The Monte Carlo method. Application to the stochastic analysis of sheet stamping processes

J. E. Hurtado[1]
F. Zárate
E.Oñate

1. National University of Colombia, Visiting Scientist at CIMNE
2. CIMNE. Universidad Politécnica de Cataluña

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International Center for Numerical Methods in Engineering
Gran Capitán s/n, 08034 Barcelona, Spain

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Preface

Stochastic Mechanics is a rapidly growing area of research, whose importance is being recognized not only in academic circles but also in industrial practice. This is no doubt due to the fact that most structural properties and loads are either random or uncertain. The first term refers to a natural chaotic variation of the parameter, while the second is associated to the human lack of knowledge about it. Both kinds of unpredictability work together in rendering doubtful the results of a (usually single) deterministic mechanical analysis. When thinking about the randomness and uncertainty linked to all physical parameters and phenomena a big question mark closes the large list of numbers produced by a finite element calculation.

In Stochastic Mechanics there are several techniques to analyse the natural scatter of strains and stresses caused by the dispersion in the given loads and/or the structural parameters. The most general one is the Monte Carlo method. However, it must be recognized that it is as well the most costly in computational terms. Nevertheless, this cost has becoming feasible with the advance in Computer Science, specially with the advent of parallel computing, due to the fact that a Monte Carlo calculation is intrinsically a task that can be performed in parallel.

The present report is intended to provide the reader an introduction to the Monte Carlo method in the context of Computational Mechanics. The technique is applied to the analysis of the uncertainty spread in a stamping process. The first chapter summarises the Monte Carlo method and its theoretical backgrounds. The second chapter is devoted to the case study, namely, the stochastic analysis of a square cup deep drawing problem. Finally, the basic equations governing the mechanical modelling of the stamping process are summarised in the appendix.

Chapter 1

Statistics, Probability and Monte Carlo

1.1 Introduction

This starting chapter of the present report summarizes the main concepts and ideas of the Monte Carlo approach to the stochastic analysis of systems. It is purported to give the reader a rapid introduction to this area. To this end it is necessary (a) to review some basic notions of Statistics and Probability theory and (b) to examine the generation of random numbers, which is the kernel of the Monte Carlo method. these two topics will be developed in what follows.

1.2 Random events

A *random event* is a physical phenomenon whose cause is uncertain, i.e. it is the result of a complex combination of unobserved (and sometimes, unobservable) causes so that it can be considered as unpredictable. For example, the motion of the gas molecules inside a recipient is the result of a large number of their mutual collisions. Despite each particle motion is, in principle, governed by the deterministic differential equations of particle dynamics, its initial conditions, however, cannot be measured. As a consequence, the only possible treatment of the subject is by having resort to probability notions and tools.

Natural phenomena such as earthquakes constitute also a good example. In fact, the physics of earthquake occurrence at the fault and the spreading, reflections and refractions of the waves across the earth layers can in principle be studied with the aid of deterministic physics. But the lack of knowledge of

the actual complex shape of the fault and earth strata, as well of their many physical parameters, suggest that the resulting seismograms be considered as random functions for the design of structures which can be subject to repeated seismic actions in the whole span of their lifetime.

Let us consider a subset A of the random event. For example, if the latter is the peak displacement X of a point of a car riding over an uneven (i.e. random) surface, the subset A can be a special range of peak displacements that is of interest for design. In general, one can group the observed peak displacements into several ranks A_i (also called *bins*) and count the number of observations falling into each of them. The plot of the number of observations versus the location of the rank of the random variable X is the well-known *histogram*, which gives origin to the the so-called *frequency definition* of probability. It states that an event A_i which has been observed to occur N_i times in N observations of the whole random phenomenon, has a probability of occurrence given by

$$P[A_i] = \frac{N_i}{N} \tag{1.1}$$

Besides the histogram, there are other measures of the scatter of the random variable, which convey some valuable statistical information. In first lieu, the *ordinary moments* are useful measures of the shape of the histogram of a random variable. They are given by

$$\nu_n = \frac{\sum_{i=1}^{N}(X_i)^n}{N} \tag{1.2}$$

The most important ordinary moment is the *mean*,

$$\hat{\mu}_X = \frac{\sum_{i=1}^{N} X_i}{N} \tag{1.3}$$

which defines the center of gravity of the histogram. The *variance*

$$\hat{\sigma}_X^2 = \frac{\sum_{i=1}^{N}(X_i - m)^2}{N-1} \tag{1.4}$$

is a measure of the spread of the data about the mean, as illustrated by its similarity with the definition of the moment of inertia in structural mechanics. The *standard deviation* is defined as its square root, $\hat{\sigma}_X$. Perhaps the most practical measure of the randomness of the variable is the *coefficient of variation* defined as

$$\eta_X = \frac{\hat{\sigma}_X}{\hat{\mu}_X} \tag{1.5}$$

so that it expresses the spread as a fraction of the mean. In general, it can be said that low and high coefficients of variation correspond to the values less than 0.05 on the one hand and higher than 0.2 on the other.

The *central moments* are quantities defined with respect to the mean in the form

$$\hat{\nu}_{n,X} = \frac{\sum_{i=1}^{N}(X_i - \hat{\mu}_X)^n}{N} \tag{1.6}$$

In particular, the normalized form of two central moments are of importance in practice. They are the *coefficient of skewness*,

$$\gamma_3 = \frac{\hat{\nu}_3}{\hat{\sigma}_X^3} \tag{1.7}$$

and the *coefficient of excess* (or *kurtosis*)

$$\gamma_4 = \frac{\hat{\nu}_4}{\hat{\sigma}_X^4} - 3 \tag{1.8}$$

The former gives a measure of the symmetry of the histogram, being positive when the dominant tail is on the right and negative in the opposite case. The latter is a measure of the flatness of the histogram, being positive for slim shapes and negative for rather flat ones.

1.3 Density and Distribution

At a difference with Statistics, Probability Theory deals not with observed data but with ideal mathematical models of randomness. Thus, its results can be considered as "exact" in the sense that they correspond to abstract random situations not corrupted by the actual, crude scatter found in nature.

In this conceptual framework, the histogram of a random variable X finds its idealisation in the *Probability Density Function* (abbreviated PDF), $f_X(x)$, which is a mathematical function expressing the concentration and spread of continuous or discrete random variables X as a function of an independent variable x, which plays here the role of the ranks used in statistics. A commonly used model is, for instance, the normal or Gaussian:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X})\mathrm{e}^{-\frac{(x-m)^2}{2\sigma_X^2}} \tag{1.9}$$

whose parameters are just the mean m and the standard deviation σ_X. The vast majority of models, however, have parameters that do not coincide with these two fundamental measures.

The density models are by definition normalized, i.e. their total area equals unity:

$$\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = 1 \tag{1.10}$$

This is a constraint imposed by the fact that the integral of the density function over the whole real axis constitute a sure event, whose probability is by definition equal to one. Random events, on the contrary, are defined over ranges of the variable, as previously noted. For calculating their probabilities, it is necessary to use the *Cumulative Distribution Function* (or CDF) $F_X(x)$, defined as the area of the density function up to a certain x and expressing the probability that the random variable X is less than or equal to x:

$$F_X(x) = P[X \leq x] = \int_{-\infty}^{x} f_X(x)\mathrm{d}x \tag{1.11}$$

As a consequence, the probability of the event $a \leq X \leq b$ is given by

$$P[a \leq X \leq b] = \int_{a}^{b} f_X(x)\mathrm{d}x = F_X(b) - F_X(a) \tag{1.12}$$

1.4 Expected values

An important concept in probability theory is the *expected value of a function* $g(X)$, defined by

$$\mathrm{E}[g(X)] = \int_{-\infty}^{x} g(x) f_X(x)\mathrm{d}x \tag{1.13}$$

meaning that it is an average of the function weighted by the density function. The *ordinary* and *central moments* mentioned before are but particular cases of this general definition. Actually, the *mean* μ_X, corresponds to $g(X) = x$ while the *variance*, σ_X^2 (also denoted as $\mathrm{Var}(X)$), to $g(X) = (x - \mu_X)^2$. Replacing this function into the general definition of the expected value, it can be observed that the variance is also equal to

$$\sigma_X^2 = \mathrm{E}[X^2] - \mu^2 \tag{1.14}$$

The value $\mathrm{E}[X^2]$ is called the *mean square*.

Note that we differentiate the notation of statistical and probabilistic measures by adding a hat to the former to indicate their empirical origin.

1.5 Some Probability models

In this section some probability models commonly applied in stochastic analysis of structures will be briefly described.

Uniform, $\mathrm{U}(a, b)$

The Uniform distribution is usually employed to model situations in which it is reasonable to assign an equal likelihood to all the values of the random variable in a given range. Thus, the density function is constant over the range. A classical example is the roulette, in which the random angle is uniformly distributed in the range $[0, 2\pi]$. The density function is

$$f_X(x) = \frac{1}{b-a} \quad a < x < b \tag{1.15}$$

Integrating, one obtains the distribution function:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases} \tag{1.16}$$

The mean and variance are

$$\mathrm{E}(X) = (a+b)/2 \tag{1.17}$$

$$\mathrm{Var}(X) = \frac{1}{12}(b-a)^2 \tag{1.18}$$

In this respect it is important to remark that most of the algorithms for simulating random variates and processes dealt with in subsequent chapters require the generation of uniform random numbers as a first step. This is perhaps the main practical application of this model in Structural Mechanics.

Gamma, $\mathrm{Gamma}(\nu, m)$

The Gamma density function is

$$f_X(x) = \frac{\nu(\nu x)^{m-1}\mathrm{e}^{-\nu x}}{(m-1)!} \quad x \geq 0 \tag{1.19}$$

The factorial in the denominator can also be expressed as

$$(m-1)! = \Gamma(m) \tag{1.20}$$

where $\Gamma(\cdot)$ is the Gamma function,

$$\Gamma(m) = \int_0^\infty \mathrm{e}^{-u} u^{m-1} \mathrm{d}u \tag{1.21}$$

The new expression for the density function is, then,

$$f_X(x) = \frac{\nu(\nu x)^{m-1}\mathrm{e}^{-\nu x}}{\Gamma(m)} \qquad x \geq 0 \tag{1.22}$$

This substitution allows the extension of the model to the general case of m a positive real number. The names *Erlang* and Gamma are conventionally assigned to the integer and non integer cases, respectively. As will be shown later on, this distinction determines different algorithms for generating random numbers from these models.

The distribution function can be found by integration. The result is

$$F_X(x) = \frac{\Gamma(m, \lambda x)}{\Gamma(m)} \tag{1.23}$$

where $\Gamma(m, y)$ is the *Incomplete Gamma* function,

$$\Gamma(m, y) = \int_0^y \mathrm{e}^{-u} u^{m-1} \mathrm{d}u \tag{1.24}$$

The mean and variance of a Gamma variate are

$$\mathrm{E}(X) = m/\nu \tag{1.25}$$

$$\mathrm{Var}(X) = m/\nu^2 \tag{1.26}$$

An important property of the Gamma distribution that is exploited in random number generation is that the sum of N independent random variables, each being Gamma(ν, m_i), is Gamma(ν, m) distributed, where

$$m = \sum_{i=1}^{N} m_i \tag{1.27}$$

A final remark about this distribution is that the Exponential model is a special case of it making $m = 1$.

Beta, Beta(a, b, q, r)

The Beta distribution is often used as a rough model for cases with scarce information. Its main advantage is the flexibility with which it can be accommodated to the skewness and kurtosis of the empirical information. It has however the disadvantage of being constrained in the range $[a, b]$ so that

it does not allow that the variable tend to infinity as the models with one or two tails do. The density function is

$$f_X(x) = \frac{1}{B(q,r)} \frac{(x-a)^{q-1}(b-x)^{r-1}}{(b-a)^{q+r-1}} \qquad a \leq x \leq b \tag{1.28}$$

where $B(q,r)$ is the *Beta function* given by

$$B(q,r) = \int_0^1 t^{q-1}(1-t)^{r-1}\mathrm{d}t \tag{1.29}$$

The distribution function can only be obtained by numerical methods. The mean and variance are

$$\mathrm{E}(X) = a + \frac{q}{q+r}(b-a) \tag{1.30}$$

$$\mathrm{Var}(X) = \frac{qr}{(q+r)^2(q+r+1)}(b-a)^2 \tag{1.31}$$

The uniform distribution $\mathrm{U}(a,b)$ is equivalent to $\mathrm{Beta}(a,b,1,1)$.

Normal (or Gaussian), $\mathrm{N}(\mu,\sigma)$

The Normal distribution arises as the limiting case (as $n \to \infty$) of a sum of n random variables, which can be either

1. Independent and identically distributed.
2. Independent but not identically distributed, if any of the summands amounts little to the total.
3. Dependent variables with zero correlation between any of them and a limited number of the rest.

These conditions are required by the *Central Limit Theorem*. The importance of the Normal model in Statistics and Probability founds an explanation in the above weak conditions, and especially in the second one. In fact, at a difference with other models resulting from sums of random numbers, the validity of the Normal distribution does not depend on the specific underlying distribution of the summands, provided their total number tends to infinity. It is by no means surprising, then, that the histograms of some random variables such as measurement errors exhibit a Normal pattern, since they can be thought of as the result of the accumulation of many random causes, all of which would show a different distribution if measured.

The equation of the Normal density function is commonly deduced in standard books of Probability by calculating the density function of the sum of an increasing number of independent, identically distributed random variables. The result is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \quad -\infty < x < \infty \tag{1.32}$$

where μ and σ happen to be the mean and standard deviation of the random variable X:

$$\mathrm{E}(X) = \mu \tag{1.33}$$

$$\mathrm{Var}(X) = \sigma^2 \tag{1.34}$$

This fact indicates that the Gaussian model is completely defined by these two low order moments. On the other hand, there is no closed form expression for the distribution function, which must be calculated by numerical integration:

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{-\infty} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \tag{1.35}$$

Both the density and the distribution functions of the so-called *standard (emphor unit) Normal variable*, defined as

$$S = \frac{X-\mu}{\sigma} \tag{1.36}$$

are widely tabulated. The standard Normal variable can be interpreted as the number of standard deviations that X differs from its own mean. Notice that $S \sim \mathrm{N}(0,1)$ and that

$$F_X(x) = P[X \leq x] = P\left[S \leq \frac{x-\mu}{\sigma}\right] = F_S(s) \tag{1.37}$$

where $s = (x-\mu)/\sigma$. Note that he symmetry of the Normal density function implies that

$$F_S(-s) = 1 - F_S(s) \tag{1.38}$$

An additional distinguishing feature of the multivariate Normal is that it is preserved under linear transformations. In other words, if $X_1, X_2, \ldots, X_n$ are Normal variables, then the weighted sum

$$Y = \alpha_1 X_1 + \alpha_2 X_2 \ldots + \alpha_n X_n \tag{1.39}$$

where $\alpha_1, i = 1, 2, \ldots n$, is also Normal. The parameters of the density of Y can be easily calculated by applying the transformation laws. In the specific case of two random variables X_1, X_2 with $\alpha_1 = \alpha_2 = 1$, we have

$$\mu_Y = \mu_{X_1} + \mu_{X_2} \tag{1.40}$$

$$\sigma_Y^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 \tag{1.41}$$

Notice that transformations like that of equation 1.39 are very common in the analysis of linear structures. For instance, if the variables X_i are Gaussian external forces and the α_i are flexibility coefficients, then the displacement Y will be also Gaussian.

Lognormal, LN(μ, σ)

Since the product of random variables $Y = Y_1 Y_2 \ldots Y_n$ can always be transformed to a sum

$$\ln Y = \ln Y_1 + \ln Y_2 + \ldots + \ln Y_n \tag{1.42}$$

it can be concluded that the variable $X = \ln Y$ is normally distributed, according to the Central Limit Theorem. The distribution of Y can then be found by applying the transformation rule for density functions, i.e.

$$f_Y(y) = |\frac{\mathrm{d}g^{-1}(y)}{\mathrm{d}(y)}| f_X(g^{-1}(y)) \tag{1.43}$$

In this case we have $g(x) = \exp(x)$ and $g^{-1}(y) = \ln y$. Since $X \sim \mathrm{N}(\mu, \sigma)$, the result is

$$f_Y(y) = \frac{1}{\sqrt{2\pi} y \sigma} \exp\left[-\frac{1}{2}(\frac{\ln y - \mu}{\sigma})^2\right] \quad y \geq 0 \tag{1.44}$$

As in the case of the Normal distribution, the distribution function can only be specified in implicit form:

$$F_Y(y) = \frac{1}{\sqrt{2\pi} y \sigma} \int_0^\infty \exp\left[-\frac{1}{2}(\frac{\ln y - \mu}{\sigma})^2\right] \quad y \geq 0 \tag{1.45}$$

The mean and standard deviation of a Lognormal variable are

$$\mathrm{E}(Y) = \exp\left(\mu + \frac{1}{2}\sigma^2\right) \tag{1.46}$$

$$\mathrm{Var}(Y) = \mathrm{E}^2(\mathrm{Y})[\mathrm{e}^{\sigma^2} - 1] \tag{1.47}$$

Extreme value distributions

There are several distributions that have been derived for modeling the largest or smallest value of a sample. They are the following:

1. Gumbel, $\mathrm{EX}-\mathrm{I}, \mathrm{l}(\alpha, \beta)$

   It is usually employed for describing the largest value of a sample without any further specification about the characteristics of the variable or the asymptotic behaviour of the density function. Its equations are:

$$f_X(x) = \alpha \mathrm{e}^{-\alpha(x-\beta)} \exp\left[-\mathrm{e}^{-\alpha(x-\beta)}\right] \quad -\infty < x < \infty \tag{1.48}$$

$$F_X(x) = \exp\left[-\mathrm{e}^{-\alpha(x-\beta)}\right] \quad -\infty < x < \infty \tag{1.49}$$

$$\mathrm{E}(X) = \beta + \frac{0.577}{\alpha} \tag{1.50}$$

$$\mathrm{Var}(X) = \frac{\pi^2}{6\alpha^2} \tag{1.51}$$

2. Fréchét, $\mathrm{EX}-\mathrm{II}, \mathrm{l}(\alpha, \beta)$

   This function is used to model the largest value of i.i.d. variables greater than zero. Its basic equations are

$$f_X(x) = \frac{\alpha}{\beta}\left(\frac{\alpha}{\beta}\right)^{\alpha+1} \mathrm{e}^{-(\beta/x)^{\alpha}} \quad x \geq 0 \tag{1.52}$$

$$F_X(x) = \mathrm{e}^{-(\beta/x)^{\alpha}} \quad x \geq 0 \tag{1.53}$$

$$\mathrm{E}(X) = \beta\Gamma\left(1 - \frac{1}{\alpha}\right) \quad \alpha > 1 \tag{1.54}$$

$$\mathrm{Var}(X) = \beta^2\left[\Gamma\left(1 - \frac{2}{\alpha}\right) - \Gamma^2\left(1 - \frac{1}{\alpha}\right)\right] \quad \alpha > 2 \tag{1.55}$$

3. Weibull, $\mathrm{EX}-\mathrm{III}, \mathrm{s}(\alpha, \beta)$

   This function is usually employed to characterize the smallest value of a sample having a distribution of a specified form in the tail. Its equations are the following:

$$f_X(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right] \quad x > 0 \tag{1.56}$$

$$F_X(x) = \begin{cases} 1 - \exp\left[-(\frac{x}{\beta})^{\alpha}\right] & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.57}$$

$$\mathrm{E}(X) = \frac{\beta}{\alpha}\Gamma(\frac{1}{\alpha}) \tag{1.58}$$

$$\mathrm{Var}(X) = \frac{(\beta)^2}{\alpha}\left[2\Gamma(\frac{2}{\alpha}) - \frac{1}{\alpha}\Gamma^2(\frac{1}{\alpha})\right] \tag{1.59}$$

An special case of the Weibull distribution is the *Rayleigh* distribution, defined as $\mathrm{Ray}(\gamma) \equiv \mathrm{EX-III,s}(2, \beta/\sqrt{2})$. Its density function, mean and variance are given by

$$f_X(x) = \frac{x}{\gamma^2} \exp\left[-\frac{1}{2}(\frac{x}{\gamma})^2\right] \quad x \geq 0 \tag{1.60}$$

$$\mathrm{E}(X) = \sqrt{\frac{\pi}{2}}\gamma \tag{1.61}$$

$$\mathrm{Var}(X) = (2 - \frac{\pi}{2})\gamma^2 \tag{1.62}$$

The Exponential distribution can also be derived from the Weibull model by setting $\nu = 1/\beta$ and $\alpha = 1$. On the other hand, the natural logarithm of a Weibull variable obeys a Gumbel model.

1.6 Notions of Statistical Inference

The passage from the empirical observations to the building of mathematical models for the random phenomena is known in statistics as *inference*, i.e. the postulating of a suitable density function and/or its main expected values out of the available information. A detailed treatment of this important subject would occupy an entire volume and thus we will restrain the attention to the following two practical situations: (a) The estimation of the mean and its confidence intervals; (b) The choice and test of a probability model. The latter, in particular, is a necessary step previous to the generation of artificial samples in Monte Carlo simulation.

Confidence interval for the mean. Having a population of observed values, the mean can be estimated by means of equation 1.3, which is repeated here for convenient reference:

$$\hat{\mu}_X = \frac{\sum_{i=1}^{N} X_i}{N} \tag{1.63}$$

We employ the verb *estimate* instead of *calculate* because the true mean of the population remains unknown, as it is also the case of other statistical measures such as the standard deviation, the skewness and, in general, the probability density function. In this regard, it is necessary to say that a basic theorem of statistics demonstrates that the statistical estimates $\hat{\mu}_X$ and $\hat{\sigma}_X^2$ obtained from a population of independent, identically distributed random variables (hereinafter abbreviated as i.i.d.) converge to their exact values μ_X and σ_X^2 as the sample size N approaches infinity. However, it is often the case that N is not so large as to allow one to state that $\mu_X = \hat{\mu}_X$ and $\hat{\sigma}_X = \sigma_X$ with certainty. It is then necessary to estimate a range that contains the true statistical measure with a high level of certainty. These are known as *confidence intervals*, which are explained next with reference to the particular case of the mean value.

It has been said that, in general, the empirical and true means, $\hat{\mu}_X$ and μ_X do not coincide, and the only statement that can be done about them is that $\hat{\mu}_X$ is an unbiased estimate of μ_X. However, a basic theorem of Statistics shows that the random variable

$$Y = \frac{\hat{\mu}_X - \mu_X}{\sigma_X \sqrt{\frac{1}{N}}} \tag{1.64}$$

is normally distributed with zero mean and unit standard deviation. In other words, Y is a standard normal variable. Using equation 1.12, this suggests that the probability that Y lies in the range $[a, b]$ is

$$P[a \leq Y \leq b] = \Phi(a) - \Phi(b) \tag{1.65}$$

where $\Phi(\cdot)$ is the standard normal variable, which is widely tabulated[1], and a and b are values normally symmetric about zero. Using the usual statistical notation, they are defined as

$$a, b = \pm k_{\alpha/2} \tag{1.66}$$

where $k_{\alpha/2}$ is the value that corresponds to a probability

$$\Phi(k_{\alpha/2}) = 1 - \frac{\alpha}{2} \tag{1.67}$$

In such a case equation 1.65 becomes

[1]This function can be easily computed after the error function: $\Phi(x) = 0.5 + 0.5\text{erf}(x/\sqrt{2})$. The error function is provided in all modern computer languages

$$P[-k_{\alpha/2} \leq Y \leq k_{\alpha/2}] = 1 - \alpha \tag{1.68}$$

or equivalently

$$P[\hat{\mu}_X - \frac{\sigma_X k_{\alpha/2}}{\sqrt{N}} \leq \mu_X \leq \hat{\mu}_X + \frac{\sigma_X k_{\alpha/2}}{\sqrt{N}}] = 1 - \alpha \tag{1.69}$$

This points out that the calculation of the confidence interval depends on the value of α, which is known as the *significance level.* Typical values used in statistical analysis are 0.01, 0.05 or 0.1, which can be interpreted in the sense that the probability that the true mean lies in the range specified by equation 1.69 is 0.99, 0.95 or 0.9. Note that the lower the value of α, the wider the confidence interval, so that α cannot be arbitrarily reduced without relaxing the bracketing of the true mean. On the other hand, the higher the value of α, the lower the probability of finding the true mean in the range. These observations indicate the need of a trade-off between the size of the interval and its associated probability of certainty. A value of α frequently used is 0.05.

Choice of a probability model. In contrast to histograms, which show a non smooth shape and are built up only with actually observed values, the mathematical density models provide smooth functions that allow to describe the randomness of the variable over the whole real space, so that the can give estimates of the probabilities corresponding to very large or very low values. Since sometimes values located at the so-called *tails* of the density functions are of importance in that they usually correspond to critical states of the variable, there is a need of passing from the actual measured data to mathematical models, i.e. from statistics to probability theory.

The fitting of probabilistic models, such as normal, lognormal, Weibull, etc. to the observed data is usually performed by postulating some of them and measuring their adequacy by means of the so-called *goodness-of-fit tests.* The most used are the *Kolmogorov-Smirnov, Anderson-Darling* or χ^2 tests. The first of them consists in calculating the maximum absolute distance from the measured and hypothesized cumulative distribution functions, $\hat{F}_X(x)$ and $F_X(x)$. The former is calculated as

$$\hat{F}_X(X_i) = \frac{i}{N} \tag{1.70}$$

where X_i is the $i-$th value of the population sorted in ascending order. The Kolmogorov-Smirnov distance is defined as

$$D = \max_{i=1}^{N} \left[|\hat{F}_X(X_i) - F_X(X_i)| \right] \tag{1.71}$$

The hypothesis that the data correspond to the underlying model $F_X(x)$ is rejected if

$$D > d \tag{1.72}$$

Otherwise it is not rejected. Here d denotes a value depending on the confidence level α and the sample size N. It is given by appropriate statistical tables. Again, a value of 0.05 is commonly used for this kind of analysis. The best model among several tested is that showing the best confidence. Finally, note that in computing the Kolmogorov - Smirnov distance account must be taken of the staircase nature of $\hat{F}_X(X_i)$, so that the selection of the maximum requires the computation at the beginning and end of each stair.

Let us now suppose that the model of a random variable, such as a material property, has been defined on the basis of experimental information. An artificial population of the random variable (also called a *random deviate* or a *realization*) can be generated from the idealized model by means of some techniques known collectively as *random number generation*, which represent the way back from the abstract models to the empirical world. This constitutes the fundamental step of the *Monte Carlo method*, as applied to the simple random variables discussed heretofore. This method constitutes the main subject of the present chapter.

1.7 Multivariate statistics

All the above said in the previous sections corresponds to the case of single random variables that behave independently to each other in the stochastic sense. In case the several random variables are dependent on each other there is a need of building up *multivariate distribution functions* of the type

$$F_{X_1,X_2}(x_1, x_2) = P([X_1 \leq x_1] \bigcap [X_2 \leq x_2]) \tag{1.73}$$

which corresponds to the case of two random variables. The distribution function for more than two variables is defined analogously. It can be shown that any multivariate distributions must satisfy the following restrictions:

$$F_{X_1,X_2,...X_n}(\infty, \infty, ...) = 1$$

$$F_{X_1,X_2,...X_n}(-\infty, -\infty, ...) = 0 \tag{1.74}$$

$$F_{X_1,X_2,...X_n}(\infty, \infty, x_j, \infty, ...) = F_{X_j}, \forall j \tag{1.75}$$

The last equation indicates that the distribution of any variable can be obtained by making the rest of variables tend to infinity. On the other hand,

the *joint density function* is defined as the partial derivative of the multivariate distribution function with respect to the implied variables. In the two dimensional case we have

$$f_{X_1,X_2}(x_1, x_2) = \frac{\partial F_{X_1,X_2}}{\partial x_1 \partial x_2} \tag{1.76}$$

The inverse relationship is then

$$F_{X_1,X_2}(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{U_1,U_2}(u_1, u_2) \mathrm{d}u_1 \mathrm{d}u_2 \tag{1.77}$$

Taking into account the first of equations 1.75, it can be demonstrated that the density of any variable (or *marginal density*) can be obtained by integrating the joint density with respect to the rest of variables:

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2) \mathrm{d}x_2 \tag{1.78}$$

Finally, it must be said that the notion of independence of several random variables is reflected mathematically by the fact that their joint density function is simply the product of the marginal densities. For instance, in the two dimensional case, the independence of variables X and Y is expressed by

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) \tag{1.79}$$

In the general multivariate case, expected values can be defined in a similar manner as in the univariate case. For instance, in the two dimensional case we have

$$\mathrm{E}[g(X, Y)] = \int_{\infty}^{\infty} \int_{\infty}^{\infty} g(x, y) f_{X,Y}(x, y) \mathrm{d}x \mathrm{d}y \tag{1.80}$$

If $g(X, Y) = XY$, its corresponding expectation is the *covariance* (denoted as $\mathrm{Cov}(X, Y)$), which is a measure of the linear relationship existing between X and Y. Its normalized form, called the *correlation coefficient*, is given by

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y}} = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} \tag{1.81}$$

Note that $-1 \leq \rho(X, Y) \leq 1$. A value of $\rho(X, Y)$ close to $+1$ (-1) indicates the existence of an almost linear positive (negative) relationship between X and Y. On the other hand, a negative or positive correlation coefficient close to zero indicate a very poor linear relationship between the implied variables. However, the possibility of nonlinear relationship should not be discarded on such basis, so that it is always necessary to examine the shape of

the plot (X, Y) to understand more deeply their relationship[2]. In Statistics, the correlation coefficient is estimated by

$$\hat{\rho}(X,Y) = \frac{1}{N}\sum_{i=1}^{N}\Big(\frac{X_i - \hat{\mu}_X}{\sigma_X}\Big)\Big(\frac{Y_i - \hat{\mu}_Y}{\sigma_Y}\Big) \tag{1.82}$$

Of great practical importance are the *mean vector* and the *covariance matrix* of a n dimensional set of random variables $\boldsymbol{X} = [X_1, X_2, \ldots, X_n]$. These are given by

$$\boldsymbol{\mu} = \mathrm{E}[\boldsymbol{X}] \tag{1.83}$$

$$\boldsymbol{\Sigma} = \mathrm{E}[(\boldsymbol{X} - \boldsymbol{\mu})^{\mathrm{T}}(\boldsymbol{X} - \boldsymbol{\mu})] \tag{1.84}$$

Note that the entries of the latter are

$$\boldsymbol{\Sigma} = \begin{pmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1, X_2) & \ldots & \mathrm{Cov}(X_1, X_n) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Var}(X_2) & \ldots & \mathrm{Cov}(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(X_n, X_1) & \mathrm{Cov}(X_n, X_2) & \ldots & \mathrm{Var}(X_n) \end{pmatrix} \tag{1.85}$$

At a difference with the univariate case, there are only a few mathematical models for describing the multivariate joint stochastic behaviour. One of the most used is the Normal or Gaussian model, due to the fact that it is completely defined in terms of the moments of the first two orders, as in the univariate case (equation 1.32).

For a set of n random variables, $\boldsymbol{X}$, the multivariate Gaussian density function is given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = (2\pi)^{-n/2}|\boldsymbol{\Sigma}|^{-1/2}\exp\Big[-\frac{1}{2}(\boldsymbol{X} - \boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{X} - \boldsymbol{\mu})\Big] \tag{1.86}$$

It must be noted that a diagonal covariance matrix $\boldsymbol{\Sigma}$ indicates a lack of correlation among the variables. In the Gaussian model it also implies their independence, as defined. In the general multivariate case, however, this particular property of the Gaussian model does not hold, i.e. lack of correlation does not imply independence.

[2]For instance, to a perfect parabola $y = x^2$ corresponds a null correlation coefficient

1.8 The Monte Carlo method

1.8.1 Introduction

Engineering and physical sciences, as well as some social sciences, employ *mathematical models* or *systems*, which describe the (linear or nonlinear) relationship existing between some given variables $\boldsymbol{X}$ and responses $\boldsymbol{Y}$. In many cases the link between the input and output variables is very complex so that it is impossible to find an exact solution and there is a need of using approximate methods. In Structural Mechanics, for example, stresses and deformations are computed from information on loads, material properties and boundary conditions using the finite element method. If at least one of the input variables X_j is random then we have a *stochastic system* in which all the responses will then be also random. For this type of systems the calculation of the exact probabilistic solution by analytical methods can only be achieved in the most simple situations. In Structural Mechanics, this plainly means that for complex systems the probability density function of the variables Y_j cannot be calculated from the density functions of the input variables X_i using the analytical input/output mapping of the theories of random vibration and stochastic finite elements unless by imposing severe restrictions.

The *Monte Carlo method* has been deemed as a mean of coping with the (usually involved) calculation of statistical measures of response random variables $\boldsymbol{Y}$ of a system, which depend on the input ones $\boldsymbol{X}$, some of which are modeled as random, in a very complex way. Instead of feeding the analytical methods with the probability densities of X_i, the idea is to generate populations of each input variable, to calculate the response of the system subject to randomly selected sets of these realisations $[\hat{X}_1, \hat{X}_2, \ldots \hat{X}_N]$ using a deterministic code and to calculate the density functions as well as other relevant statistical measures of the responses Y_j.

It is evident that the method supposes a considerable computational labor which has discouraged many researches and engineers for using it in the past. In fact, one can find some technical literature on stochastic mechanics written before the nineties not recommending its use. However, the fast development of high speed computers occurring continuously nowadays makes easier the application of the method for gaining insight into the sometimes bizarre and surprising random response of stochastic systems, which cannot otherwise be anticipated, with the final aim of making structures more reliable.

1.8.2 Generation of random numbers

Since the methods for generating samples of random numbers corresponding to non uniform distributions rely on the previous generation of a uniform random deviates in the range $(0, 1)$, it is important to mention the way this is performed these days in computers. The most commonly used technique is that known as *linear congruential method*, which generates the random variates X_i sequentially in the form

$$X_{i+1} = aX_i \text{mod} m \tag{1.87}$$

where a is a multiplier and the operation mod$m$ means the removing of m till the residual be less than or equal to m. For instance, the recursion

$$X_{i+1} = 3X_i \text{mod} 7 \tag{1.88}$$

using $X_1 = 1$ as starting value, produces the sequence 1 3 2 6 4 5 1. Notice that m represents also the maximum achievable value, so that the random deviates in the range $(0, 1)$ can be obtained by dividing into m. Also note that m determines the cycle length, i.e. the period of the sequence after which the same sequence is generated anew. This indicates the importance of having a large m for avoiding periodicities in the random stream. A commonly used value of m is $2^{31} - 1$. On the other hand, the following values of a have been recommended (Bratley 1987): 742938285, 950706376, 1226874159, 62089911, 1343714438.

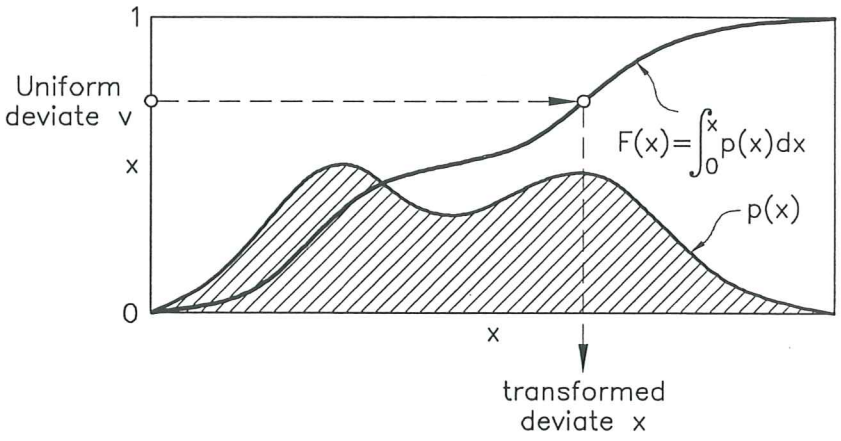

Figure 1.1: Inversion method.

There are basically two methods for generating a random deviate from a given non uniform probability distribution: the *Inversion* and the *Rejection*

methods. In the *Inversion Method* (see figure 1) uniform random numbers u_i are generated in the range $(0, 1)$, so that they can be associated to the values of the distribution function, and the corresponding variates are calculated by inversion of the latter. Since the steep part of the distribution function corresponds to the higher concentration of values, it is evident from the figure that even in case of a regular partition of the $(0, 1)$ probability range the inversion will give a mass of points on the $x-$ axis more crowded in the central part than in the tails. The general algorithm of inversion is the following:

1. Generate $U \sim \mathrm{U}(0, 1)$.
2. Return $X = F^{-1}(U)$.

The main disadvantage of this method is the requirement of inverting the distribution function, for which in some cases (such as the Normal distribution) there is no explicit expression. However, the method is especially suited for the application of some economical sampling techniques, such as Stratified Sampling, whose aim is to diminish the scatter in the statistical estimates of the random variables (i.e. moments an probabilities) with the maximum possible parsimony in the generation of the random number samples. The importance of such economies in Finite Element - based Probabilistic Structural Analysis need not be emphasized.

The *Rejection Method* is based on the following idea (figure 2): a random variate is generated using a fictitious density $g(x)$ that majorizes the target density $f(x)$. Note that its area will be greater than one, because

$$a = \int_{-\infty}^{\infty} g(x)\mathrm{d}x \geq \int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1 \tag{1.89}$$

This indicates that $g(x)$ is not a true density function but also that $h(x) = g(x)/c$ is. The density $h(x)$ is selected in such a way that it be easy to generate random numbers Y satisfying it. Accordingly, one can write the distribution function of X as a conditional probability on Y as follows:

$$P(X \leq x) = P(Y \leq x|A) \tag{1.90}$$

where A denotes the event that Y can accepted as a valid random deviate for X. This can take place if the above conditional probability is the same as that of X:

$$P(Y \leq x|A) = \int_{-\infty}^{x} f(x)\mathrm{d}x \tag{1.91}$$

It is easy to demonstrate that this equation effectively holds. In fact, using the definition of conditional probability, the latter can be expressed as

$$P[Y \leq x|A] = \frac{P(A, Y \leq x)}{P(A)} \tag{1.92}$$

On the other hand the conditional probability $P(A|Y = y)$ is given by

$$P(A|Y = y) = P\Big(U \leq \frac{f(y)}{g(y)}\Big) = \frac{f(y)}{g(y)} \tag{1.93}$$

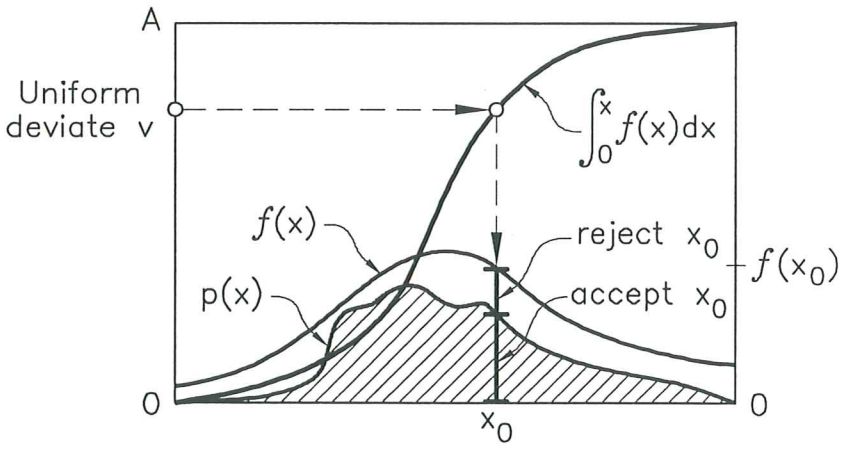

Figure 1.2: Rejection method.

In other words, the probability of accepting Y as a valid X (the event A), given that $Y = y$, is the same as the ratio of the densities of X and Y evaluated at $Y = y$. This implies that if $U \leq \frac{f(y)}{g(y)}$, where U is a uniform random number independent from Y, can be taken as a test whether Y can be accepted as a valid X. The joint probability required in equation 1.92 is then

$$\begin{aligned} P(A, Y \leq x) &= \int_{-\infty}^{x} P(A, Y \leq x|Y = y)h(y)\mathrm{d}y \\ &= \int_{-\infty}^{x} P(A|Y = y)\frac{g(y)}{a}\mathrm{d}y = \frac{1}{a}\int_{-\infty}^{x} f(y)\mathrm{d}y \end{aligned} \tag{1.94}$$

On the other hand

$$P(A) =) = \int_{-\infty}^{\infty} P(A|Y = y)h(y)\mathrm{d}y == \frac{1}{a} \tag{1.95}$$

Substituting these results into equation 1.92 on obtains that equation 1.91 effectively holds. Thus, the algorithm of the rejection method is the following:

1. Generate Y from density $h(x)$.
2. Generate $U \sim \mathrm{U}(0,1)$, independent of Y.
3. If $U \leq f(Y)/g(Y)$, return $X = Y$. Otherwise repeat the process.

Notice that this method depends on the selection of a fictitious density $g(x)$ close to the target one $f(x)$ in order to reduce the number of iterations. a common method of doing this is to specify $h(x)$ to be the Normal density and to calculate the smallest a such that $ah(x) \geq f(x)$. On the other hand, a disadvantage of the rejection method lies in the fact that the final value of a valid U escapes from the control of the analyst. This makes complicated (or, at least, time-consuming) the linking of this method to some powerful algorithms for economical Monte Carlo simulation such as Latin Hypercube or Stratified Sampling described later on.

Finally, it should be added that in many instances methods for generating variates that are specific for each type of density are preferred. For instance, such is the case of the Normal and Gamma distributions. In the following section some specific algorithms for generating random numbers satisfying the density functions most commonly applied in Structural Mechanics are described.

1.8.3 Specific methods for non uniform random number generation

In this section the generation of random numbers after some distributions are briefly revised.

Uniform

The inversion method can be easily applied to generate a random deviate X such that $X \sim \mathrm{U}(a,b)$, i.e

$$F(x) = \frac{1}{b-a}, a \leq x \leq b \tag{1.96}$$

In fact, if U is a uniform random number generated in the range $(0,1)$, then

$$X = F^{-1}(U) = a + (b-a)U \tag{1.97}$$

Therefore, the procedure is the following:

1. Generate $U \sim \mathrm{U}(0,1)$.

2. Deliver $X = a + (b-a)U$.

Erlang and Gamma

These probability models differ in that the parameter m in the expression of the density function

$$f_X(x) = \frac{\lambda(\lambda x)^{m-1}\mathrm{e}^{-\lambda x}}{\Gamma(m)} \qquad x \geq 0 \tag{1.98}$$

is a positive integer or any positive real value in the Erlang and the Gamma cases, respectively. This difference implies that the algorithm for generating random variates from the former model is much simpler than from the latter. In fact, an Erlang random variable X can be written as a sum of m independent, exponentially distributed variables Y_i with common mean $1/\lambda m$. This implies that a realization of X can be constructed in the form

$$X = \sum_{i=1}^{m} Y_i = \sum_{i=1}^{m} -\frac{1}{\lambda k}\ln U_i = -\frac{1}{\lambda k}\ln \prod_{i=1}^{m} U_i \tag{1.99}$$

The algorithm is then the following:

1. Generate $U_i \sim \mathrm{U}(0,1), i = 1, 2, \ldots, m$.

2. Deliver $X = -\frac{1}{\lambda k}\ln \prod_{i=1}^{m} U_i$

The generation of Gamma deviates is a good example of the application of the rejection method. First of all, let us point out that if $Y \sim \mathrm{Gamma}(m,1)$ a variate $X \sim \mathrm{Gamma}(m,\lambda)$ can be obtained as $X = Y/\lambda$. Thus we can restrict our attention to the case of $\lambda = 1$. For $0 < m < 1$ the majorizing function is

$$g(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^{m-1}}{\Gamma(m)} & 0 \leq x \leq 1 \\ \frac{\mathrm{e}^{-x}}{\Gamma(m)} & 1 < x \end{cases} \tag{1.100}$$

The area under this function is $a = b/[m\Gamma(m)]$, in which $b = (\mathrm{e}+m)/\mathrm{e}$. the sampling density is, then,

$$h(x) = \begin{cases} 0 & x \leq 0 \\ \frac{mx^{m-1}}{b} & 0 \leq x \leq 1 \\ \frac{m\mathrm{e}^{-x}}{b} & 1 < x \end{cases} \tag{1.101}$$

and the corresponding distribution is

$$H(x) = \int_0^x h(x)\mathrm{d}x = \begin{cases} \frac{x^m}{b} & 0 \le x \le 1 \\ \frac{1-me^{-x}}{b} & 1 < x \end{cases} \tag{1.102}$$

The inversion of this function yields

$$H(u)^{-1} = \begin{cases} (bu)^{\frac{1}{m}} & u \le \frac{1}{b} \\ -\ln\frac{b(1-u)}{m} & \text{otherwise} \end{cases} \tag{1.103}$$

The basic criterion for the acceptance or the rejection of the deviate generated from distribution $H(x)$ is the ratio $f(x)/g(x)$, as stated previously. It is given in this case by

$$\frac{f(x)}{g(x)} = \begin{cases} \exp(-x) & 0 \le x \le 1 \\ x^{m-1} & 1 < x \end{cases} \tag{1.104}$$

According to this, an algorithm for generating $Y \sim \text{Gamma}(m, 1)$ when $0 < m < 1$ is the following:

1. Generate $U_1, U_2 \sim \text{U}(0, 1)$. If $V = bU_1 > 1$, skip step 2.
2. Set $Z = V^{\frac{1}{m}}$. If $U_2 \le \exp(-Z)$, deliver $Y = Z$. Otherwise go back to step 1.
3. Set $Z = -\ln\frac{b-V}{m}$. If $U_2 \le Z^{m-1}$, deliver $Y = Z$. Otherwise go back to step 1.

In case $m > 1$ an efficient algorithm uses as majorizing function the following:

$$h(x) = \begin{cases} \frac{\beta m^\beta x^{\beta-1}}{(m^\beta + x^\beta)^2} & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.105}$$

which is linked to function $g(x)$ by the constant $a = 4m^m\exp(-m)/[\beta\Gamma(m)]$. The sampling distribution function is

$$H(x) = \begin{cases} \frac{x^\beta}{m^\beta + x^\beta} & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.106}$$

whose inverse is

$$H(u)^{-1} = \left(\frac{m^\beta u}{1-u}\right)^{\frac{1}{\beta}} \tag{1.107}$$

Normal distribution

The generation of random numbers from the Normal distribution faces the problem of the absence of an explicit equation giving the distribution function as well as its inverse. Hence, the inversion method can only be applied by approximating the latter using numerical techniques. Due to the importance of the Normal distribution in statistical analysis, more direct methods have been developed to generate Normal random deviates making use of some statistical principles. One of the most popular methods of this kind is the Box-Muller method, which has the advantage that can be linked to the variance reduction and optimal sampling algorithms explained in the following chapter, whose applications is crucial for the sake of simulation economy in Probabilistic Mechanics. The algorithm implies the generation of pairs of uniform numbers U_1 and U_2, which are transformed to the Normal numbers $Z_1, Z_2 \sim \mathrm{N}(0,1)$ by the following relationships:

$$\begin{aligned} Z_1 &= -2\ln(U_1)^{\frac{1}{2}}\cos 2\pi U_2 \\ Z_2 &= -2\ln(U_1)^{\frac{1}{2}}\sin 2\pi U_2 \end{aligned} \tag{1.108}$$

For the general case of a variable $X \sim \mathrm{N}(\mu, \sigma)$ the linear transformation

$$X = \sigma Z + mu \tag{1.109}$$

must be applied.

The joint normality of Z_1 and Z_2 can be easily demonstrated. In fact, setting $V = -\ln U_1$ and $U = U_2$, the above pair satisfy the following relationships:

$$\begin{aligned} Z_1^2 + Z_2^2 &= 2V \\ \frac{Z_2}{Z_1} &= \tan 2\pi U \end{aligned} \tag{1.110}$$

The Jacobian of the transformation is

$$J = \left\| \begin{matrix} \frac{\partial u}{\partial z_1} & \frac{\partial u}{\partial z_2} \\ \frac{\partial v}{\partial z_1} & \frac{\partial v}{\partial z_2} \end{matrix} \right\| = \left\| \begin{matrix} -\frac{z_2 \cos^2 2\pi u}{2\pi z_1^2} & \frac{\cos^2 2\pi u}{2\pi z_1} \\ z_1 & z_2 \end{matrix} \right\| = -\frac{1}{2\pi} \tag{1.111}$$

Therefore

$$f_{Z_1,Z_2}(z_1, z_2) = |J| f_{U,V}(u, v) = \frac{1}{2\pi}\exp\Big(-\frac{z_1^2 + z_2^2}{2}\Big) \tag{1.112}$$

which is the joint density of two independent normal variables having zero mean and unit standard deviation. Since uncorrelated Gaussian variables are also uncorrelated, the algorithm is especially suitable for the generation of the random process known as *Gaussian white noise*, which requires the latter property (see Chapter 5). The algorithm is, then,

1. Generate two independent variates $U_1, U_2 \sim \mathrm{U}(0,1)$.

2. Deliver Z_1, Z_2 given by the above equations.

Figure 1.3 shows some variables generated from a standard Normal function.

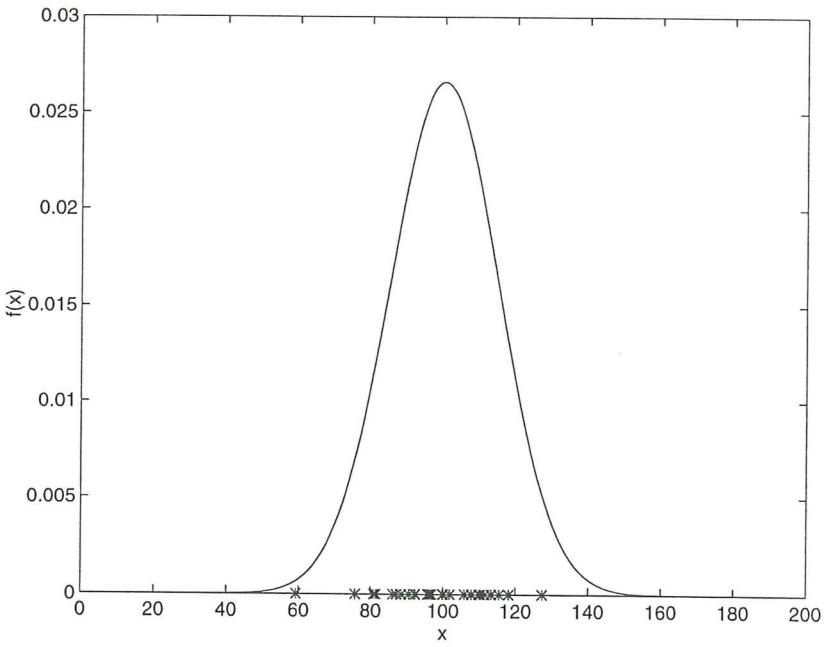

Figure 1.3: Random Normal variates.

Lognormal distribution

On the basis of the relationship existing between the Normal and Lognormal distributions it is possible to derive a simple algorithm for the generations of Lognormal deviates. In fact, if $Z \sim \mathrm{N}(\mu, \sigma)$, then $\exp Y \sim \mathrm{LN}(\mu, \sigma)$. Consequently, the algorithm is the following:

1. Generate $Z \sim \mathrm{N}(\mu, \sigma)$.

2. Deliver $X = \exp Y$.

Notice that μ and σ are not the mean and standard deviation of X but merely parameters. Their true values are given by

$$\mu_X = \exp(\mu + \frac{\sigma^2}{2}),\ \sigma_X = \exp(\mu + \frac{\sigma^2}{2}) \tag{1.113}$$

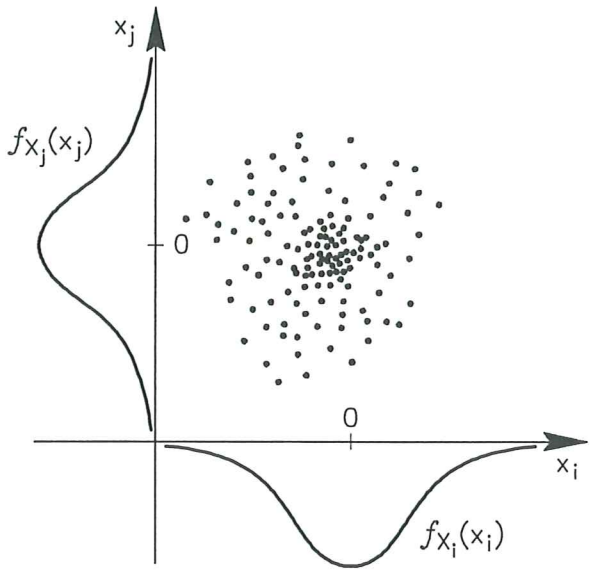

Figure 1.4: Simple random sampling.

1.8.4 Combination of random numbers

The random numbers of all the input variables, generated according to the above quoted techniques, should be combined to produce the samples of the output ones. The algorithm using the crude random numbers of the input variables and combining them haphazardly without any constraint is known as *Simple Random Sampling* . In this case no optimization is applied to the population of input data with the aim of reducing its size without sacrifying the quality of the statistical description of the structural behaviour. Figure 1.4 shows a typical cloud of input points obtained by this technique. Note that in spite of the uniform generation of the probabilities, there is a high concentration of points around the mean values, which is in contrast to the sparseness appearing towards the tails. For a Monte Carlo analysis this would

imply that many analysis would be performed with very close data in the mean zone, with the consequence that their results would be rather similar. While this method can be fast for reliability analysis of single structural components or even structures with a reduced number of degrees of freedom, it is too costly for analyzing large structures. In such a case the repetition of calculations with close data should be avoided by sampling the probabilities of the input data in a controlled (less random) way.

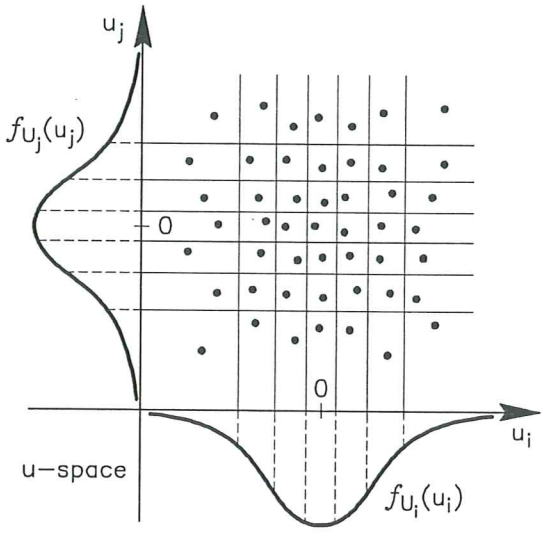

Figure 1.5: Stratified sampling.

The following are some popular methods for sample reduction:

1. Stratified Sampling

   In order to sample the whole range of the input variables, it has been proposed to that the whole space of each variable be divided into subsets of equal probability. Then an outcome is generated from each subset and the analysis is performed with the corresponding sets of points.

2. Latin Hypercube Sampling (Iman and Canover 1980; Bazant and Liu 1985) This method represents an even more drastic reduction of the sampled population because each subset number (identified with a Latin letter, whence the name of the algorithm) of each random variable is combined with other subset numbers of the rest of the variables

only once in a randomized way. A variant of the method has been proposed (Florian 1992). In it a Cholesky factorization is applied to the correlation matrix of the permuted ranks of the input variables, in order to diminish the relatively high correlation resulting from the drastic reduction of the population with respect to the conventional method.

3. Descriptive Sampling (Ziha 1995). This method is quite similar to the previous one, the difference between them lying in the way of generating the permutation matrix of the ranks. Figures 1.5 and 1.6 illustrate the way in which samples are generated by Stratified Sampling and the other two methods. It is important to say that in all of them the sample inside a rank is taken either from the its middle or randomly from it.

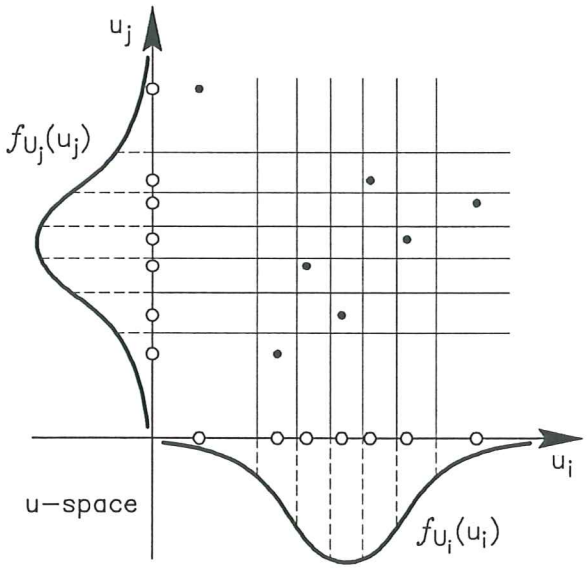

Figure 1.6: Descriptive sampling.

Chapter 2

Stochastic stamping analysis

2.1 Introduction

Sheet metal forming is one of the most widely used process in manufacturing. The goals of the sheet metal forming process are to minimize the time and cost of the manufacturing process while optimizing the quality of the parts. The deformation modes of metal sheet, during forming process are very complex. Without properly designed tools and process parameters, final products may be subjected to various types of defects such as wrinkles, fractures and dimensional inaccuracies. A great deal of time and money in industry is thus consumed by finding appropriate tool geometries and manufacturing parameters by trial and error, whereby physical experiments have to be performed and tools are repeatedly modified according to the experimental results.

In order to overcome the expensive trial and error practice, the simulation technique based on FEM becomes popular to optimize the tool design i.e. to minimize time and cost in the design phase. Such simulation is most useful and efficient when it is performed in the early stage of the design by designers rather than by analysis specialists after the detailed design is complete. It is however clear how little information is conveyed by classical single-shot analysis and how this information may be often misleading. The necessity to reach higher level of predictability, is pushing engineers to introduce scatter as integrating part of their everyday practice.

As stated in the introduction to this report, uncertainty and scatter constitute important physical dimensions of almost any engineering problem which often become the driving mechanism behind the behavior and performance of complex large systems. It is however of paramount importance to identify the dominating parameters early in the design phase since this will have direct repercussions on the subsequent engineering cost.

Friction plays an important role in sheet metal forming process and is therefore of major importance for reliable finite element simulations of these processes. For these simulations a frequently used friction model is the Coulomb model in which the friction coefficient is constant parameter. From a wider point of view, this is not satisfying because the friction coefficient depends on local contact conditions like pressure, combined surface roughness of the sheet and the tools, lubricant viscosity and sum velocity of the surfaces.

During sheet forming process, the friction between sheet and tools has an active role in affecting the material flow, the strain distribution and the forming force. It also takes part in determining the forming failures (e.g. wrinkling, tearing and surface distortion) and the location of these failures. Hence, an accurate simulation of the sheet forming process requires a detailed understanding of frictional behavior under actual forming condition. Since the frictional force is known to be a complicated function of material properties and process parameters, it is unrealistic to use a single test to accurately represent the sheet metal forming process.

In the frictional problem, two cases can be distinguished. At the early stage of the process where a full stick condition between the sheet and tools is verified, a tangential force appears opposing to the relative slip. At this phase the active friction coefficient is the so-called static one. Once a threshold value in the modulus of the force is reached a slip condition is verified, thus the dynamic friction coefficient becomes dominating.

In this chapter a classical problem in stamping analysis is examined from the stochastic perpective. After a brief description of the modelling details, the results will show the relevance of this kind of approach.

2.2 Case study

2.2.1 Problem description

The square cup deep drawing problem is one of the most practical industrial application of sheet stamping. This problem has been the subject of much research and represents a benchmark test that was proposed by the NUMISHEET'93. The NUMISHEET conferences are designed to offer engineers and scientist the opportunity to compare state-of-the art simulations with careful experimental data, each contributed by the leading sheet forming research and production groups world-wide.

The benchmark test reported herein features an the ability to simulate a large amount of material draw-in with a nearly vertical wall and the ability

to efficiently treating the frictional contact. It consists of the analysis of a deep drawing process of a square sheet into a square cup. The emphasis of the chapter, however, lies on the discussion of the implications of the stochastic viewpoint adopted in the present study. The probabilistic analysis was performed via Monte Carlo simulation, briefly summarized in the previous chapter, by means of the computer code PROMENVIR (CASA, 1997), which is a general PRObabilistic MEchanical design ENVIRonment. This environment allows the calculation of a Monte Carlo task in several computer machines at a time using the same deterministic solver installed on each of them (i.e., coarse-grain parallelism) under the control of a master node. The generality of the program lies in its solver-independence. In other words, it does not need any interface with the determinsitic solver, as the input and output variables are simply highlighted on the ASCII files on which they appear. Once this is done, it is only needed to give the probabilistic information of the input random variables (i.e., the density function $f_X(x)$ and its parameters), as well as the number of samples, the constituents of the paralel virtual machine and other details. The execution can be stopped and reinitiated at any time if needed. Finally, the statistics of the output variables can be visualised on-line as the solution progresses and can also be exported to other programs via ASCII files.

2.2.2 Finite element model

Geometry

The geometries of the punch, die and blankholder are shown in Figure 2.1. For this deep drawing problem, the punch travel is $40mm$.

Mesh The sheet was modelled using triangular shell layered elements. The

blankholder, punch and die were modelled using rigid elements. The initial geometry of the tools as well as the sheet geometry after deformations were complete are shown in Figures 1.2 and 1.3.

Materials

The sheet material was mild steel using a non-linear hardening elasto-plastic model with the following properties: Young's modulus $E = 206\text{GPa}$, Poisson's ratio $\nu = 0.3$, mass density $\rho = 7800\text{kg/m}^3$ and the initial yield

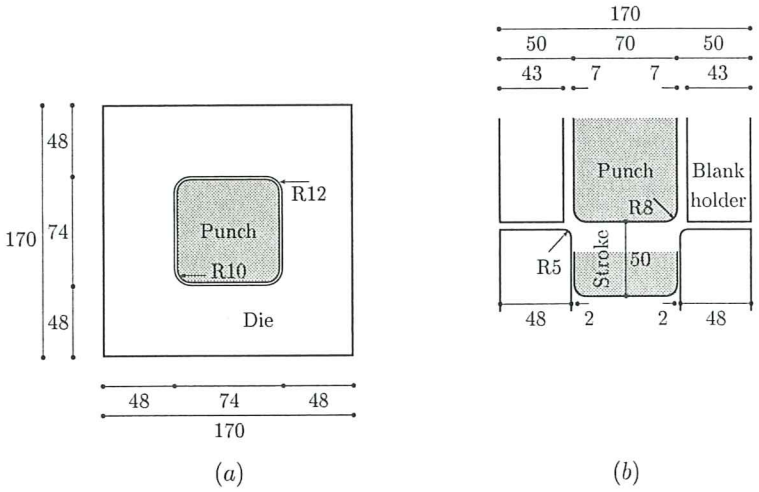

Figure 2.1: Square cup deep drawing. (a): Upper view. (b): Frontal view

value C=565.3MPa. The blankholder material was steel using a linear elastic model. with the following properties: Young's modulus $E = 210\text{GPa}$, Poisson's ratio $\nu = 0.3$ and mass density $\rho = 7800\text{kg/m}^3$.

Boundary conditions

The die was held in place while the punch was moved vertically. Contact pair surfaces between the tools and sheet, die - sheet and blankholder - sheet contact pairs, were modelled using a constant friction coefficient. Contact pair surface punch - sheet was modelled using different values of friction coefficient.

Loading

A constant blankholding force of $19.6kN$ was applied to the surface of the blankholder as a surface load on those elements. The punch was moved vertically for $40mm$ by applying a sinusoidal velocity over the total analy-

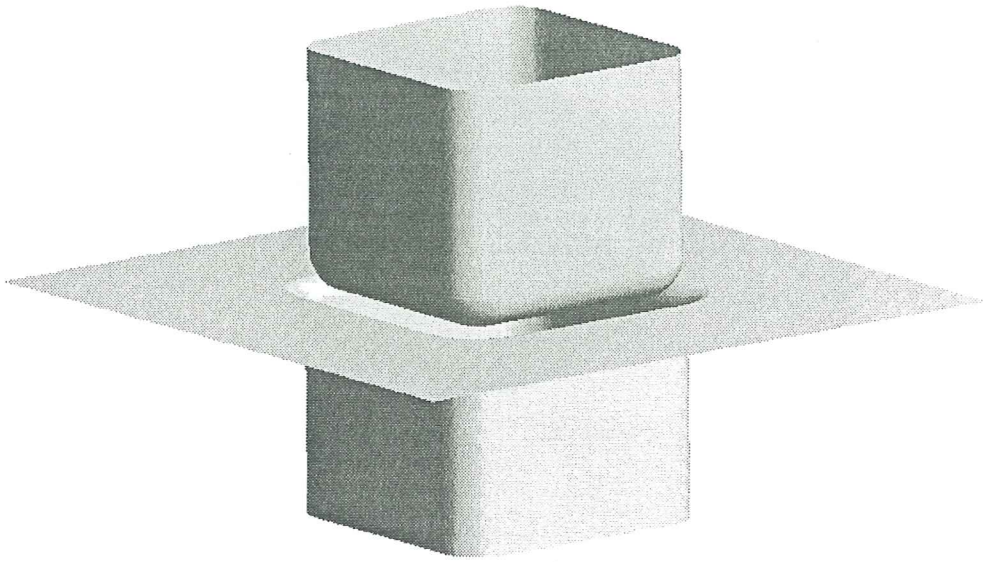

Figure 2.2: Initial geometry of the tools.

sis time period. The total response time was obtained using an automatic time stepping calculation and was terminated when the punch stroke was completed. Material non-linearities were considered to be rate independent.

2.2.3 Probabilistic model

A crucial stage in any probabilistic problem is the selection of the random variables together with their probability density functions. In the present case, the static and dynamic friction coefficients were selected as the input random variables, as they are those appearing as the most uncertain of all the implied variables. Since knowledge of the probabilistic information regarding these variables is rather scarce, it was assumed that they obey a lognormal distribution, due to the fact that they are always positive. The parameters of the density function, which is repeated here for convenient reference,

$$f_Y(y) = \frac{1}{\sqrt{2\pi}y\sigma} \exp\left[-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2\right] \quad y \geq 0 \tag{2.1}$$

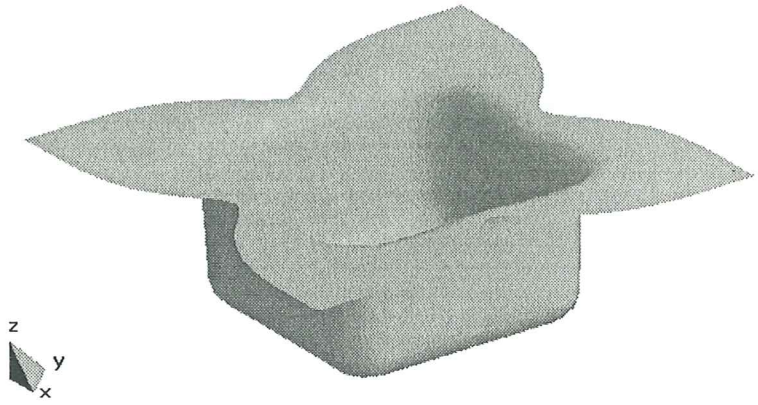

Figure 2.3: End sheet state.

were set equal to $\mu = 0.753$ and $\sigma = -2.45569$. The mentioned random input values are applied on the contact surface pair sheet-punch and were limited to be in the range 0.05 to 0.3, as the PROMENVIR code allows working with truncated probabilistic distributions. The output random variables selected are the minimum, maximum and medium thicknesses of the sheet.

The deterministic Finite Element code STAMPACK (CIMNE, 1997) was used as the work-horse for performing the Monte Carlo analyses under the control of the PROMENVIR environment. The STAMPACK code uses the new finite elements described in Cendoya (1996), Oñate *et al.* (1997), Duffet *et al.* (1997), Oñate *et al.* (1999). The number of Monte Carlo samples was 150, which were run at CIMNE on a single SGI processor, Origin 2000 computer, in 4 hours and 45 minutes.

2.3 Results and discussion

As said before, the large amount of information given by PROMENVIR can be visualised either on real time or after the completion of the whole Monte Carlo task. For the present case it is summarized in the following.

Histogram of inputs

The histograms of the input random variables are displayed as a data check and should coincide with the assumed lognormal distribution. As an illustration, the histogram for the dynamic friction coefficient is presented in Figure 2.4. Notice that as an effect of the truncation just mentioned, the mean value in the histogram should not coincide with that of the theoretical one used for the derivation of the parameters ξ and λ.

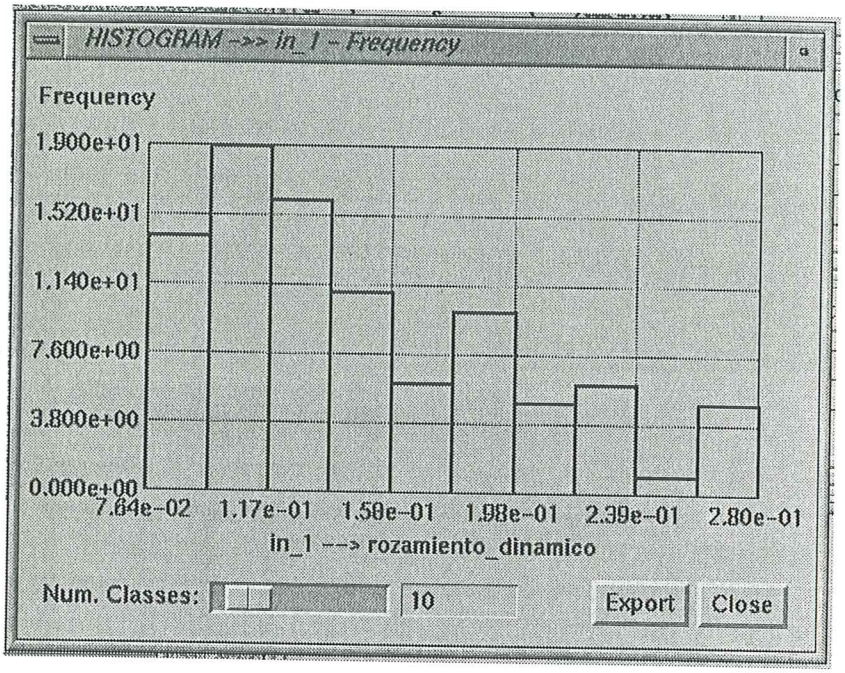

Figure 2.4: Histogram of the dynamic friction coefficient.

Mean

The mean μ_X of the minimum thickness was used as a measure of the convergence of the solution with respect to the requested sample size (see Figure 2.5). This information is of great engineering relevance since it provides directly the most likely value of the minimum thickness. PROMENVIR also computes two important additional curves, defining the $5-95\%$ confidence interval of the mean (see Chapter 1). It can be observed in the figure that the both the mean and the confidence interval curves stabilize at a number of samples roughly equal to 50.

Standard deviation

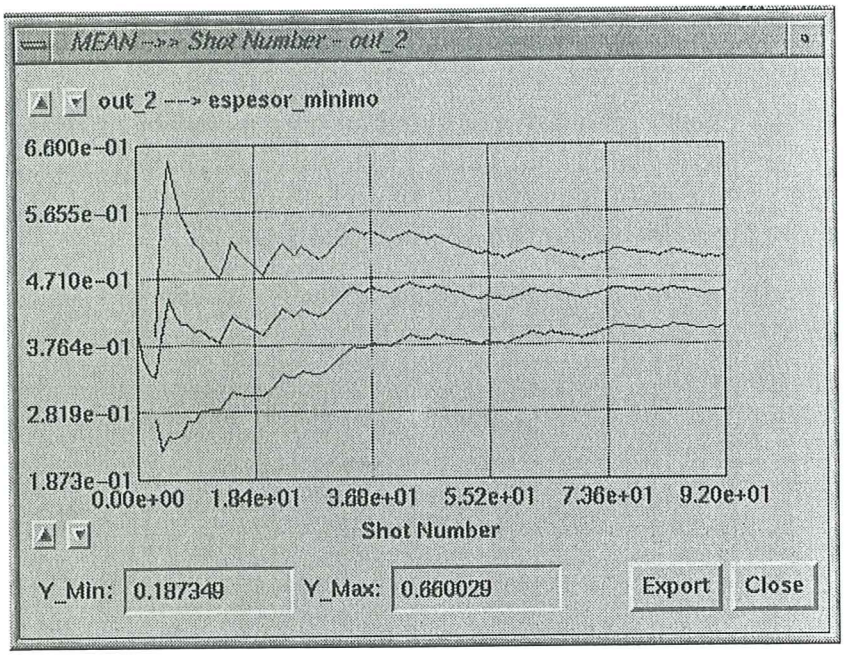

Figure 2.5: Minimum thickness - Mean value.

The standard deviation σ_X of the minimum thickness is computed in order to assess the amount of scatter which, in its turn, is a reflection of the robustness of the product. It is plotted in Figure 2.6. In general, large scatter is undesired since it can lead, under unfavorable circumstances, to unexpected failures. As explained in the first chapter, a convenient Quality Assurance measure is the coefficient of variation, given by

$$\eta = \frac{\sigma_X}{\mu_X} \tag{2.2}$$

Two additional curves that define the width of the confidence interval of the standard deviation are also plotted in the figure. Once more, the width of the confidence interval appears as stable after approximately 50 shots.

Thickness' histogram

The histogram of the most important output random variable, i.e. minimum thickness, is displayed in Figure 2.7. The values greater then 0.6 are considered to be in the safety zone. It is clear from the histogram that there are three clusters of possible values, i.e. around 0.28, 0.42 and 0.70. The same information may be inferred from the ant-hill plot in Figure 2.10. This result

clearly reflects the relevance of the stochastic regard in stamping analysis. As a mater of course, the existence of several clusters (which correponds to what is called *multimodality* in Statistics) implies that a single deterministic analysis is far from being a sufficient basis for assuring the quality of this highly nonlinear problem. Moreover, the existence of a large cluster falling into the cathegory of unsafety (as it is the case in the present example) shows the risk of having a large percent of products not matching the quality standards which cannot be detected by conventional deterministic calculations.

Cumulative Distribution Function

The Cumulative Distribution Function (CDF) of the minimum thickness is displayed in Figure 2.8. Again, the CDF clearly reflects the existence of the three clusters of thickness. From this figure it can be concluded that the probability of having unsafe products (under the assumed distributions for the friction coefficients) is very high (about 71 %).

Ant-hill plot

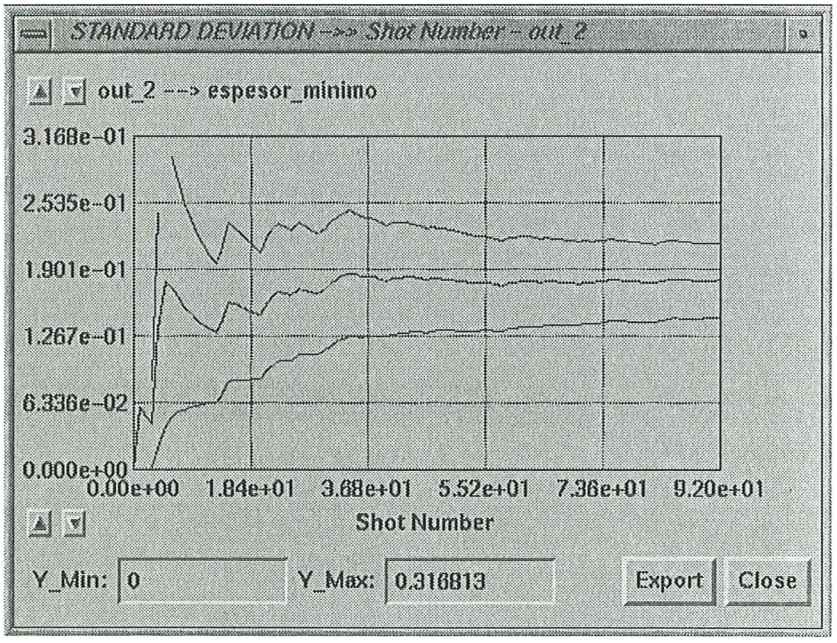

Figure 2.6: Minimum thickness - Standard deviation.

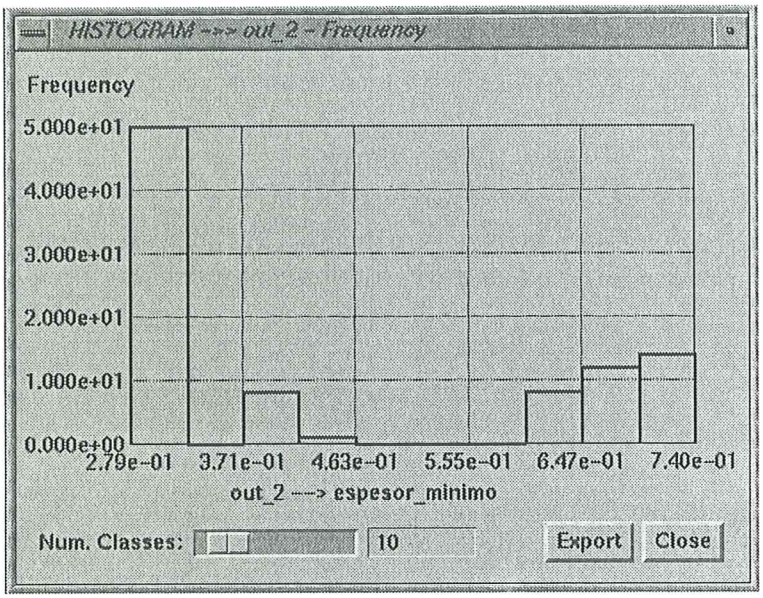

Figure 2.7: Histogram of the minimum thickness.

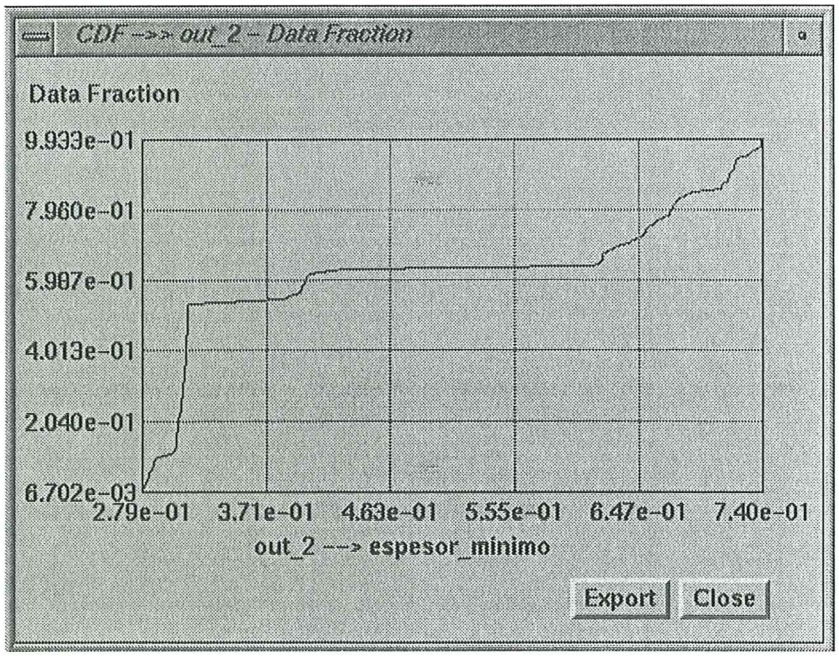

Figure 2.8: CDF of the minimum thickness.

This PROMENVIR functionality plots the relationship between two random variables. The term ant-hill is due to the characteristic shape that the plot assumes when the two variables are uncorrelated. The two random variables are maximum and minimum thicknesses, presented in Figure 2.9. The left bottom cloud shows clearly that this is a risky zone where all products have fallen with some possible concentration of material and the right top cloud is a safety zone. The correlation coefficient of value $\rho = 0.98$ indicates that there is a strong association between maximum and minimum thicknesses. This means, for example, that if the minimum thickness increases, so will the maximum thickness. However, the figure warns the danger of interpreting this statement too loosely, as it is evident that there is a deadzone in which the minimimum nor the maximum thicknesses take values.

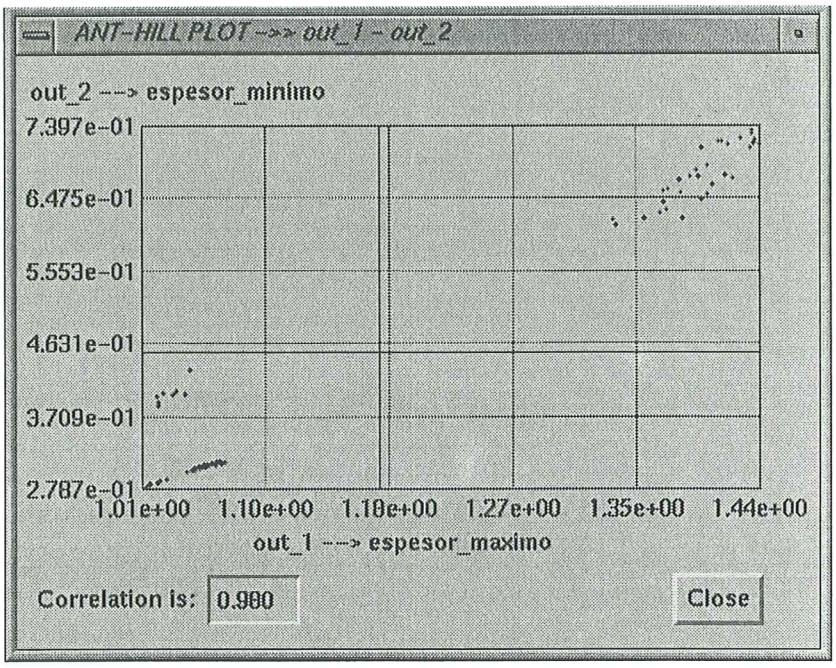

Figure 2.9: Ant-hill plot - maximum versus minimum thickness.

Figure 2.10 presents ant-hill plot of the dynamic friction versus the minimum thickness. It has been especially interesting to investigate how far the dynamic friction coefficient can be considered as responsible for the successful performance of the metal forming process. The left top cloud is in a safety zone and the recommendable dynamic friction coefficient is less than 0.117, if this variable could ever be under control. Greater values of dynamic friction coefficient lead to the forming failures. The correlation coefficient between

two random variables is -0.712. After the comparison of the results obtained from the stochastic simulation, it becomes clear that the influence of the dynamic friction coefficient is very high and therefore has to be measured and treated with care in the present context.

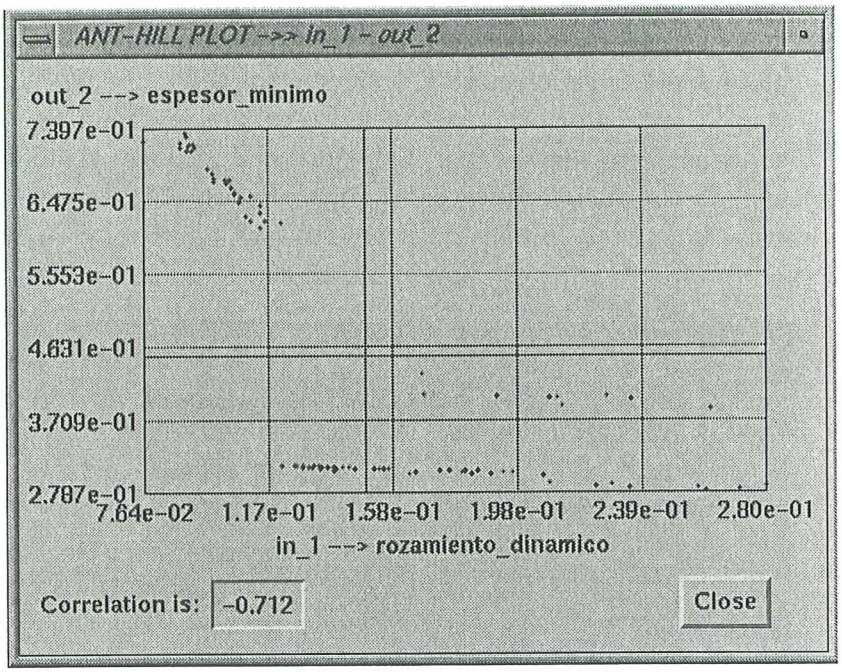

Figure 2.10: Dynamic friction coefficient vs. minimum thickness.

Figure 2.11 presents the ant-hill plot of the static friction coefficient versus minimum thickness. The correlation coefficient is -0.052 - a value that reflects an almost total independence of the minimum thickness on the static friction coefficient. The reason for this is that the static friction really acts as such for a very short time. In fact, once the material starts to flow, it is the dynamic friction that controls the response. Therefore the temporal correlation of the static friction coefficient with the minimum thickness decays very rapidly and the entire phenomenon appears as independent on this particular variable.

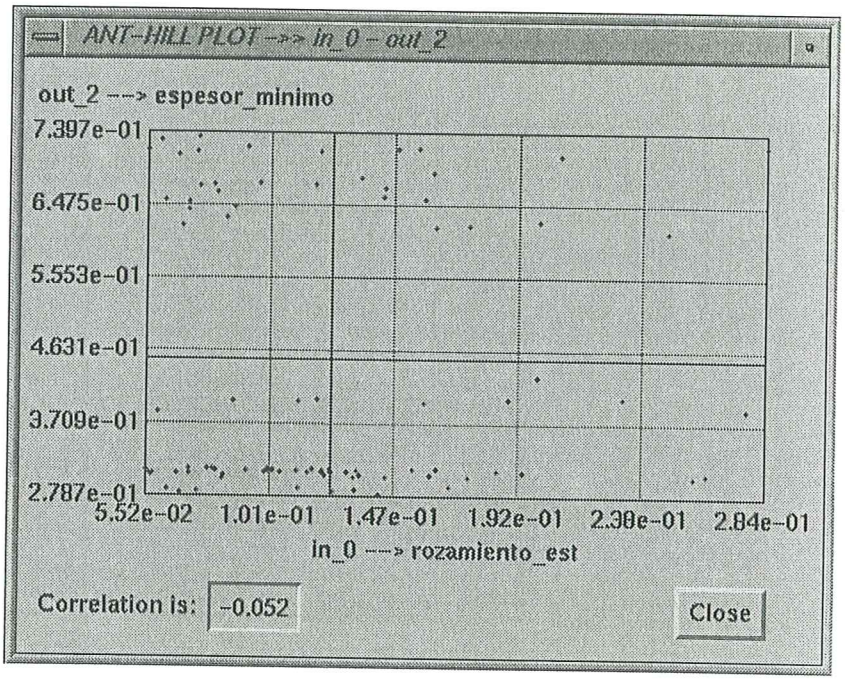

Figure 2.11: Static friction coefficient vs. minimum thickness.

Appendix A

Mechanics of the stamping process

This appendix summarizes the main concepts and equations involved in the mechanical analysis of an stamping problem, as it is needed for a more complete description of the study reported herein.

As is well known, this mechanical problem must be dealt with in both Eulerian (or spatial) and Lagrangian (or material) coordinate system, denoted respectively as $\boldsymbol{X}$ and $\boldsymbol{x}$ in the sequel. The displacement field is defined either in the material and spatial systems as

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t) \\ \boldsymbol{X} &= \boldsymbol{x} - \boldsymbol{u}(\boldsymbol{X}, t) \end{aligned} \tag{A.1}$$

Other important concepts that must be taken into account are those concerning of the *deformation gradient tensor*, defined in tensor notation as

$$F_{iK} = \frac{\partial x_i}{\partial X_K} \tag{A.2}$$

In terms of the tensor $\boldsymbol{F}$ the *left* and *right Cauchy-Green tensors*, $\boldsymbol{P}$ and $\boldsymbol{Q}$ are defined as

$$\begin{aligned} P_{ij}^{-1} &= X_{K,i} X_{K,j} \\ Q_{KL} &= x_{i,K} x_{i,L} \end{aligned} \tag{A.3}$$

These tensors allow the definition of the *Green-Lagrange* $\boldsymbol{E}$ and the *Almansi* $\boldsymbol{\epsilon}$ strain tensors, which are used in the following section, as

$$\boldsymbol{E} = \frac{1}{2}\big(\boldsymbol{Q} - \boldsymbol{I}\big)$$
$$\boldsymbol{\epsilon} = \frac{1}{2}\big(\boldsymbol{I} - \boldsymbol{P}^{-1}\big) \tag{A.4}$$

Other notations used are introduced on due places.

A.1 Equations of motion

In analyzing a stamping problem it is necessary to consider both the equation of motion in spatial coordinates and material coordinates. The first one reads

$$\nabla_{\boldsymbol{x}}\boldsymbol{\sigma} + \rho\boldsymbol{b} = \rho\boldsymbol{a}, \qquad \boldsymbol{x} \in \Omega^t \tag{A.5}$$

with boundary and initial conditions, given respectively by

$$\begin{aligned} \boldsymbol{n}\cdot\boldsymbol{\sigma} &= \boldsymbol{t}, && \boldsymbol{x} \in \Gamma^t \\ \boldsymbol{u} = \boldsymbol{u}_0, \boldsymbol{v} &= \boldsymbol{v}_0, , && \boldsymbol{x} \in \Omega^t \end{aligned} \tag{A.6}$$

In material coordinates the equation of motion is

$$\nabla_{\boldsymbol{x}}\boldsymbol{T} + \rho_0\boldsymbol{b}_0 = \rho\boldsymbol{A}, \qquad \boldsymbol{x} \in \Omega^0 \tag{A.7}$$

with the following boundary and initial conditions:

$$\begin{aligned} \boldsymbol{N}\cdot\boldsymbol{T} &= \boldsymbol{T}_\Gamma, && \boldsymbol{x} \in \Gamma^0 \\ \boldsymbol{u} = \boldsymbol{u}_0, \boldsymbol{v} &= \boldsymbol{v}_0, , && \boldsymbol{x} \in \Omega^0 \end{aligned} \tag{A.8}$$

where $\boldsymbol{T}$ is the first stress tensor of Piola-Kirchoff and $\boldsymbol{N}$ is a unit normal vector. The weak form of the equilibrium equations is

$$\begin{aligned} \int_{\Omega^t} \boldsymbol{\sigma} : \delta\boldsymbol{\epsilon}\cdot \mathrm{d}\Omega &= \int_{\Omega^t} \rho(\boldsymbol{b}-\boldsymbol{a})\delta\boldsymbol{u}\cdot \mathrm{d}\Omega + \int_{\Gamma^t} \boldsymbol{t}\delta\boldsymbol{u}\cdot \mathrm{d}\Omega \\ \int_{\Omega^0} \boldsymbol{T} : \delta\boldsymbol{F}\cdot \mathrm{d}\Omega &= \int_{\Omega^0} \rho_0(\boldsymbol{b}_0-\boldsymbol{A})\delta\boldsymbol{u}\cdot \mathrm{d}\Omega + \int_{\Gamma^0} \boldsymbol{T}\delta\boldsymbol{u}\cdot \mathrm{d}\Omega \\ \int_{\Omega^0} \boldsymbol{S} : \nabla\times\delta\boldsymbol{E}\cdot \mathrm{d}\Omega &= \int_{\Omega^0} \rho_0(\boldsymbol{b}_0-\boldsymbol{A})\delta\boldsymbol{u}\cdot \mathrm{d}\Omega + \int_{\Gamma^0} \boldsymbol{T}\delta\boldsymbol{u}\cdot \mathrm{d}\Omega \end{aligned} \tag{A.9}$$

where $\boldsymbol{T}$ and $\boldsymbol{S}$ are the first and second Piola-Kirchoff tensors, respectively, $\boldsymbol{F}$ is the gradient tensor, $\boldsymbol{E}$ is the Green-Lagrange tensor and $\boldsymbol{\epsilon}$ the Almansi strain tensor. The constitutive equations are those of an hypoelastic material (Khan and Huang, 1995). The form of the constitutive tensor depends on the modeling of the material as anisotropic, orthotropic or isotropic.

A.2 Finite element equations

Using an updated Lagrangian formulation, the first of the equations (A.10), written in matrix form, is

$$\int_{\Omega^t} \delta \boldsymbol{u}^{\mathrm{T}} \rho \ddot{\boldsymbol{u}} \cdot \mathrm{d}\Omega + \int_{\Omega^t} \delta \boldsymbol{\epsilon}^{\mathrm{T}} \cdot \boldsymbol{\sigma} \cdot \mathrm{d}\Omega - \int_{\Omega^t} \delta \boldsymbol{u}^{\mathrm{T}} \cdot \rho \boldsymbol{b} \mathrm{d}\Omega - \int_{\Gamma^t} \delta \boldsymbol{u}^{\mathrm{T}} \boldsymbol{t} \cdot \mathrm{d}\Omega = 0 \quad \text{(A.10)}$$

where

$$\begin{aligned} \boldsymbol{\sigma} &= [\sigma_{11}\sigma_{22}\sigma_{33}\sigma_{12}\sigma_{13}\sigma_{23}]^{\mathrm{T}} \\ \boldsymbol{\epsilon} &= [\epsilon_{11}\epsilon_{22}\epsilon_{33}2\epsilon_{12}2\epsilon_{13}2\epsilon_{23}]^{\mathrm{T}} \end{aligned} \quad \text{(A.11)}$$

the conventional elasticity definitions of the components of $\boldsymbol{\epsilon}$ being applied. Supposing that the kinematics of body is compatible with the displacement field, the following finite element discretization is adopted:

$$\boldsymbol{u}(X) = \boldsymbol{N}(X)\boldsymbol{r} \quad \text{(A.12)}$$

where $\boldsymbol{N}(X)$ is the shape function matrix and $\boldsymbol{r}$ is the vector of nodal displacements. The first variation of the displacement field and that of the Green-Lagrange tensor are

$$\delta \boldsymbol{u}(X) = \boldsymbol{N}(X)\delta \boldsymbol{r} \delta \boldsymbol{\epsilon}(X) = \boldsymbol{B}(X)\delta \boldsymbol{r} \quad \text{(A.13)}$$

where $\boldsymbol{B}$ is the deformation matrix, given in this case by

$$\boldsymbol{B} = \begin{pmatrix} \left(\frac{\partial x}{\partial X_1}\right)^{\mathrm{T}} \left(\frac{\partial N}{\partial X_1}\right) \\ \left(\frac{\partial x}{\partial X_2}\right)^{\mathrm{T}} \left(\frac{\partial N}{\partial X_2}\right) \\ \left(\frac{\partial x}{\partial X_3}\right)^{\mathrm{T}} \left(\frac{\partial N}{\partial X_3}\right) \\ \left(\frac{\partial x}{\partial X_1}\right)^{\mathrm{T}} \left(\frac{\partial N}{\partial X_2}\right) + \left(\frac{\partial x}{\partial X_2}\right)^{\mathrm{T}} \left(\frac{\partial N}{\partial X_2}\right) \\ \left(\frac{\partial x}{\partial X_1}\right)^{\mathrm{T}} \left(\frac{\partial N}{\partial X_3}\right) + \left(\frac{\partial x}{\partial X_3}\right)^{\mathrm{T}} \left(\frac{\partial N}{\partial X_3}\right) \\ \left(\frac{\partial x}{\partial X_2}\right)^{\mathrm{T}} \left(\frac{\partial N}{\partial X_3}\right) + \left(\frac{\partial x}{\partial X_3}\right)^{\mathrm{T}} \left(\frac{\partial N}{\partial X_3}\right) \end{pmatrix} \quad \text{(A.14)}$$

The application of this discretization leads eventually to a matrix differential equation of the form

$$\boldsymbol{M}\ddot{\boldsymbol{r}} + \boldsymbol{F}^{\text{int}} - \boldsymbol{F}^{\text{ext}} = \boldsymbol{0} \tag{A.15}$$

where the matrices $\boldsymbol{M}$, $\boldsymbol{F}^{\text{int}}$ and $\boldsymbol{F}^{\text{ext}}$ result from the assemblage of the element matrices

$$\begin{aligned} \boldsymbol{M}^{(e)} &= \int_{\Omega^e} \rho \boldsymbol{N}^{\mathrm{T}} \boldsymbol{N} \mathrm{d}\Omega \\ \boldsymbol{F}^{\text{int},(e)} &= \int_{\Omega^e} \boldsymbol{B}^{\mathrm{T}} \boldsymbol{\sigma} \mathrm{d}\Omega \\ \boldsymbol{F}^{\text{ext},(e)} &= \int_{\Omega^e} \boldsymbol{N}^{\mathrm{T}} \rho \boldsymbol{b} \mathrm{d}\Omega + \int_{\Gamma^e} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{t} \mathrm{d}\Gamma \end{aligned} \tag{A.16}$$

Since the mass matrix appearing in the above differential equation is not diagonal, thus making the solution of the problem cumbersome, a modification of the system is in order. To this purpose, an equivalent solution of the form

$$\boldsymbol{M}^*\ddot{\boldsymbol{r}} + \boldsymbol{C}\dot{\boldsymbol{r}} + \boldsymbol{F}^{\text{int}} + \boldsymbol{F}^{\text{cont}} - \boldsymbol{F}^{\text{ext}} = \boldsymbol{0} \tag{A.17}$$

where $\boldsymbol{M}^*$ is a diagonal mass matrix and $\boldsymbol{C}$ is a damping matrix, which is introduced in order to reduce the vibrations. the term $\boldsymbol{F}^{\text{cont}}$ is also introduced to represent the contact force.

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