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D2Q9 MODEL OF UPWIND LATTICE BOLTZMANN SCHEME FOR HYPERBOLIC SCALAR CONSERVATION LAWS

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Abstract. A D2Q9 model of upwind lattice Boltzmann scheme that captures 45° discontinuities exactly is discussed in this paper.

1 INTRODUCTION

Lattice Boltzmann Method (LBM) is a popular kinetic theory based mesoscopic incompressible Navier-Stokes solver at low Mach number limit. Its popularity is due to the algorithmic simplicity involving streaming and collision operators. Development of LBM for compressible flow problems, with dominant hyperbolic convection terms, is non-trivial and is an active area of research. A novel LBM which is equivalent to an upwind method at macroscopic level is introduced in [1] for hyperbolic scalar conservation laws with and without stiff source terms. While the equivalence of that Lattice Boltzmann (LB) algorithm with macroscopic Computational Fluid Dynamics (CFD) algorithms is an advantage, it is based on a simple D2Q5-plus model. In this work, the upwind LBM in [1] is extended to D2Q9 model which results in an LBM that varies based on the way the total flux gets decomposed between coordinate and diagonal-to-coordinate (45° from coordinate) directions, thereby becoming the superset of D2Q5-plus and D2Q5-cross models. A test problem involving multi-directional discontinuities has been solved using the D2Q9 model of upwind LBM. It is observed that the scheme captures the 45° discontinuity exactly for a specific partition of total flux between coordinate and diagonal-to-coordinate directions, a remarkable feature of genuinely multi-dimensional modelling.

2 UPWIND LATTICE BOLTZMANN SCHEME

The two dimensional description of upwind LBM developed in [1] based on flux decomposed equilibrium distribution functions of [2], is as follows. The lattice Boltzmann equation (LBE),

$$f_n\left(x_1 + v_n^{(1)}\Delta t, x_2 + v_n^{(2)}\Delta t, t + \Delta t\right) = (1-\omega) f_n\left(x_1, x_2, t\right) + \omega f_n^{eq}\left(u\left(x_1, x_2, t\right)\right); n \in \{1,2,3,4,5\} \tag{1}$$

split-up into collision and streaming steps as,

$$\text{Collision:} \quad f_n^* = (1-\omega) f_n\left(x_1, x_2, t\right) + \omega f_n^{eq}\left(u\left(x_1, x_2, t\right)\right) \tag{2}$$

$$\text{Streaming:} \quad f_n\left(x_1 + v_n^{(1)}\Delta t, x_2 + v_n^{(2)}\Delta t, t + \Delta t\right) = f_n^* \tag{3}$$

guarantees exact streaming when $v_n^{(k)}$ is either 0 or $\frac{\Delta x_k}{\Delta t}, \forall k \in \{1, 2\}$. More specifically, the discrete velocities take the form,

$$v_1^{(1)} = \lambda \qquad v_1^{(2)} = 0 \tag{4}$$
$$v_2^{(1)} = 0 \qquad v_2^{(2)} = \lambda \tag{5}$$
$$v_3^{(1)} = 0 \qquad v_3^{(2)} = 0 \tag{6}$$
$$v_4^{(1)} = -\lambda \qquad v_4^{(2)} = 0 \tag{7}$$
$$v_5^{(1)} = 0 \qquad v_5^{(2)} = -\lambda \tag{8}$$

with $\lambda = \frac{\Delta x_1}{\Delta t} = \frac{\Delta x_2}{\Delta t}$ on a uniform structured lattice. The flux decomposed equilibrium distribution functions f_n^{eq} in (1) are:

$$f_1^{eq} = \frac{g_1^+}{\lambda} \tag{9}$$
$$f_2^{eq} = \frac{g_2^+}{\lambda} \tag{10}$$
$$f_3^{eq} = u - \frac{1}{\lambda}\left(\left(g_1^+ + g_2^+\right) + \left(g_1^- + g_2^-\right)\right) \tag{11}$$
$$f_4^{eq} = \frac{g_1^-}{\lambda} \tag{12}$$
$$f_5^{eq} = \frac{g_2^-}{\lambda} \tag{13}$$

Here, u, g_1 and g_2 are conserved variable and components of flux in x_1 and x_2 directions respectively, of the two-dimensional hyperbolic scalar conservation law,

$$\frac{\partial u}{\partial t} + \frac{\partial g_1(u)}{\partial x_1} + \frac{\partial g_2(u)}{\partial x_2} = 0 \tag{14}$$

with

$$g_k^+ = \begin{cases} g_k & \text{if } \partial_u g_k > 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } g_k^- = \begin{cases} -g_k & \text{if } \partial_u g_k < 0 \\ 0 & \text{otherwise} \end{cases}, \text{ for } k \in \{1, 2\} \tag{15}$$

These give $g_k^+ - g_k^- = g_k$ and $g_k^+ + g_k^- = |g_k|$ for $k \in \{1, 2\}$. In [1], the LBE (1) with relaxation factor ω such that $0 < \omega < 2$, conserved moment $\sum_{n=1}^{5} f_n = \sum_{n=1}^{5} f_n^{eq} = u$ and non-conserved moment $\sum_{n=1}^{5} v_n^{(k)} f_n^{eq} = g_k(u)$ for $k \in \{1, 2\}$, was shown to be equivalent to the upwind Engquist-Osher scheme [3] for (14) upto second order in time,

$$\begin{aligned} u(x_1, x_2, t + \Delta t) - u(x_1, x_2, t) = &- \frac{\Delta t}{\Delta x}\left(g_1^+(x_1, x_2, t) - g_1^+(x_1 - \lambda \Delta t, x_2, t)\right) \\ &- \frac{\Delta t}{\Delta x}\left(g_1^-(x_1 + \lambda \Delta t, x_2, t) - g_1^-(x_1, x_2, t)\right) \\ &- \frac{\Delta t}{\Delta x}\left(g_2^+(x_1, x_2, t) - g_2^+(x_1, x_2 - \lambda \Delta t, t)\right) \\ &- \frac{\Delta t}{\Delta x}\left(g_2^-(x_1, x_2 + \lambda \Delta t, t) - g_2^-(x_1, x_2, t)\right) \end{aligned} \tag{16}$$

where $\Delta x = \Delta x_1 = \Delta x_2$. In particular, for $\omega = 1$, LBE is equivalent, irrespective of the order in time, to the upwind Engquist-Osher scheme. However, this is limited to D2Q5-plus model as the flux decomposed equilibrium distribution functions involve splitting of components of flux vector along x_1, x_2 directions (or coordinate directions), as shown in figure 1a. In this paper, the upwind LBM is extended to D2Q5-cross and D2Q9 models by introducing novel modifications to flux decomposed equilibrium distribution functions.

3 D2Q5-CROSS MODEL OF UPWIND LBM

In two dimensions, any flux has two basis vectors. In terms of canonical bases, the flux is represented as $\vec{g} = g_1\vec{i} + g_2\vec{j}$, where $\vec{i}$ and $\vec{j}$ are unit vectors in x_1 and x_2 directions respectively (or coordinate directions). Further, the flux $\vec{g}$ can be written as a linear combination of any two independent basis vectors $\vec{\xi}$ and $\vec{\eta}$ as $\vec{g} = g_\xi\vec{\xi} + g_\eta\vec{\eta}$. Hence, this representation is not limited to canonical bases, and thereby, the upwind LBM can be easily extended to D2Q5-cross model with particles streaming along diagonal-to-coordinate (45° from the coordinate) directions by taking the basis vectors to be vectors that are inclined 45° from $\vec{i}$ and $\vec{j}$ vectors. The equilibrium distribution functions for D2Q5-cross model as shown in figure 1b are constructed as follows:

$$f_1^{eq} = \frac{g_\xi^+}{\lambda} \tag{17}$$

$$f_2^{eq} = \frac{g_\eta^+}{\lambda} \tag{18}$$

$$f_3^{eq} = u - \frac{1}{\lambda}\left(\left(g_\xi^+ + g_\eta^+\right) + \left(g_\xi^- + g_\eta^-\right)\right) \tag{19}$$

$$f_4^{eq} = \frac{g_\xi^-}{\lambda} \tag{20}$$

$$f_5^{eq} = \frac{g_\eta^-}{\lambda} \tag{21}$$

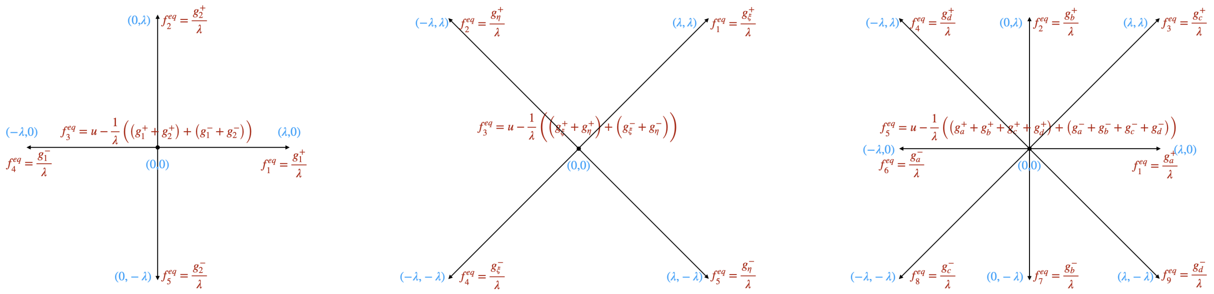
with

$$g_l^+ = \begin{cases} g_l & \text{if } \partial_u g_l > 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } g_l^- = \begin{cases} -g_l & \text{if } \partial_u g_l < 0 \\ 0 & \text{otherwise} \end{cases}, \text{ for } l \in \{\xi, \eta\} \tag{22}$$

These give $g_l^+ - g_l^- = g_l$ and $g_l^+ + g_l^- = |g_l|$, for $l \in \{\xi, \eta\}$. The equilibrium distribution functions satisfy the conserved moment relation, $\sum_{n=1}^{5} f_n = \sum_{n=1}^{5} f_n^{eq} = u$. The non-conserved moments become $\sum_{n=1}^{5} v_n^{(1)} f_n^{eq} = g_\xi - g_\eta$ and $\sum_{n=1}^{5} v_n^{(2)} f_n^{eq} = g_\xi + g_\eta$. In order to satisfy non-conserved moment relations, the following must be ensured: $g_\xi - g_\eta = g_1$ and $g_\xi + g_\eta = g_2$. Hence, $g_\xi = \frac{g_2+g_1}{2}$ and $g_\eta = \frac{g_2-g_1}{2}$. With these equilibrium distribution functions, LBE (1) is equivalent, upto second order in time, to macroscopic scheme with pure upwinding along diagonal-to-coordinate directions,

$$u(x_1,x_2,t+\Delta t)-u(x_1,x_2,t)=$$
$$-\frac{\Delta t}{\Delta x}\left(g_\xi^+(x_1,x_2,t)-g_\xi^+(x_1-\lambda\Delta t,x_2-\lambda\Delta t,t)\right)-\frac{\Delta t}{\Delta x}\left(g_\xi^-(x_1+\lambda\Delta t,x_2+\lambda\Delta t,t)-g_\xi^-(x_1,x_2,t)\right)$$
$$-\frac{\Delta t}{\Delta x}\left(g_\eta^+(x_1,x_2,t)-g_\eta^+(x_1+\lambda\Delta t,x_2-\lambda\Delta t,t)\right)-\frac{\Delta t}{\Delta x}\left(g_\eta^-(x_1-\lambda\Delta t,x_2+\lambda\Delta t,t)-g_\eta^-(x_1,x_2,t)\right) \tag{23}$$

where $\Delta x=\Delta x_1=\Delta x_2$. In particular, for $\omega=1$, LBE is equivalent, irrespective of the order in time, to the above macroscopic upwind scheme.



(a) D2Q5-plus model (b) D2Q5-cross model (c) D2Q9 model

Figure 1: Different models of upwind LBM (Text in red: equilibrium distribution functions; Text in blue: velocities corresponding to equilibrium distribution functions)

4 D2Q9 MODEL OF UPWIND LBM

As seen in the previous section, two dimensional flux has two basis vectors, and hence flux can be represented as a linear combination of any two basis vectors. However, in order to extend the scheme to D2Q9 model, it is required to represent the flux in terms of four different vectors ($\vec{i},\vec{j}, 45°$ anti-clockwise from $\vec{i}$, and $45°$ anti-clockwise from $\vec{j}$). At this point, it is worth noting that any number of vectors greater than two (extended from any basis set) can span the two dimensional flux space. Hence, flux can be written as a linear combination of any number of vectors greater than two (for example four, as $\vec{g}=g_a\vec{a}+g_b\vec{b}+g_c\vec{c}+g_d\vec{d}$). Splitting the fluxes into positive and negative components along each of these four directions as

$$g_l^+=\begin{cases} g_l & \text{if } \partial_u g_l>0 \\ 0 & \text{otherwise} \end{cases} \text{ and } g_l^-=\begin{cases} -g_l & \text{if } \partial_u g_l<0 \\ 0 & \text{otherwise} \end{cases}, \text{ for } l\in\{a,b,c,d\} \tag{24}$$

(which give $g_l^+ - g_l^- = g_l$ and $g_l^+ + g_l^- = |g_l|$ for $l \in \{a, b, c, d\}$), the equilibrium distribution functions as shown in figure 1c are constructed as,

$$f_1^{eq} = \frac{g_a^+}{\lambda} \tag{25}$$

$$f_2^{eq} = \frac{g_b^+}{\lambda} \tag{26}$$

$$f_3^{eq} = \frac{g_c^+}{\lambda} \tag{27}$$

$$f_4^{eq} = \frac{g_d^+}{\lambda} \tag{28}$$

$$f_5^{eq} = u - \frac{1}{\lambda}\left(\left(g_a^+ + g_b^+ + g_c^+ + g_d^+\right) + \left(g_a^- + g_b^- + g_c^- + g_d^-\right)\right) \tag{29}$$

$$f_6^{eq} = \frac{g_a^-}{\lambda} \tag{30}$$

$$f_7^{eq} = \frac{g_b^-}{\lambda} \tag{31}$$

$$f_8^{eq} = \frac{g_c^-}{\lambda} \tag{32}$$

$$f_9^{eq} = \frac{g_d^-}{\lambda} \tag{33}$$

The equilibrium distribution functions satisfy the conserved moment relation $\sum_{n=1}^{9} f_n = \sum_{n=1}^{9} f_n^{eq} = u$, and the non-conserved moments become $\sum_{n=1}^{9} v_n^{(1)} f_n^{eq} = g_a + g_c - g_d$ and $\sum_{n=1}^{9} v_n^{(2)} f_n^{eq} = g_b + g_c + g_d$. To satisfy the non-conserved moment relations, $g_a + g_c - g_d = g_1$ and $g_b + g_c + g_d = g_2$ must be ensured. Therefore,

$$g_c = \frac{g_2 + g_1}{2} - \frac{g_b + g_a}{2} \text{ and } g_d = \frac{g_2 - g_1}{2} - \frac{g_b - g_a}{2} \quad \forall g_a, g_b \in \mathbb{R} \tag{34}$$

With these equilibrium distribution functions, LBE (1) is equivalent, upto second order in time, to the macroscopic scheme with upwinding along both coordinate and diagonal-to-coordinate directions,

$$\begin{aligned}
&u(x_1, x_2, t + \Delta t) - u(x_1, x_2, t) = \\
&\quad -\frac{\Delta t}{\Delta x}\left(g_a^+(x_1, x_2, t) - g_a^+(x_1 - \lambda\Delta t, x_2, t)\right) - \frac{\Delta t}{\Delta x}\left(g_a^-(x_1 + \lambda\Delta t, x_2, t) - g_a^-(x_1, x_2, t)\right) \\
&\quad -\frac{\Delta t}{\Delta x}\left(g_b^+(x_1, x_2, t) - g_b^+(x_1, x_2 - \lambda\Delta t, t)\right) - \frac{\Delta t}{\Delta x}\left(g_b^-(x_1, x_2 + \lambda\Delta t, t) - g_b^-(x_1, x_2, t)\right) \\
&-\frac{\Delta t}{\Delta x}\left(g_c^+(x_1, x_2, t) - g_c^+(x_1 - \lambda\Delta t, x_2 - \lambda\Delta t, t)\right) - \frac{\Delta t}{\Delta x}\left(g_c^-(x_1 + \lambda\Delta t, x_2 + \lambda\Delta t, t) - g_c^-(x_1, x_2, t)\right) \\
&-\frac{\Delta t}{\Delta x}\left(g_d^+(x_1, x_2, t) - g_d^+(x_1 + \lambda\Delta t, x_2 - \lambda\Delta t, t)\right) - \frac{\Delta t}{\Delta x}\left(g_d^-(x_1 - \lambda\Delta t, x_2 + \lambda\Delta t, t) - g_d^-(x_1, x_2, t)\right)
\end{aligned} \tag{35}$$

with $\Delta x = \Delta x_1 = \Delta x_2$. In particular, for $\omega = 1$, LBE is equivalent, irrespective of the order in time, to the above macroscopic upwind scheme. If $g_a = g_1$ and $g_b = g_2$, then $g_c = g_d = 0$. This results in $g_c^+ = g_c^- = g_d^+ = g_d^- = 0$, and hence 35 becomes 16 resulting in a D2Q5-plus model. Similarly, if $g_a = 0$ and $g_b = 0$, then $g_c = \frac{g_2+g_1}{2}$ and $g_d = \frac{g_2-g_1}{2}$. This results in $g_a^+ = g_a^- = g_b^+ = g_b^- = 0, g_c = g_\xi$ and $g_d = g_\eta$, and hence 35 becomes 23 resulting in a D2Q5-cross model. Therefore, D2Q9 model is a superset of D2Q5-plus and D2Q5-cross models.

4.1 Numerical diffusion and stability

Chapman-Enskog analysis as in [1] gives the modified PDE,

$$\partial_t u + \sum_{k=1}^{2} \partial_{x_k} g_k(u) = \Delta t \left(\frac{1}{\omega} - \frac{1}{2}\right) \sum_{k=1}^{2} \partial_{x_k} \left(\sum_{i=1}^{2} \partial_{x_i} \left(\sum_{n=1}^{9} v_n^{(k)} v_n^{(i)} f_n^{eq} \right) - \partial_u g_k \left(\sum_{i=1}^{2} \partial_u g_i \partial_{x_i} u \right) \right) + O(\Delta t^2) \quad (36)$$

For stability of numerical scheme, it is required to have $0 < \omega < 2$ and positive-semidefiniteness of the following matrix consisting of numerical diffusion coefficients of (36).

$$\begin{bmatrix} \lambda\Big(\partial_u\left(|g_a| + |g_c| + |g_d|\right)\Big) - (\partial_u g_1)^2 & \lambda\partial_u\Big(|g_c| - |g_d|\Big) - \partial_u g_1 \partial_u g_2 \\ \lambda\partial_u\Big(|g_c| - |g_d|\Big) - \partial_u g_2 \partial_u g_1 & \lambda\Big(\partial_u\left(|g_b| + |g_c| + |g_d|\right)\Big) - (\partial_u g_2)^2 \end{bmatrix} \quad (37)$$

Therefore, upon ensuring positive-semidefiniteness of the matrix, following condition on λ is obtained for stability.

$$\lambda \geq \frac{\partial_u|g_c|\left(\partial_u g_1 - \partial_u g_2\right)^2 + \partial_u|g_d|\left(\partial_u g_1 + \partial_u g_2\right)^2 + \partial_u|g_a|\left(\partial_u g_2\right)^2 + \partial_u|g_b|\left(\partial_u g_1\right)^2}{\partial_u\left(|g_a| + |g_b|\right)\partial_u\left(|g_c| + |g_d|\right) + \partial_u|g_a|\partial_u|g_b| + 4\partial_u|g_c|\partial_u|g_d|} \quad (38)$$

4.2 Boundary conditions

At boundary, the macroscopic variables u, g_a, g_b, g_c and g_d are usually known through Dirichlet or extrapolation-from-inside boundary conditions. From these, the split fluxes $g_a^\pm, g_b^\pm, g_c^\pm$ and $g_d^\pm$ can be found. Using these split fluxes, equilibrium distribution functions can be evaluated at the boundary. Define $f_n^{neq} = f_n - f_n^{eq}$, $\forall n \in \{1, 2, .., 9\}$, and from the definition of conserved moment $\sum_{n=1}^{9} f_n = \sum_{n=1}^{9} f_n^{eq} = u$, it is inferred that $\sum_{n=1}^{9} f_n^{neq} = 0$.

4.2.1 Left boundary

At any point on left boundary, f_2, f_4, f_5, f_6, f_7 and f_8 are known from the computational domain (refer to 1c), as these particle distribution functions from neighbouring points (from inside) hop to points on left boundary. Let $\mathcal{I}$ be the set of these known particle distribution functions. The unknowns at left boundary are f_1, f_3 and f_9 (as shown in figure 2a), as these particle distributions must come from the outside of computational domain to left boundary. Let $\mathcal{J}$ be the set of these unknown particle distribution functions. Since f_n^{eq} can be evaluated $\forall n \in \{1, 2, .., 9\}$ and f_n is known $\forall n \in \mathcal{I}$, $f_n^{neq} = f_n - f_n^{eq}$ can be found $\forall n \in \mathcal{I}$ (as $\mathcal{I} \subset \{1, 2, .., 9\}$). Then $f_n^{neq}, \forall n \in \mathcal{J}$ can be written as,

$$f_3^{neq} = -f_8^{neq} - \frac{f_2^{neq} + f_5^{neq} + f_7^{neq}}{3} \tag{39}$$

$$f_1^{neq} = -f_6^{neq} - \frac{f_2^{neq} + f_5^{neq} + f_7^{neq}}{3} \tag{40}$$

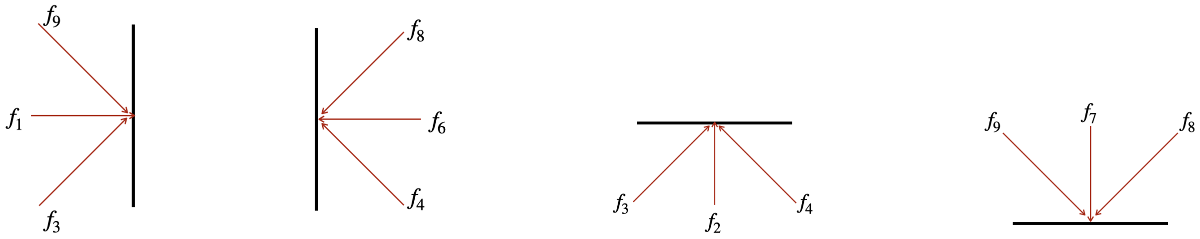
$$f_9^{neq} = -f_4^{neq} - \frac{f_2^{neq} + f_5^{neq} + f_7^{neq}}{3} \tag{41}$$

satisfying $\sum_{n=1}^{9} f_n^{neq} = 0$. Now, $f_n = f_n^{eq} + f_n^{neq}\ \forall n \in \mathcal{J}$ can be found to be,

$$f_3 = \frac{|g_c|}{\lambda} + \frac{u}{3} - \frac{|g_a| + |g_c| + |g_d|}{3\lambda} - f_8 - \frac{f_2 + f_5 + f_7}{3} \tag{42}$$

$$f_1 = \frac{|g_a|}{\lambda} + \frac{u}{3} - \frac{|g_a| + |g_c| + |g_d|}{3\lambda} - f_6 - \frac{f_2 + f_5 + f_7}{3} \tag{43}$$

$$f_9 = \frac{|g_d|}{\lambda} + \frac{u}{3} - \frac{|g_a| + |g_c| + |g_d|}{3\lambda} - f_4 - \frac{f_2 + f_5 + f_7}{3} \tag{44}$$



(a) Left (b) Right (c) Bottom (d) Top

Figure 2: Boundary conditions (Dark black lines indicate boundaries; red arrows indicate unknown distribution functions at each boundary)

4.2.2 Right boundary

By following a similar procedure, the unknown distribution functions at right boundary (as shown in figure 2b) can be found as,

$$f_4 = \frac{|g_d|}{\lambda} + \frac{u}{3} - \frac{|g_a| + |g_c| + |g_d|}{3\lambda} - f_9 - \frac{f_2 + f_5 + f_7}{3} \tag{45}$$

$$f_6 = \frac{|g_a|}{\lambda} + \frac{u}{3} - \frac{|g_a| + |g_c| + |g_d|}{3\lambda} - f_1 - \frac{f_2 + f_5 + f_7}{3} \tag{46}$$

$$f_8 = \frac{|g_c|}{\lambda} + \frac{u}{3} - \frac{|g_a| + |g_c| + |g_d|}{3\lambda} - f_3 - \frac{f_2 + f_5 + f_7}{3} \tag{47}$$

4.2.3 Bottom boundary

The unknown distribution functions at bottom boundary (as shown in figure 2c) can be found as,

$$f_3 = \frac{|g_c|}{\lambda} + \frac{u}{3} - \frac{|g_b| + |g_c| + |g_d|}{3\lambda} - f_8 - \frac{f_1 + f_5 + f_6}{3} \tag{48}$$

$$f_2 = \frac{|g_b|}{\lambda} + \frac{u}{3} - \frac{|g_b| + |g_c| + |g_d|}{3\lambda} - f_7 - \frac{f_1 + f_5 + f_6}{3} \tag{49}$$

$$f_4 = \frac{|g_d|}{\lambda} + \frac{u}{3} - \frac{|g_b| + |g_c| + |g_d|}{3\lambda} - f_9 - \frac{f_1 + f_5 + f_6}{3} \tag{50}$$

4.2.4 Top boundary

The unknown distribution functions at top boundary (as shown in figure 2d) can be found as,

$$f_9 = \frac{|g_d|}{\lambda} + \frac{u}{3} - \frac{|g_b| + |g_c| + |g_d|}{3\lambda} - f_4 - \frac{f_1 + f_5 + f_6}{3} \tag{51}$$

$$f_7 = \frac{|g_b|}{\lambda} + \frac{u}{3} - \frac{|g_b| + |g_c| + |g_d|}{3\lambda} - f_2 - \frac{f_1 + f_5 + f_6}{3} \tag{52}$$

$$f_8 = \frac{|g_c|}{\lambda} + \frac{u}{3} - \frac{|g_b| + |g_c| + |g_d|}{3\lambda} - f_3 - \frac{f_1 + f_5 + f_6}{3} \tag{53}$$

4.2.5 Bottom-left corner

At bottom left corner, the known equilibrium distribution functions are f_7, f_8, f_5 and f_6 (refer to 1c). The unknown equilibrium distribution functions are f_1, f_3, f_2, f_4 and f_9 (refer to 3a). Since f_4 and f_9 do not enter or leave the computational domain, evaluation of them is not needed. Hence, it can be assumed that $f_9^{neq} + f_4^{neq} + f_5^{neq} = 0$. Then f_n^{neq} for other unknown equilibrium distribution functions can be written as,

$$f_1^{neq} = -f_6^{neq} \tag{54}$$

$$f_3^{neq} = -f_8^{neq} \tag{55}$$

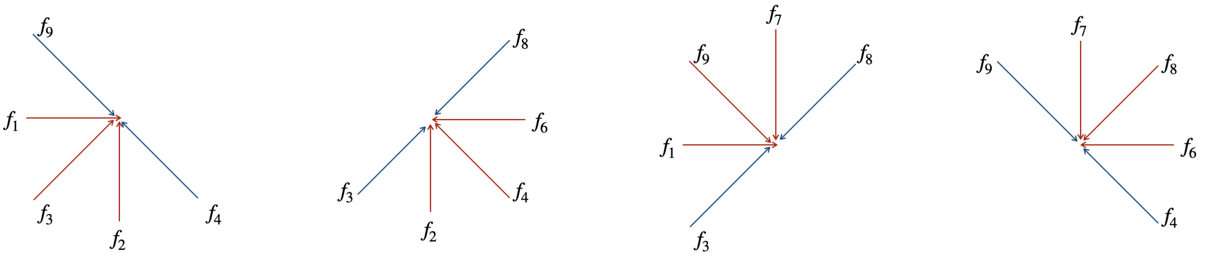
$$f_2^{neq} = -f_7^{neq} \tag{56}$$

satisfying $\sum_{n=1}^{9} f_n^{neq} = 0$. Now, $f_n = f_n^{eq} + f_n^{neq}$ can be found to be,

$$f_1 = \frac{|g_a|}{\lambda} - f_6 \tag{57}$$

$$f_3 = \frac{|g_c|}{\lambda} - f_8 \tag{58}$$

$$f_2 = \frac{|g_b|}{\lambda} - f_7 \tag{59}$$



(a) Bottom-left (b) Bottom-right (c) Top-left (d) Top-right

Figure 3: Corner conditions (Red arrows indicate unknown distribution functions that are evaluated; blue arrows indicate unknown distribution functions that are not evaluated)

4.2.6 Bottom-right corner

By following the similar procedure, the bottom-right corner conditions (as shown in figure 3b) are found to be,

$$f_2 = \frac{|g_b|}{\lambda} - f_7 \tag{60}$$

$$f_4 = \frac{|g_d|}{\lambda} - f_9 \tag{61}$$

$$f_6 = \frac{|g_a|}{\lambda} - f_1 \tag{62}$$

4.2.7 Top-left corner

The top-left corner conditions (as shown in figure 3c) are,

$$f_1 = \frac{|g_a|}{\lambda} - f_6 \tag{63}$$

$$f_9 = \frac{|g_d|}{\lambda} - f_4 \tag{64}$$

$$f_7 = \frac{|g_b|}{\lambda} - f_2 \tag{65}$$

4.2.8 Top-right corner

The top-right corner conditions (as shown in figure 3d) are,

$$f_6 = \frac{|g_a|}{\lambda} - f_1 \tag{66}$$

$$f_7 = \frac{|g_b|}{\lambda} - f_2 \tag{67}$$

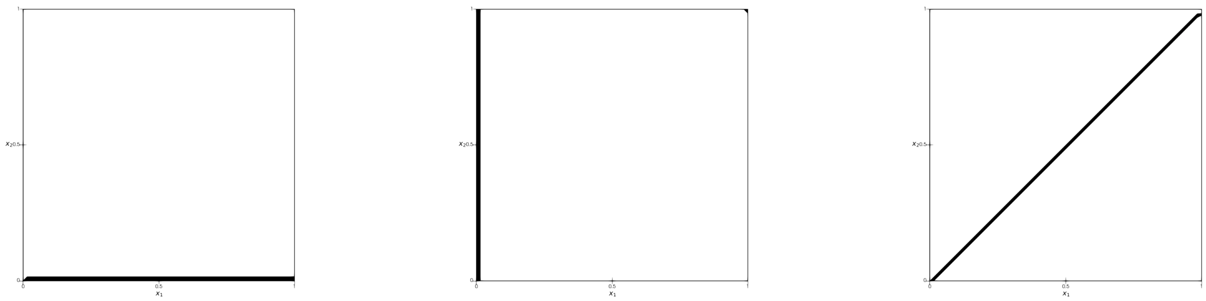
$$f_8 = \frac{|g_c|}{\lambda} - f_3 \tag{68}$$

4.3 Numerical results

The novel D2Q9 model for upwind LBM has been used to simulate a standard test problem from [4]. The problem is governed by

$$\partial_t u + \partial_{x_1} g_1(u) + \partial_{x_2} g_2(u) = 0 \text{ with } g_1(u) = au,\ g_2(u) = bu \tag{69}$$

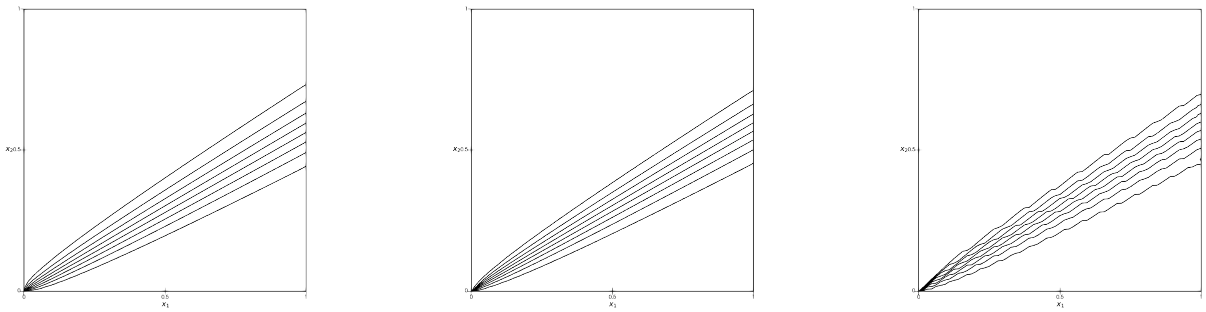
on the domain $[0,1] \times [0,1]$. Here, $a = cos\ \theta$, $b = sin\ \theta$ with $\theta \in \left(0, \frac{\pi}{2}\right)$. Boundary conditions of the steady-state problem are $u(0, x_2) = 1$ for $0 < x_2 < 1$ and $u(x_1, 0) = 0$ for $0 < x_1 < 1$. Exact solution of the steady-state problem is, $u(x_1, x_2) = 1$ for $bx_1 - ax_2 < 0$ and $u(x_1, x_2) = 0$ for $bx_1 - ax_2 > 0$. The numerical solutions for $\theta = 0$ and $\frac{\pi}{2}$ obtained by choosing the fluxes along diagonal-to-coordinate directions in D2Q9 model as 0 (*i.e.*, $g_c = g_d = 0$), thereby replicating a D2Q5-plus model, are shown in figures 4a and 4b respectively. The numerical solution for $\theta = \frac{\pi}{4}$ obtained by choosing the fluxes along coordinate directions in D2Q9 model as 0 (*i.e.*, $g_a = g_b = 0$), thereby replicating a D2Q5-cross model, is shown in figure 4c. It can be seen from these results that, for some specific partition of total flux between coordinate and diagonal-to-coordinate directions, the D2Q9 model captures discontinuities aligned with x_1, x_2 and diagonal directions exactly.



(a) 0° discontinuity (b) 90° discontinuity (c) 45° discontinuity

Figure 4: Discontinuities at different angles captured exactly

For discontinuities that are not aligned along x_1, x_2 and diagonal directions, partition of total flux between coordinate and diagonal-to-coordinate directions can be made by doing a linear interpolation (LI) between D2Q5-plus and D2Q5-cross models for the angle of discontinuity. Figures 5a, 5b and 5c show 30° discontinuity captured with D2Q5-plus, LI between D2Q5-plus and D2Q5-cross, and D2Q5-cross representations of D2Q9 model respectively. It can be seen that the linear interpolation model has minimal diffusion, as this model ensures that upwinding happens along 30° (upto the error due to LI).
Figures 6a, 6b and 6c show 15° discontinuity captured with D2Q5-plus, LI between D2Q5-plus and D2Q5-cross, and D2Q5-cross representations of D2Q9 model respectively. As expected, the D2Q5-cross representation of D2Q9 model gives the most diffusion, while the other two are similar.
All numerical results are obtained for $\omega = 1$, when LBE is exactly equivalent (irrespective of the order in time) to macroscopic upwind scheme.

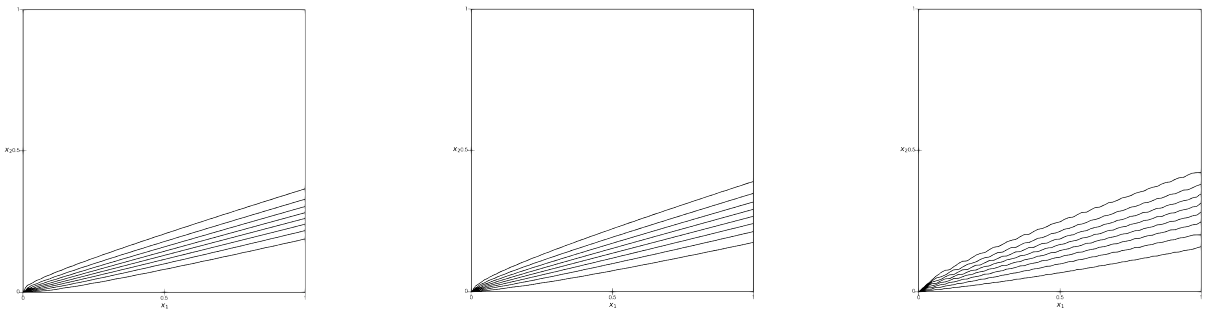


(a) 30° discontinuity with D2Q5-plus representation

(b) 30° discontinuity with LI

(c) 30° discontinuity with D2Q5-cross representation

Figure 5: 30° discontinuity with different representations of D2Q9 model



(a) 15° discontinuity with D2Q5-plus representation

(b) 15° discontinuity with LI

(c) 15° discontinuity with D2Q5-cross representation

Figure 6: 15° discontinuity with different representations of D2Q9 model

5 CONCLUSIONS

The novel D2Q9 model of upwind LBM for hyperbolic scalar conservation laws captures discontinuities aligned with x_1, x_2 and diagonal directions exactly, thereby making LBM competitive with genuinely multi-dimensional CFD algorithms. The numerical results shown are solved for a linear advection problem, while smooth flux splitting (in D2Q9 model) required for non-linear problems are being explored, and will be presented in future works.

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