Free vibration analysis of plates, bridges and axisymmetric shells using a thick finite strip method

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ABSTRACT

A unified approach for the vibration analysis of curved or straight prismatic plates and bridges and axisymmetric shells using a finite strip method based in Reissner–Mindlin shell theory is presented. Details of obtaining all relevant strip matrices and vectors are given. It is also shown how the use of the simple linear two node strip with reduced integration leads to direct explicit forms of all relevant matrices. Examples of application which show the accuracy of the linear strip for free vibration analysis of structures are presented.

INTRODUCTION

The finite strip method for the analysis of prismatic structures combines the use of Fourier expansions and one-dimensional finite elements to model the longitudinal and transverse behaviour of the structure, respectively. The finite strip method was initially developed by Cheung[1–5] and Loo and Cusens[6,7] who presented a wide range of solutions for static and dynamic analysis of plates and bridges using Kirchhoff's thin plate theory[8].

Applications of the finite strip method for the static and vibration analysis of thick plates using Reissner–Mindlin plate theory[9], have been reported by Benson and Hinton[10], Dawe[11] and Ronfacil and Dawe[12]. Extensions of the 'thick strip' theory for bridge deck analysis were carried out by Oñate[13]. More recently, Oñate and Suárez[14,15] have shown that the simple two node thick strip with a single point reduced integration is a valuable element for static analysis of a wide range of prismatic shell type structures.

The work presented in this paper can be considered as an extension of that presented by the authors[15]. It will be shown here that the free vibration analysis of bridges, plates and shells by the finite strip method can be treated in a uniform manner under the general framework of tronconoincal Reissner–Mindlin shell theory. Also, the advantages of using the simple linear strip element in order to obtain explicit forms of all the relevant matrices and vectors will be detailed. Finally, the accuracy of the linear strip for practical free vibration analysis of structures will be checked out with some examples of application.

Before going any further, the basic ideas of the finite strip method for free vibration analysis of structures will be briefly presented in the next section. More detailed information about the subject can be found in References 1 and 7.

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BASIC FINITE STRIP EQUATIONS

In the finite strip method the displacement field is first expressed in terms of a Fourier expansion along the longitudinal direction of the structure. For example, in a rectangular plate the displacement vector $\mathbf{u}$ is a function of the two coordinates of the point x, y and of time t. Thus, we can expand $\mathbf{u}$ in the longitudinal direction y as:

$$\mathbf{u}(x,y,t)=\sum_{l=0}^{n}\left(\bar{\mathbf{u}}^l(x,t)\sin\frac{l\pi}{b}y+\bar{\bar{\mathbf{u}}}^l(x,t)\cos\frac{l\pi}{b}y\right) \quad (1)$$

where $\bar{\mathbf{u}}^l(x,t)$ and $\bar{\bar{\mathbf{u}}}^l(x,t)$ are the amplitudes for the lth harmonic term. The trigonometric functions are chosen so that the boundary conditions at both ends of the structure are automatically satisfied.

The second step implies the finite element discretization along the transverse direction. Here the amplitudes are expressed in terms of its values at the strip nodes in a standard finite element form[16], i.e.

$$\begin{aligned}\bar{\mathbf{u}}^l(x,t)&=\sum_{i=1}^{k}\mathbf{N}_i(x)\bar{\mathbf{a}}_i^l(t)\\ \bar{\bar{\mathbf{u}}}^l(x,t)&=\sum_{i=1}^{k}\mathbf{N}_i(x)\bar{\bar{\mathbf{a}}}_i^l(t)\end{aligned} \quad (2)$$

where $\mathbf{N}_i(x)$ is the one-dimensional finite element shape function of node i, $\bar{\mathbf{a}}_i^l(t)$, $\bar{\bar{\mathbf{a}}}_i^l(t)$ are the amplitudes of node i for the lth harmonic term and k is the number of nodes of the strip. For static problems the nodal amplitudes are independent of time t.

Substituting (2) in (1) we get:

$$\begin{aligned}\mathbf{u}(x,y,t)&=\sum_{l=1}^{n}\sum_{i=1}^{k}\mathbf{N}_i(x)\left(\bar{\mathbf{a}}_i^l\sin\frac{l\pi}{b}y+\bar{\bar{\mathbf{a}}}_i^l\cos\frac{l\pi}{b}y\right)\\ &=\sum_{l=1}^{n}\sum_{i=1}^{k}\mathbf{N}_i^l\mathbf{a}_i^l\end{aligned} \quad (3)$$

Equation (3) allows us to express the generalized strains and resultant stresses in terms of the nodal amplitudes as:

$$\boldsymbol{\varepsilon}=\sum_{l=1}^{n}\sum_{i=1}^{k}\mathbf{B}_i^l\mathbf{a}_i^l \quad (4)$$

$$\boldsymbol{\sigma}=\mathbf{D}\sum_{l=1}^{n}\sum_{i=1}^{k}\mathbf{B}_i^l\mathbf{a}_i^l \quad (5)$$

where $\mathbf{B}_i^l$ is the strain matrix for node i and the lth harmonic term and $\mathbf{D}$ is the standard constitutive elasticity matrix[16].

The virtual work expressions for dynamic analysis can be written in the absence of external loads, as,

$$\iint_A \delta\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma}\,\mathrm{d}A=-\iint_A \delta\mathbf{u}^{\mathrm{T}}(\mathbf{P}\ddot{\mathbf{u}})\,\mathrm{d}A \quad (6)$$

where $-\mathbf{P}\ddot{\mathbf{u}}$ represent the inertia forces and A is the area of the midsurface of the plate. Differentiating (3) with respect to time we get:

$$\ddot{\mathbf{u}}=\sum_{l=1}^{n}\sum_{i=1}^{k}\mathbf{N}_i^l\ddot{\mathbf{a}}_i^l \quad (7)$$

where $\ddot{} = \partial^2/\partial t^2$. Also from (3) and (4) we have:

$$\delta\mathbf{u}=\sum_{l=1}^{n}\sum_{i=1}^{k}\mathbf{N}_i^l\delta\mathbf{a}_i^l;\qquad \delta\boldsymbol{\varepsilon}=\sum_{l=1}^{n}\sum_{i=1}^{k}\mathbf{B}_i^l\delta\mathbf{a}_i^l \quad (8)$$

Substituting (5), (7) and (8) in (6) and taking into account the orthogonality properties of the trigonometric

functions chosen, i.e.

$$\left.\begin{array}{l}\displaystyle\int_0^b \sin\frac{l\pi}{b}y\,\sin\frac{m\pi}{b}y\,\mathrm{d}y\\ \displaystyle\int_0^b \cos\frac{l\pi}{b}y\,\cos\frac{m\pi}{b}y\,\mathrm{d}y\end{array}\right\}=\begin{array}{ll}\dfrac{b}{2} & \text{if } l=m\\ 0 & \text{if } l\neq m\end{array} \tag{9}$$

we can easily get the following system of equations

$$\begin{bmatrix}\mathbf{K}^{11} & & & \mathbf{0}\\ & \mathbf{K}^{22} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{K}^{nn}\end{bmatrix}\begin{Bmatrix}\mathbf{a}^1\\ \mathbf{a}^2\\ \vdots\\ \mathbf{a}^n\end{Bmatrix}+\begin{bmatrix}\mathbf{M}^{11} & & & \mathbf{0}\\ & \mathbf{M}^{22} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{M}^{nn}\end{bmatrix}\begin{Bmatrix}\ddot{\mathbf{a}}^1\\ \ddot{\mathbf{a}}^2\\ \vdots\\ \ddot{\mathbf{a}}^n\end{Bmatrix}=\mathbf{0} \tag{10}$$

Details of matrices $\mathbf{K}^{ll}$ and $\mathbf{M}^{ll}$ will be given in the next sections.

Equation (10) is an uncoupled system of matrix equations which can be solved independently for each harmonic term. Thus for the lth harmonic we have:

$$\mathbf{K}^{ll}\mathbf{a}^l+\mathbf{M}^{ll}\ddot{\mathbf{a}}^l=\mathbf{0} \tag{11}$$

The solution of each uncoupled matrix equation is obtained using standard procedures of structural dynamics. Thus, the general solution of (11) is written as:

$$\mathbf{a}^l=\boldsymbol{\phi}^l e^{iw^l t} \tag{12}$$

where $e^{iw^l t}=\cos w^l t+i\sin w^l t$ and w^l and $\boldsymbol{\phi}^l$ are the natural frequency and vibration mode (eigenvector) for the lth harmonic term. Substituting (12) in (11) we obtain the classical eigenvalue equation:

$$[\mathbf{K}^{ll}-(w^l)^2\mathbf{M}^{ll}]\boldsymbol{\phi}^l=\mathbf{0} \tag{13}$$

The nontrivial solution of (13) implies:

$$|\mathbf{K}^{ll}-(w^l)^2\mathbf{M}^{ll}|=0 \tag{14}$$

from which the natural frequencies can be obtained. Note that the number of natural frequencies is equal to the rank of matrix $\mathbf{K}^{ll}$. Finally, the corresponding eigenvectors $\boldsymbol{\phi}^l$ for each frequency are obtained from (13).

In conclusion, the free vibration analysis of structures using the finite strip method implies the solution of n uncoupled eigenvalue problems, n being the number of harmonic terms used in the analysis. The natural frequencies and the corresponding vibration modes for each harmonic term are obtained from (13) and (12), respectively. In the next sections, the detailed expressions of all the matrices and vectors shown above are given for various structures.

CURVED BRIDGES AND PLATES: TRONCONICAL SHELL THEORY

We start here from the most general case of a curved bridge with circular plant (see *Figure 1*), formed by assembly of tronconical shell elements like that shown in *Figure 2*. It will be shown how the formulation for straight bridges and plates and axisymmetric shells can be considered as a particular case of the formulation presented here below.

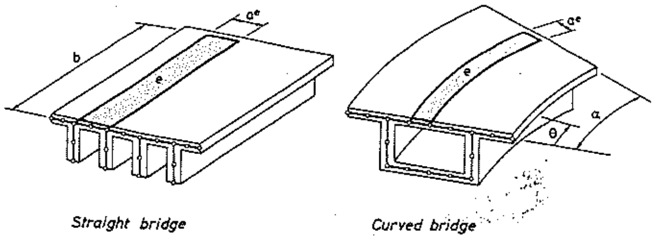

Figure 1 Finite strip discretizations for curved and straight bridges

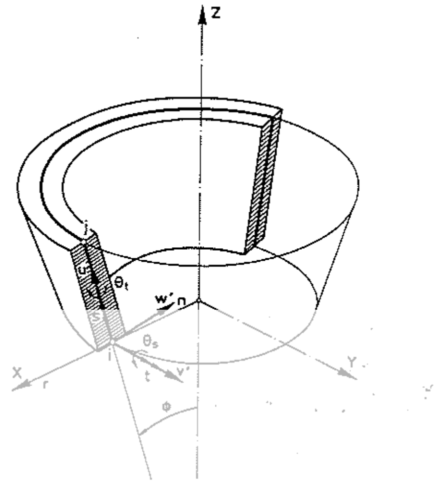

Figure 2 Sign convention for displacements in a tronconical shell

The finite strip formulation will be based in classical tronconical shell theory with Reissner–Mindlin assumptions. Details of that particular shell theory can be found in Reference 14.

Basic finite strip equations

For the finite strip analysis the bridge is divided into circular k noded strips as shown in *Figure 1*. The displacement field inside the strip e can be expressed, following the steps explained in the previous section, as products of polynomial one-dimensional shape functions in the transverse direction of the structure and Fourier expansions in the circular one. Thus, we can write

$$\mathbf{u}'=\sum_{l=1}^{n}\sum_{i=1}^{k}\mathbf{N}_i^l\mathbf{a}_i'^l \tag{15}$$

where

$$\mathbf{u}'=[u_0', v_0', w_0', \theta_s, \theta_t]^{\mathrm{T}} \tag{16a}$$

is the local displacement vector of a strip point with u_0', v_0', w_0' being the local displacements of the points over the midsurface of the strip and θ_s, θ_t the two local rotations of the midsurface normal. For sign convention see *Figure 2*.

$$\mathbf{a}_i'^l=[u_{0_i}'^l, v_{0_i}'^l, w_{0_i}'^l, \theta_{s_i}^l, \theta_{t_i}^l]^{\mathrm{T}} \tag{16b}$$

is the vector of local nodal displacement amplitudes for node i and the lth harmonic term, and

$$\mathbf{N}_i^l=\begin{bmatrix}N_iS_l & & & & \mathbf{0}\\ & N_iC_l & & & \\ & & N_iS_l & & \\ & & & N_iS_l & \\ \mathbf{0} & & & & N_iC_l\end{bmatrix} \tag{17}$$

is the generalized shape function matrix associated with node i for the lth harmonic term. In (17) $S_l=\sin(l\pi/\alpha)\theta$, $C_l=\cos(l\pi/\alpha)\theta$, and α is the bridge angle (see *Figure 1*).

It can easily be checked that the harmonic expansions chosen satisfy the conditions of simply supported strip for $\theta=0$ and $\theta=\alpha$. Thus, this formulation is valid for simply supported bridges with rigid diaphragms at the two ends. Other boundary conditions can be reproduced by an appropriate choice of the Fourier-like expansions of (17)[5]. However, not all the expansions chosen lead to the uncoupled system of equations of (10). The expansions chosen here are the most simple and useful in practice since they reproduce the boundary conditions of most real bridge and plate structures.

The generalized strain vector can be obtained from tronconical shell theory[14] as:

$$\boldsymbol{\varepsilon}=\begin{Bmatrix}\boldsymbol{\varepsilon}_m\\ \boldsymbol{\varepsilon}_b\\ \boldsymbol{\varepsilon}_s\end{Bmatrix} \tag{18a}$$

with

$$\boldsymbol{\varepsilon}_m=\begin{Bmatrix}\dfrac{\partial u_0'}{\partial s}\\ \dfrac{1}{r}\dfrac{\partial v_0'}{\partial\theta}+\dfrac{u_0'}{r}\sin\phi-\dfrac{w_0'}{r}\cos\phi\\ \dfrac{\partial v_0'}{\partial s}+\dfrac{1}{r}\dfrac{\partial u_0'}{\partial\theta}-\dfrac{v_0'}{r}\sin\phi\end{Bmatrix}$$

$$\boldsymbol{\varepsilon}_b=\begin{Bmatrix}\dfrac{\partial\theta_s}{\partial s}\\ \dfrac{1}{r}\dfrac{\partial\theta_t}{\partial\theta}+\dfrac{\theta_s}{r}\sin\phi\\ \dfrac{\partial\theta_t}{\partial s}+\dfrac{1}{r}\dfrac{\partial\theta_s}{\partial\theta}-\dfrac{\theta_t}{r}\sin\phi-\dfrac{\cos\phi}{r}\dfrac{\partial v_0'}{\partial s}\end{Bmatrix} \tag{18b}$$

$$\boldsymbol{\varepsilon}_s=\begin{Bmatrix}\theta_s+\dfrac{\partial w_0'}{\partial s}\\ \theta_t+\dfrac{1}{r}\dfrac{\partial w_0'}{\partial\theta}+\dfrac{v_0'}{r}\cos\varphi\end{Bmatrix}$$

where $\boldsymbol{\varepsilon}_m$, $\boldsymbol{\varepsilon}_b$ and $\boldsymbol{\varepsilon}_s$ are generalized membrane, bending and shear strain vectors, respectively. Substituting (15) in (18) we obtain:

$$\boldsymbol{\varepsilon}=\begin{Bmatrix}\boldsymbol{\varepsilon}_m\\ \boldsymbol{\varepsilon}_b\\ \boldsymbol{\varepsilon}_s\end{Bmatrix}=\sum_{l=1}^{n}\sum_{i=1}^{k}\begin{Bmatrix}\mathbf{B}_{m_i}^l\\ \mathbf{B}_{b_i}^l\\ \mathbf{B}_{s_i}^l\end{Bmatrix}\mathbf{a}_i'^l=\sum_{l=1}^{n}\sum_{i=1}^{k}\mathbf{B}_i^l\mathbf{a}'^l \tag{19}$$

where $\mathbf{B}_i^l$ is the generalized strain matrix of node i for the lth harmonic term and $\mathbf{B}_{m_i}^l$, $\mathbf{B}_{b_i}^l$ and $\mathbf{B}_{s_i}^l$ are the generalized strain matrices due to membrane, bending and shear effects respectively. These matrices are given by:

$$\mathbf{B}_{m_i}^l=\begin{bmatrix}\dfrac{\partial N_i}{\partial s}S_l & 0 & 0 & 0 & 0\\ \dfrac{N_i}{r}\sin\phi S_l & -\dfrac{N_i}{r}\gamma S_l & -\dfrac{N_i}{r}\sin\phi S_l & 0 & 0\\ \dfrac{N_i}{r}\gamma C_l & \left(\dfrac{\partial N_i}{\partial s}-\dfrac{N_i}{r}\sin\phi\right)C_l & 0 & 0 & 0\end{bmatrix}$$

$$\mathbf{B}_{b_i}^l=\begin{bmatrix}0 & 0 & 0 & \dfrac{\partial N_i}{\partial s}S_l & 0\\ 0 & 0 & 0 & \dfrac{N_i}{r}\sin\phi S_l & -\dfrac{N_i}{r}\gamma S_l\\ 0 & -\dfrac{\cos\phi}{r}\dfrac{\partial N_i}{\partial s}C_l & 0 & \dfrac{N_i}{r}\gamma C_l & \left(\dfrac{\partial N_i}{\partial s}-\dfrac{N_i}{r}\cos\phi\right)C_l\end{bmatrix}$$

$$\mathbf{B}_{s_i}^l=\begin{bmatrix}0 & 0 & \dfrac{\partial N_i}{\partial s}S_l & N_iS_l & 0\\ 0 & \dfrac{N_i}{r}\cos\phi C_l & \dfrac{N_i}{r}\gamma C_l & 0 & N_iC_l\end{bmatrix} \tag{20}$$

where $\gamma=l\pi/\alpha$ and ϕ is the angle that the strip forms with the global z axis (see *Figure 2*). The resultant stress vector $\boldsymbol{\sigma}$ is related to the generalized strains by:

$$\boldsymbol{\sigma}=\mathbf{D}\boldsymbol{\varepsilon} \tag{21}$$

where

$$\boldsymbol{\sigma}=[N_s, N_t, N_{st}, M_s, M_t, M_{st}, Q_s, Q_t]^{\mathrm{T}} \tag{22}$$

and the constitutive elasticity matrix $\mathbf{D}$ can be written for an isotropic elastic material as:

$$\mathbf{D}=\begin{bmatrix}\mathbf{D}_m & & \mathbf{0}\\ & \mathbf{D}_b & \\ \mathbf{0} & & \mathbf{D}_s\end{bmatrix} \tag{23}$$

where $\mathbf{D}_m$, $\mathbf{D}_b$, and $\mathbf{D}_s$ correspond to membrane, bending and shear effects with:

$$\mathbf{D}_m=\frac{Et}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \dfrac{1-\nu}{2}\end{bmatrix} \tag{24}$$

$$\mathbf{D}_b=\frac{t^2}{12}\mathbf{D}_m \quad \text{and} \quad \mathbf{D}_s=\frac{Et\beta}{2(1+\nu)}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

where E is the Young's modulus, ν the Poisson's coefficient, t the shell thickness and β a coefficient to take into account the warping of the section ($\beta=5/6$ for rectangular sections).

The expression of virtual work (6) can now be written as:

$$\iint_A \delta\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma}r\,\mathrm{d}A=-\iint_A \delta\mathbf{u}^{\mathrm{T}}\mathbf{P}\ddot{\mathbf{u}}r\,\mathrm{d}A \tag{25}$$

where

$$\mathbf{P}=\rho t\begin{bmatrix}1 & & & & \mathbf{0}\\ & 1 & & & \\ & & 1 & & \\ & & & \dfrac{t^2}{12} & \\ \mathbf{0} & & & & \dfrac{t^2}{12}\end{bmatrix} \tag{26}$$

where ρ is the density of the material. Substituting (17), (19) and (23) in (25) and making use of (8) and (9) the uncoupled systems of (10) can be obtained. The stiffness and mass matrices of the strip *in the local coordinate system* are given by:

$$\mathbf{K}_{ij}'^{ll}=\frac{\alpha}{2}\int_0^{a^e}[\bar{\mathbf{B}}_i^l]^{\mathrm{T}}\hat{\mathbf{D}}\bar{\mathbf{B}}_j^l r\,\mathrm{d}s \tag{27a}$$

$$\mathbf{M}_{ij}'^{ll} = \frac{\alpha}{2}\int_0^{a^e} \bar{\mathbf{N}}_i^{\mathrm{T}} \mathbf{P}\bar{\mathbf{N}}_j r \, \mathrm{d}s = \mathbf{M}_{ij}' \tag{27b}$$

$$\mathbf{K}_{ij}'^{lm} = \mathbf{M}_{ij}'^{lm} = 0 \qquad \text{for } l \neq m \tag{27c}$$

where a^e is the width of strip e and $\bar{\mathbf{B}}_i^l$ and $\bar{\mathbf{N}}_i$ are directly obtained from (20) and (17) making $S_l = C_l = 1$. Note that matrix $\mathbf{M}_{ij}'^{ll}$ is independent of the harmonic number l and, therefore, the same matrix can be used for all the different harmonic equations. In (27) the primes mean that the respective matrices are computed in the local coordinate system of each strip. The stiffness matrix of (27a) can be rewritten using (19) and (23) as:

$$\mathbf{K}_{ij}'^{ll} = \frac{\alpha}{2}\int_0^{a^e} ([\bar{\mathbf{B}}_{m_i}^l]^{\mathrm{T}}\mathbf{D}_m\bar{\mathbf{B}}_{m_j}^l + [\bar{\mathbf{B}}_{b_i}^l]^{\mathrm{T}}\mathbf{D}_b\bar{\mathbf{B}}_{b_j}^l + [\bar{\mathbf{B}}_{s_i}^l]^{\mathrm{T}}\mathbf{D}_s\bar{\mathbf{B}}_{s_j}^l) r \, \mathrm{d}s = \mathbf{K}_{m_{ij}}'^{ll} + \mathbf{K}_{b_{ij}}'^{ll} + \mathbf{K}_{s_{ij}}'^{ll} \tag{28}$$

where $\mathbf{K}_{m_{ij}}'^{ll}$, $\mathbf{K}_{b_{ij}}'^{ll}$ and $\mathbf{K}_{s_{ij}}'^{ll}$ are respectively the membrane, bending and shear contributions to the stiffness matrix. The independent evaluation of each of these matrices is very advantageous from the computational point of view.

Assembly of the stiffness and mass matrices, coordinate transformations

To assemble the complete stiffness and mass matrices of the structures from the individual strip matrices, all nodal forces and displacements must be expressed in a common and uniquely defined coordinate system. This can be easily done using the coordinate transformation matrix relating displacement and forces in the local and global systems. We have to note that when several strips meeting at a common node lie in different planes it is necessary to include a third global rotation, θ_z, and bending moment, M_z, for a consistent transformation of displacements and forces from the local to global coordinate axes. Thus, in general, we can write:

$$\mathbf{a}_i^l = \mathbf{T}_i^{(e)}\mathbf{a}_i'^l \tag{29a}$$

and

$$\mathbf{f}_i^l = \mathbf{T}_i^{(e)}\mathbf{f}_i'^l \tag{29b}$$

where

$$\mathbf{a}_i^l = [u_{0_i}^l, v_{0_i}^l, w_{0_i}^l, \theta_{x_i}^l, \theta_{y_i}^l, \theta_{z_i}^l]^{\mathrm{T}} \tag{30a}$$

and

$$\mathbf{f}_i^l = [F_{x_i}^l, F_{y_i}^l, F_{z_i}^l, M_{x_i}^l, M_{y_i}^{lz}, M_{z_i}^l]^{\mathrm{T}} \tag{30b}$$

are the displacement and force amplitude vectors in the global coordinate systems x, y, z (see *Figure 2*). The transformation matrix $\mathbf{T}_i^{(e)}$ of (29a) is given by:

$$\mathbf{T}_i^{(e)} = \begin{bmatrix} \sin\phi^{(e)} & 0 & -\cos\phi^{(e)} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \cos\phi^{(e)} & 0 & \sin\phi^{(e)} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \sin\phi^{(e)} \\ 0 & 0 & 0 & 0 & -\cos\phi^{(e)} \end{bmatrix} \tag{31}$$

After standard transformations the strip stiffness and mass matrices in the global system can be written as:

$$\mathbf{K}_{ij}^{ll} = \mathbf{T}_i^{(e)}\mathbf{K}_{ij}'^{ll}[\mathbf{T}_i^{(e)}]^{\mathrm{T}} \tag{32}$$

$$\mathbf{M}_{ij} = \mathbf{T}_i^{(e)}\mathbf{M}_{ij}'[\mathbf{T}_i^{(e)}]^{\mathrm{T}} = \mathbf{M}_{ij}' \tag{33}$$

Note that in (33) the superscript l has been omitted, due to the constant value of the mass matrix for all the harmonic terms. Equation (32) can be written in a more useful alternative form, using (27a) as:

$$\mathbf{K}_{ij}^{ll} = \frac{\alpha}{2}\int_0^{a^e} [\mathbf{B}_i^{*l}]^{\mathrm{T}}\mathbf{D}\mathbf{B}_j^{*l} r \, \mathrm{d}s \tag{34}$$

where

$$\mathbf{B}_i^{*l} = \bar{\mathbf{B}}_i^l[\mathbf{T}_i^{(e)}]^{\mathrm{T}} \tag{35}$$

Equation (34) can now be expressed in terms of the membrane, bending and shear contributions to matrix $\mathbf{K}_{ij}^{ll}$, similarly as in (28).

We have to point out here that the inclusion of the sixth rotational degree of freedom in the global stiffness equations may lead to singularity of the stiffness matrix if all the strips meeting at a node are laying in the same plane (coplanar node). Such a singularity can be avoided either at the equation solution level or introducing *a priori* a 'spring' coefficient in the stiffness diagonal position corresponding to the extra sixth degree of freedom[12,16]. In the examples presented later in the chapter we have chosen the first procedure consisting in simply not assembling the equations corresponding to the sixth degree of freedom in the coplanar nodes. This avoids the appearance of spurious natural frequencies and vibration modes associated to that degree of freedom, which, due to the coplanarity of the strips, lack any physical meaning.

It is also worth noting here that (27b) leads to a full 'consistent' mass matrix. However, diagonalization of the mass matrix can be easily made following any of the well known existing procedures. For the examples shown later we have used a diagonal mass matrix obtained by taking as diagonal values the sum of the coefficients of each row of the consistent mass matrix of (27b).

Linear strip element with reduced integration

For practical applications of the above strip formulation any of the well known one-dimensional finite elements from the Lagrangean family can be used. However, the success of the Reissner–Mindlin formulation presented for thick and thin shell/plate analyses lies in the use of reduced integration techniques for the numerical computation of the stiffness matrix. This simply implies that the shear terms contributing to the stiffness matrix are numerically integrated with a Gaussian quadrature order—less than that needed for its exact computation, whereas the rest of the stiffness matrix can be exactly calculated. Details of this technique can be found in many references[13,16].

It has been recently shown by Oñate and Suárez[13,14] that the simple two noded linear strip element with *a single integration point* for the numerical computation of all terms of the stiffness matrix has an excellent behaviour for thin and thick plate and shell analysis in comparison with other elements of higher order. In addition, the single point integration implies that a direct explicit form of the element matrices can be directly obtained simply evaluating the terms of all integrals at the element midpoint. Thus we can write from (34) and (33):

$$\mathbf{K}_{ij}^{ll} = \frac{a^e\alpha}{2}([\mathbf{B}_i^{*l}]^{\mathrm{T}}\mathbf{D}\mathbf{B}_j^{*l} r)_0$$
$$\mathbf{M}_{ij} = \frac{a^e\alpha}{2}(\bar{\mathbf{N}}_i^{\mathrm{T}}\mathbf{P}\bar{\mathbf{N}}_j r)_0 \tag{36}$$

where the index 0 implies that all the terms inside the parentheses are evaluated at the strip midpoint. The explicit form of the stiffness and mass matrices is simply obtained noting that in $\mathbf{B}_i^{*l}$ and $\bar{\mathbf{N}}_i$ the terms $(N_i)_0=\frac{1}{2}$ and $(\partial N_i/\partial s)_0=(-1)^i/a^e$.

FINITE STRIP FORMULATION FOR STRAIGHT-FOLDED PLATE STRUCTURES

The general expressions for straight bridges or similar folded plate structures can be directly obtained from those of curved bridges presented in the previous section. In *Figure 1* is shown the geometry of a straight bridge and its discretization in straight strips. All the expressions for the displacement, strain and stress vectors are directly deduced from the corresponding ones of the curved formulation, simply substituting the coordinates s, t and n by x, y, z; the derivatives $\partial/r\,\partial\theta$ by $\partial/\partial y$; the terms A/r (where A is a displacement) by zero and the bridge angle α by the bridge length b.

With the above considerations the membrane, bending and shear generalized strain matrices are obtained from (19) and (20) as:

$$\mathbf{B}_{m_i}^l=\begin{bmatrix} \frac{\partial N_i}{\partial x}S_l & 0 & 0 & 0 & 0 \\ 0 & -N_i\bar{\gamma}S_l & 0 & 0 & 0 \\ N_i\bar{\gamma}C_l & \frac{\partial N_i}{\partial x}C_l & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}_{b_i}^l=\begin{bmatrix} 0 & 0 & 0 & \frac{\partial N_i}{\partial x}S_l & 0 \\ 0 & 0 & 0 & 0 & -N_i\bar{\gamma}S_l \\ 0 & 0 & 0 & N_i\bar{\gamma}C_l & \frac{\partial N_i}{\partial x}C_l \end{bmatrix} \quad (37)$$

and

$$\mathbf{B}_{s_i}^l=\begin{bmatrix} 0 & 0 & \frac{\partial N_i}{\partial x}S_l & N_iS_l & 0 \\ 0 & 0 & N_i\bar{\gamma}C_l & 0 & N_iC_l \end{bmatrix}$$

with

$$\bar{\gamma}=\frac{l\pi}{b}$$

It can be checked that the above matrices can be directly obtained simply making $\gamma=r\bar{\gamma}$ in (20) and r equal to a large number (such that $1/r\to 0$). The uncoupled stiffness and mass matrices for the lth harmonic term are obtained in *local strip* axes as:

$$\mathbf{K}_{ij}'^{ll}=\frac{b}{2}\int_0^{a^e}[\bar{\mathbf{B}}_i^l]^{\mathrm{T}}\mathbf{D}\bar{\mathbf{B}}_j^l\,\mathrm{d}x \quad (38a)$$

and

$$\mathbf{M}_{ij}'=\frac{b}{2}\int_0^{a^e}\bar{\mathbf{N}}_i^{\mathrm{T}}\mathbf{P}\bar{\mathbf{N}}_j\,\mathrm{d}x \quad (38b)$$

In (38a) $\bar{\mathbf{B}}_i^l$ is directly obtained from (37) making $S_l=C_l=1$. All other matrices in (38) are identical to those for the curved formulation.

The transformation of the stiffness and mass matrices to global axes follows exactly the same steps explained earlier for curved bridges. On the other hand, if linear strip elements are used, the explicit form of those matrices are obtained by:

$$\mathbf{K}_{ij}^{ll}=\frac{ba^e}{2}([\mathbf{B}_i^{*l}]^{\mathrm{T}}\mathbf{D}\mathbf{B}_j^{*l})_0 \quad (39a)$$

$$\mathbf{M}_{ij}=\frac{ba^e}{2}(\bar{\mathbf{N}}_i^{\mathrm{T}}\mathbf{P}\bar{\mathbf{N}}_j)_0 \quad (39b)$$

where $\mathbf{B}_i^{*l}$ is obtained by (35) and the index 0 indicates values of the matrices at the strip midpoint. All the remarks indicated for the case of curved bridges are again valid for the straight formulation.

CURVED AND STRAIGHT PLATES

The formulation for curved and straight plates (see *Figure 3*) can be easily derived from the general formulations presented in earlier sections by simply neglecting in all relevant vectors and matrices the contributions of the membrane terms. Thus, the displacement vectors and shape function matrix are now given by:

curved plate   *straight plate*

$$\mathbf{u}=[w_0,\theta_s,\theta_t]^{\mathrm{T}} \qquad \mathbf{u}=[w_0,\theta_x,\theta_y]^{\mathrm{T}} \quad (40)$$

$$\mathbf{a}_i^l=[w_{0_i}^l,\theta_{s_i}^l,\theta_{t_i}^l]^{\mathrm{T}} \qquad \mathbf{a}_i^l=[w_{0_i}^l,\theta_{x_i}^l,\theta_{y_i}^l]^{\mathrm{T}}$$

$$\mathbf{N}_i^l=\begin{bmatrix} N_iS_l & & \mathbf{0} \\ & N_iS_l & \\ \mathbf{0} & & N_iC_l \end{bmatrix}$$

On the other hand, the strain and constitutive matrices are given by:

$$\mathbf{B}_i^l=\begin{Bmatrix}\mathbf{B}_{b_i}^l \\ \mathbf{B}_{s_i}^l\end{Bmatrix} \qquad \text{and} \qquad \mathbf{D}=\begin{bmatrix}\mathbf{D}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_s\end{bmatrix} \quad (41)$$

where the corresponding terms of matrix $\mathbf{B}_i^l$ are given by (20) and (37) for the curved and straight plate formulation, respectively, and matrix $\mathbf{D}_b$ and $\mathbf{D}_s$ are given by (24).

Finally, the *global* stiffness and mass matrices for curved and straight plates are given by (27a) and (38a), respectively. Note, however, that matrix $\mathbf{P}$ in the mass matrix is now given by:

$$\mathbf{P}=\rho t\begin{bmatrix}1 & & \mathbf{0} \\ & \frac{t^2}{12} & \\ \mathbf{0} & & \frac{t^2}{12}\end{bmatrix} \quad (42)$$

AXISYMMETRIC SHELLS

The formulation for axisymmetric shells (see *Figure 4*) follows identically the steps of the curved bridge formulation. The displacement field within each strip can be expressed in terms of the symmetric and antisymmetric contributions of the shell deformation with respect to a meridional plane, as:

$$\mathbf{u}'=\sum_{l=0}^{n}\sum_{i=1}^{k}({}_S\mathbf{N}_i^l\,{}_S\mathbf{a}_i'^l+{}_A\mathbf{N}_i^l\,{}_A\mathbf{a}_i'^l) \quad (43)$$

where $\mathbf{u}'$ is given by (16a) and the first and second terms

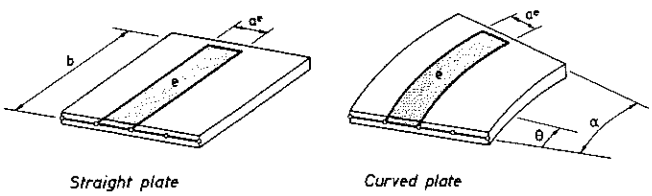

Figure 3 Finite strip discretizations for curved and straight plates

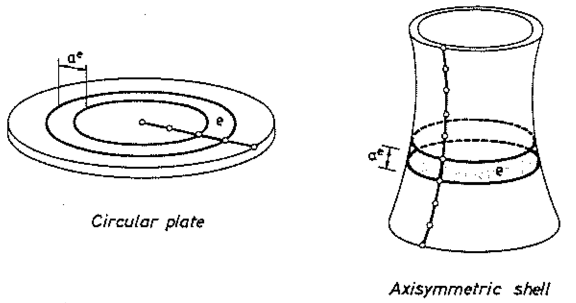

Figure 4 Finite strip discretizations for axisymmetric shells

in the right-hand side of (43) correspond to symmetric and antisymmetric components of the displacement field. Matrix ${}_A\mathbf{N}_i^l$ coincides with (17) and matrix ${}_S\mathbf{N}_i^l$ is obtained directly from ${}_A\mathbf{N}_i^l$ interchanging the terms S_l by C_l and *vice versa*. On the other hand, in this case $S_l = \sin l\theta$ and $C_l = \cos l\theta$. In practice it is usual to take as reference symmetry plane that of $\theta = 0$.

To simplify the computations it is more convenient to compute separately the contributions of the symmetric and antisymmetric displacement fields. An arbitrary deformation can thus be obtained as the sum of its symmetric and antisymmetric parts. On the other hand, it can be noticed that in (43) the harmonic zero has been included. This term has a clear physical meaning and it corresponds to an axisymmetric displacement field.

The expressions for the generalized strain and resultant stress vectors and the constitutive matrix are identical to those studied for the curved bridge case. The only difference in the computation of the stiffness and mass matrices with respect to the curved bridge formulation is that the integrals over the 'length' of the structure are now performed over a whole circumference. The local stiffness matrix of the strip is obtained as:

$$\mathbf{K}_{ij}^{\prime ll} = C \int_0^{a^e} [\bar{\mathbf{B}}_i^l]^{\mathrm{T}} \mathbf{D} \bar{\mathbf{B}}_j^l r \, ds \qquad (44)$$

where C is equal to 2π or π for $l=0$ and $l \neq 0$, respectively.

It is interesting that matrix $\bar{\mathbf{B}}_i^l$ in (44) can be directly obtained from (20) simply making

$$\begin{array}{cc} \textit{symmetric case} & \textit{antisymmetric case} \\ \gamma = -l & \gamma = l \end{array} \qquad (45)$$

On the other hand, the mass matrix for the strip is obtained by (27b) substituting the value of $\alpha/2$ by the constant C of (44).

Finally, the transformation of the stiffness and mass matrices to global axes follows *exactly* earlier steps and they will not be repeated here. Also, if linear strip elements are used an explicit form of both matrices can be obtained similarly as in (36).

EXAMPLES

All the examples shown next have been analysed using the linear strip element with a single integrating point for the evaluation of all terms of the stiffness and mass matrices[13,14].

Circular box bridge

In this example a two cells box bridge of circular plant is analysed. The geometry and material properties of the problem are shown in *Figure 5*. 36 linear strip elements have been used to discretize the transverse section of the structure. Different deformation modes are shown in *Figure 5*. Results for the values of the natural frequencies for the first four harmonics are shown in *Figure 5*, where some numerical results obtained by Cheung[5] have also been plotted.

Cylindrical shell

Details of the geometry and material properties of the shell are shown in *Figure 6*. The curved geometry of the shell has been modelled using 48 (flat) linear strip elements. The first three vibration modes for the first harmonic term are plotted in *Figure 6* where a plot of the natural frequencies for the two first harmonics is also shown. Comparison of results with those obtained by Morris and Dawe using curved strips[17], also plotted in the same figure, is good.

Cylindrical tank with spherical dome

The final example analysed is a cylindrical tank with a spherical dome (*Figure 7*). The meridional section of the structure has been discretized using 15 and 34 strip elements for the cylindrical wall and the dome respectively. Results for the first vibration modes corresponding to the harmonic terms zero and one, have been plotted in *Figure 7*. The natural frequencies corresponding to the first mode for different harmonic terms are presented in *Figure 7* where results obtained by Feijoo *et al.*[18] using curved elements have also been plotted. Good agreement between both sets of results is again obtained.

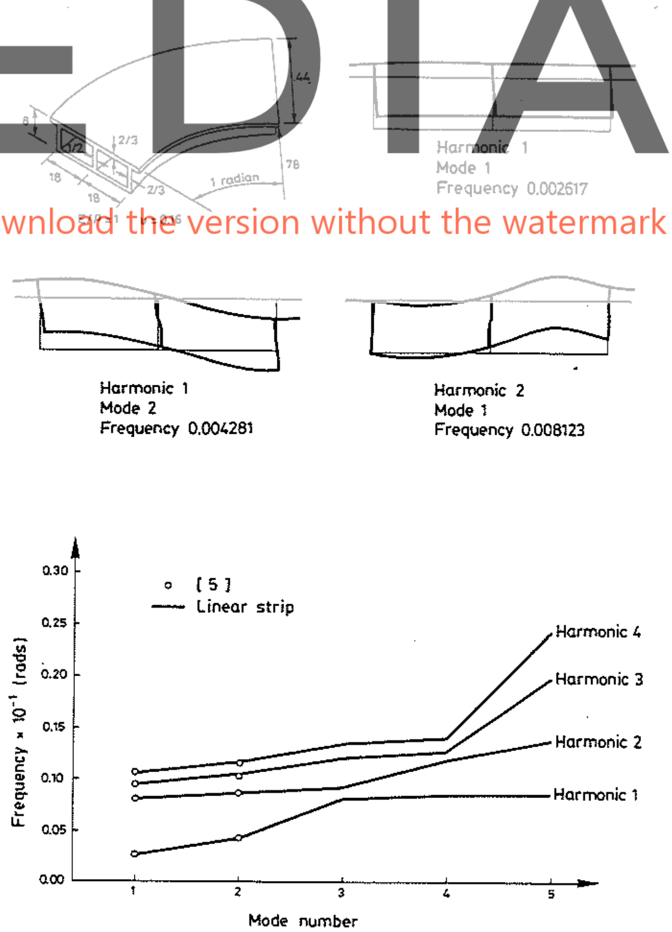

Figure 5 Circular box bridge. Modal amplitudes and frequencies for various harmonic terms

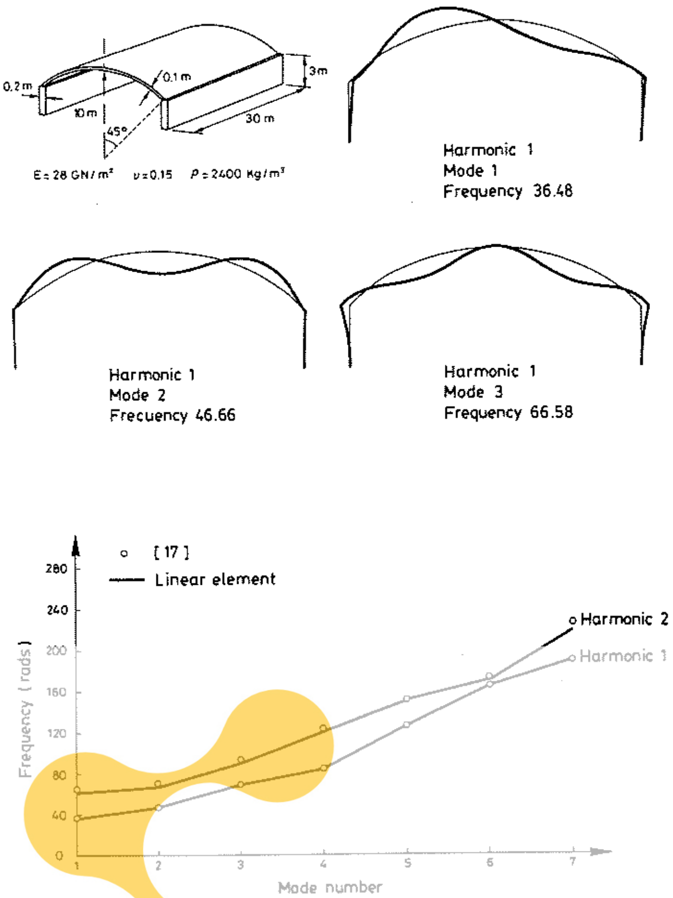

Figure 6 Circular shell. Modal amplitudes and frequencies for different harmonic terms

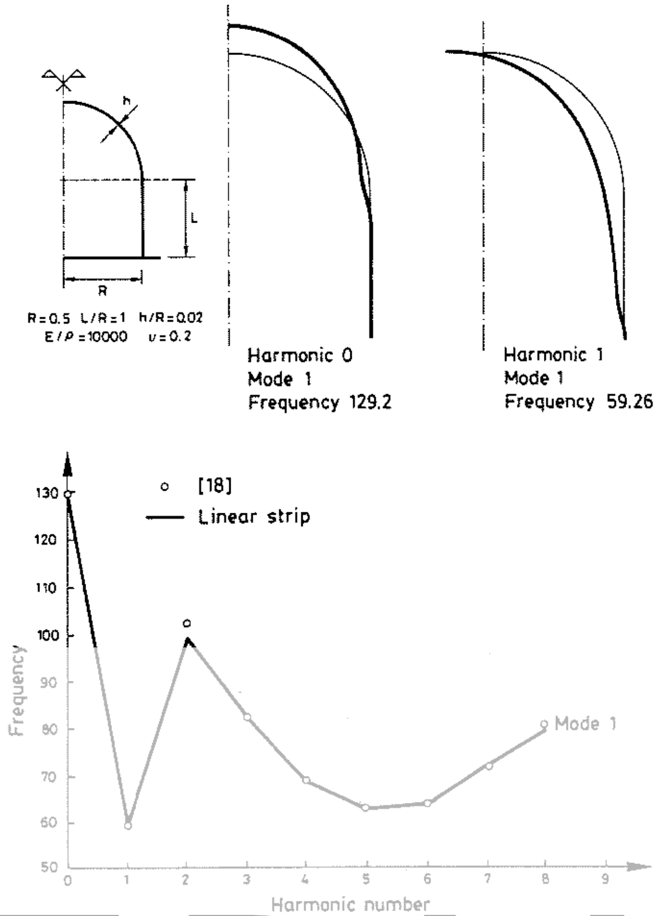

Figure 7 Cylindrical tank with spherical dome. Modal shapes and frequencies for mode 1 for different harmonic terms

CONCLUSIONS

In this paper, a finite strip formulation for free vibration analysis of prismatic shell type structures has been presented. It has been shown how the different finite strip matrices for the analysis of bridges, plates and axisymmetric shells can be easily derived from the general expressions for the circular bridge case. In particular, the single point integrated linear strip allows to obtain simple explicit forms of all matrices and vectors which make the formulation particularly attractive for use in small computers. The examples analysed show the accuracy of the linear strip element for free vibration structural analysis.

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