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**ABSTRACT**

A highly adaptable censoring framework, known as the unified Type-I progressive hybrid censoring scheme, has been recently introduced as an enhancement to the unified hybrid censoring approach. However, this censoring method has a significant limitation: it assumes that the removal pattern is fixed and established prior to conducting the experiment, which may lack practical applicability. This study presents, for the first time, the unified Type-I progressive hybrid censoring scheme incorporating binomial removal, which enables the random withdrawal of surviving units following each failure, making it more suitable for survival studies. Assuming that the underlying distribution of the test units follows the Nadarajah–Haghighi distribution, we examine both point and interval estimation problems for the model parameters, binomial parameter, and two key survival measures. Maximum likelihood estimation serves as the classical methodology for obtaining point estimates and approximate confidence intervals. For Bayesian estimation, we employ the squared error loss function in conjunction with Markov Chain Monte Carlo sampling techniques. A comprehensive simulation analysis is conducted to assess the performance of the various point and interval estimators. In addition, to demonstrate the practical relevance of the proposed methodologies in survival analysis, a real-world data application involving the death times of a group of 45 gastric cancer patients is investigated.

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## 1 Introduction

In statistical modeling, a critical task is choosing a suitable probability distribution that effectively represents the behavior of the variable under study. The literature offers a wide range of distributions for this purpose, with the exponential, Weibull, gamma, and generalized exponential distributions being among the most frequently employed. Notably, the Weibull, gamma, and generalized exponential distributions can be viewed as generalizations of the classical exponential distribution. A key limitation of the exponential distribution is its inability to capture varying hazard rate functions

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(HRFs), as it only supports a constant HRF. Its generalizations, however, offer greater flexibility in modeling diverse survival patterns. An important alternative within this context is the Nadarajah–Haghighi (NH) distribution, introduced by Nadarajah and Haghighi [1] for modeling rainfall data in Florida. Proposed as a generalization of the exponential distribution, the NH model provides improved adaptability in fitting real-world data. It can accommodate increasing, decreasing, and constant hazard rates, much like the Weibull, gamma, and generalized exponential distributions. Let the random variable $Y$ follow the NH distribution, denoted by $Y \sim NH(\mu, \phi)$, where $\mu > 0$ is the scale parameter and $\phi > 0$ is the shape parameter. Then, the probability density function (PDF) and cumulative distribution function (CDF) of $Y$ are given as follows

$$g(y;\, \boldsymbol{\theta}) = \mu\phi(1+\mu y)^{\phi-1} \exp\left\{1-(1+\mu y)^{\phi}\right\},\ y>0,\ \mu,\ \phi>0 \tag{1}$$

and

$$G(y;\, \boldsymbol{\theta}) = 1-\exp\left\{1-(1+\mu y)^{\phi}\right\}, \tag{2}$$

respectively, where $\boldsymbol{\theta} = (\mu,\ \phi)^{\top}$. It is worth noting that the NH distribution is sometimes referred to as the extended exponential distribution in some studies, see for example Kumar et al. [2]. Two important survival measures are the survival function (SF) and the HRF. For the NH distribution, these functions are given, respectively, by

$$R(y;\, \boldsymbol{\theta}) = \exp\left\{1-(1+\mu y)^{\phi}\right\} \tag{3}$$

and

$$H(y;\, \boldsymbol{\theta}) = \mu\phi(1+\mu y)^{\phi-1}. \tag{4}$$

Several studies have addressed statistical inference for the NH distribution. For instance, Wu and Gui [3] investigated parameter estimation and prediction for the NH distribution based on progressively Type-II censored data. Almarashi et al. [4] considered survival estimation under adaptive Type-I progressive censoring. Further contributions include Elshahhat and Abu El Azm [5], who studied inference under generalized progressively hybrid censoring and applied the NH distribution to model the failure times of electronic devices; Abushal [6], who used unified hybrid censored competing risks data; Hasaballah et al. [7], who examined joint progressive Type-II censoring; and Abushal and AL-Zaydi [8], who focused on Type-II generalized hybrid censoring with competing risks, among others. These works provide a growing body of literature on the analysis of the NH distribution under various censoring schemes. Alotaibi et al. [9] employed the NH distribution to model the failure times of light bulbs subjected to ramp-voltage tests. Shi et al. [10] analyzed both biaxial fatigue-life data for metal specimens (measured in cycles) and the failure times of an aircraft air-conditioning system using the NH distribution. In addition, Newer et al. [11] investigated the lifetimes of small electronic appliances with known failure causes. Collectively, these studies highlight the superior flexibility of the NH distribution compared to several well-known lifetime models, demonstrating its effectiveness across diverse application areas, including engineering and medical fields.

A review of the literature on the estimation of the NH distribution, whether for model parameters or survival metrics, reveals that most existing studies are conducted under censoring schemes. This trend arises from the impracticality of using complete sample data in modern life testing experiments. With the advancement of technology, products have become increasingly reliable, making it inefficient and costly to wait for all items to fail. As a result, the use of censored samples has become a practical approach, allowing researchers to reduce both testing time and associated expenses. In the field of life testing and survival analysis, various censoring schemes are commonly used. Among the

most widely adopted are the Type-I and Type-II censoring plans. In Type-I censoring, the test is terminated at a pre-specified time, whereas in Type-II censoring, the experiment ends after a fixed number of failures have been observed. Several extensions of these basic censoring schemes have been proposed in the literature. For instance, Epstein [12] introduced the Type-I hybrid censoring (HC) plan, Childs et al. [13] proposed the Type-II HC plan, and Balakrishnan et al. [14] developed the unified HC scheme. These extensions aim to enhance flexibility in experimental design by combining features of both time- and failure-based termination rules.

The aforementioned censoring schemes, along with others not discussed here, are classified as single-stage censoring plans. A key drawback of these schemes is that they do not allow for the removal of surviving test units before the experiment concludes. To address this drawback, the conventional progressive Type-II censoring plan is often used. This approach allows for the removal of surviving units at each failure time, according to a pre-specified removal scheme. Several generalizations of the progressive Type-II censoring plan have also been proposed in the literature, including the progressive Type-I HC plan introduced by Kundu and Joarder [15], and the adaptive progressive Type-II HC plan proposed by Ng et al. [16]. For a comprehensive overview of various censoring schemes, refer to the work by Balakrishnan and Cramer [17]. To enhance flexibility in life-testing experiments, Górny and Cramer [18] introduced the unified progressive HC (UPHC) approach. This methodology integrates features from both the unified HC scheme and the progressive Type-II censoring scheme. In their study, four variations of UPHC were presented: unified Type-I, Type-II, Type-III, and Type-IV progressive HC schemes. The present study specifically concentrates on the unified Type-I progressive HC (UTIPHC) scheme.

Under the UTIPHC censoring scheme, the researcher establishes several parameters in advance: two integers $k$ and $m$, with $k < m$, two time boundaries $T_1$ and $T_2$ with $T_1 < T_2$, and a progressive censoring plan $\boldsymbol{S} = (S_1, \ldots, S_m)$ satisfies $n = m + \sum_{i=1}^{m} S_i$, with $n$ representing the total number of units under test. Following the same approach as conventional progressive Type-II censoring, when the $i$-th failure occurs at time $Y_{i:n}$, the researcher withdraws $S_i$ units that are still functioning from the remaining sample. The experiment's conclusion time under the UTIPHC plan depends on predetermined criteria that incorporate the values of $k$, $m$, $T_1$, and $T_2$, and is specified as follows:

$$\tau = max\left\{min(Y_{k:n},\ T_2),\ min(Y_{m:n},\ T_1)\right\}.$$

According to the work of Górny and Cramer [18], the UTIPHC scheme can result in six different experimental scenarios. Nevertheless, when analogous situations are consolidated together, this censoring approach yields four primary cases that can be outlined as follows:

- **Case I:** If $Y_{m:n} < T_1$, the test is stopped at $Y_{m:n}$;
- **Case II:** If $Y_{k:n} < T_1 < Y_{m:n}$, end the test at $T_1$;
- **Case III:** If $T_1 < Y_{k:n} < T_2$, the test is terminated at $Y_{k:n}$;
- **Case IV:** If $T_2 < Y_{k:n}$, stop the test at $T_2$.

Based on an observed UTIPHC sample, denoted by $\mathbf{y}$, the likelihood function of the observed data, combining the four possible cases and ignoring constant terms, can be written as follows

$$\mathcal{L}_1(\boldsymbol{\theta};\ \mathbf{y}) = \prod_{i=1}^{d} g(y_i)[1 - G(y_i)]^{S_i}[1 - G(\tau)]^{S^*}, \tag{5}$$

where $y_i = y_{i:n}$ for simplicity, $d, \tau$ and $S^*$ are given by

$$(d,\ \tau,\ S^*) = \begin{cases} \left(m,\ 0,\ n-m-\sum_{i=1}^{m-1} S_i\right), & \text{for Case I,} \\ \left(q_1,\ T_1,\ n-q_1-\sum_{i=1}^{q_1} S_i\right), & \text{for Case II,} \\ \left(k,\ 0,\ n-k-\sum_{i=1}^{k-1} S_i\right), & \text{for Case III,} \\ \left(q_2,\ T_2,\ n-q_2-\sum_{i=1}^{q_2} S_i\right), & \text{for Case IV,} \end{cases}$$

where $q_j, j = 1, 2$ denotes the observed number of failures before $T_j$. In recent years, several authors have employed the UTIPHC scheme to investigate statistical inference for various lifetime models. Examples include the study by Anwar et al. [19] on stress-strength survival estimation for the inverted exponentiated Rayleigh distribution, the work by Lone et al. [20] on parameter estimation for the gamma-mixed Rayleigh distribution, the investigation by Dutta and Kayal [21] into estimation and prediction for the Burr Type-III distribution, and the analysis by Dutta et al. [22] on inferences for the Kumaraswamy-G family of distributions under partially observed competing risks data. Recently, Prakash et al. [23] compared statistical and machine learning approaches for reliability estimation using the UTIPHC scheme.

One of the key limitations of the existing studies on the UTIPHC scheme is the assumption that unit removals follow a fixed and preassigned pattern throughout the experiment. While analytically convenient, this assumption is often unrealistic in real survival or reliability testing environments. In practical scenarios, removals may occur unpredictably due to factors such as safety constraints, administrative decisions, cost considerations, or the inability to continue monitoring particular units. As noted by Yuen and Tse [24], researchers may choose to discontinue observation of certain individuals regardless of their current survival status. In such situations, the random withdrawal of surviving units following each failure represents a more realistic operational mechanism. The literature has considered several probabilistic structures for modeling random removals, including the discrete uniform, binomial, and beta-binomial approaches; see Tse et al. [25], Singh et al. [26], Elbatal et al. [27], and Alqasem et al. [28]. However, no existing work has integrated such random-removal mechanisms into the UTIPHC framework. Moreover, to the best of our knowledge, no previous study has investigated reliability analysis under the NH distribution using the UTIPHC scheme, nor extended this plan to incorporate a binomial removal mechanism. The present work fills this methodological gap by introducing, for the first time, the UTIPHC design with binomially distributed removals, allowing the number of surviving units withdrawn at each failure time to vary randomly. This modification substantially enhances the flexibility and practical relevance of the UTIPHC scheme. From an inferential perspective, incorporating binomial removal introduces additional uncertainty into the pattern of observed and censored data, which leads to more robust and realistic estimation of both the model parameters and the associated survival measures.

Motivated by the need for a more realistic censoring mechanism and by the modeling flexibility of the NH distribution, the main objective of this study is to develop and investigate parameter and survival estimation for the NH model under the UTIPHC scheme with binomial removal. The estimation targets include the scale and shape parameters of the NH distribution, the binomial removal parameter, and two key survival characteristics: the survival function (SF) and the hazard rate function (HRF). To accomplish this, we employ two complementary inferential approaches. The first is maximum likelihood estimation, which yields point estimates and approximate confidence intervals (ACIs). The second is a Bayesian framework under a squared error loss function, providing Bayes point estimators and Bayes credible intervals (BCIs) via Markov Chain Monte Carlo (MCMC) sampling. The performance of these estimation methods is systematically assessed through an extensive

simulation study conducted under multiple experimental settings and evaluated using several precision criteria. Finally, to highlight the practical importance and applicability of the proposed methodology, we analyze a real-world cancer dataset, demonstrating the usefulness of the extended UTIPHC scheme in survival analysis.

The remainder of this paper is structured as follows: Section 2 focuses on deriving the joint probability function for unit removals under binomial removal assumption, obtaining maximum likelihood estimators (MLEs), and constructing the ACIs for different parameters and survival measures. Section 3 addresses Bayesian estimation methodology. Section 4 presents the simulation analysis. Section 5 analyzes the lifetimes of cancer patients. Finally, Section 6 provides concluding observations.

## 2 Likelihood Estimation

This section examines the classical estimation approach for the NH distribution parameters, the binomial distribution parameter, and two survival measures. We begin by establishing the joint probability function for unit removals based on the binomial removal framework. Assume that the removal $S_i$ occurring at the $i$-th failure follows a binomial distribution with probability $\vartheta$. This implies that each surviving unit has an equal probability $\vartheta$ of being withdrawn from the experiment. Therefore, the probability mass function for the random removal $S_i$, characterized by parameters $n_i^\star = n - m - \sum_{j=1}^{i-1} S_j$ and $\vartheta$, is written as

$$P(S_i = s_i | S_{i-1} = s_{i-1}, \ldots, S_1 = s_1) = \binom{n_i^\star}{s_i} \vartheta^{s_i} (1-\vartheta)^{n-m-\sum_{j=1}^{i} s_j}, \tag{6}$$

where $0 \le s_i \le n_i^\star$, $i = 1, \ldots, r$, and $r = m - 1$, $q_1$, $k - 1$ and $q_2$ for Cases I, II, III, and IV, respectively. Accordingly, the joint probability corresponding to $S_i = s_i$ can be expresses as follows

$$\mathcal{L}_2(\mathbf{S} = \mathbf{s};\, \vartheta) = P(S_1 = s_1) P(S_2 = s_2 | S_1 = s_1) \ldots P(S_r = s_r | S_{r-1} = s_{r-1}, \ldots, S_1 = s_1). \tag{7}$$

It follows from (6) and (7) that

$$\mathcal{L}_2(\mathbf{S} = \mathbf{s}; \vartheta) = \left[ \prod_{i=1}^{r} \binom{n_i^\star}{s_i} \right] \vartheta^{\sum_{i=1}^{r} s_i} (1-\vartheta)^{r(n-m) - \sum_{i=1}^{r} (r+1-i) s_j}. \tag{8}$$

Using (5) and (8), we can formulate the full likelihood function of the observed UTIPHC sample $\mathbf{y}$ with binomial removal as follows

$$\mathcal{L}(\boldsymbol{\theta},\, \vartheta;\, \mathbf{y},\, \mathbf{s}) = \mathcal{L}_1(\boldsymbol{\theta};\, \mathbf{y} | \mathbf{S} = \mathbf{s}) \mathcal{L}_2(\mathbf{S} = \mathbf{s};\, \vartheta). \tag{9}$$

The full likelihood function in (9) can be used to derive estimators for the model parameters, binomial parameters, and survival measures. Alternatively, by utilizing the independence between $S_i$ and $Y_i$, we can readily obtain both classical and Bayesian estimators for the shape and scale parameters, or any function of them, by directly utilizing the likelihood function in Eq. (5). Meanwhile, the binomial parameter estimation can be accomplished using the objective function presented in Eq. (8).

### *2.1 Point Estimation*

Based on an observed UTIPHC sample $\mathbf{y}$ with binomial removal selected from the NH population, we can write the likelihood function using (1), (2) and (5), without constant terms, as follows

$$\mathcal{L}_1(\boldsymbol{\theta};\mathbf{y}|\mathbf{S}=\mathbf{s}) = (\mu\phi)^d \exp\left\{(\phi-1)\sum_{i=1}^{d}\log(1+\mu y_i) - \psi(\mathbf{y};\boldsymbol{\theta})\right\}, \tag{10}$$

where $\psi(\mathbf{y};\boldsymbol{\theta}) = \sum_{i=1}^{d}(1+s_i)(1+\mu y_i)^{\phi} + s^*(1+\mu\tau)^{\phi}$. The natural logarithm of (10) is now acquired as

$$l_1(\boldsymbol{\theta};\mathbf{y}|\mathbf{S}=\mathbf{s}) = d\log(\mu\phi) + (\phi-1)\sum_{i=1}^{d}\log(1+\mu y_i) - \psi(\mathbf{y};\boldsymbol{\theta}). \tag{11}$$

By taking the first partial derivatives of Eq. (11) with respect to $\mu$ and $\phi$ and setting them equal to zero, we obtain the normal equations that must be solved simultaneously to derive the MLEs of $\mu$ and $\phi$, denoted by $\hat{\mu}$ and $\hat{\phi}$. These normal equations are expressed as

$$\frac{\partial l_1(\boldsymbol{\theta};\mathbf{y}|\mathbf{S}=\mathbf{s})}{\partial\mu} = \frac{d}{\mu} + (\phi-1)\sum_{i=1}^{d}\frac{y_i}{1+\mu y_i} - \psi_1(\mathbf{y};\boldsymbol{\theta}) = 0 \tag{12}$$

and

$$\frac{\partial l_1(\boldsymbol{\theta};\mathbf{y}|\mathbf{S}=\mathbf{s})}{\partial\phi} = \frac{d}{\phi} + \sum_{i=1}^{d}\log(1+\mu y_i) - \psi_2(\mathbf{y};\boldsymbol{\theta}) = 0, \tag{13}$$

where

$$\psi_1(\mathbf{y};\boldsymbol{\theta}) = \phi\sum_{i=1}^{d} y_i(1+s_i)(1+\mu y_i)^{\phi-1} + \phi\tau s^*(1+\mu\tau)^{\phi-1},$$

and

$$\psi_2(\mathbf{y};\boldsymbol{\theta}) = \sum_{i=1}^{d}(1+s_i)(1+\mu y_i)^{\phi}\log(1+\mu y_i) + s^*(1+\mu\tau)^{\phi}\log(1+\mu\tau).$$

By solving the two normal equations presented in (12) and (13), the MLEs $\hat{\mu}$ and $\hat{\phi}$ can be determined. It is clear that these MLEs cannot be obtained in closed form. Consequently, numerical methods such as Newton's algorithm can be applied to solve these equations and derive the MLEs. On the other hand, the MLEs of the survival metrics in (3) and (4), at mission time $t$, can be computed through the invariance feature of the MLEs as follows

$$\widehat{R}(t) = \exp\left\{1-(1+\hat{\mu}t)^{\hat{\phi}}\right\}$$

and

$$\widehat{H}(t) = \hat{\mu}\hat{\phi}(1+\hat{\mu}t)^{\hat{\phi}-1}.$$

With regard to estimating the binomial parameter, the MLE of $\vartheta$, denoted by $\hat{\vartheta}$, can be obtained by maximizing the objective function given in (8). Accordingly, the MLE $\hat{\vartheta}$ is derived as follows

$$\hat{\vartheta} = \frac{\sum_{i=1}^{r} s_i}{r(n-m) - \sum_{i=1}^{r}(r-i)s_j}.$$

### *2.2 Interval Estimation*

In this part, we utilize the asymptotic normality property of the MLEs to construct ACIs for the model parameters, binomial parameters, and the two survival measures. The crucial step in obtaining these interval estimates is determining the variance-covariance matrix. This matrix is derived by taking the inverse of the Fisher information matrix. However, due to the analytical intractability involved in our case, obtaining exact expressions for the Fisher information matrix is not feasible. As an alternative approach, we estimate the variance-covariance matrix by inverting the observed Fisher information matrix. To implement this approach, we first compute the estimated variance-covariance matrix of the MLEs for the model parameters. This matrix serves as the foundation for quantifying the uncertainty associated with the point estimates and is essential for constructing ACIs for all parameters and survival measures of interest. We begin by obtaining the second partial derivatives of the log-likelihood function given in (11) as follows

$$\frac{\partial^2 l_1(\boldsymbol{\theta};\, \mathbf{y}|\mathbf{S}=\mathbf{s})}{\partial \mu^2} = -\frac{d}{\mu^2} - (\phi-1)\sum_{i=1}^{d}\left(\frac{y_i}{1+\mu y_i}\right)^2 - \psi_{11}(\mathbf{y};\, \boldsymbol{\theta}),$$

$$\frac{\partial^2 l_1(\boldsymbol{\theta};\, \mathbf{y}|\mathbf{S}=\mathbf{s})}{\partial \phi^2} = -\frac{d}{\phi^2} - \psi_{22}(\mathbf{y};\, \boldsymbol{\theta})$$

and

$$\frac{\partial^2 l_1(\boldsymbol{\theta};\mathbf{y}|\mathbf{S}=\mathbf{s})}{\partial \mu \partial \phi} = \sum_{i=1}^{d}\frac{y_i}{1+\mu y_i} - \psi_{12}(\mathbf{y};\boldsymbol{\theta}),$$

where

$$\psi_{11}(\mathbf{y};\, \boldsymbol{\theta}) = \phi(\phi-1)\sum_{i=1}^{d} y_i^2(1+s_i)(1+\mu y_i)^{\phi-2} + \phi(\phi-1)\tau^2 s^*(1+\mu\tau)^{\phi-1},$$

$$\psi_{22}(\mathbf{y};\, \boldsymbol{\theta}) = \sum_{i=1}^{d}(1+s_i)(1+\mu y_i)^{\phi}\log^2(1+\mu y_i) + s^*(1+\mu\tau)^{\phi}\log^2(1+\mu\tau)$$

and

$$\psi_{12}(\mathbf{y};\, \boldsymbol{\theta}) = \sum_{i=1}^{d} y_i(1+s_i)(1+\mu y_i)^{\phi-1}[1+\phi\log(1+\mu y_i)] + \tau s^*(1+\mu\tau)^{\phi-1}[1+\phi\log(1+\mu\tau)].$$

Thus, the estimated variance-covariance matrix of the MLEs $\hat{\boldsymbol{\theta}} = (\hat{\mu}, \hat{\phi})^\top$, can be obtained as follows

$$\begin{aligned}\boldsymbol{I}^{-1}(\hat{\boldsymbol{\theta}}) &= \begin{bmatrix} -\dfrac{\partial^2 l_1(\boldsymbol{\theta};\mathbf{y}\mid\mathbf{S}=\mathbf{s})}{\partial\mu^2} & -\dfrac{\partial^2 l_1(\boldsymbol{\theta};\mathbf{y}\mid\mathbf{S}=\mathbf{s})}{\partial\mu\partial\phi} \\ & -\dfrac{\partial^2 l_1(\boldsymbol{\theta};\mathbf{y}\mid\mathbf{S}=\mathbf{s})}{\partial\phi^2} \end{bmatrix}^{-1}_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}} \\ &= \begin{bmatrix} \widehat{\mathcal{V}}_{11} & \widehat{\mathcal{V}}_{12} \\ & \widehat{\mathcal{V}}_{22} \end{bmatrix}. \end{aligned} \tag{14}$$

By employing the asymptotic normality of the MLEs, i.e., $\hat{\boldsymbol{\theta}} \sim N(\boldsymbol{\theta}, \boldsymbol{I}^{-1}(\hat{\boldsymbol{\theta}}))$. Then, for a specific level of significance $\alpha$, the $(1-\alpha)\%$ ACIs for $\mu$ and $\phi$, can be obtained as

$$\hat{\mu} \pm z_{\alpha/2}\sqrt{\widehat{\mathcal{V}}_{11}}, \quad \text{and} \quad \hat{\phi} \pm z_{\alpha/2}\sqrt{\widehat{\mathcal{V}}_{22}},$$

respectively, where $z_{\alpha/2}$ refers to the upper $\alpha/2$ percentile of $N(0,1)$.

To construct the ACIs for the two survival metrics, we must first determine the associated variances of their MLEs. For this purpose, we employ the delta method, See for more detail Greene [29], and Ahmed [30]. The delta method is a widely used technique for approximating variances of functions involving MLEs. It works by creating a linear approximation of a complex function and then calculating the variance of this simplified linear function, which can then be applied for large-sample statistical inference. In order to approximate the variances of the MLEs of the SF and HRF, the following two vectors are needed:

$$\hat{\Delta}_R = -(1+\hat{\mu}t)^{\hat{\phi}} \exp\left\{1-(1+\hat{\mu}t)^{\hat{\phi}}\right\} \begin{bmatrix} \hat{\phi}t(1+\hat{\mu}t)^{-1} \\ \log(1+\hat{\mu}t) \end{bmatrix}$$

and

$$\hat{\Delta}_H = (1+\hat{\mu}t)^{\hat{\phi}-1} \begin{bmatrix} \hat{\phi}(1+\hat{\mu}t)^{-1}\left(1+\hat{\mu}\hat{\phi}t\right) \\ \hat{\mu}\left(1+\hat{\phi}\log(1+\hat{\mu}t)\right) \end{bmatrix}.$$

Then, the approximated variances of both the MLEs of the SF and HRF, can be computed, respectively, as follows

$$\mathcal{V}_{\hat{R}} \approx \hat{\Delta}_R^{\top} \boldsymbol{I}^{-1}(\hat{\boldsymbol{\theta}}) \hat{\Delta}_R \quad \text{and} \quad \mathcal{V}_{\hat{H}} \approx \hat{\Delta}_H^{\top} \boldsymbol{I}^{-1}(\hat{\boldsymbol{\theta}}) \hat{\Delta}_H.$$

Thus, the $(1-\alpha)\%$ ACIs for the two survival metrics are as follows

$$\widehat{R} \pm z_{\alpha/2}\sqrt{\mathcal{V}_{\hat{R}}}, \quad \text{and} \quad \widehat{\phi} \pm z_{\alpha/2}\sqrt{\mathcal{V}_{\hat{R}}}.$$

For the binomial parameter $\vartheta$, to construct the associated $(1-\alpha)\times 100\%$ ACI, we can estimate the variance of its MLE $\hat{\vartheta}$, as follows

$$\mathcal{V}_{\hat{\vartheta}} = \left[-\frac{d^2 \log \mathcal{L}_2(\mathbf{S}=\mathbf{s};\vartheta)}{d\vartheta^2}\right]^{-1}_{\vartheta=\hat{\vartheta}},$$

where

$$\frac{d^2 \log \mathcal{L}_2(\mathbf{S}=\mathbf{s};\vartheta)}{d\vartheta^2} = -\frac{\sum_{i=1}^{r} s_i}{\vartheta^2} - \frac{r(n-m)-\sum_{i=1}^{r}(r+1-i)s_j}{(1-\vartheta)^2}.$$

As a result, the $(1-\alpha)\%$ ACI of $\vartheta$ can be acquired as

$$\hat{\vartheta} \pm z_{\alpha/2}\sqrt{\mathcal{V}_{\hat{\vartheta}}}.$$

## 3 Bayesian Estimation

The Bayesian estimation method offers a powerful alternative to classical estimation techniques. While classical estimators typically improve with larger sample sizes, they may lack accuracy in scenarios involving small samples, such as those commonly encountered in censored data from highly reliable products. The Bayesian approach addresses this limitation by integrating prior knowledge

about the unknown parameters with the observed data, resulting in more robust and informative inferences, especially in small-sample settings. In this section, we consider the Bayesian estimation for the NH distribution parameters, binomial parameter, SF, and HRF under the UTIPHC methodology with binomial removal.

### *3.1 Prior Information and Posterior Formulation*

The Bayesian approach commences with the selection of suitable prior distributions for all unknown parameters. For the scale and shape parameters of the NH distribution, we begin by examining the structure of the likelihood function presented in (10). This analysis demonstrates that conjugate priors cannot be employed for these parameters. Additionally, obtaining non-informative Jeffreys' priors, which depend on the Fisher information matrix, proves computationally challenging due to the complex nature of the variance-covariance matrix. In light of these constraints, we implement informative gamma priors for both parameters, taking advantage of their versatility and compatibility with the positive domain of the NH distribution's parameters, see Dey et al. [31], and Newer et al. [11]. Assuming independence between the two random variables $\mu$ and $\phi$, where $\mu \sim \text{Gamma}(b_1, c_1)$ and $\phi \sim \text{Gamma}(b_2, c_2)$, the joint prior distribution of $\boldsymbol{\theta}$ is given by

$$p(\boldsymbol{\theta}) \propto \mu^{b_1-1}\phi^{b_2-1} \exp\{-c_1\mu - c_2\phi\}, \ \mu, \phi > 0, \tag{15}$$

with $b_j, c_j > 0$ are known hyper-parameters. By merging the observed data given by (10) with the prior knowledge provided by (15), and applying the Bayes rule, we can formulate the posterior distribution of $\boldsymbol{\theta}$ given the data as follows

$$\mathcal{P}_1(\boldsymbol{\theta}|\mathbf{y}, \mathbf{S} = \mathbf{s}) = \frac{1}{\mathcal{C}}\, \mu^{d+b_1-1}\phi^{d+b_2-1} \exp\left\{(\phi - 1)\sum_{i=1}^{d} \log(1 + \mu y_i) - \psi(\mathbf{y}; \boldsymbol{\theta}) - c_1\mu - c_2\phi\right\}, \tag{16}$$

where $\mathcal{C}$ is provided by

$$\mathcal{C} = \int_0^\infty \int_0^\infty p(\boldsymbol{\theta})\mathcal{L}_1(\boldsymbol{\theta}; \mathbf{y}|\mathbf{S} = \mathbf{s}) d\mu\, d\phi.$$

Let $\xi(\boldsymbol{\theta})$ denote any parametric function of the unknown parameters to be estimated, and let $\tilde{\xi}(\boldsymbol{\theta})$ represent its Bayes estimator. Then, under the squared error loss function, $\tilde{\xi}(\boldsymbol{\theta})$ can be obtained as follows:

$$\tilde{\xi}(\boldsymbol{\theta}) = \frac{\int_0^\infty \int_0^\infty \xi(\boldsymbol{\theta})p(\boldsymbol{\theta})\mathcal{L}_1(\boldsymbol{\theta}; \mathbf{y}|\mathbf{S} = \mathbf{s}) d\mu\, d\phi}{\int_0^\infty \int_0^\infty p(\boldsymbol{\theta})\mathcal{L}_1(\boldsymbol{\theta}; \mathbf{y}|\mathbf{S} = \mathbf{s}) d\mu\, d\phi}. \tag{17}$$

The Bayesian estimator presented in (17) involves computing integrals that cannot be solved analytically, mainly because of the high-dimensional and non-conjugate characteristics of the posterior distribution. To overcome this challenge, we implement the MCMC techniques to generate samples directly from the posterior distribution described in (16). These generated samples are subsequently employed to compute Bayes point estimates and to develop the BCIs for the NH distribution parameters and the two survival metrics.

With regard to the binomial distribution parameter $\vartheta$, it is well known that the associated conjugate prior is the beta distribution, see Dey and Dey [32]. Assume that $\vartheta \sim B(a_1, a_2)$. Then, using (8), the posterior distribution of $\vartheta$ can be expressed as follows:

$$\mathcal{P}_2(\mathbf{S} = \mathbf{s}; \vartheta) = \frac{1}{B(a_1^\star, a_2^\star)} \vartheta^{a_1^\star - 1}(1 - \vartheta)^{a_2^\star - 1}, \tag{18}$$

where $a_1^\star = \sum_{i=1}^{r} s_i + a_1$ and $a_2^\star = r(n-m) - \sum_{i=1}^{r}(r+1-i)s_j + a_2$. Under the squared error loss function, the Bayes estimator of $\vartheta$, denoted by $\tilde{\vartheta}$, can be obtained from (20) as follows

$$\tilde{\vartheta} = \frac{1}{B(a_1^\star, a_2^\star)} \int_0^1 \vartheta^{a_1^\star}(1-\vartheta)^{a_2^\star - 1} d\vartheta$$
$$= \frac{a_1^\star}{a_1^\star + a_2^\star}.$$

Furthermore, using the posterior distribution of $\vartheta$ in (20), the $100(1-\alpha)\%$ BCI of $\vartheta$ can be obtained as follows

$$\left[B(a_1^*, a_2^*, \alpha/2),\ B(a_1^*, a_2^*, 1-\alpha/2)\right],$$

where $B(.,.,.)$ is the quantile function of the beta distribution.

### *3.2 MCMC and Posterior Analysis*

As previously mentioned, obtaining explicit Bayes estimators for the NH distribution parameters or the two survival metrics is not feasible due to the complex form of the posterior distribution in (16). Therefore, we utilize the efficiency of MCMC sampling methods to first generate samples from this posterior distribution and subsequently use these samples to calculate the desired point estimates and interval estimates. The initial step in this process involves deriving the full conditional distributions for $\mu$ and $\phi$, respectively, which are given as follows

$$\mathcal{P}_{11}(\mu|\phi, \mathbf{y}, \mathbf{S}=\mathbf{s}) \propto \mu^{d+b_1-1} \exp\left\{(\phi-1)\sum_{i=1}^{d}\log(1+\mu y_i) - \psi(\mathbf{y};\boldsymbol{\theta}) - c_1\mu\right\} \tag{19}$$

and

$$\mathcal{P}_{12}(\phi|\mu, \mathbf{y}, \mathbf{S}=\mathbf{s}) \propto \phi^{d+b_2-1} \exp\left\{\phi\sum_{i=1}^{d}\log(1+\mu y_i) - \psi(\mathbf{y};\boldsymbol{\theta}) - c_2\phi\right\}. \tag{20}$$

It is evident that the two conditional distributions presented in (19) and (20) do not correspond to standard distributions, therefore, the sampling process will be implemented using the Metropolis-Hastings (MH) algorithm with an accept/reject mechanism. This algorithm requires a proposal distribution to generate candidate values. In our analysis, we employ the normal distribution as the proposal distribution for each parameter The sample generation procedure follows the steps described Algorithm 1.

**Algorithm 1:** MCMC procedure

1: Initialize the starting values $\mu^{(0)}$ and $\phi^{(0)}$, using the MLEs as initial estimates.
2: Set $j = 1$.
3: Execute the following MH procedure to generate $\mu^{(j)}$:
   1. Draw $\mu^\star \sim \mathrm{N}(\hat{\mu}, \widehat{\mathcal{V}}_{11})$.
   2. Calculate the acceptance probability ($\mathcal{A}_\mu$):

$$\mathcal{A}_\mu = \min\left(1, \frac{\mathcal{P}_{11}(\mu^\star|\phi^{(j-1)}, \mathbf{y}, \mathbf{S}=\mathbf{s})}{\mathcal{P}_{11}(\mu^{(j-1)}|\phi^{(j-1)}, \mathbf{y}, \mathbf{S}=\mathbf{s})}\right).$$

   3. Accept the candidate value $\mu^*$ as $\mu^{(j)}$ with probability $\mathrm{A}_\mu$, otherwise retain $\mu^{(j-1)}$.

(Continued)

**Algorithm 1** (continued)

4: Generate $\phi^{(j)}$ using the following MH procedure:
1. Simulate $\phi^{\star} \sim \mathrm{N}(\hat{\phi}, \widehat{\mathcal{V}}_{22})$.
2. Compute $\mathcal{A}_{\phi}$:

$$\mathcal{A}_{\phi} = \min\left(1, \frac{\mathcal{P}_{12}(\phi^{\star}|\mu^{(j)}, \mathbf{y}, \mathbf{S}=\mathbf{s})}{\mathcal{P}_{12}(\phi^{(j-1)}|\mu^{(j)}, \mathbf{y}, \mathbf{S}=\mathbf{s})}\right)$$

3. Set the obtained value $\phi^{\star}$ as $\phi^{(j)}$ with probability $\mathcal{A}_{\mu}$, otherwise set $\phi^{(j)}$ as $\mu^{(j-1)}$ .
5: Use $(\mu^{(j)}, \phi^{(j)})$ to obtain the SF and HRF at iteration $j$ as:
$R^{(j)} = \exp\left\{1-(1+\mu^{(j)}t)^{\phi^{(j)}}\right\}$ and $H^{(j)} = \mu^{(j)}\phi^{(j)}(1+\mu^{(j)}t)^{\phi^{(j)}-1}$.
6: Update $j$ to $j+1$.
7: Repeat steps 3 through 6, $\mathcal{M}$ times.
8: Save the generated sequence as

$$\left\{\mu^{(j)}, \phi^{(j)}, R^{(j)}, H^{(j)}\right\}, j = 1, \ldots, \mathcal{M}. \tag{21}$$

Then, based on the generated MCMC samples presented in (21), we can simply obtain the Bayes estimates of the NH distribution parameters as well as the two survival metrics under the squared error loss function as follows

$$\tilde{\mu} = \frac{1}{\mathcal{M}^{\star}}\sum_{j=\mathcal{B}+1}^{\mathcal{M}}\mu^{(j)}, \ \tilde{\phi} = \frac{1}{\mathcal{M}^{\star}}\sum_{j=\mathcal{B}+1}^{\mathcal{M}}\phi^{(j)},$$

$$\tilde{R}(t) = \frac{1}{\mathcal{M}^{\star}}\sum_{j=\mathcal{B}+1}^{\mathcal{M}}R^{(j)}, \text{ and } \tilde{H}(t) = \frac{1}{\mathcal{M}^{\star}}\sum_{j=\mathcal{B}+1}^{\mathcal{M}}H^{(j)},$$

where $\mathcal{B}$ denotes the number of iterations discarded as a burn-in period, and $\mathcal{M}^{\star} = \mathcal{M} - \mathcal{B}$ represents the effective number of MCMC samples used for inference. In addition, by sorting the MCMC samples in (21) in ascending order as $\mu^{[\mathcal{B}+1]} < \ldots < \mu^{[\mathcal{M}]}$, $\phi^{[\mathcal{B}+1]} < \ldots < \phi^{[\mathcal{M}]}$, $R^{[\mathcal{B}+1]} < \ldots < R^{[\mathcal{M}]}$ and $H^{[\mathcal{B}+1]} < \ldots < H^{[\mathcal{M}]}$. The the $100(1-\alpha)\%$ BCIs can be calculated as follows

$$\left\{\mu^{[\alpha\mathcal{M}^{\star}/2]}, \mu^{[(1-\alpha/2)\mathcal{M}^{\star}]}\right\}, \ \left\{\phi^{[\alpha\mathcal{M}^{\star}/2]}, \phi^{[(1-\alpha/2)\mathcal{M}^{\star}]}\right\},$$

$$\left\{R^{[\alpha\mathcal{M}^{\star}/2]}, R^{[(1-\alpha/2)\mathcal{M}^{\star}]}\right\}, \text{ and } \left\{H^{[\alpha\mathcal{M}^{\star}/2]}, H^{[(1-\alpha/2)\mathcal{M}^{\star}]}\right\}.$$

## 4 Simulation Comparison

To assess the performance and inferential accuracy of the proposed estimators for the parameters $\mu$, $\phi$, $R(t)$, $H(t)$, and $\vartheta$, we conducted an extensive Monte Carlo simulation study. Following Algorithm 2, from the NH (0.5, 1.5) model, we replicated the UTIPHC 1000 times based on several configurations of $\vartheta$ (binomial probability), $T_i$, $i = 1, 2$ (thresholds), $n$ (total testing units), and $(k, m)$ (total failed units). Specifically, $\vartheta$ (= 0.3, 0.6, 0.9), $n$ (= 20, 40, 80), and $(k, m)$ are determined as a failure present for each $n$ such as $\frac{k}{n} \times 100$ (= 25%, 50%) and $\frac{m}{n} \times 100$ (= 50%, 75%). To highlight the impact of the thresholds $T_i$, $i = 1, 2$ on inferential calculations, for each given of $\vartheta$, $n$, $m$, and $k$, two different levels of $T_i$, $i = 1, 2$ are utilized, namely $T_1$(= 0.5, 1) and $T_2$(= 1, 2). At $t = 0.1$, the actual values of $R(t)$ and $H(t)$ are taken as 0.93017 and 0.69937, respectively.

**Algorithm 2:** Generation steps of UTIPHC scheme

**Input:** Set of NH ($\boldsymbol{\theta}$) population values
**Output:** Simulate $\kappa$ independent uniform random variables $\kappa_1, \kappa_2, \ldots, \kappa_m \sim U(0,1)$
Compute $\zeta_i = \kappa_i^{\left(i+\sum_{v=m-i+1}^{m} S_v\right)^{-1}}$, for $i = 1, \ldots, m$
Obtain $U_i = 1 - \prod_{s=m-i+1}^{m} \zeta_s$, for $i = 1, \ldots, m$
Transform $y_i = G^{-1}(u_i; \boldsymbol{\theta})$, for $i = 1, \ldots, m$
Identify $q_i$ at $T_i$ for $i = 1, 2$
**Compute remaining survival units $S^\star$:**
**if** $y_k < y_m < T_1 < T_2$ **then**
$S^\star = n - m - \sum_{i=1}^{m-1} S_i$
**end if**
**if** $y_k < T_1 < y_m < T_2$ **or** $y_k < T_1 < T_2 < y_m$ **then**
$S^\star = n - q_1 - \sum_{i=1}^{q_1} S_i$
**end if**
**if** $T_1 < y_k < y_m < T_2$ **or** $T_1 < y_k < T_2 < y_m$ **then**
$S^\star = n - k - \sum_{i=1}^{k-1} S_i$
**end if**
**if** $T_1 < T_2 < y_k < y_m$ **then**
$S^\star = n - q_2 - \sum_{i=1}^{q_2} S_i$
**end if**
**Obtain the sample type:**
**if** $y_k < y_m < T_1 < T_2$ **then**
Stop the test at $y_m$
**end if**
**if** $y_k < T_1 < y_m < T_2$ **or** $y_k < T_1 < T_2 < y_m$ **then**
Stop the test at $T_1$
**end if**
**if** $T_1 < y_k < y_m < T_2$ **or** $T_1 < y_k < T_2 < y_m$ **then**
Stop the test at $y_k$
**end if**
**if** $T_1 < T_2 < y_k < y_m$ **then**
Stop the test at $T_2$
**end if**

To see the effects of gamma prior assumptions on Bayesian inference (MCMC and BCI) of the NH model parameters $\mu$ and $\phi$ as well as the associated survival indices $R(t)$ and $H(t)$, we propose two different informative sets of the hyperparameters $(b_i,\ c_i),\ i = 1,\ 2$, namely:

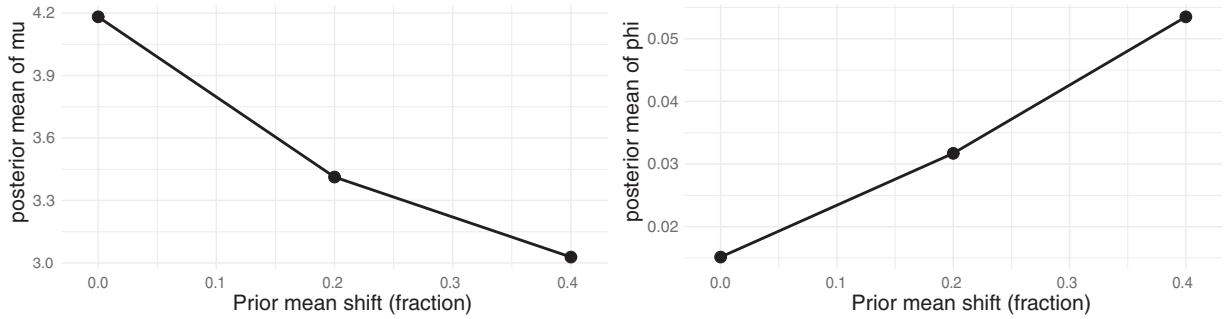
- Prior-1: $(b_1,\ b_2,\ c_1,\ c_2) = (2.5,\ 7.5,\ 5,\ 5)$;
- Prior-2: $(b_1,\ b_2,\ c_1,\ c_2) = (5,\ 15,\ 10,\ 10)$.

Again, to explore the importance of Bayesian inference to beta prior assumptions, particularly for the binomial parameter $\vartheta$, we proposed two different informative sets of the hyperparameters $(a_1,\ a_2)$, namely:

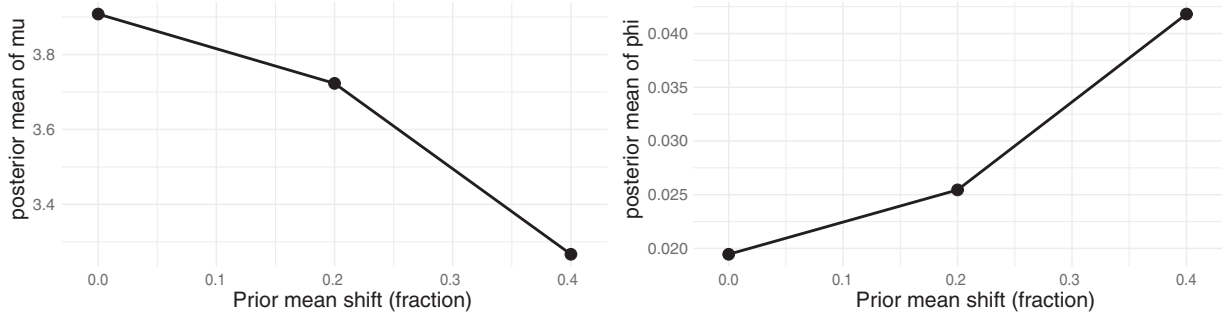
- Prior-1: $(a_1,\ a_2) = (1.5,\ 3.5)$, $(3,\ 2)$, and $(4.5,\ 0.5)$ for $\vartheta (= 0.3,\ 0.6,\ 0.9)$;
- Prior-2: $(a_1,\ a_2) = (3,\ 7)$, $(6,\ 4)$, and $(9,\ 1)$ for $\vartheta (= 0.3, 0.6, 0.9)$.

It should be mentioned here that the proposed priors 1 and 2 of $\mu$, $\phi$, or $\vartheta$ use the methodology of prior mean and prior variance outlined by Kundu [33]. This approach enables the suggestion of prior parameters in a manner that ensures the prior expectation of each specified informative set matches the predetermined actual value of each parameter.

To examine the robustness of the Bayesian estimators to prior misspecification, we implemented a prior sensitivity experiment in which the hyperparameters of the proposed gamma priors assigned to $\mu$ or $\phi$ were intentionally shifted by 20% and 40% away from the values used in data generation. Fig. 1 displays the posterior mean of the induced lifetime mean of each parameter and its precision under these perturbations. The results show a clear and smooth monotonic response and reveal that the posterior mean of $\mu$ decreases as the prior moves further from the truth, whereas the corresponding posterior precision $\phi$ increases. This reciprocal pattern is theoretically coherent since reductions in $\mu$ are accompanied by tighter posterior spread. Importantly, although the posterior summaries do shift in the direction of the prior distortion, the changes remain moderate and without instability or abrupt behavior, demonstrating that the proposed Bayesian estimation scheme retains good qualitative robustness to reasonable levels of prior misspecification. These findings provide empirical support that, under the UTIPHC framework with binomial removals, the posterior inference for the NH model remains stable even when the prior information is only approximately specified.




(a) Using Prior-1




(b) Using Prior-2

**Figure 1:** Posterior means of $\mu$ and $\phi$ under (**a**) Prior-1, and (b) Prior-2

To perform Bayesian estimation, we implemented the MCMC procedure discussed in Section 3.2. For each simulated dataset, we generated $\mathcal{M} = 12{,}000$ MCMC samples after discarding the initial $\mathcal{B} = 2000$ iterations as burn-in. Posterior summaries, including point estimates and 95% BCIs, are computed for all model parameters. Initial values for the MCMC chains are set to the corresponding MLEs derived from frequentist analysis. Optimization techniques are

essential to statistical modeling because they provide the mathematical machinery needed to estimate parameters that best explain observed data, even in complex or high-dimensional settings. For this purpose, we considered the `maxLik` package (Henningsen and Toomet [34]) streamlines likelihood estimation in R version 4.2.2, enabling flexible specification of likelihood functions and efficient optimization even in high-dimensional or non-standard models. Complementing this, the `coda` package (Plummer et al. [35]) provides robust diagnostics and output analysis for MCMC algorithms, ensuring that Bayesian estimators are based on well-converged posterior samples.

For each calculation setup, the average point estimate (APE) of $\vartheta$ (as an example) is determined as:

$$\text{APE} = \frac{1}{1000}\sum_{\varrho=1}^{1000}\grave{\vartheta}^{[\varrho]},$$

where $\grave{\vartheta}^{[\varrho]}$ represents the estimate of $\vartheta$ for $\varrho$th sample. To assess the precision of point estimates, the root mean squared error (RMSE) and average relative absolute bias (ARAB) are used as follows:

$$\text{RMSE}\left(\grave{\vartheta}\right) = \sqrt{\frac{1}{1000}\sum_{\varrho=1}^{1000}\left(\grave{\vartheta}^{[\varrho]} - \vartheta\right)^2},$$

and

$$\text{ARAB}\left(\grave{\vartheta}\right) = \frac{1}{1000}\sum_{\varrho=1}^{1000}\vartheta^{-1}\left|\grave{\vartheta}^{[\varrho]} - \vartheta\right|.$$

Furthermore, two key interval-based accuracy measures are considered, called average interval length (AIL) and coverage probability (CP) at a 95% confidence level, as follows:

$$\text{AIL}_{95\%}(\vartheta) = \frac{1}{1000}\sum_{\varrho=1}^{1000}\left(\mathcal{U}_{\grave{\vartheta}[\varrho]}(\vartheta) - \mathcal{L}_{\grave{\vartheta}[\varrho]}(\vartheta)\right),$$

and

$$\text{CP}_{95\%}(\vartheta) = \frac{1}{1000}\sum_{\varrho=1}^{1000}\Lambda_{\left(\mathcal{L}_{\grave{\vartheta}[\varrho]}(\vartheta),\mathcal{U}_{\grave{\vartheta}[\varrho]}(\vartheta)\right)}(\vartheta),$$

where $\Lambda$ denotes the indicator function, and $\mathcal{L}(\cdot)$ and $\mathcal{U}(\cdot)$ represent of 95% interval bounds.

All point simulation results for $\mu$, $\phi$, $R(t)$, $H(t)$, and $\vartheta$ are presented in Tables 1–10; specifically, Tables 1–5 display the APEs, RMSEs, and ARABs in the first, second, and third columns, respectively, whereas Tables 6–10 summarize AILs and CPs in the first and second columns, respectively.

**Table 1:** The point inferential results of $\mu$

| $n\left[\frac{k}{n}\%\right]$ | $n\left[\frac{m}{n}\%\right]$ | $\vartheta$ | MLE | | | MCMC | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Prior → | | | | | | 1 | | | 2 | | |
| | | | | | | $(T_1, T_2) = (0.5, 1)$ | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 2.091 | 2.512 | 1.791 | 1.655 | 0.853 | 0.350 | 1.189 | 0.453 | 0.254 |
| | | 0.6 | 2.051 | 2.129 | 1.667 | 1.564 | 0.543 | 0.299 | 1.592 | 0.386 | 0.206 |
| | | 0.9 | 1.209 | 1.934 | 1.523 | 1.348 | 0.504 | 0.260 | 1.522 | 0.324 | 0.178 |
| 20 [50%] | 20 [75%] | 0.3 | 1.972 | 1.814 | 1.046 | 1.695 | 0.438 | 0.239 | 1.606 | 0.262 | 0.146 |
| | | 0.6 | 1.250 | 1.681 | 0.873 | 1.444 | 0.423 | 0.233 | 1.516 | 0.245 | 0.128 |
| | | 0.9 | 1.452 | 1.217 | 0.603 | 1.351 | 0.410 | 0.221 | 1.715 | 0.222 | 0.113 |
| 40 [25%] | 40 [50%] | 0.3 | 2.175 | 2.424 | 1.727 | 2.175 | 0.596 | 0.324 | 1.318 | 0.431 | 0.241 |
| | | 0.6 | 2.063 | 2.098 | 1.620 | 1.321 | 0.535 | 0.288 | 1.169 | 0.347 | 0.198 |
| | | 0.9 | 1.699 | 1.931 | 1.347 | 1.477 | 0.471 | 0.244 | 1.374 | 0.282 | 0.165 |
| 40 [50%] | 40 [75%] | 0.3 | 1.341 | 1.764 | 1.014 | 1.405 | 0.433 | 0.239 | 1.480 | 0.255 | 0.142 |
| | | 0.6 | 1.283 | 1.568 | 0.872 | 1.706 | 0.418 | 0.230 | 1.364 | 0.240 | 0.125 |
| | | 0.9 | 1.643 | 1.211 | 0.587 | 1.591 | 0.396 | 0.219 | 1.337 | 0.212 | 0.109 |
| 80 [25%] | 80 [50%] | 0.3 | 2.235 | 2.393 | 1.681 | 1.763 | 0.552 | 0.310 | 1.213 | 0.420 | 0.224 |
| | | 0.6 | 1.870 | 2.046 | 1.595 | 1.221 | 0.511 | 0.271 | 1.211 | 0.334 | 0.180 |
| | | 0.9 | 1.731 | 1.841 | 1.329 | 1.316 | 0.442 | 0.241 | 1.283 | 0.275 | 0.153 |
| 80 [50%] | 80 [75%] | 0.3 | 1.654 | 1.759 | 0.952 | 1.330 | 0.430 | 0.238 | 1.506 | 0.254 | 0.132 |
| | | 0.6 | 1.588 | 1.288 | 0.837 | 1.591 | 0.416 | 0.226 | 1.430 | 0.230 | 0.118 |
| | | 0.9 | 1.816 | 1.074 | 0.579 | 1.672 | 0.358 | 0.186 | 1.602 | 0.187 | 0.095 |
| | | | | | | $(T_1, T_2) = (1, 2)$ | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 1.425 | 2.143 | 1.693 | 1.707 | 0.785 | 0.327 | 1.561 | 0.445 | 0.245 |
| | | 0.6 | 1.294 | 1.943 | 1.479 | 1.638 | 0.516 | 0.280 | 1.597 | 0.347 | 0.181 |
| | | 0.9 | 1.934 | 1.714 | 1.333 | 1.278 | 0.467 | 0.248 | 1.525 | 0.284 | 0.159 |
| 20 [50%] | 20 [75%] | 0.3 | 1.455 | 1.305 | 0.863 | 1.745 | 0.424 | 0.233 | 1.609 | 0.259 | 0.142 |
| | | 0.6 | 1.880 | 1.229 | 0.816 | 1.453 | 0.416 | 0.226 | 1.537 | 0.240 | 0.125 |
| | | 0.9 | 1.683 | 1.079 | 0.600 | 1.351 | 0.400 | 0.220 | 1.715 | 0.217 | 0.111 |
| 40 [25%] | 40 [50%] | 0.3 | 1.411 | 2.117 | 1.613 | 2.073 | 0.566 | 0.317 | 1.337 | 0.428 | 0.239 |
| | | 0.6 | 1.700 | 1.913 | 1.419 | 1.321 | 0.491 | 0.276 | 1.169 | 0.328 | 0.178 |
| | | 0.9 | 1.651 | 1.624 | 1.243 | 1.407 | 0.458 | 0.241 | 1.337 | 0.272 | 0.155 |
| 40 [50%] | 40 [75%] | 0.3 | 1.553 | 1.240 | 0.860 | 1.494 | 0.422 | 0.233 | 1.495 | 0.254 | 0.132 |
| | | 0.6 | 1.317 | 1.193 | 0.805 | 1.660 | 0.407 | 0.224 | 1.353 | 0.237 | 0.120 |
| | | 0.9 | 1.433 | 1.064 | 0.583 | 1.652 | 0.367 | 0.202 | 1.341 | 0.201 | 0.107 |
| 80 [25%] | 80 [50%] | 0.3 | 1.270 | 2.041 | 1.539 | 1.747 | 0.535 | 0.303 | 1.224 | 0.376 | 0.199 |
| | | 0.6 | 1.814 | 1.757 | 1.364 | 1.270 | 0.484 | 0.262 | 1.199 | 0.326 | 0.176 |
| | | 0.9 | 1.851 | 1.498 | 1.168 | 1.273 | 0.439 | 0.235 | 1.284 | 0.271 | 0.144 |
| 80 [50%] | 80 [75%] | 0.3 | 1.532 | 1.233 | 0.847 | 1.330 | 0.418 | 0.230 | 1.506 | 0.244 | 0.127 |
| | | 0.6 | 1.283 | 1.153 | 0.749 | 1.675 | 0.403 | 0.221 | 1.404 | 0.220 | 0.113 |
| | | 0.9 | 1.489 | 0.900 | 0.569 | 1.672 | 0.316 | 0.176 | 1.602 | 0.177 | 0.091 |

**Table 2:** The point inferential results of $\phi$

| $n\left[\frac{k}{n}\%\right]$ | $n\left[\frac{m}{n}\%\right]$ | $\vartheta$ | MLE | | | MCMC | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Prior → | | | | | | 1 | | | 2 | | |
| $(T_1, T_2) = (0.5, 1)$ | | | | | | | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 0.696 | 0.511 | 0.404 | 0.674 | 0.210 | 0.306 | 0.560 | 0.166 | 0.257 |
| | | 0.6 | 0.748 | 0.637 | 0.651 | 0.483 | 0.376 | 0.538 | 0.540 | 0.302 | 0.411 |
| | | 0.9 | 0.580 | 0.457 | 0.339 | 0.454 | 0.159 | 0.235 | 0.401 | 0.139 | 0.224 |
| 20 [50%] | 20 [75%] | 0.3 | 0.585 | 0.388 | 0.225 | 0.692 | 0.149 | 0.211 | 0.668 | 0.134 | 0.163 |
| | | 0.6 | 0.609 | 0.207 | 0.209 | 0.456 | 0.123 | 0.186 | 0.429 | 0.116 | 0.154 |
| | | 0.9 | 0.552 | 0.190 | 0.189 | 0.548 | 0.109 | 0.145 | 0.479 | 0.093 | 0.111 |
| 40 [25%] | 40 [50%] | 0.3 | 0.621 | 0.619 | 0.609 | 0.603 | 0.285 | 0.401 | 0.681 | 0.214 | 0.321 |
| | | 0.6 | 0.621 | 0.495 | 0.402 | 0.452 | 0.197 | 0.263 | 0.445 | 0.163 | 0.250 |
| | | 0.9 | 0.796 | 0.432 | 0.333 | 0.525 | 0.157 | 0.233 | 0.546 | 0.137 | 0.211 |
| 40 [50%] | 40 [75%] | 0.3 | 0.908 | 0.383 | 0.224 | 0.624 | 0.137 | 0.198 | 0.559 | 0.121 | 0.160 |
| | | 0.6 | 0.886 | 0.196 | 0.200 | 0.553 | 0.119 | 0.182 | 0.608 | 0.115 | 0.124 |
| | | 0.9 | 0.459 | 0.178 | 0.168 | 0.573 | 0.104 | 0.137 | 0.612 | 0.081 | 0.110 |
| 80 [25%] | 80[50%] | 0.3 | 0.726 | 0.520 | 0.472 | 0.453 | 0.255 | 0.332 | 0.548 | 0.207 | 0.299 |
| | | 0.6 | 0.666 | 0.475 | 0.366 | 0.731 | 0.167 | 0.249 | 0.634 | 0.153 | 0.243 |
| | | 0.9 | 0.743 | 0.428 | 0.235 | 0.512 | 0.154 | 0.225 | 0.506 | 0.136 | 0.207 |
| 80 [50%] | 80 [75%] | 0.3 | 0.788 | 0.301 | 0.224 | 0.571 | 0.130 | 0.189 | 0.492 | 0.120 | 0.159 |
| | | 0.6 | 0.860 | 0.191 | 0.199 | 0.571 | 0.118 | 0.170 | 0.564 | 0.111 | 0.122 |
| | | 0.9 | 0.603 | 0.177 | 0.154 | 0.438 | 0.098 | 0.107 | 0.440 | 0.067 | 0.107 |
| $(T_1, T_2) = (1, 2)$ | | | | | | | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 0.628 | 0.427 | 0.615 | 0.442 | 0.323 | 0.500 | 0.413 | 0.258 | 0.285 |
| | | 0.6 | 0.690 | 0.285 | 0.335 | 0.501 | 0.188 | 0.263 | 0.465 | 0.162 | 0.193 |
| | | 0.9 | 0.674 | 0.174 | 0.249 | 0.511 | 0.151 | 0.230 | 0.438 | 0.134 | 0.096 |
| 20 [50%] | 20 [75%] | 0.3 | 0.537 | 0.153 | 0.213 | 0.605 | 0.136 | 0.198 | 0.607 | 0.120 | 0.054 |
| | | 0.6 | 0.552 | 0.134 | 0.201 | 0.424 | 0.123 | 0.163 | 0.403 | 0.105 | 0.049 |
| | | 0.9 | 0.705 | 0.117 | 0.168 | 0.548 | 0.093 | 0.145 | 0.479 | 0.068 | 0.018 |
| 40 [25%] | 40 [50%] | 0.3 | 0.675 | 0.412 | 0.412 | 0.758 | 0.254 | 0.354 | 0.823 | 0.190 | 0.276 |
| | | 0.6 | 0.621 | 0.280 | 0.294 | 0.452 | 0.175 | 0.252 | 0.445 | 0.157 | 0.174 |
| | | 0.9 | 0.581 | 0.167 | 0.242 | 0.642 | 0.149 | 0.228 | 0.645 | 0.129 | 0.090 |
| 40 [50%] | 40 [75%] | 0.3 | 0.483 | 0.152 | 0.210 | 0.572 | 0.130 | 0.198 | 0.532 | 0.120 | 0.053 |
| | | 0.6 | 0.562 | 0.130 | 0.184 | 0.540 | 0.118 | 0.158 | 0.591 | 0.104 | 0.048 |
| | | 0.9 | 0.863 | 0.104 | 0.167 | 0.545 | 0.081 | 0.137 | 0.593 | 0.051 | 0.018 |
| 80 [25%] | 80 [50%] | 0.3 | 0.617 | 0.375 | 0.400 | 0.475 | 0.210 | 0.331 | 0.567 | 0.184 | 0.248 |
| | | 0.6 | 0.661 | 0.254 | 0.276 | 0.677 | 0.154 | 0.246 | 0.615 | 0.152 | 0.171 |
| | | 0.9 | 0.479 | 0.166 | 0.230 | 0.643 | 0.138 | 0.219 | 0.611 | 0.122 | 0.071 |
| 80 [50%] | 80 [75%] | 0.3 | 0.679 | 0.136 | 0.203 | 0.571 | 0.124 | 0.185 | 0.492 | 0.111 | 0.050 |
| | | 0.6 | 0.719 | 0.125 | 0.183 | 0.525 | 0.116 | 0.153 | 0.543 | 0.093 | 0.048 |
| | | 0.9 | 0.603 | 0.098 | 0.154 | 0.438 | 0.067 | 0.107 | 0.440 | 0.051 | 0.018 |

**Table 3:** The point inferential results of $R(t)$

| $n\left[\frac{k}{n}\%\right]$ | $n\left[\frac{m}{n}\%\right]$ | $\vartheta$ | MLE | | | MCMC | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Prior → | | | | | | 1 | | | 2 | | |
| $(T_1, T_2) = (0.5, 1)$ | | | | | | | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 0.917 | 0.067 | 0.073 | 0.929 | 0.046 | 0.042 | 0.942 | 0.036 | 0.033 |
| | | 0.6 | 0.925 | 0.048 | 0.036 | 0.908 | 0.026 | 0.028 | 0.918 | 0.021 | 0.020 |
| | | 0.9 | 0.922 | 0.045 | 0.031 | 0.945 | 0.023 | 0.020 | 0.944 | 0.019 | 0.017 |
| 20 [50%] | 20 [75%] | 0.3 | 0.929 | 0.040 | 0.025 | 0.897 | 0.021 | 0.018 | 0.903 | 0.016 | 0.014 |
| | | 0.6 | 0.928 | 0.029 | 0.023 | 0.941 | 0.017 | 0.015 | 0.940 | 0.014 | 0.012 |
| | | 0.9 | 0.929 | 0.024 | 0.020 | 0.932 | 0.013 | 0.012 | 0.925 | 0.012 | 0.010 |
| 40 [25%] | 40 [50%] | 0.3 | 0.916 | 0.056 | 0.040 | 0.886 | 0.039 | 0.039 | 0.918 | 0.027 | 0.030 |
| | | 0.6 | 0.926 | 0.047 | 0.034 | 0.948 | 0.025 | 0.023 | 0.953 | 0.020 | 0.018 |
| | | 0.9 | 0.922 | 0.042 | 0.031 | 0.929 | 0.022 | 0.019 | 0.930 | 0.018 | 0.017 |
| 40 [50%] | 40 [75%] | 0.3 | 0.929 | 0.038 | 0.025 | 0.922 | 0.019 | 0.017 | 0.924 | 0.015 | 0.013 |
| | | 0.6 | 0.926 | 0.028 | 0.022 | 0.917 | 0.017 | 0.014 | 0.923 | 0.014 | 0.012 |
| | | 0.9 | 0.930 | 0.024 | 0.018 | 0.919 | 0.011 | 0.010 | 0.924 | 0.011 | 0.009 |
| 80 [25%] | 80 [50%] | 0.3 | 0.916 | 0.052 | 0.037 | 0.928 | 0.029 | 0.033 | 0.939 | 0.024 | 0.028 |
| | | 0.6 | 0.925 | 0.046 | 0.034 | 0.921 | 0.024 | 0.022 | 0.931 | 0.019 | 0.017 |
| | | 0.9 | 0.923 | 0.041 | 0.028 | 0.939 | 0.021 | 0.019 | 0.939 | 0.017 | 0.016 |
| 80 [50%] | 80 [75%] | 0.3 | 0.929 | 0.030 | 0.024 | 0.932 | 0.018 | 0.016 | 0.932 | 0.015 | 0.013 |
| | | 0.6 | 0.929 | 0.028 | 0.021 | 0.920 | 0.016 | 0.013 | 0.925 | 0.013 | 0.011 |
| | | 0.9 | 0.930 | 0.021 | 0.017 | 0.934 | 0.010 | 0.009 | 0.935 | 0.010 | 0.008 |
| $(T_1, T_2) = (1, 2)$ | | | | | | | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 0.918 | 0.056 | 0.051 | 0.933 | 0.039 | 0.037 | 0.941 | 0.033 | 0.031 |
| | | 0.6 | 0.923 | 0.043 | 0.034 | 0.927 | 0.024 | 0.021 | 0.932 | 0.019 | 0.020 |
| | | 0.9 | 0.922 | 0.034 | 0.030 | 0.941 | 0.021 | 0.017 | 0.939 | 0.016 | 0.015 |
| 20 [50%] | 20 [75%] | 0.3 | 0.927 | 0.030 | 0.025 | 0.907 | 0.018 | 0.016 | 0.911 | 0.015 | 0.013 |
| | | 0.6 | 0.927 | 0.026 | 0.022 | 0.945 | 0.016 | 0.013 | 0.943 | 0.012 | 0.011 |
| | | 0.9 | 0.928 | 0.021 | 0.018 | 0.932 | 0.013 | 0.012 | 0.925 | 0.011 | 0.010 |
| 40 [25%] | 40 [50%] | 0.3 | 0.915 | 0.044 | 0.040 | 0.863 | 0.029 | 0.027 | 0.900 | 0.025 | 0.028 |
| | | 0.6 | 0.923 | 0.040 | 0.033 | 0.948 | 0.023 | 0.019 | 0.953 | 0.018 | 0.017 |
| | | 0.9 | 0.922 | 0.033 | 0.030 | 0.918 | 0.020 | 0.017 | 0.920 | 0.016 | 0.014 |
| 40 [50%] | 40 [75%] | 0.3 | 0.927 | 0.028 | 0.024 | 0.924 | 0.017 | 0.015 | 0.927 | 0.014 | 0.013 |
| | | 0.6 | 0.925 | 0.025 | 0.021 | 0.920 | 0.016 | 0.013 | 0.926 | 0.012 | 0.011 |
| | | 0.9 | 0.929 | 0.021 | 0.018 | 0.920 | 0.011 | 0.009 | 0.926 | 0.010 | 0.009 |
| 80 [25%] | 80 [50%] | 0.3 | 0.915 | 0.044 | 0.036 | 0.925 | 0.025 | 0.022 | 0.936 | 0.023 | 0.024 |
| | | 0.6 | 0.922 | 0.035 | 0.032 | 0.925 | 0.022 | 0.018 | 0.933 | 0.016 | 0.015 |
| | | 0.9 | 0.923 | 0.031 | 0.027 | 0.927 | 0.018 | 0.017 | 0.927 | 0.015 | 0.014 |
| 80 [50%] | 80 [75%] | 0.3 | 0.928 | 0.026 | 0.024 | 0.932 | 0.017 | 0.014 | 0.932 | 0.013 | 0.012 |
| | | 0.6 | 0.927 | 0.022 | 0.020 | 0.923 | 0.015 | 0.013 | 0.929 | 0.011 | 0.010 |
| | | 0.9 | 0.929 | 0.020 | 0.016 | 0.934 | 0.010 | 0.009 | 0.935 | 0.010 | 0.008 |

**Table 4:** The point inferential results of $H(t)$

| $n\left[\frac{k}{n}\%\right]$ | $n\left[\frac{m}{n}\%\right]$ | $\vartheta$ | MLE | | | MCMC | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Prior → | | | | | | 1 | | | 2 | | |
| $(T_1, T_2) = (0.5, 1)$ | | | | | | | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 0.781 | 0.784 | 0.794 | 0.712 | 0.430 | 0.455 | 0.584 | 0.313 | 0.350 |
| | | 0.6 | 0.731 | 0.374 | 0.410 | 0.948 | 0.269 | 0.313 | 0.827 | 0.226 | 0.233 |
| | | 0.9 | 0.748 | 0.319 | 0.353 | 0.543 | 0.240 | 0.281 | 0.556 | 0.194 | 0.196 |
| 20 [50%] | 20 [75%] | 0.3 | 0.702 | 0.270 | 0.306 | 1.067 | 0.216 | 0.246 | 1.003 | 0.176 | 0.182 |
| | | 0.6 | 0.708 | 0.234 | 0.267 | 0.581 | 0.184 | 0.206 | 0.592 | 0.153 | 0.150 |
| | | 0.9 | 0.706 | 0.196 | 0.224 | 0.680 | 0.139 | 0.158 | 0.747 | 0.125 | 0.134 |
| 40 [25%] | 40 [50%] | 0.3 | 0.773 | 0.432 | 0.535 | 1.179 | 0.382 | 0.420 | 0.837 | 0.263 | 0.329 |
| | | 0.6 | 0.723 | 0.368 | 0.407 | 0.517 | 0.267 | 0.302 | 0.471 | 0.205 | 0.226 |
| | | 0.9 | 0.745 | 0.312 | 0.346 | 0.716 | 0.226 | 0.263 | 0.705 | 0.192 | 0.194 |
| 40 [50%] | 40 [75%] | 0.3 | 0.707 | 0.263 | 0.298 | 0.794 | 0.197 | 0.231 | 0.766 | 0.161 | 0.169 |
| | | 0.6 | 0.724 | 0.229 | 0.263 | 0.834 | 0.176 | 0.203 | 0.782 | 0.151 | 0.145 |
| | | 0.9 | 0.701 | 0.192 | 0.217 | 0.823 | 0.121 | 0.131 | 0.770 | 0.117 | 0.131 |
| 80 [25%] | 80 [50%] | 0.3 | 0.781 | 0.381 | 0.459 | 0.718 | 0.367 | 0.400 | 0.615 | 0.254 | 0.312 |
| | | 0.6 | 0.729 | 0.367 | 0.401 | 0.814 | 0.250 | 0.284 | 0.706 | 0.198 | 0.210 |
| | | 0.9 | 0.738 | 0.285 | 0.318 | 0.607 | 0.218 | 0.259 | 0.610 | 0.183 | 0.184 |
| 80 [50%] | 80 [75%] | 0.3 | 0.700 | 0.262 | 0.298 | 0.685 | 0.196 | 0.219 | 0.679 | 0.160 | 0.155 |
| | | 0.6 | 0.700 | 0.215 | 0.242 | 0.808 | 0.163 | 0.198 | 0.758 | 0.139 | 0.139 |
| | | 0.9 | 0.700 | 0.190 | 0.215 | 0.649 | 0.110 | 0.123 | 0.646 | 0.110 | 0.123 |
| $(T_1, T_2) = (1, 2)$ | | | | | | | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 0.780 | 0.616 | 0.645 | 0.665 | 0.417 | 0.450 | 0.588 | 0.287 | 0.415 |
| | | 0.6 | 0.749 | 0.367 | 0.398 | 0.732 | 0.245 | 0.286 | 0.681 | 0.211 | 0.255 |
| | | 0.9 | 0.752 | 0.314 | 0.344 | 0.586 | 0.212 | 0.242 | 0.610 | 0.168 | 0.231 |
| 20 [50%] | 20 [75%] | 0.3 | 0.716 | 0.267 | 0.302 | 0.945 | 0.183 | 0.217 | 0.904 | 0.157 | 0.197 |
| | | 0.6 | 0.718 | 0.228 | 0.257 | 0.547 | 0.165 | 0.184 | 0.562 | 0.131 | 0.178 |
| | | 0.9 | 0.715 | 0.195 | 0.222 | 0.680 | 0.129 | 0.158 | 0.747 | 0.120 | 0.145 |
| 40 [25%] | 40 [50%] | 0.3 | 0.783 | 0.396 | 0.426 | 1.444 | 0.309 | 0.370 | 1.045 | 0.261 | 0.329 |
| | | 0.6 | 0.743 | 0.361 | 0.396 | 0.517 | 0.240 | 0.272 | 0.471 | 0.191 | 0.246 |
| | | 0.9 | 0.754 | 0.311 | 0.341 | 0.837 | 0.209 | 0.238 | 0.821 | 0.168 | 0.230 |
| 40 [50%] | 40 [75%] | 0.3 | 0.720 | 0.259 | 0.293 | 0.766 | 0.182 | 0.199 | 0.734 | 0.150 | 0.184 |
| | | 0.6 | 0.735 | 0.227 | 0.256 | 0.800 | 0.164 | 0.178 | 0.752 | 0.125 | 0.170 |
| | | 0.9 | 0.708 | 0.190 | 0.216 | 0.808 | 0.114 | 0.129 | 0.747 | 0.110 | 0.129 |
| 80 [25%] | 80 [50%] | 0.3 | 0.790 | 0.373 | 0.413 | 0.747 | 0.254 | 0.302 | 0.643 | 0.248 | 0.296 |
| | | 0.6 | 0.747 | 0.356 | 0.387 | 0.768 | 0.236 | 0.249 | 0.677 | 0.171 | 0.232 |
| | | 0.9 | 0.739 | 0.282 | 0.316 | 0.748 | 0.186 | 0.228 | 0.744 | 0.167 | 0.221 |
| 80 [50%] | 80 [75%] | 0.3 | 0.713 | 0.258 | 0.289 | 0.685 | 0.177 | 0.197 | 0.679 | 0.141 | 0.180 |
| | | 0.6 | 0.716 | 0.211 | 0.235 | 0.775 | 0.156 | 0.168 | 0.714 | 0.122 | 0.152 |
| | | 0.9 | 0.705 | 0.188 | 0.214 | 0.649 | 0.104 | 0.115 | 0.646 | 0.097 | 0.115 |

**Table 5:** The point inferential results of $\vartheta$

| $n\left[\frac{k}{n}\%\right]$ | $n\left[\frac{m}{n}\%\right]$ | $\vartheta$ | MLE | | | MCMC | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Prior → | | | | | | 1 | | | 2 | | |
| $(T_1, T_2) = (0.5, 1)$ | | | | | | | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 0.313 | 0.170 | 0.413 | 0.293 | 0.154 | 0.342 | 0.285 | 0.103 | 0.214 |
| | | 0.6 | 0.621 | 0.154 | 0.244 | 0.606 | 0.131 | 0.225 | 0.608 | 0.056 | 0.145 |
| | | 0.9 | 0.910 | 0.119 | 0.215 | 0.903 | 0.108 | 0.185 | 0.904 | 0.036 | 0.091 |
| 20 [50%] | 20 [75%] | 0.3 | 0.346 | 0.099 | 0.162 | 0.322 | 0.089 | 0.146 | 0.317 | 0.023 | 0.051 |
| | | 0.6 | 0.640 | 0.085 | 0.100 | 0.616 | 0.082 | 0.080 | 0.611 | 0.015 | 0.029 |
| | | 0.9 | 0.913 | 0.077 | 0.076 | 0.907 | 0.074 | 0.067 | 0.903 | 0.011 | 0.017 |
| 40[25%] | 40 [50%] | 0.3 | 0.307 | 0.166 | 0.330 | 0.302 | 0.145 | 0.293 | 0.298 | 0.087 | 0.192 |
| | | 0.6 | 0.606 | 0.128 | 0.230 | 0.612 | 0.121 | 0.208 | 0.604 | 0.051 | 0.109 |
| | | 0.9 | 0.905 | 0.113 | 0.209 | 0.904 | 0.105 | 0.178 | 0.900 | 0.034 | 0.089 |
| 40 [50%] | 40 [75%] | 0.3 | 0.326 | 0.098 | 0.143 | 0.319 | 0.085 | 0.134 | 0.316 | 0.021 | 0.038 |
| | | 0.6 | 0.620 | 0.085 | 0.088 | 0.613 | 0.080 | 0.076 | 0.604 | 0.014 | 0.024 |
| | | 0.9 | 0.907 | 0.063 | 0.070 | 0.900 | 0.060 | 0.057 | 0.902 | 0.011 | 0.014 |
| 80 [25%] | 80 [50%] | 0.3 | 0.305 | 0.159 | 0.325 | 0.306 | 0.139 | 0.283 | 0.304 | 0.066 | 0.148 |
| | | 0.6 | 0.607 | 0.124 | 0.222 | 0.606 | 0.111 | 0.191 | 0.602 | 0.044 | 0.101 |
| | | 0.9 | 0.903 | 0.111 | 0.165 | 0.904 | 0.093 | 0.158 | 0.903 | 0.027 | 0.056 |
| 80 [50%] | 80 [75%] | 0.3 | 0.310 | 0.092 | 0.114 | 0.308 | 0.083 | 0.110 | 0.310 | 0.015 | 0.034 |
| | | 0.6 | 0.613 | 0.084 | 0.087 | 0.608 | 0.076 | 0.073 | 0.606 | 0.013 | 0.020 |
| | | 0.9 | 0.905 | 0.062 | 0.057 | 0.906 | 0.057 | 0.035 | 0.900 | 0.009 | 0.009 |
| $(T_1, T_2) = (1, 2)$ | | | | | | | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 0.320 | 0.164 | 0.372 | 0.308 | 0.130 | 0.309 | 0.300 | 0.092 | 0.182 |
| | | 0.6 | 0.623 | 0.140 | 0.214 | 0.611 | 0.118 | 0.200 | 0.610 | 0.041 | 0.104 |
| | | 0.9 | 0.910 | 0.108 | 0.190 | 0.903 | 0.096 | 0.166 | 0.904 | 0.029 | 0.081 |
| 20 [50%] | 20 [75%] | 0.3 | 0.347 | 0.090 | 0.148 | 0.324 | 0.081 | 0.132 | 0.319 | 0.022 | 0.051 |
| | | 0.6 | 0.640 | 0.078 | 0.091 | 0.616 | 0.073 | 0.073 | 0.611 | 0.013 | 0.026 |
| | | 0.9 | 0.913 | 0.070 | 0.069 | 0.907 | 0.067 | 0.061 | 0.903 | 0.010 | 0.014 |
| 40 [25%] | 40 [50%] | 0.3 | 0.312 | 0.154 | 0.295 | 0.310 | 0.128 | 0.258 | 0.307 | 0.079 | 0.174 |
| | | 0.6 | 0.607 | 0.112 | 0.209 | 0.612 | 0.110 | 0.189 | 0.604 | 0.040 | 0.083 |
| | | 0.9 | 0.905 | 0.103 | 0.189 | 0.904 | 0.095 | 0.156 | 0.900 | 0.025 | 0.065 |
| 40 [50%] | 40 [75%] | 0.3 | 0.326 | 0.089 | 0.136 | 0.319 | 0.079 | 0.129 | 0.316 | 0.020 | 0.042 |
| | | 0.6 | 0.620 | 0.078 | 0.080 | 0.613 | 0.071 | 0.069 | 0.604 | 0.012 | 0.025 |
| | | 0.9 | 0.907 | 0.057 | 0.064 | 0.900 | 0.054 | 0.059 | 0.902 | 0.009 | 0.013 |
| 80 [25%] | 80 [50%] | 0.3 | 0.306 | 0.143 | 0.277 | 0.306 | 0.120 | 0.251 | 0.305 | 0.052 | 0.132 |
| | | 0.6 | 0.607 | 0.111 | 0.202 | 0.606 | 0.105 | 0.174 | 0.602 | 0.032 | 0.082 |
| | | 0.9 | 0.903 | 0.101 | 0.149 | 0.904 | 0.084 | 0.142 | 0.903 | 0.024 | 0.054 |
| 80 [50%] | 80 [75%] | 0.3 | 0.310 | 0.081 | 0.103 | 0.308 | 0.077 | 0.100 | 0.310 | 0.015 | 0.035 |
| | | 0.6 | 0.613 | 0.076 | 0.079 | 0.608 | 0.069 | 0.066 | 0.606 | 0.011 | 0.018 |
| | | 0.9 | 0.905 | 0.056 | 0.051 | 0.906 | 0.052 | 0.048 | 0.900 | 0.006 | 0.008 |

**Table 6:** The 95% interval estimation results of $\mu$

| $n\left[\frac{k}{n}\%\right]$ | $n\left[\frac{m}{n}\%\right]$ | $\vartheta$ | ACI | | BCI | | | |
|---|---|---|---|---|---|---|---|---|
| Prior → | | | | | 1 | | 2 | |
| | | | $(T_1, T_2) = (0.5, 1)$ | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 2.137 | 0.902 | 1.989 | 0.917 | 1.170 | 0.931 |
| | | 0.6 | 1.876 | 0.914 | 1.514 | 0.931 | 1.042 | 0.939 |
| | | 0.9 | 1.728 | 0.921 | 1.391 | 0.937 | 1.004 | 0.941 |
| 20 [50%] | 20 [75%] | 0.3 | 1.567 | 0.928 | 1.084 | 0.946 | 0.937 | 0.945 |
| | | 0.6 | 1.483 | 0.932 | 0.810 | 0.959 | 0.678 | 0.960 |
| | | 0.9 | 1.422 | 0.935 | 0.683 | 0.966 | 0.593 | 0.965 |
| 40 [25%] | 40 [50%] | 0.3 | 2.100 | 0.904 | 1.847 | 0.923 | 1.107 | 0.935 |
| | | 0.6 | 1.828 | 0.916 | 1.430 | 0.935 | 1.023 | 0.940 |
| | | 0.9 | 1.676 | 0.923 | 1.245 | 0.938 | 0.998 | 0.941 |
| 40 [50%] | 40 [75%] | 0.3 | 1.538 | 0.929 | 0.950 | 0.953 | 0.906 | 0.947 |
| | | 0.6 | 1.445 | 0.934 | 0.792 | 0.960 | 0.636 | 0.963 |
| | | 0.9 | 1.411 | 0.935 | 0.657 | 0.967 | 0.538 | 0.968 |
| 80 [25%] | 80 [50%] | 0.3 | 2.002 | 0.908 | 1.750 | 0.927 | 1.086 | 0.936 |
| | | 0.6 | 1.780 | 0.918 | 1.404 | 0.936 | 1.007 | 0.941 |
| | | 0.9 | 1.613 | 0.926 | 1.213 | 0.939 | 0.974 | 0.943 |
| 80 [50%] | 80 [75%] | 0.3 | 1.524 | 0.930 | 0.838 | 0.958 | 0.785 | 0.954 |
| | | 0.6 | 1.442 | 0.934 | 0.733 | 0.963 | 0.617 | 0.964 |
| | | 0.9 | 1.325 | 0.939 | 0.633 | 0.968 | 0.480 | 0.972 |
| | | | $(T_1, T_2) = (1, 2)$ | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 2.064 | 0.904 | 1.898 | 0.919 | 1.107 | 0.933 |
| | | 0.6 | 1.859 | 0.914 | 1.225 | 0.948 | 1.042 | 0.937 |
| | | 0.9 | 1.628 | 0.924 | 1.152 | 0.951 | 0.985 | 0.940 |
| 20 [50%] | 20 [75%] | 0.3 | 1.550 | 0.928 | 0.971 | 0.959 | 0.697 | 0.958 |
| | | 0.6 | 1.465 | 0.932 | 0.800 | 0.964 | 0.650 | 0.961 |
| | | 0.9 | 1.405 | 0.935 | 0.664 | 0.970 | 0.577 | 0.965 |
| 40 [25%] | 40 [50%] | 0.3 | 1.987 | 0.908 | 1.397 | 0.941 | 1.088 | 0.934 |
| | | 0.6 | 1.828 | 0.915 | 1.173 | 0.950 | 1.004 | 0.939 |
| | | 0.9 | 1.596 | 0.926 | 1.084 | 0.954 | 0.969 | 0.941 |
| 40 [50%] | 40 [75%] | 0.3 | 1.524 | 0.929 | 0.943 | 0.960 | 0.688 | 0.958 |
| | | 0.6 | 1.443 | 0.933 | 0.767 | 0.966 | 0.627 | 0.962 |
| | | 0.9 | 1.363 | 0.937 | 0.654 | 0.971 | 0.494 | 0.970 |
| 80 [25%] | 80 [50%] | 0.3 | 1.949 | 0.909 | 1.289 | 0.945 | 1.054 | 0.936 |
| | | 0.6 | 1.699 | 0.921 | 1.154 | 0.951 | 0.994 | 0.940 |
| | | 0.9 | 1.561 | 0.927 | 1.025 | 0.956 | 0.954 | 0.942 |
| 80 [50%] | 80 [75%] | 0.3 | 1.490 | 0.931 | 0.825 | 0.962 | 0.655 | 0.960 |
| | | 0.6 | 1.420 | 0.934 | 0.733 | 0.967 | 0.605 | 0.963 |
| | | 0.9 | 1.225 | 0.943 | 0.624 | 0.972 | 0.440 | 0.973 |

**Table 7:** The 95% interval estimation results of $\phi$

| $n\left[\frac{k}{n}\%\right]$ | $n\left[\frac{m}{n}\%\right]$ | $\vartheta$ | ACI | | BCI | | | |
|---|---|---|---|---|---|---|---|---|
| Prior → | | | | | 1 | | 2 | |
| | | | $(T_1, T_2) = (0.5, 1)$ | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 2.588 | 0.891 | 1.382 | 0.903 | 0.712 | 0.914 |
| | | 0.6 | 1.829 | 0.923 | 0.995 | 0.930 | 0.576 | 0.932 |
| | | 0.9 | 1.286 | 0.946 | 0.875 | 0.939 | 0.479 | 0.944 |
| 20 [50%] | 20 [75%] | 0.3 | 1.105 | 0.954 | 0.716 | 0.950 | 0.430 | 0.949 |
| | | 0.6 | 0.963 | 0.960 | 0.666 | 0.953 | 0.360 | 0.958 |
| | | 0.9 | 0.742 | 0.969 | 0.430 | 0.970 | 0.228 | 0.973 |
| 40 [25%] | 40 [50%] | 0.3 | 2.406 | 0.899 | 1.215 | 0.915 | 0.603 | 0.929 |
| | | 0.6 | 1.689 | 0.929 | 0.939 | 0.934 | 0.532 | 0.937 |
| | | 0.9 | 1.183 | 0.950 | 0.859 | 0.940 | 0.442 | 0.948 |
| 40 [50%] | 40 [75%] | 0.3 | 1.074 | 0.955 | 0.709 | 0.950 | 0.408 | 0.952 |
| | | 0.6 | 0.910 | 0.962 | 0.626 | 0.956 | 0.315 | 0.963 |
| | | 0.9 | 0.719 | 0.970 | 0.396 | 0.972 | 0.197 | 0.977 |
| 80 [25%] | 80 [50%] | 0.3 | 2.129 | 0.911 | 1.170 | 0.918 | 0.597 | 0.930 |
| | | 0.6 | 1.487 | 0.938 | 0.884 | 0.938 | 0.489 | 0.942 |
| | | 0.9 | 1.167 | 0.951 | 0.823 | 0.942 | 0.431 | 0.949 |
| 80 [50%] | 80 [75%] | 0.3 | 1.060 | 0.956 | 0.697 | 0.951 | 0.371 | 0.956 |
| | | 0.6 | 0.804 | 0.966 | 0.543 | 0.962 | 0.284 | 0.967 |
| | | 0.9 | 0.676 | 0.972 | 0.319 | 0.978 | 0.167 | 0.980 |
| | | | $(T_1, T_2) = (1, 2)$ | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 2.134 | 0.901 | 1.034 | 0.910 | 0.696 | 0.916 |
| | | 0.6 | 1.722 | 0.920 | 0.817 | 0.929 | 0.440 | 0.947 |
| | | 0.9 | 1.341 | 0.938 | 0.754 | 0.934 | 0.408 | 0.951 |
| 20 [50%] | 20 [75%] | 0.3 | 0.842 | 0.961 | 0.699 | 0.939 | 0.388 | 0.953 |
| | | 0.6 | 0.480 | 0.978 | 0.518 | 0.955 | 0.276 | 0.967 |
| | | 0.9 | 0.372 | 0.983 | 0.397 | 0.971 | 0.177 | 0.979 |
| 40 [25%] | 40 [50%] | 0.3 | 1.943 | 0.910 | 0.933 | 0.919 | 0.574 | 0.930 |
| | | 0.6 | 1.697 | 0.921 | 0.803 | 0.930 | 0.420 | 0.949 |
| | | 0.9 | 1.332 | 0.938 | 0.737 | 0.936 | 0.407 | 0.951 |
| 40 [50%] | 40 [75%] | 0.3 | 0.730 | 0.966 | 0.682 | 0.941 | 0.370 | 0.955 |
| | | 0.6 | 0.478 | 0.978 | 0.496 | 0.958 | 0.262 | 0.968 |
| | | 0.9 | 0.347 | 0.984 | 0.360 | 0.974 | 0.135 | 0.984 |
| 80 [25%] | 80 [50%] | 0.3 | 1.727 | 0.920 | 0.840 | 0.927 | 0.562 | 0.932 |
| | | 0.6 | 1.590 | 0.926 | 0.787 | 0.932 | 0.414 | 0.950 |
| | | 0.9 | 1.246 | 0.942 | 0.712 | 0.938 | 0.400 | 0.951 |
| 80 [50%] | 80 [75%] | 0.3 | 0.674 | 0.969 | 0.673 | 0.942 | 0.360 | 0.956 |
| | | 0.6 | 0.431 | 0.980 | 0.457 | 0.964 | 0.239 | 0.971 |
| | | 0.9 | 0.321 | 0.985 | 0.309 | 0.979 | 0.127 | 0.985 |

**Table 8:** The 95% interval estimation results of $R(t)$

| $n\left[\frac{k}{n}\%\right]$ | $n\left[\frac{m}{n}\%\right]$ | $\vartheta$ | ACI | | BCI | | | |
|---|---|---|---|---|---|---|---|---|
| Prior → | | | | | 1 | | 2 | |
| | | | $(T_1, T_2) = (0.5, 1)$ | | | | | |
| 20 [25%] | 20[50%] | 0.3 | 0.190 | 0.962 | 0.116 | 0.971 | 0.079 | 0.975 |
| | | 0.6 | 0.156 | 0.969 | 0.092 | 0.976 | 0.066 | 0.979 |
| | | 0.9 | 0.134 | 0.973 | 0.082 | 0.979 | 0.060 | 0.981 |
| 20 [50%] | 20[75%] | 0.3 | 0.119 | 0.976 | 0.062 | 0.981 | 0.056 | 0.983 |
| | | 0.6 | 0.100 | 0.980 | 0.057 | 0.985 | 0.046 | 0.986 |
| | | 0.9 | 0.088 | 0.982 | 0.043 | 0.987 | 0.039 | 0.988 |
| 40 [25%] | 40[50%] | 0.3 | 0.181 | 0.964 | 0.099 | 0.975 | 0.075 | 0.977 |
| | | 0.6 | 0.151 | 0.970 | 0.091 | 0.977 | 0.064 | 0.980 |
| | | 0.9 | 0.133 | 0.973 | 0.068 | 0.979 | 0.059 | 0.981 |
| 40 [50%] | 40 [75%] | 0.3 | 0.111 | 0.978 | 0.060 | 0.983 | 0.053 | 0.983 |
| | | 0.6 | 0.097 | 0.981 | 0.053 | 0.985 | 0.044 | 0.986 |
| | | 0.9 | 0.084 | 0.983 | 0.039 | 0.988 | 0.038 | 0.988 |
| 80 [25%] | 80 [50%] | 0.3 | 0.168 | 0.966 | 0.095 | 0.976 | 0.067 | 0.979 |
| | | 0.6 | 0.149 | 0.970 | 0.082 | 0.977 | 0.062 | 0.981 |
| | | 0.9 | 0.129 | 0.974 | 0.064 | 0.980 | 0.058 | 0.982 |
| 80 [50%] | 80 [75%] | 0.3 | 0.107 | 0.979 | 0.058 | 0.984 | 0.049 | 0.985 |
| | | 0.6 | 0.093 | 0.981 | 0.046 | 0.986 | 0.043 | 0.987 |
| | | 0.9 | 0.082 | 0.984 | 0.035 | 0.988 | 0.036 | 0.989 |
| | | | $(T_1, T_2) = (1, 2)$ | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 0.184 | 0.963 | 0.103 | 0.972 | 0.075 | 0.977 |
| | | 0.6 | 0.153 | 0.969 | 0.077 | 0.979 | 0.059 | 0.982 |
| | | 0.9 | 0.134 | 0.973 | 0.070 | 0.981 | 0.057 | 0.983 |
| 20 [50%] | 20 [75%] | 0.3 | 0.111 | 0.978 | 0.058 | 0.984 | 0.052 | 0.984 |
| | | 0.6 | 0.099 | 0.980 | 0.055 | 0.985 | 0.044 | 0.987 |
| | | 0.9 | 0.079 | 0.984 | 0.041 | 0.988 | 0.039 | 0.988 |
| 40 [25%] | 40 [50%] | 0.3 | 0.175 | 0.965 | 0.094 | 0.975 | 0.062 | 0.981 |
| | | 0.6 | 0.150 | 0.970 | 0.074 | 0.980 | 0.059 | 0.982 |
| | | 0.9 | 0.131 | 0.974 | 0.067 | 0.982 | 0.055 | 0.983 |
| 40 [50%] | 40 [75%] | 0.3 | 0.110 | 0.978 | 0.057 | 0.985 | 0.046 | 0.986 |
| | | 0.6 | 0.097 | 0.981 | 0.050 | 0.986 | 0.043 | 0.987 |
| | | 0.9 | 0.080 | 0.984 | 0.038 | 0.989 | 0.037 | 0.989 |
| 80 [25%] | 80 [50%] | 0.3 | 0.165 | 0.967 | 0.086 | 0.977 | 0.061 | 0.982 |
| | | 0.6 | 0.148 | 0.970 | 0.071 | 0.981 | 0.058 | 0.982 |
| | | 0.9 | 0.127 | 0.975 | 0.062 | 0.983 | 0.053 | 0.984 |
| 80 [50%] | 80 [75%] | 0.3 | 0.105 | 0.979 | 0.057 | 0.985 | 0.046 | 0.986 |
| | | 0.6 | 0.092 | 0.982 | 0.045 | 0.988 | 0.041 | 0.988 |
| | | 0.9 | 0.079 | 0.984 | 0.035 | 0.990 | 0.034 | 0.990 |

**Table 9:** The 95% interval estimation results of $H(t)$

| $n\left[\frac{k}{n}\%\right]$ | $n\left[\frac{m}{n}\%\right]$ | $\vartheta$ | ACI | | BCI | | | |
|---|---|---|---|---|---|---|---|---|
| Prior → | | | | | 1 | | 2 | |
| | $(T_1, T_2) = (0.5, 1)$ | | | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 1.562 | 0.908 | 1.237 | 0.918 | 0.896 | 0.927 |
| | | 0.6 | 1.333 | 0.922 | 0.987 | 0.936 | 0.716 | 0.942 |
| | | 0.9 | 1.160 | 0.932 | 0.862 | 0.944 | 0.651 | 0.947 |
| 20 [50%] | 20 [75%] | 0.3 | 1.001 | 0.941 | 0.634 | 0.953 | 0.577 | 0.958 |
| | | 0.6 | 0.876 | 0.948 | 0.583 | 0.961 | 0.481 | 0.961 |
| | | 0.9 | 0.743 | 0.956 | 0.432 | 0.967 | 0.399 | 0.971 |
| 40 [25%] | 40 [50%] | 0.3 | 1.487 | 0.913 | 1.117 | 0.926 | 0.826 | 0.933 |
| | | 0.6 | 1.312 | 0.923 | 0.969 | 0.937 | 0.675 | 0.945 |
| | | 0.9 | 1.145 | 0.933 | 0.718 | 0.949 | 0.623 | 0.952 |
| 40 [50%] | 40 [75%] | 0.3 | 0.993 | 0.942 | 0.622 | 0.953 | 0.573 | 0.959 |
| | | 0.6 | 0.869 | 0.949 | 0.547 | 0.963 | 0.452 | 0.964 |
| | | 0.9 | 0.714 | 0.958 | 0.399 | 0.969 | 0.375 | 0.972 |
| 80 [25%] | 80 [50%] | 0.3 | 1.414 | 0.917 | 1.012 | 0.933 | 0.726 | 0.941 |
| | | 0.6 | 1.296 | 0.924 | 0.902 | 0.942 | 0.664 | 0.946 |
| | | 0.9 | 1.100 | 0.935 | 0.678 | 0.950 | 0.616 | 0.955 |
| 80 [50%] | 80 [75%] | 0.3 | 0.970 | 0.943 | 0.605 | 0.957 | 0.522 | 0.960 |
| | | 0.6 | 0.832 | 0.951 | 0.478 | 0.963 | 0.448 | 0.968 |
| | | 0.9 | 0.686 | 0.960 | 0.361 | 0.972 | 0.346 | 0.972 |
| | | | | $(T_1, T_2) = (1, 2)$ | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 1.550 | 0.911 | 1.097 | 0.920 | 0.833 | 0.932 |
| | | 0.6 | 1.275 | 0.927 | 0.806 | 0.941 | 0.642 | 0.948 |
| | | 0.9 | 1.130 | 0.935 | 0.747 | 0.946 | 0.606 | 0.951 |
| 20 [50%] | 20 [75%] | 0.3 | 0.983 | 0.944 | 0.594 | 0.957 | 0.555 | 0.955 |
| | | 0.6 | 0.871 | 0.950 | 0.565 | 0.959 | 0.466 | 0.962 |
| | | 0.9 | 0.733 | 0.958 | 0.430 | 0.968 | 0.391 | 0.969 |
| 40 [25%] | 40 [50%] | 0.3 | 1.450 | 0.917 | 0.917 | 0.933 | 0.766 | 0.938 |
| | | 0.6 | 1.248 | 0.929 | 0.760 | 0.945 | 0.618 | 0.950 |
| | | 0.9 | 1.128 | 0.936 | 0.694 | 0.950 | 0.594 | 0.952 |
| 40 [50%] | 40 [75%] | 0.3 | 0.972 | 0.944 | 0.585 | 0.957 | 0.491 | 0.960 |
| | | 0.6 | 0.848 | 0.952 | 0.547 | 0.960 | 0.452 | 0.963 |
| | | 0.9 | 0.731 | 0.958 | 0.380 | 0.969 | 0.375 | 0.972 |
| 80 [25%] | 80 [50%] | 0.3 | 1.395 | 0.920 | 0.895 | 0.935 | 0.655 | 0.947 |
| | | 0.6 | 1.247 | 0.929 | 0.754 | 0.945 | 0.615 | 0.950 |
| | | 0.9 | 1.094 | 0.937 | 0.622 | 0.955 | 0.586 | 0.952 |
| 80 [50%] | 80 [75%] | 0.3 | 0.951 | 0.946 | 0.567 | 0.959 | 0.481 | 0.961 |
| | | 0.6 | 0.820 | 0.953 | 0.459 | 0.965 | 0.428 | 0.967 |
| | | 0.9 | 0.723 | 0.959 | 0.361 | 0.972 | 0.346 | 0.973 |

**Table 10:** The 95% interval estimation results of $\vartheta$

| $n\left[\frac{k}{n}\%\right]$ | $n\left[\frac{m}{n}\%\right]$ | $\vartheta$ | ACI | | BCI | | | |
|---|---|---|---|---|---|---|---|---|
| Prior → | | | | | 1 | | 2 | |
| | | | $(T_1, T_2) = (0.5, 1)$ | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 0.629 | 0.952 | 0.461 | 0.968 | 0.392 | 0.974 |
| | | 0.6 | 0.452 | 0.961 | 0.357 | 0.975 | 0.334 | 0.981 |
| | | 0.9 | 0.386 | 0.964 | 0.326 | 0.977 | 0.302 | 0.984 |
| 20 [50%] | 20 [75%] | 0.3 | 0.333 | 0.972 | 0.275 | 0.981 | 0.240 | 0.986 |
| | | 0.6 | 0.245 | 0.973 | 0.242 | 0.983 | 0.232 | 0.990 |
| | | 0.9 | 0.224 | 0.980 | 0.174 | 0.988 | 0.170 | 0.991 |
| 40 [25%] | 40 [50%] | 0.3 | 0.617 | 0.954 | 0.436 | 0.969 | 0.387 | 0.974 |
| | | 0.6 | 0.452 | 0.962 | 0.344 | 0.976 | 0.322 | 0.981 |
| | | 0.9 | 0.358 | 0.965 | 0.281 | 0.980 | 0.297 | 0.985 |
| 40 [50%] | 40 [75%] | 0.3 | 0.312 | 0.972 | 0.265 | 0.981 | 0.239 | 0.987 |
| | | 0.6 | 0.234 | 0.976 | 0.221 | 0.985 | 0.200 | 0.990 |
| | | 0.9 | 0.174 | 0.980 | 0.170 | 0.988 | 0.169 | 0.993 |
| 80 [25%] | 80 [50%] | 0.3 | 0.470 | 0.958 | 0.371 | 0.974 | 0.353 | 0.980 |
| | | 0.6 | 0.387 | 0.963 | 0.335 | 0.976 | 0.315 | 0.984 |
| | | 0.9 | 0.343 | 0.966 | 0.281 | 0.980 | 0.287 | 0.986 |
| 80 [50%] | 80 [75%] | 0.3 | 0.272 | 0.973 | 0.254 | 0.982 | 0.233 | 0.989 |
| | | 0.6 | 0.225 | 0.979 | 0.180 | 0.987 | 0.177 | 0.991 |
| | | 0.9 | 0.157 | 0.986 | 0.124 | 0.991 | 0.123 | 0.993 |
| | | | $(T_1, T_2) = (1, 2)$ | | | | | |
| 20 [25%] | 20 [50%] | 0.3 | 0.588 | 0.956 | 0.431 | 0.963 | 0.367 | 0.973 |
| | | 0.6 | 0.423 | 0.961 | 0.346 | 0.970 | 0.326 | 0.980 |
| | | 0.9 | 0.356 | 0.966 | 0.304 | 0.974 | 0.282 | 0.983 |
| 20 [50%] | 20 [75%] | 0.3 | 0.325 | 0.973 | 0.232 | 0.980 | 0.224 | 0.985 |
| | | 0.6 | 0.229 | 0.974 | 0.222 | 0.981 | 0.213 | 0.989 |
| | | 0.9 | 0.208 | 0.981 | 0.163 | 0.988 | 0.159 | 0.990 |
| 40 [25%] | 40 [50%] | 0.3 | 0.577 | 0.957 | 0.408 | 0.965 | 0.356 | 0.973 |
| | | 0.6 | 0.382 | 0.962 | 0.321 | 0.972 | 0.313 | 0.982 |
| | | 0.9 | 0.346 | 0.967 | 0.262 | 0.977 | 0.275 | 0.984 |
| 40 [50%] | 40 [75%] | 0.3 | 0.303 | 0.973 | 0.231 | 0.980 | 0.224 | 0.986 |
| | | 0.6 | 0.218 | 0.977 | 0.192 | 0.984 | 0.187 | 0.990 |
| | | 0.9 | 0.161 | 0.981 | 0.158 | 0.989 | 0.157 | 0.993 |
| 80 [25%] | 80 [50%] | 0.3 | 0.439 | 0.959 | 0.365 | 0.968 | 0.341 | 0.980 |
| | | 0.6 | 0.362 | 0.964 | 0.313 | 0.973 | 0.294 | 0.983 |
| | | 0.9 | 0.331 | 0.970 | 0.262 | 0.977 | 0.248 | 0.985 |
| 80 [50%] | 80 [75%] | 0.3 | 0.231 | 0.974 | 0.227 | 0.980 | 0.217 | 0.989 |
| | | 0.6 | 0.209 | 0.980 | 0.180 | 0.986 | 0.165 | 0.990 |
| | | 0.9 | 0.147 | 0.986 | 0.116 | 0.992 | 0.115 | 0.993 |

Based on minimizing RMSE, ARAB, and AIL values while maximizing CP yields the following key findings:

- The proposed inferential frameworks exhibit good performance in estimating $\mu$, $\phi$, $R(t)$, $H(t)$, and $\vartheta$.
- Increasing values of $n$, $k$, or $m$ contribute to improved precision in both point frequentist/Bayesian estimates and both asymptotic/credible interval estimates. This result thereby leads to more accurate and stable inferential outcomes.
- Bayes MCMC estimators and corresponding BCIs for $\mu$, $\phi$, $R(t)$, $H(t)$, and $\vartheta$ demonstrate superior performance over their frequentist counterparts, particularly due to the incorporation of informative gamma and beta prior knowledge.
- As $\vartheta$ increases, the following patterns are observed:
  - The RMSEs, ARABs, and AILs for $\mu$, $\phi$, $R(t)$, $H(t)$, or $\vartheta$ tend to decrease;
  - The CPs for $\mu$, $\phi$, $R(t)$, $H(t)$, or $\vartheta$ improve by grow.
- Since increasing thresholds $T_i$, $i = 1, 2$ offers more information about the parameters of interest in life-testing experiments, we noticed that
  - The RMSEs, ARABs, and AILs for $\mu$, $\phi$, $R(t)$, $H(t)$, or $\vartheta$ tend to decrease;
  - The CPs for $\mu$, $\phi$, $R(t)$, $H(t)$, or $\vartheta$ improve by grow.
- Across most given setups, the calculated CPs of both ACI/BCI results for $\mu$, $\phi$, $R(t)$, $H(t)$, or $\vartheta$ closely align with the nominal 95% confidence level.
- As a summary, the Bayesian approach, integrating the MCMC approach with independent gamma and beta priors, emerges as the preferred method for accurately analyzing component lifetimes from the NH population under the suggested binomial-removal-based censoring scheme.

## 5 Gastric Cancer Data Analysis

Gastric cancer patients who receive chemotherapy and radiation treatment often experience improved survival rates, especially when these therapies are combined either before or after surgery. Chemotherapy helps shrink tumors and target cancer cells systemically, while radiation focuses on destroying localized cancerous tissues. However, the treatments can lead to significant side effects such as fatigue, nausea, and reduced immune function, requiring careful management and supportive care. From the objective of the gastric cancer, this application analyzes survival data, considering the survival times (in years) of a group of gastric cancer patients given chemotherapy and radiation treatment. In Table 11, for calculation purposes, each survival time was multiplied by 100, and items marked with an asterisk represent patients who are still censored. This dataset was reported analyzed by Bekker et al. [36].

**Table 11:** Survival times (in years) of a group of 45 gastric cancer patients

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 4.7 | 11.5 | 12.1 | 13.2 | 16.4 | 19.7 | 20.3 | 26.0 | 28.2 | 29.6 |
| 33.4 | 39.5 | 45.8 | 46.6 | 50.1 | 50.7 | 52.9 | 53.4 | 54.0 | 57.0 |
| 64.1 | 64.4 | 69.6 | 84.1 | 86.3 | 109.9 | 121.9 | 127.1 | 132.6 | 144.7 |
| 148.5 | 153.3 | 158.1 | 158.9 | 217.8 | 234.3 | 241.6* | 244.4* | 282.5* | 283.0* |
| 357.8* | 365.8* | 374.3* | 397.8* | 403.3* | | | | | |

Note: *Refers to censored observations.

Before applying the theoretical estimation procedures, it is essential to assess the adequacy of the NH $(\mu, \phi)$ distribution to the gastric cancer dataset provided in Table 11. After discarding the lifetimes of patients still under observation, the MLEs (with their standard errors (Std.Errs)) of $\mu$ and $\phi$ are obtained first to get the Kolmogorov-Smirnov (KS) statistic (with its $p$-value). As a result, the MLE (Std.Err) of $\mu$ is 0.0027 (0.0007) and $\phi$ is 3.3658 (0.8223); subsequently, the KS ($p$-value) statistic becomes 0.0857 (0.9337). Since the computed $p$-value is much higher than the desired significance percentage ($\alpha = 5\%$), this fact confirms that the NH lifespan distribution fits the gastric cancer dataset satisfactorily.

Using five data representation tools in Fig. 2, namely: (i) fitted survival line; (ii) fitted probability-probability (PP) line; (iii) fitted quantile-quantile (QQ) line; (iv) scaled-TTT line; and (v) log-likelihood contour for $\mu$ and $\phi$ of the NH model are also provided to highlight the fitting results. The subplots in Fig. 2a–c indicate that the fitted SF, PP, and QQ lines of the proposed NH distribution captured the corresponding empirical lines adequately; Fig. 2d reveals that the fitted TTT transforms indicate that the gastric cancer dataset provides an increasing failure rate, which matches one of the theoretical NH failure rates; and Fig. 2e shows that the estimated MLEs $\hat{\mu} \simeq 0.0027$ and $\hat{\phi} \simeq 3.3658$ exist and are unique. Moreover, all visualization facts presented in Fig. 2 confirm the same fitting results obtained numerically. Consequently, we recommend utilizing these estimated values as starting points in subsequent computational iterations.

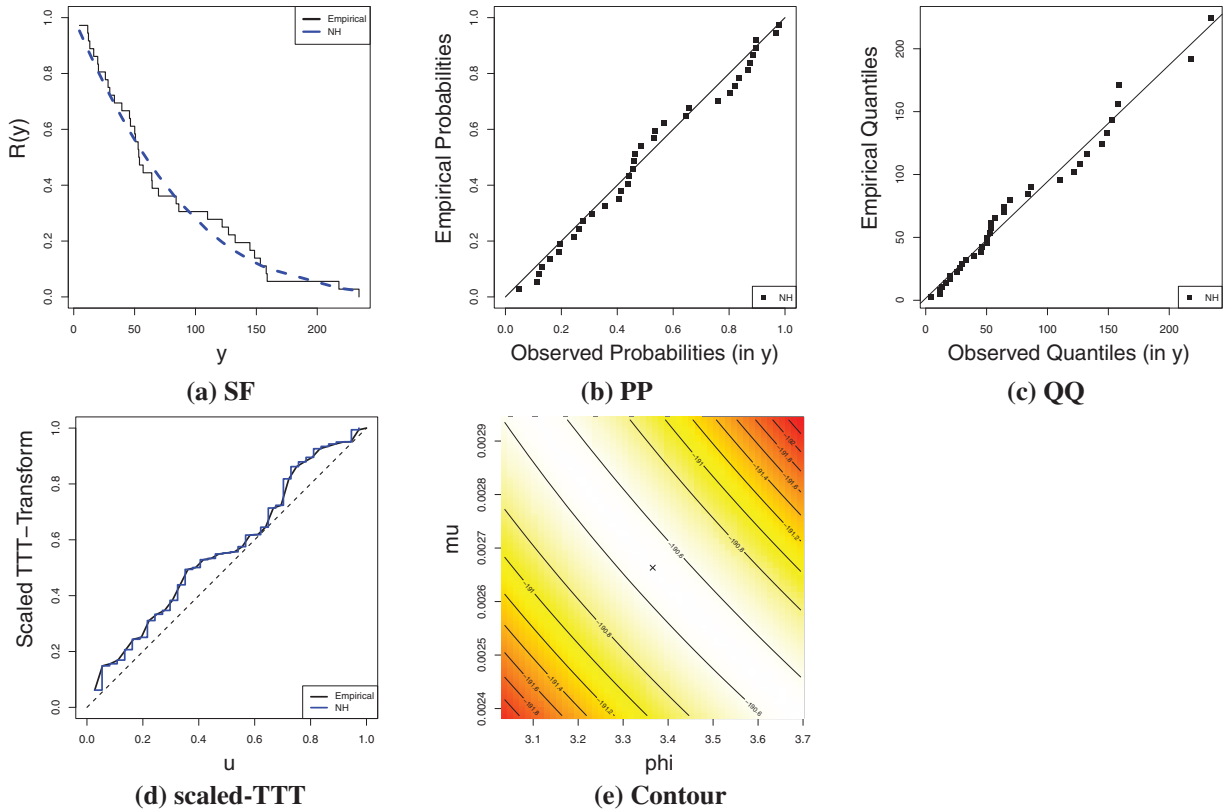



**Figure 2:** Five fitting diagrams for the NH model from gastric cancer data. (**a**) SF, (**b**) PP, (**c**) QQ, (**d**) scaled-TT, (**e**) Contour

Briefly, to show the utility of the proposed NH model, we compare its fit results with existing lifetime models featuring flexible and discrete failure rates, namely exponentiated-Weibull (EW ($\delta$, $\phi$, $\mu$)) by Mudholkar and Srivastava [37], Weibull-exponential (WE ($\delta$, $\phi$, $\mu$)) by Oguntunde et al. [38], alpha-power exponential (APE($\phi$, $\mu$)) by Mahdavi and Kundu [39], and log-normal (LN($\phi$, $\mu$)) by Crow and Shimizu [40] distributions. Besides the KS ($p$-value), this comparison is performed based on different criteria, including (i) Akaike information criterion (ACI), (ii) Bayesian information criterion (BIC), (iii) consistent AIC (CAIC), and (iv) Hannan–Quinn information criterion (HQIC); see Table 12. It shows that the NH model produces the smallest values of AIC, CAIC, BIC, HQIC, and KS, as well as the largest $p$-value; thus, the proposed NH model outperforms all competing alternatives, and it is the most suitable choice for the gastric cancer dataset.

**Table 12:** Fitting outcomes of the NH model and its competitors from gastric cancer data

| Model | $\delta$ | | $\phi$ | | $\mu$ | | NLL | AIC | CAIC | BIC | HQIC | KS ($p$-value) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Est. | Std.Err | Est. | Std.Err | Est. | Std.Err | | | | | | |
| NH | – | – | 3.3658 | 0.8223 | 0.0027 | 0.0007 | 190.71 | 385.42 | 385.78 | 388.59 | 386.53 | 0.0857 (0.9337) |
| EW | 1.0589 | 0.5465 | 0.0160 | 0.0124 | 1.4421 | 1.3590 | 190.93 | 386.26 | 387.01 | 391.01 | 387.92 | 0.0929 (0.8871) |
| WE | 0.0049 | 0.0022 | 1.9147 | 1.3321 | 1.0628 | 0.1665 | 190.84 | 386.88 | 387.63 | 391.63 | 388.53 | 0.1217 (0.6174) |
| APE | – | – | 5.5416 | 5.5399 | 0.0191 | 0.0041 | 190.71 | 385.43 | 385.79 | 388.59 | 386.53 | 0.0924 (0.8907) |
| LN | – | – | 3.9772 | 0.1532 | 0.9193 | 0.1083 | 191.23 | 386.46 | 386.82 | 389.63 | 387.57 | 0.1006 (0.8240) |

From the entire gastric cancer data (provided in Table 11), based on several configurations of $\vartheta$, $k$, $m$, and $T_i(q_i)$, $i = 1, 2$, four UTIPHC samples (symbolized by $\mathcal{S}[i]$, $i = 1, 2, 3, 4$) are created; see Table 13. For $\mathcal{S}[i]$, $i = 1, 2, 3, 4$, the point maximum likelihood and Bayes estimations (along with their Std.Errs) and 95% asymptotic and credible interval estimates (with their interval lengths (ILs)) of $\mu$, $\phi$, $R(t)$, $H(t)$ (at $t = 0.1$), and $\vartheta$ are calculated; see Table 14. Because we lacked the prior knowledge about the NH($\boldsymbol{\theta}$) parameters from the gastric cancer data, all Bayes MCMC and credible interval calculations are developed through the improper gamma and beta priors. To achieve these objectives, without loss of generality, we assign $\mathcal{M} = 40{,}000$ and $\mathcal{B} = 10{,}000$ to develop all acquired Bayes' inferences of $\mu, \phi, R(t), H(t)$, and $\vartheta$. In this part, the classical estimated value of each unknown subject (in Table 14) is considered as a starting point in Markov iterations. It is noted, from Table 14, that the estimated values of $\mu$, $\phi$, $R(t)$, $H(t)$, or $\vartheta$ obtained from the Bayes approach using the MCMC-based method behaved satisfactorily compared to those obtained from the likelihood approach in terms of the smallest estimated Std.Err values. Similar behavior, in terms of the smallest IL values, is also reached when comparing the credible intervals with the asymptotic intervals.

**Table 13:** Four UTIPHC samples from gastric cancer data

| Sample | | $(\vartheta, k, m) = (0.4, 5, 10)$, $\{T_1(q_1), T_2(q_2)\} = \{1.25(10), 1.35(10)\}$, and $(\tau, S^*) = (1.219, 0)$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{S}1$ | $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | $s_i$ | 11 | 6 | 4 | 2 | 2 | 1 | 0 | 0 | 0 | 0 |
| | $y_i$ | 0.047 | 0.132 | 0.203 | 0.296 | 0.395 | 0.458 | 0.534 | 0.641 | 0.863 | 1.219 |

(Continued)

**Table 13 (continued)**

| Sample | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $(\vartheta, k, m) = (0.4, 5, 10)$, $\{T_1(q_1), T_2(q_2)\} = \{1.25(10), 1.35(10)\}$, and $(\tau, S^*) = (1.219, 0)$ | | | | | | | | | |
| | | $(\vartheta, k, m) = (0.8, 5, 10)$, $\{T_1(q_1), T_2(q_2)\} = \{1.30(8), 1.35(8)\}$, and $(\tau, S^*) = (1.30, 2)$ | | | | | | | | | |
| $\mathcal{S}2$ | $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | $s_i$ | 23 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | | |
| | $y_i$ | 0.047 | 0.132 | 0.296 | 0.458 | 0.534 | 0.641 | 0.863 | 1.271 | | |
| | | $(\vartheta, k, m) = (0.4, 15, 20)$, $\{T_1(q_1), T_2(q_2)\} = \{1.25(11), 1.50(15)\}$, and $(\tau, S^*) = (1.485, 5)$ | | | | | | | | | |
| $\mathcal{S}3$ | $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | $s_i$ | 8 | 7 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $y_i$ | 0.047 | 0.121 | 0.132 | 0.260 | 0.395 | 0.466 | 0.507 | 0.529 | 0.534 | 0.696 |
| | $i$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| | $s_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $y_i$ | 0.863 | 1.271 | 1.326 | 1.447 | 1.485 | | | | | |
| | | $(\vartheta, k, m) = (0.8, 15, 20)$, $\{T_1(q_1), T_2(q_2)\} = \{0.85(12), 0.95(12)\}$, and $(\tau, S^*) = (0.85, 8)$ | | | | | | | | | |
| $\mathcal{S}4$ | $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | $s_i$ | 14 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $y_i$ | 0.047 | 0.132 | 0.164 | 0.260 | 0.334 | 0.395 | 0.458 | 0.507 | 0.641 | 0.696 |
| | $i$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| | $s_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $y_i$ | 0.841 | 1.099 | | | | | | | | |

**Table 14:** Estimates of $\mu$, $\phi$, $R(t)$, $H(t)$, and $\vartheta$ from gastric cancer data

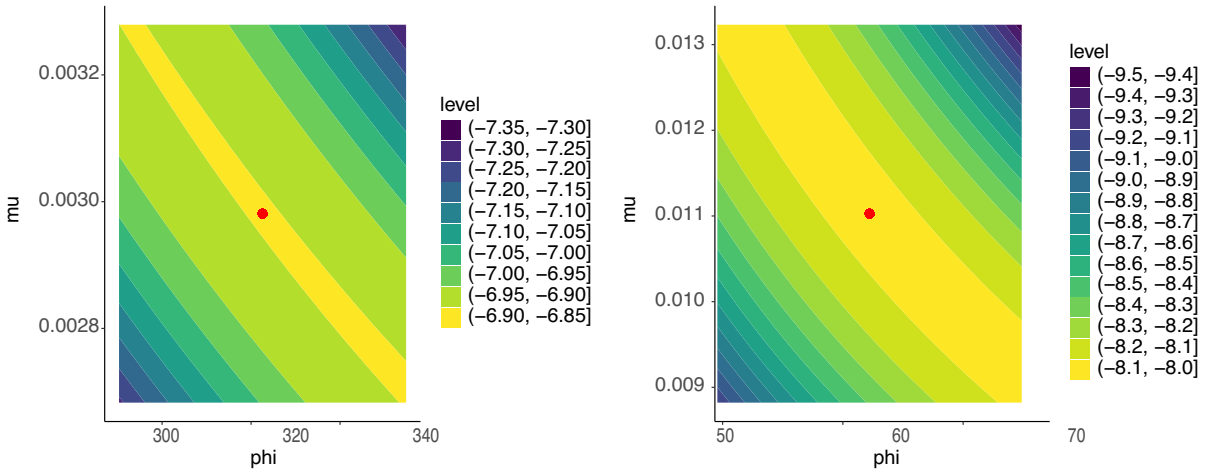
| Sample | Par. | MLE | | Bayes | | ACI | | | BCI | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Est. | Std.Err | Est. | Std.Err | Low. | Upp. | IL | Low. | Upp. | IL |
| $\mathcal{S}1$ | $\mu$ | 0.0030 | 0.0007 | 0.0029 | 0.0004 | 0.0016 | 0.0043 | 0.0027 | 0.0021 | 0.0037 | 0.0016 |
| | $\phi$ | 322.58 | 15.3257 | 322.58 | 0.0010 | 292.54 | 352.62 | 60.0757 | 322.58 | 322.58 | 0.0040 |
| | $R(t)$ | 0.9040 | 0.0219 | 0.9055 | 0.0131 | 0.8610 | 0.9470 | 0.0860 | 0.8798 | 0.9311 | 0.0513 |
| | $H(t)$ | 1.0584 | 0.2659 | 1.0421 | 0.1582 | 0.5372 | 1.5796 | 1.0424 | 0.7390 | 1.3591 | 0.6202 |
| | $\vartheta$ | 0.4407 | 0.0557 | 0.4406 | 0.0645 | 0.3315 | 0.5498 | 0.2183 | 0.3170 | 0.5697 | 0.2527 |
| $\mathcal{S}2$ | $\mu$ | 0.0110 | 0.0038 | 0.0110 | 0.0005 | 0.0037 | 0.0184 | 0.0147 | 0.0100 | 0.0120 | 0.0020 |
| | $\phi$ | 63.279 | 14.067 | 63.279 | 0.0010 | 35.708 | 90.851 | 55.143 | 63.277 | 63.281 | 0.0039 |
| | $R(t)$ | 0.9303 | 0.0179 | 0.9305 | 0.0031 | 0.8953 | 0.9653 | 0.0700 | 0.9243 | 0.9366 | 0.0123 |
| | $H(t)$ | 0.7474 | 0.2052 | 0.7458 | 0.0360 | 0.3451 | 1.1496 | 0.8044 | 0.6755 | 0.8169 | 0.1413 |

(Continued)

**Table 14 (continued)**

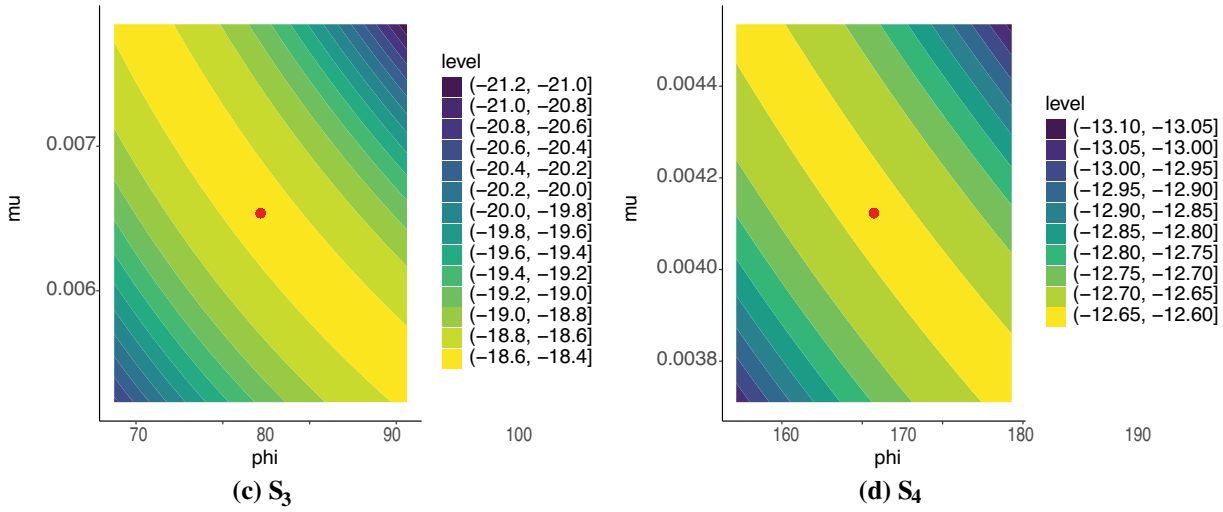
| Sample | Par. | MLE | | Bayes | | ACI | | | BCI | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Est. | Std.Err | Est. | Std.Err | Low. | Upp. | IL | Low. | Upp. | IL |
| | $\vartheta$ | 0.8667 | 0.0241 | 0.8641 | 0.0612 | 0.8194 | 0.9139 | 0.0945 | 0.7238 | 0.9592 | 0.2354 |
| $\mathcal{S}3$ | $\mu$ | 0.0065 | 0.0019 | 0.0065 | 0.0005 | 0.0029 | 0.0102 | 0.0073 | 0.0056 | 0.0074 | 0.0018 |
| | $\phi$ | 84.358 | 18.063 | 84.358 | 0.0010 | 48.956 | 119.76 | 70.804 | 84.356 | 84.360 | 0.0039 |
| | $R(t)$ | 0.9449 | 0.0103 | 0.9452 | 0.0039 | 0.9247 | 0.9651 | 0.0404 | 0.9374 | 0.9528 | 0.0154 |
| | $H(t)$ | 0.5823 | 0.1150 | 0.5794 | 0.0438 | 0.3568 | 0.8077 | 0.4509 | 0.4947 | 0.6670 | 0.1724 |
| | $\vartheta$ | 0.2941 | 0.0602 | 0.2952 | 0.0633 | 0.1761 | 0.4121 | 0.2361 | 0.1794 | 0.4263 | 0.2469 |
| $\mathcal{S}4$ | $\mu$ | 0.0041 | 0.0010 | 0.0041 | 0.0004 | 0.0021 | 0.0061 | 0.0040 | 0.0032 | 0.0050 | 0.0017 |
| | $\phi$ | 171.46 | 18.7178 | 171.46 | 0.0001 | 134.77 | 208.15 | 73.372 | 171.46 | 171.46 | 0.0004 |
| | $R(t)$ | 0.9294 | 0.0156 | 0.9301 | 0.0076 | 0.8988 | 0.9599 | 0.0611 | 0.9152 | 0.9447 | 0.0296 |
| | $H(t)$ | 0.7584 | 0.1794 | 0.7504 | 0.0868 | 0.4067 | 1.1100 | 0.7033 | 0.5842 | 0.9244 | 0.3402 |
| | $\vartheta$ | 0.8000 | 0.0436 | 0.7968 | 0.0872 | 0.7145 | 0.8855 | 0.1711 | 0.6015 | 0.9367 | 0.3352 |

Fig. 3 shows the log-likelihood contours for the NH($\boldsymbol{\theta}$), derived from the gastric cancer datasets $\mathcal{S}_i$, $i = 1, 2, 3, 4$, as outlined in Table 13. The subplots clearly corroborate the parameter estimates reported in Table 14 as well as demonstrate both the existence and uniqueness of the fitted MLEs of $\mu$, $\phi$, $R(t)$, $H(t)$, and $\vartheta$, providing strong empirical support for the adaptability of the proposed frequentist estimation procedure. To examine the convergence of the MCMC chains corresponding to $\mu$, $\phi$, $R(t)$, $H(t)$, and $\vartheta$, Gaussian kernel density estimates and trace plots are generated from the posterior samples based on $\mathcal{S}_1$ (as a representative case), using the last 30,000 iterations. These diagnostic plots, presented in Fig. 4, indicate satisfactory mixing and convergence of the MCMC chains. Moreover, the estimated marginal posterior estimates of $\mu$, $\phi$, $R(t)$, $H(t)$, or $\vartheta$ appear nearly symmetric.



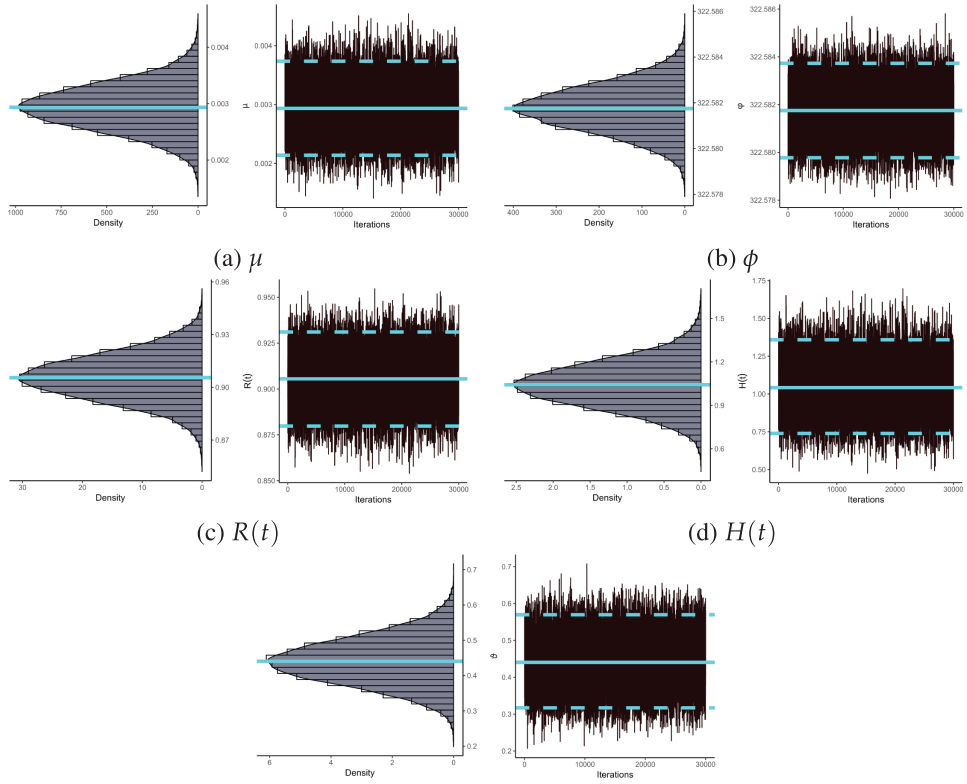

**(a) $S_1$** **(b) $S_2$**

**Figure 3:** (Continued)




**(c) $S_3$** **(d) $S_4$**

**Figure 3:** Contours of NH($\boldsymbol{\theta}$) parameters from gastric cancer data. (**a**) $\mathcal{S}_1$, (**b**) $\mathcal{S}_2$, (**c**) $\mathcal{S}_3$, (**d**) $\mathcal{S}_4$,




(a) $\mu$ (b) $\phi$

(c) $R(t)$ (d) $H(t)$

(e) $\vartheta$

**Figure 4:** The density and trace plots of (**a**) $\mu$, (**b**) $\phi$, (**c**) $R(t)$, (**d**) $H(t)$, and (**e**) $\vartheta$ from gastric cancer data

Overall, analyzing gastric cancer data helps identify trends, assess treatment outcomes, and inform future clinical decisions to improve patient care. Additionally, understanding survival patterns aids in

patient prognosis, resource allocation, and the development of more targeted treatment strategies in the presence of observed data gathered from the proposed censoring strategy. Furthermore, the findings advocate for the use of the Bayesian MCMC approach in analyzing such censored data, as it yields more reliable and efficient parameter estimates.

## 6 Concluding Remarks

This study introduced the unified Type-I progressive hybrid censoring framework with random binomial removal to address the limitations of fixed removal patterns, which may lack practical feasibility in survival analysis applications. Using this enhanced censoring scheme, we explored both classical and Bayesian estimation approaches for the Nadarajah–Haghighi distribution, which represents an important and extensively utilized model. The estimation procedures encompassed the model parameters, two key survival measures, and the binomial distribution parameter. We derived maximum likelihood estimators and asymptotic confidence intervals for all unknown parameters. Furthermore, to mitigate the challenges posed by limited data, Bayesian estimation methods were employed to derive both point estimates and Bayesian credible intervals. The Bayesian estimates were developed by sampling from the posterior distribution using Markov Chain Monte Carlo techniques. It should be noted that both classical estimators for the model parameters and Bayesian estimators cannot be obtained in closed form due to the complex structure of the normal equations and posterior distribution, respectively. Therefore, we performed an extensive simulation study across various experimental conditions to numerically evaluate and compare different estimators using specific performance measures. Posterior results show that the MCMC summaries remain stable under mild-to-moderate prior misspecification and retain strong qualitative robustness to reasonable levels of prior misspecification. From a practical standpoint, we also analyzed a genuine cancer application to illustrate the effectiveness and utility of the proposed estimation methods in both statistical estimation theory and survival analysis.

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**Availability of Data and Materials:** The data that support the findings of this study are available from the corresponding author, upon reasonable request.

**Ethics Approval:** Not applicable.

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