functions.

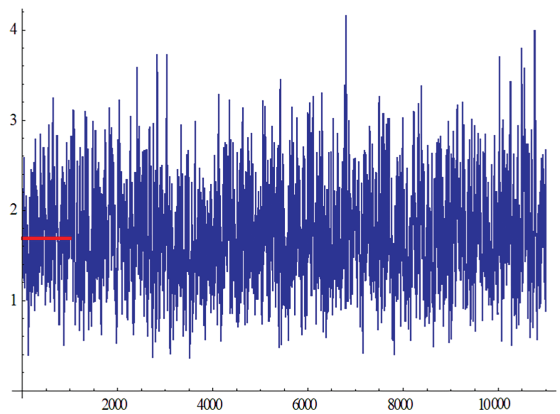

Figure 3: Simulation number of $\alpha = 1.8$ by MCMC method

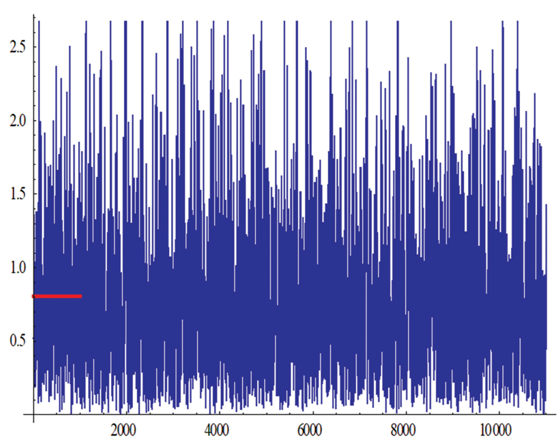

Figure 4: Simulation number of $\beta = 0.8$ by MCMC method

Thus, it is possible to estimate the unknown parameters and create the approximate CIs using MCMC samples.

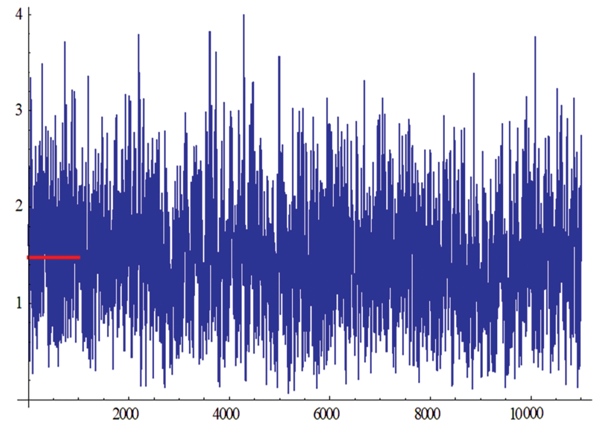

Figure 5: Simulation number of $\lambda = 1.5$ by MCMC method

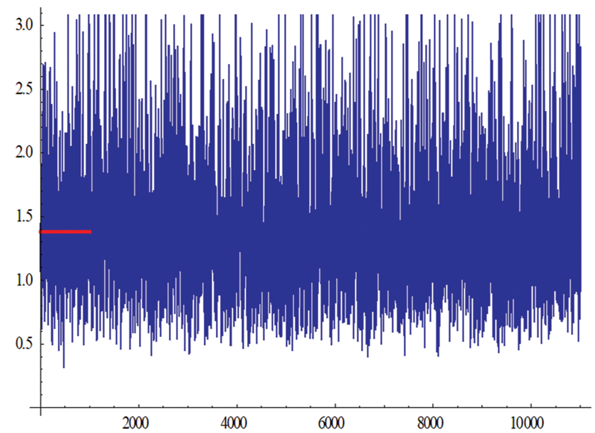

Figure 6: Simulation number of $\alpha = 1.4$ by the MCMC method

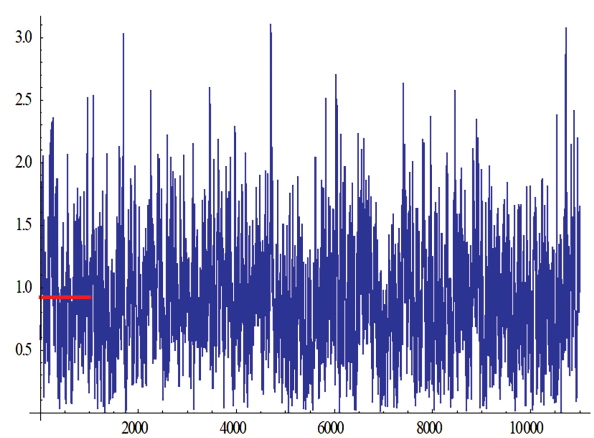

Figure 7: Simulation number of $\beta = 0.9$ by the MCMC method

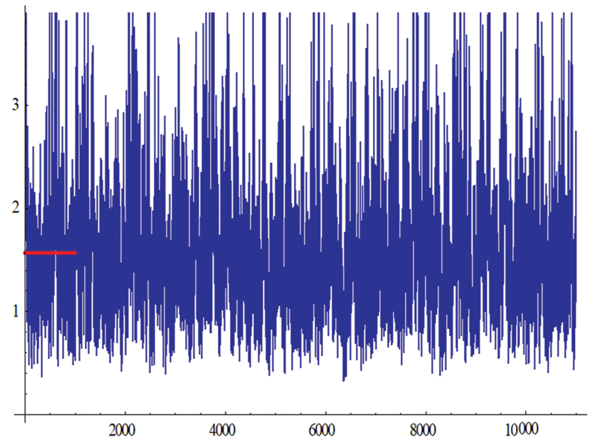

Figure 8: Simulation number of $\lambda = 1.6$ by the MCMC method

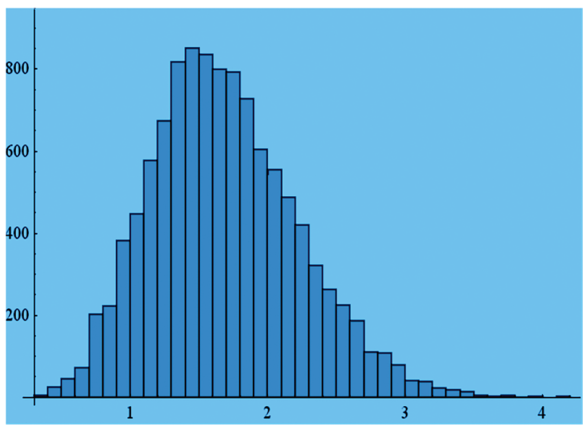

Figure 9: Histogram of $\alpha = 1.8$ by MCMC method

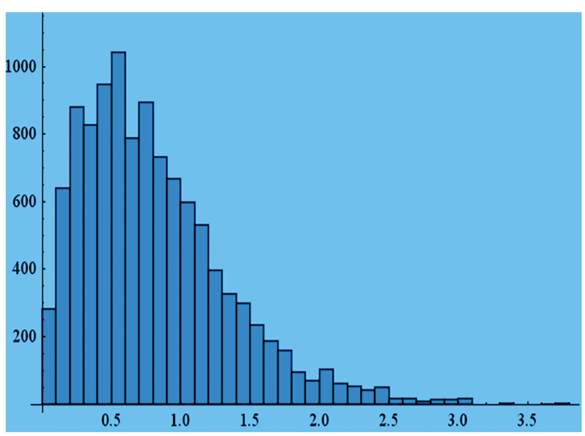

Figure 10: Histogram of $\beta = 0.8$ by MCMC method

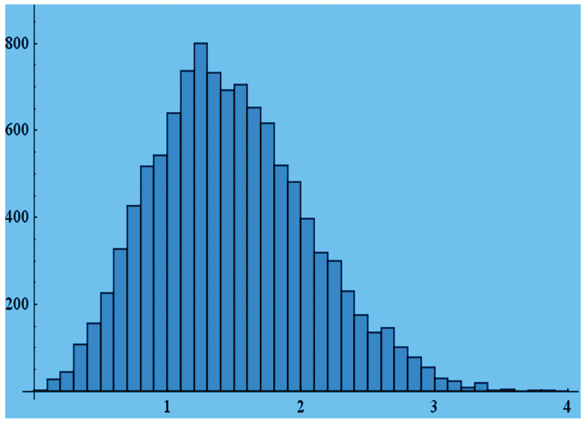

Figure 11: Histogram of $\lambda = 1.5$ by MCMC method

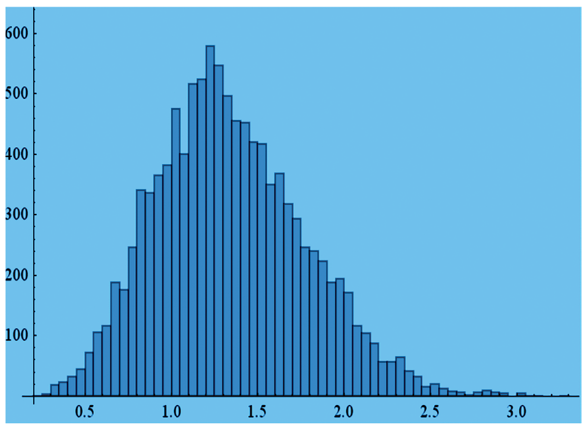

Figure 12: Histogram of $\alpha = 1.4$ by the MCMC method

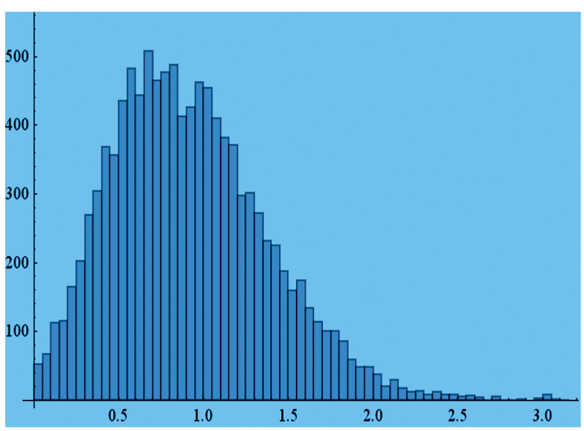

Figure 13: Histogram of $\beta = 0.9$ by the MCMC method

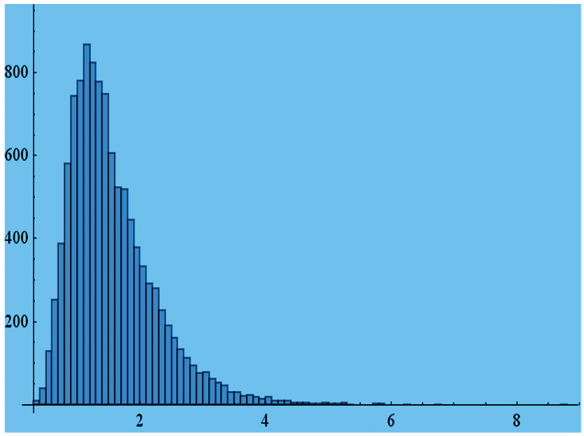

Figure 14: Histogram of $\lambda = 1.6$ by the MCMC method

8 Conclusion

This article uses Type-I hybrid censored data to estimate the unknown parameters of the exponentiated Rayleigh distribution under a step-stress model. We present two techniques of estimation: Maximum likelihood (ML) and Bayes methods. Point and interval estimates are found. The Bayes estimates of the parameters are calculated for the squared error and the linear exponential loss functions by the MCMC technique. A real-life data example and simulation study comparing all suggested estimating techniques is provided. Tables 3–6 show the following:

- Bayes estimates are better than MLEs in small samples, while in large samples, MLEs are the same quality of Bayes estimates or better.
- For the parameters α, β, the Bayes estimates under the squared error loss (SEL) function outperform the LINEX loss function; however, for λ, the LINEX loss function outperforms the SEL function.
- MSE for estimation of parameters decreases with increasing n.
- For a fixed n the MSE and the length of CIs decrease with increasing censoring size.
- As the sample size increases, the length of credible CIs for α, β and λ reduces. Over time, credible CIs provide more accurate results than MLE CIs in small samples, but in large samples, the opposite is often true.

As future work, for the exponentiated Rayleigh distribution under a step-stress model by using Type-I hybrid censored data, using one-sample and two-sample prediction methodologies, the problem of Bayesian prediction boundaries for the future observations can be identified.

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